NDA SHAURYA

FOR NDA 2, 2024

MATHEMATICS

Lecture - 01

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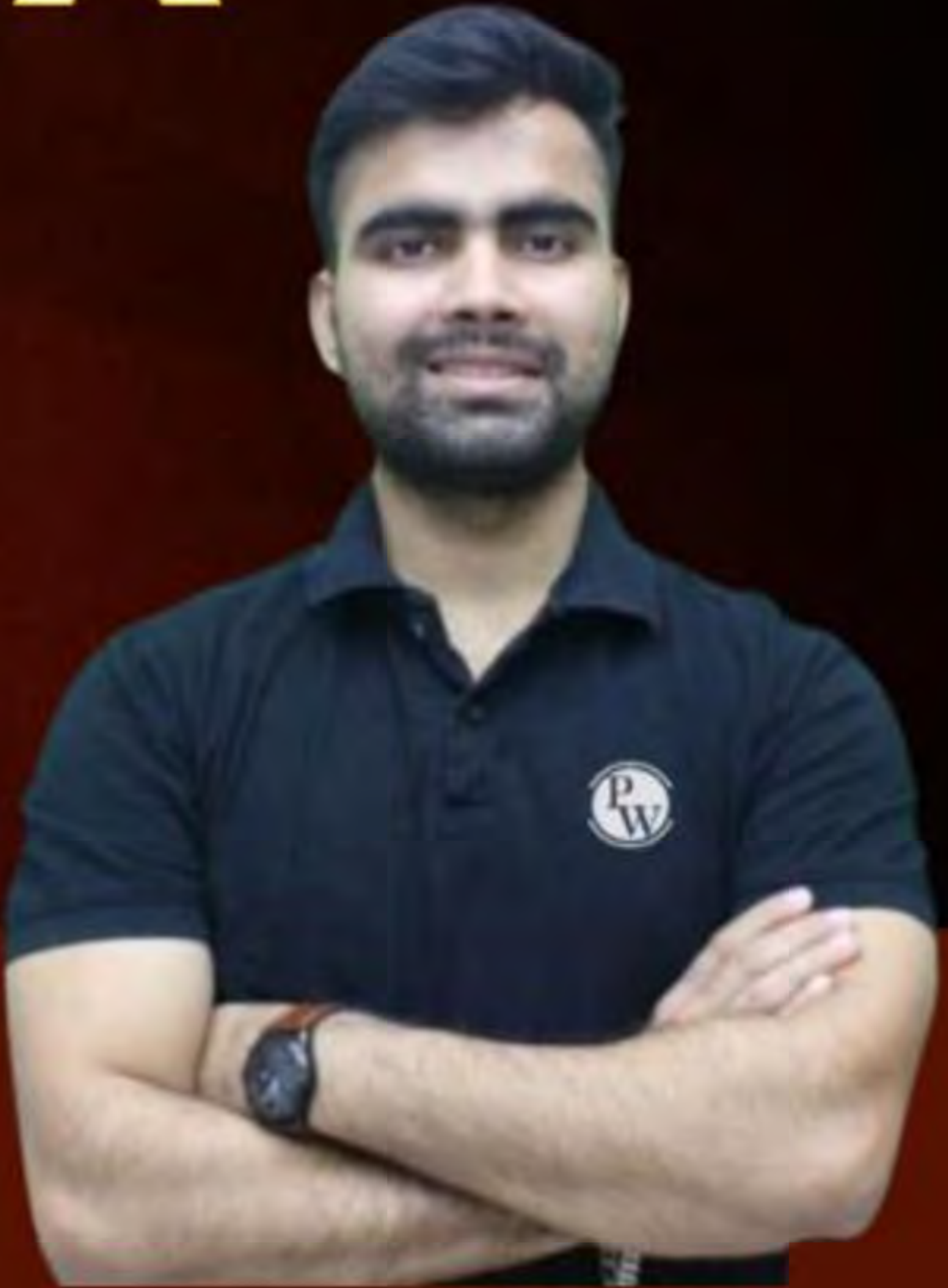

CHAPTER NAME

Definite Integrals

TOPICS *to be covered*

1. Basics of Definite Integrals
2. Important result

Topic : Definition Definite Integrals

If $\int f(x)dx = \boxed{F(x) + c}$ then

$[F(x)]_a^b$ $\qquad |F(x)|_a^b$

$$\int_a^b f(x)dx = F(b) - F(a) = |F(x) + c|_a^b = |F(x)|_a^b$$

is called Definite Integral of f(x) w.r.t. 'x' from x = a to x = b.

