The contents of this note are from the book

Nodal Discontinuous Galerkin Methods by *Jan S. Hesthaven*

1. FDM vs. FVM vs. FEM vs. DG

Consider the one-dimensional scalar conservation law for the solution $u(x,t)$

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g, \quad x \in \Omega$$

subject to an appropriate set of initial conditions and boundary conditions. Here $f(u)$ is the flux, and $g(x,t)$ is some prescribed forcing function.

- How does one represent the solution $u(x,t)$ by an approximate solution $u_h(x,t)$?
- In which sense will the approximate solution $u_h(x,t)$ satisfy the partial differential equation?

$\Longrightarrow$ These two choices separate the different methods and define the properties of the methods.

§1.1. FDM

Let $x^k, k = 1, \cdots, K$ be a grid in space. Consider a semidiscrete problem (discretize only in space):

$$\frac{du_h(x,t)}{dt} + \frac{f_h(x^{k+1},t) - f_h(x^{k-1},t)}{h^k + h^{k-1}} = g(x^k,t),$$

where u_h and f_h are the numerical approximations to the solution and the flux, respectively.
Define the residual $x \in [x^{k-1}, x^{k+1}] : R_h(x,t) = \frac{\partial u_h}{\partial t} + \frac{\partial f_h}{\partial x} - g(x,t)$.
If we have a total of K grid points and thus, K unknown grid point values $u_h(x^k,t)$, a natural choice is to require that the residual vanishes exactly at these grid points.
$\Longrightarrow$ K different equations

- Good!
  - simple
  - explicit semidiscrete form gives flexibility in the choice of time stepping-method

- extension to the high-order approximation is straightforward
- supported by an extensive body of theories

• Bad!

- ill-suited to deal with complex geometries, both in terms of general computational domains and internal discontinuities as well as for local order and grid size changes to reflect local features of the solution

§1.2. FVM

$u(x,t)$ is approximated on the element by a constant, $\bar{u}^k(t)$, at the center, x^k, of the element

$\Longrightarrow$ cellwise residual : $x \in D^k = [x^{k-\frac{1}{2}}, x^{k+\frac{1}{2}}]$ with $x^{k+\frac{1}{2}} = \frac{1}{2}(x^k + x^{k+1})$

$$R_h(x,t) = \frac{\partial \bar{u}^k}{\partial t} + \frac{\partial f(\bar{u}^k)}{\partial x} - g(x,t)$$

In FVM, we require that the cell average of the residual vanishes identically, leading to the scheme

$$\Longrightarrow \frac{d\bar{u}^k}{dt} + \frac{f^{k+\frac{1}{2}} - f^{f-\frac{1}{2}}}{h^k} = \bar{g}^k, \quad \text{for each cell.}$$

Note that the approximation and the scheme is purely local, and thus, imposes no conditions on the grid structure.

Flux $\Longrightarrow$ divergence theorem $\Longrightarrow$ a pure surface term

that is, introduces the need to evaluate the fluxes on the boundaries $\Longrightarrow$ different FVMs.

The need for a high-order reconstruction reintroduces the need for a particular grid structure and thus destroys the geometric flexibility of the finite volume method in high dimensions. This defeats our initial motivation for considering the FVM.

Main limitation : inability to extend to high-order accuracy on general unstructured grids.

§1.3. FEM

K elements, $D^k = [x^k, x^{k+1}],\ k = 0, \cdots, K.$

$$x \in D^k \ : \ u_h(x) = \sum_{n=1}^{N_p} b_n \psi_n(x),$$

where $\psi_n(x)$ is a locally defined basis functions.
For example, $x \in D^k$,

$$u_h(x) = u(x^k)\frac{x - x^{k+1}}{x^k - x^{k+1}} + u(x^{k+1})\frac{x - x^k}{x^{k+1} - x^k}.$$

To recover the scheme to solve, we define a space of test functions, V_h, and require the residual is orthogonal to all test functions in this space as

$$\int_\Omega \left(\frac{\partial u_h}{\partial t} + \frac{\partial f_h}{\partial x} - g_h \right) \phi_h(x) dx = 0, \quad \forall \phi_h \in V_h$$
$$\Longrightarrow M\frac{du_h}{dt} + Sf_h = Mg_h,$$

where $M_{ij} = \int_\Omega \phi_i(x)\phi_j(x)dx$, $S_{ij} = \int_\Omega \phi_i(x)\frac{d\phi_j(x)}{dx}dx$.
: Semidiscrete system becomes implicit and M must be inverted. For time depending problem, that is a clear disadvantage compared to FDM and FVM.

