L15: Proof by Diagonalization that ATM (Halting Problem) is not Decidable

Let $T = \{\langle M \rangle \mid M$ is a TM that accepts w^R whenever it accepts $w\}$.
Show that T is undecidable.

Let $T = \{\langle M \rangle \mid M$ is a TM that accepts w^R whenever it accepts $w\}$.
It is already known that $L = \{(w, M): w$ is accepted by $M\}$ is undecidable.

Assume that T is decidable, then there must exist a TM by which T can be decided.
Lets say p is the Turing Machine that decides T.

For any input (w, M), M' can be constructed as follows:

If $w = w^R$, simulate M on w. The Σ is the alphabet set of M and let $a, b \notin M$.

Let $\Sigma \cup \{a, b\}$ be the alphabet set of M'. Then for input ab, M' will reject all the other strings except ab.

Now simulate M on w.

if M accepts w, M' rejects.
If M rejects w, M' accepts.

Claim: P accepts M' iff M accepts w.

Proof: If p accepts M'. Since M' rejects all the other strings which include ba also, then M' rejects ab which implies M accepts w.

If w is accepted by M, then M' rejects ab. Since M' rejects all the other strings, M' is accepted p.

Now construct a TM, Q for L, construct M' on input (w, M) and run p on it. Q accepts iff p accepts.

This contradict the fact that L is undecidable.

$\therefore$ T is undecidable.