Here 'a' is called lower limit and 'b' is called upper limit.

$$\int_2^3 x^2 dx$$

$$\left[\frac{x^3}{3}\right]_2^3$$

$$\Rightarrow \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$$

Example (i)

$$\int_0^2 x^5 dx = \left|\frac{x^6}{6}\right|_0^2 = \frac{1}{6}\left(2^6 - 0^6\right) = \frac{64}{6} = \frac{32}{3}$$

Example (ii)

$$\int_a^b \cos x dx = |\sin x|_a^b = \sin b - \sin a$$

Example (iii)

$$\int_{-4}^{-1} \frac{dx}{x} = [\log|x|]_{-4}^{-1}$$

$= \log 1 - \log 4 = -2\log 2$

$\hookrightarrow \log\left|\frac{1}{4}\right|$

$\Rightarrow \log 2^{-2} = -2\log 2$

Example (iv)

$$\int_0^1 \frac{dx}{\sqrt{1+x}+\sqrt{x}} = \int_0^1 \frac{(\sqrt{1+x}-\sqrt{x})dx}{1+\cancel{x}-\cancel{x}}$$

$$= \int_0^1 ((1+x)^{1/2} - x^{1/2})dx = \left|\frac{2(1+x)^{3/2}}{3} - \frac{2}{3}x^{3/2}\right|_0^1$$

$$= \frac{2}{3}[(2)^{3/2} - 1 - 1] = \frac{2}{3}[2^{3/2} - 2] = \frac{4}{3}[\sqrt{2} - 1$$

Example (v)

$$\int_0^{\pi} \sin^3 x dx = \int_0^{\pi} \frac{3\sin x - \sin 3x}{4} dx$$

[$\because \sin 3x = 3\sin x - 4\sin^3 x$]

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right]_0^{\pi} = \frac{1}{4}\left[-3\cos\pi + \frac{\cos 3\pi}{3} + 3\cos 0 - \frac{\cos 0}{3}\right]$$

$$= \frac{1}{4}\left[3 - \frac{1}{3} + 3 - \frac{1}{3}\right] = \frac{1}{2}\left[3 - \frac{1}{3}\right] = \frac{4}{3}$$

Example (vi)

$$\int_0^1 e^x dx = |e^x|_0^1 = e^1 - e^0 = e - 1$$

Example (vii)

$$\int_0^1 a^x dx = \left|\frac{a^x}{\log a}\right|_0^1 = \frac{1}{\log a}\left[a^1 - a^0\right] = \frac{a-1}{\log a}$$

Example (viii)

$$\int_0^{\pi/4} \sqrt{1+\sin 2x}\,dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{(\cos x + \sin x)^2}\,dx = \int_0^{\frac{\pi}{4}} (\cos x + \sin x)\,dx$$

$= |\sin x - \cos x|$

$$= \left(\sin\frac{\pi}{4} - \cos\frac{\pi}{4}\right) - (\sin 0 - \cos 0)$$

$= 0 - (0 - 1) = 1$

Example (ix)

$$\int_0^{\pi/2} \sqrt{1-\cos 2x}\,dx = \int_0^{\pi/2} \sqrt{2\sin^2 x}\,dx$$

$$= \sqrt{2}\int_0^{\pi/2} \sin x\,dx = \sqrt{2}[-\cos x]_0^{\pi/2}$$

$$= \sqrt{2}\left[-\cos\frac{\pi}{2} + \cos 0\right] = \sqrt{2}$$

Topic : Definite Integrals

#Q. $\int_{\pi/6}^{\pi/2} \frac{\cos x}{\sin^2 x} dx =$

$$\int_{\pi/6}^{\pi/2} \cot x \,\text{cosec}\, x$$

$$-\left[\text{cosec}\, x\right]_{\pi/6}^{\pi/2}$$

$$= -\left[\text{cosec}\frac{\pi}{2} - \text{cosec}\frac{\pi}{6}\right] = -\left[1-2\right]$$

$$= -(-1) = 1$$

(A) 0

(B) 1

(C) 2

(D) −1

Topic : Definite Integrals

#Q. $\int_0^{\pi/2} \cos^2 x dx =$

A $\frac{\pi}{3}$

B $\frac{\pi}{2}$

C $\frac{\pi}{4}$

D 0

$1+\cos 2x = 2\cos^2 x$

$\Rightarrow \cos^2 x = \boxed{\frac{1+\cos 2x}{2}}$

$\int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx$

$\frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/2}$

$\frac{1}{2}\left[\frac{\pi}{2} - (0+0)\right] = \frac{1}{2} \times \frac{\pi}{2} = \pi/4$

