Mathematical Background

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The Polynomial Remainder Theorem

We will develop the constructive proof briefly mentioned in the wikipedia article on polynomial remainder theorem.

Let

$$f(x) \quad = \quad a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_1\, x + a_0$$

be a polynomial with coefficients in a ring, and let r be any element of the ring. Then there exists a polynomial $Q(x)$ of degree at most $(n-1)$ such that

$$f(x) \quad = \quad (x-r)\; Q(x) \;+\; f(r).$$

The remainder of the division of $f(x)$ by the linear polynomial $(x-r)$ is exactly $f(r)$.

Proof

The polynomial and its evaluation at r

Let

$$f(x) \quad = \quad a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

Evaluate $f(r)$:

$$f(r) \quad = \quad a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0.$$

The difference $f(x) - f(r)$

$$\begin{aligned}
f(x) - f(r) &= (a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0) - (a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0). \\
&= (a_n x^n - a_n r^n) + (a_{n-1} x^{n-1} - a_{n-1} r^{n-1}) + \ldots + (a_1 x - a_1 r) + (a_o - a_0) \\
&= \underbrace{a_n (x^n - r^n)}_{\text{TERM } n} + \underbrace{a_{n-1} x^{n-1} - r^{n-1}}_{\text{TERM } n-1} + \;\cdots + \underbrace{a_1 (x - r)}_{\text{TERM } 1}
\end{aligned}$$

Recall the algebraic identity

$$\begin{aligned}
x^k - r^k &= (x-r)(x^{k-1} + x^{k-2} r + \cdots + x r^{k-2} + r^{k-1}) \quad \textit{for an integer } k \geq 1. \\
a_k (x^k - r^k) &= a_k\, (x-r)\, (x^{k-1} + x^{k-2} r + \cdots + x r^{k-2} + r^{k-1}) \quad \textit{for } k = 1, \ldots, n.
\end{aligned}$$

Apply the identity to each term in $f(x) - f(r)$

Consider the term $a_n(x^n - r^n)$ in the above equation, and substitute

$$a_k(x^k - r^k) = (x - r)\, a_k(x^{k-1} + x^{k-2}r + \cdots + r^{k-1}) \quad 1 \leq k \leq n.$$

Rewriting the terms in (5) gives

$$\begin{array}{lcl} a_n(x^n - r^n) & = & (x - r)\, a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) \\ a_n(x^{n-1} - r^{n-1}) & = & (x - r)\, a_{n-1}(x^{n-2} + x^{n-3}r + \cdots + r^{n-2}) \\ \vdots & & \\ a_1(x - r) & = & (x - r)\, a_1 \end{array}$$

Factor out $(x - r)$.

$$\begin{aligned} f(x) - f(r) \quad &= (x - r)\, a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) \\ &\quad +(x - r)\, a_{n-1}(x^{n-2} + x^{n-3}r + \cdots + r^{n-2}) \\ &\quad + \cdots + (x - r)\, a_1. \end{aligned}$$

Factor $(x - r)$ from the entire sum (the right-hand side expression):

$$f(x) - f(r) = (x - r)\Big(a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) + \cdots + a_1\Big).$$

Define the polynomial $Q(x)$ and Complete the Proof

Let

$$Q(x) = a_n(x^{n-1} + x^{n-2}r + \cdots + r^{n-1}) + a_{n-1}(x^{n-2} + x^{n-3}r + \cdots + r^{n-2}) + \cdots + a_1.$$

where

$$\deg(Q) \leq n - 1$$

We use polynomial Q, and write

$$f(x) - f(r) = (x - r)\, Q(x).$$

from which it follows that

$$f(x) = (x - r)\, Q(x) + f(r).$$

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