L17: Using Reductions to Prove Language Undecidable

Problem 5.9 from the book
This problem was assigned as homework in the lecture.

5.9) Let $T = \{\langle M \rangle \mid M$ is a TM that accepts $w^R$ whenever it accepts $w\}$

Show that T is undecidable.

Assume that T is decidable, then there must exist a TM by which T can be decided. Suppose p is the Turing Machine that decides T.

For any input $(w, M)$, $M'$ can be constructed as follows:

If $w = w^R$, simulate M on w. The $\Sigma$ is the alphabet set of M and let $a, b \notin M$.

Let $\Sigma \cup \{a, b\}$ be the alphabet set of $M'$. Then for input ab, $M'$ will reject all the other strings except ab.

Now, simulate M on w.

- If M accepts w, $M'$ rejects.
- If M rejects w, $M'$ accepts.

Claim: p accepts $M'$ iff M accepts w.

Proof:
If p accepts $M'$. Since $M'$ rejects all the other strings which include ba also, then $M'$ rejects ab which implies M accepts w.

If w is accepted by M, then $M'$ rejects ab. Since, $M'$ rejects all the other strings, $M'$ is accepted by p.

Now, construct a TM, Q for L. Construct $M'$ on input $(w, M)$ and run p on it. Q accepts iff p accept.

This contradict that the fact that L is undecidable.

Therefore, T is undecidable.