$$\int_0^{\pi/2} \sin x \cos x \, dx = \int_0^1 t \, dt = \left[\frac{t^2}{2}\right]_0^1 \rightarrow \frac{1}{2}$$

$$\boxed{\sin x = t}$$

$$\cos x \, dx = dt$$

$\sin 0 = t$

$t = 0$

$\sin \pi/2 = t$

$1 = t$

Topic : Definite Integrals

#Q. $\int_0^1 x e^{x^2} dx =$

(A) $\frac{e+1}{2}$

(B) $\frac{e-1}{2}$

(C) $\frac{e}{2}$

(D) $\frac{1}{2}$

$x^2 = t$

$\Rightarrow 2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$t = 0$

$1 = t$

$\frac{1}{2}\int_0^1 e^t\,dt$

$\frac{1}{2}\left[e^t\right]_0^1$

$\frac{1}{2}\left[e^1 - e^0\right]$

$= \frac{e-1}{2}$

Topic : Definite Integrals

#Q. $\int_0^{\pi} \frac{dx}{1+\sin x} =$

(A) 1

(B) 2

(C) 3

(D) 0

$\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}$

$\int_0^{\pi} (\sec^2 x - \sec x \tan x)\,dx$

$[\tan x]_0^{\pi} - [\sec x]_0^{\pi}$

$\Rightarrow 0 - [-1-1] = 2$

$\frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x \cos x}$

$\Rightarrow \tan x \sec x$

Topic : Definite Integrals

#Q. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx =$

(A) $\frac{\pi}{3}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{2}$

(D) 0

$\sin x = t$

$\cos x\, dx = dt$

$\sin 0 = t$

$t = 0$

$\sin \pi/2 = t$

$1 = t$

$\int_0^1 \frac{dt}{1+t^2}$

$\left[\tan^{-1} t\right]_0^1$

$\Rightarrow \left[\tan^{-1} 1 - \tan^{-1} 0\right]$

$= \pi/4 - 0$

$\Rightarrow \pi/4$

Topic : Some Important Result

$$\boxed{\int_a^b f(x)dx = -\int_b^a f(x)dx}$$

$$\int_1^3 x^3 dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = \frac{80}{4} = 20$$

$$\int_3^1 x^3 dx = \left[\frac{x^4}{4}\right]_3^1 = \frac{1}{4} - \frac{81}{4} = -20$$

Topic : Some Important Result

$$\int_a^a f(x)dx = 0$$

[i.e., if b = a, $\int_a^b f(x)dx = 0$]

Topic : Some Important Result

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$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

[If a < c < b]

↳ Mod. func

Topic : Definite Integrals

#Q. $\int_{-5}^{5} |x + 2| dx =$

A 15

B 40

C 29

D 10

$$\int_{-5}^{-2} |x+2| dx + \int_{-2}^{5} |x+2| dx$$

$$\Rightarrow \int_{-5}^{-2} -(x+2) + \int_{-2}^{5} (x+2) dx$$

$$= -\left[\frac{x^2}{2} + 2x\right]_{-5}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{5}$$

$$\Rightarrow -\left[\left(\frac{4}{2} - 4\right) - \left(\frac{25}{2} - 10\right)\right] + \left[\frac{25}{2} + 10 - \left(\frac{4}{2} - 4\right)\right]$$

$$2 + \frac{5}{2} + \frac{45}{2} + 2 \Big]$$

$$\Rightarrow 4 + 25 = 29$$

Topic : Definite Integrals

#Q. $\int_0^3 [x]dx =$

$[\cdot] \Rightarrow$ Greatest integ function.

$$\int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx$$

$$\Rightarrow [x]_1^2 + 2[x]_2^3$$

$$\Rightarrow 1 + 2 = 3$$

A) 1

B) 3

C) 2

D) 4

Topic : Some Important Result

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

** $I = \int_0^{\pi} \cos x \, dx$, $I = \int_0^{\pi} \cos(\pi - x) dx$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$I \Rightarrow \int_0^{\pi} -\cos x \, dx$

$2I = \int_0^{\pi} \cancel{\cos x} - \int_0^{\pi} \cancel{\cos x}$

$I = 0$

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Topic : Some Important Result

$\int_{-a}^{a} f(x)dx = 0$ if f is an odd function $= 2\int_{0}^{a} f(x)dx$ if f is an even function.

What is the value of $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} (\sin x + \tan x)dx$?