	Complex geometries	High-order accuracy and hp-adaptivity	Explicit semidiscrete form	Conservation laws	Elliptic problems
FDM	×	◯	◯	◯	◯
FVM	◯	×	◯	◯	(◯)
FEM	◯	◯	×	(◯)	◯
DG-FEM	◯	◯	◯	◯	(◯)

Table 1: (·)-with modifications

An intelligent combination of the finite element and the finite volume methods, utilizing a space of basis and test functions that mimics the finite element method but satisfying the equations in a sense closer to the finite volume method, appears to offer many of the desired properties. This combination is exactly what leads to the discontinuous Galerkin finite element method (DGFEM).

§1.4. DG-FEM

Elements : $D^k = [x^k, x^{k+1}]$
To ensure the locality of the scheme, we duplicate the variables located at nodes x^k.
$\Longrightarrow$ unknown vector : $u_h - [u^1, u^2, u^2, u^3, u^3, \cdots, u^{K-1}, u^{K-1}, u^K]^t$.
$\Longrightarrow$ $2K$ components (instead of $K+1$-components)

The space of basis functions :

$$V_h = \oplus_k = 1^K \{l_i^k\}_{i=0}^1 \ : \text{ as the space of piecewise polynomial functions}$$

that is, for $x \in D^k$,

$$\begin{aligned} u_h^k(x) &= u^k \frac{x - x^{k+1}}{x^k - x^{k+1}} + u^{k+1} \frac{x - x^k}{x^{k+1} - x^k} = \sum_{i=0}^{1} u^{k+i} l_i^k(x) \\ f_h^k(x) &= f^k \frac{x - x^{k+1}}{x^k - x^{k+1}} + f^{k+1} \frac{x - x^k}{x^{k+1} - x^k} = \sum_{i=0}^{1} f^{k+i} l_i^k(x) \end{aligned}$$

Now we assume that the local solution can be well represented by a linear approximation $u_h \in V_h$ and form the local residual

$$x \in D^k, \ R_h(x,t) = \frac{\partial u_h^k}{\partial t} + \frac{\partial f_h^k}{\partial x} - g(x,t)$$

for each element. Then we require that the residual is orthogonal to all test functions $\phi_h \in V_h$ leading to

$$\int_{D^k} R_h(x,t) l_j^k(x) dx = 0, \quad \forall l_j^k(x).$$

Green's theorem:

$$\int_{D^k} \frac{\partial u_h^k}{\partial t} l_j^k - f_h^k \frac{dl_j^k}{dx} - g l_j^k dx = -[f_h^k l_j^k]_{x^k}^{x^{k+1}}.$$

What does the right-hand-side mean?
$\Longrightarrow$ consider if l_j^k is a constant !
$\Longrightarrow$ RHS connects the elements!!
(Since both element D^k and element D^{k+1} depends on the flux evaluation at the point, x^{k+1}, shared among two elements. That is, in fact, an identical situation to the reconstruction problem discussed previously for the FVM where the interface flux is recovered by combining the information of the two cell averages appropriately.)

At this point it suffices to introduce the numerical flux, f^*, as the unique value to be used at the interface and obtained by combining information from both elements. With this we recover the scheme

$$\int_{D^k} \frac{\partial u_h^k}{\partial t} l_j^k - f_h^k \frac{dl_j^k}{dx} - g l_j^k dx = -[f^* l_j^k]_{x^k}^{x^{k+1}}$$

or, by applying Green's theorem once again,

$$\int_{D^k} R_h(x,t) l_j^k(x) = [(f_h^k - f^*) l_j^k]_{x^k}^{x^{k+1}}.$$

: These two formulations are the discontinuous Galerkin finite element schemes for the scalar conservation law in weak and strong form, respectively.