(a) $-\frac{1}{\sqrt{2}} + \ell n\left(\frac{1}{\sqrt{2}}\right)$

(b) $\frac{1}{\sqrt{2}}$

(c) 0

(d) $\sqrt{2}$

$\sin(-x) + \tan(-x)$

$= -\sin x - \tan x$

$-(\sin x + \tan x)$

$= -f(x)$

odd $f(-x) = -f(x)$

even $f(-x) = f(x)$

Topic : Some Important Result

$$\boxed{\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx}$$

if $f(2a - x) = f(x)$

$= 0$ if $f(2a - x) = -f(x)$

Topic : Some Important Result

$$\int_0^{na} f(x)dx = n\int_0^a f(x)dx$$

$\boxed{\text{if } f(a+x) = f(x)}$

Topic : Some Important Result

$$\int_0^{\frac{\pi}{2}} \log \sin x \, dx = \int_0^{\frac{\pi}{2}} \log \cos x \, dx = -\frac{\pi}{2} \log 2$$

Topic : Some Important Result

$$\int_0^{\pi} x\phi(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} \phi(\sin x)dx = \pi \int_0^{\pi/2} \phi(\sin x)dx$$

$$\int_0^{a} x\phi(x)dx = \frac{1}{2}a \int_0^{a} \phi(x)dx$$

provided $\phi(a - x) = \phi(x)$

Topic : Some Important Result

If $f(x)$ is a periodic function with period T, then $\int_a^{a+T} f(x)dx$ is independent of a.

Example: $\int_a^{a+\frac{\pi}{2}} (\sin^4 x + \cos^4 x)dx$ is independent of a.

[$\because \sin^4 x + \cos^4 x$ is a periodic function with period $\frac{\pi}{2}$]

Topic : Some Important Result

If f(x) is a periodic function with period T, then

$$\int_0^{nT} f(x)dx = \boxed{n\int_0^{T} f(x)dx}$$

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and further if $a \in R^+$, then

$$\int_{nT}^{a+nT} f(x)dx = \int_0^{a} f(x)dx$$

$$\int_0^{8\pi} |\sin x|\,dx$$

$$\Rightarrow 8\int_0^{\pi} |\sin x|\,dx = 8\int_0^{\pi} \sin x = 8\left[-\cos x\right]_0^{\pi}$$

$$-8[-1-1] = 16$$

e.g. Since x - [x] is a periodic function with period 1.

$$\therefore \int_0^8 (x-[x])dx = 8\int_0^1 (x-[x])dx$$

$$= 8\left[\int_0^1 x dx - \int_0^1 [x]dx\right] = 8\left[\left|\frac{x^2}{2}\right|_0^1 - 0\right] = 4$$

Thus, $\int_0^8 (x-[x])dx = 4$

Similarly, $\int_0^{10} (x-[x])dx = 5$

$\int_0^{12} (x-[x])dx = 6$ and so on.

Thus $\int_0^{2K} (x-[x])dx = K$ where K is an integer.

Topic : Shortcut Method

$$\int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x-\alpha(\beta-x)}} [\beta > \alpha] = \pi$$

e.g., $\int_{2}^{3} \frac{dx}{\sqrt{(x-2)(3-x)}} = \pi$

$$\int_{1}^{100} \frac{dx}{\sqrt{(x-1)(100-x)}} = \pi$$

Topic : Shortcut Method

$$\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}dx = \frac{\pi}{8(\beta-\alpha)^2}$$

e.g., $$\int_{1}^{2} \sqrt{(x-1)(2-x)}dx = \frac{\pi}{8(2-1)^2} = \frac{\pi}{8}$$

Topic : Shortcut Method

$$\int_a^b \sqrt{\frac{x-a}{b-x}}\,dx = \frac{\pi}{2}(b-a)$$

e.g., $\int_1^2 \sqrt{\frac{x-1}{2-x}}\,dx = \frac{\pi}{2(2-1)} = \frac{\pi}{2}$

Topic : Shortcut Method

(a) $\int_a^b \frac{f(x)dx}{f(x)+f(a+b-x)} = \frac{1}{2}(b-a)$.

(b) $\int_0^b \frac{f(x)dx}{f(x)+f(b-x)} = \frac{1}{2}b$

Topic : Shortcut Method

$\not{x}$

$$\int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{a\sin x + b\,\mathrm{cosec}\,x}{\sin x + \mathrm{cosec}\,x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{a\tan x + b\cot x}{\tan x + \cot x}\,dx = \frac{\pi}{4(a+b)}$$

Topic : Shortcut Method

$\int_a^b (|x-a| + |x-b|)dx = (b-a)^2$

e.g. $\int_2^3 (|x-2| + |x-3|)dx = (3-2)^2 = 1.$

$\int_{-2}^{5} |x+2| + |x-5|$

$a = -2$ $b = 5$

Topic : Shortcut Method

$$\int_0^{\frac{\pi}{2}} \sin^n x dx \text{ or } \int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 1}$$

if n is odd. and

$$= \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2} \cdot \frac{\pi}{2}$$

if n is even.

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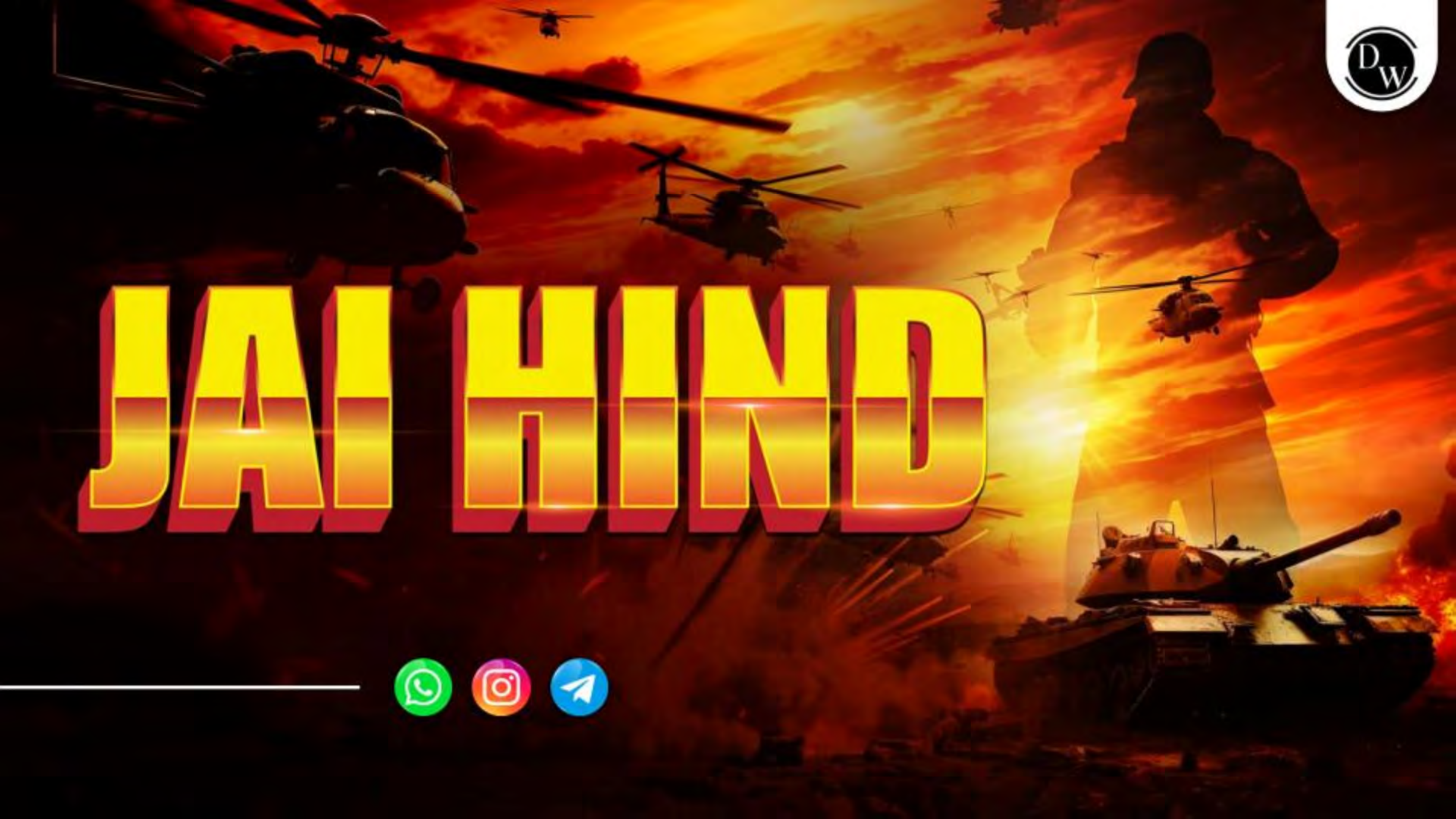
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