Note that the choice of the numerical flux, f^*, is a central element of the scheme and is also where on can introduce knowledge of the dynamics of the problem.

$$\Longrightarrow \quad M^k \frac{du_h^k}{dt} - (S^k)^T f_h^k - M^k g_h^k = \quad -f^*(x^{k+1}) l^k(x^{k+1}) + f^*(x^k) l^k(x^k)$$

and

$$M^k \frac{du_h^k}{dt} + S^k f_h^k - M^k g_h^k = \quad (f_h^k(x^{k+1}) - f^*(x^{k+1})) l^k(x^{k+1}) \\ -(f_h^k(x^k) - f^*(x^k)) l^k(x^k)$$

Vectors of local unknowns u_h^k, of fluxes f_h^k.
Known g_h^k : forcing

$$M_{ij}^k = \int_{D^k} l_i^k(x) l_j^k(x) dx, \quad S_{ij}^k = \int_{D^k} l_i^k(x) \frac{dl_j^k}{dx} dx.$$

While the structure of the DG-FEM is very similar to that of the FEM, there are several fundamental differences.

- The mass matrix is local rather than global.
  Thus can be inverted at very little cost.
- By carefully designing the numerical flux to reflect the underlying dynamics, one has more flexibility that in the classical FEM to ensure stability for wave-dominated problems.
- DG-FEM tends to be more sparse than for a FEM (because of the doubling of degrees of freedom).

Compared to the FVM, DG-FEM is capable of high-order accurate approximations.

All of this comes at a price : increase in the total degrees of freedom.

2. Discontinuous Galerkin (Finite Element) Method

§2.1. Notations

$$(u,v)_{D^k} = \int_{D^k} uvdx, \quad \|u\|^2_{D^k} = (u,u)_{D^k}$$
$$(u,v)_{\Omega} = \int_{\Omega} uvdx, \quad \|u\|^2_{\Omega} = (u,u)_{\Omega}$$

Let $\Omega = \bigcup_{k=1}^{K} D^k$, then $(u,v)_\Omega = \sum_{k=1}^{K}(u,v)_{D^k}$.

- $\{\{u\}\} = (u^- + u^+)/2$
- Jumps at the interface
  - u : scalar $\Longrightarrow [[u]] = \hat{n}^- u^- + \hat{n}^+ u^+$
  - $\mathbf{u}$: scalar $\Longrightarrow [[\mathbf{u}]] = \hat{n}^- \cdot \mathbf{u}^- + \hat{n}^+ \cdot \mathbf{u}^+$

§2.2. Basic elements of the scheme

Consider the linear scalar wave equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad x \in [L,R] \equiv \Omega,$$

where the linear flux is given as $f(u) = au$.
Initial condition : $u(x,0) = u_0(x)$
Boundary conditions(inflow condition) :

$$\begin{aligned} u(L,t) &= g(t) \quad \text{if } a > 0 \\ u(R,t) &= g(t) \quad \text{if } a < 0 \end{aligned}$$

- Local solution

$$x \in D^k, \quad u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\psi_n(x) = \sum_{i=1}^{N_p} u_h^k(x_i^k,t) l_i^k(x),$$

  where $\psi_n(x)$ are modal basis and $l_i^k(x)$ are nodal basis.

- Global solution: approximated by the piecewise N-th order polynomial approximation $u_h(x,t)$

$$u(x,t) \simeq u_h(x,t) = \oplus_{k=1}^{K} u_h^k(x,t).$$

Define the residual $R_h(x,t) = \frac{\partial u_h}{\partial t} + \frac{\partial a u_h}{\partial x}$ and we must decide in which sense this should vanish!

V_h : test function space = the space of piecewise smooth functions defined on Ω.

$$V_h = \oplus_{k=1}^{K} V_h^k,$$

where $V_h^k = span\{\psi_n(x)\}_{n=1}^{N_p}$ and for $\phi_h^k \in V_h^k$,

$$x \in D^k, \quad \phi_h^k(x) = \sum_{n=1}^{N_p} \hat{\phi}_n^k \psi_n(x).$$

We now require that the residual is orthogonal to all test functions in V_h resulting in the local statement

$$\int_{D^k} R_h(x,t)\psi_n(x)dx = 0, \quad 1 \le n \le N_p,$$

on all K elements.

$$\begin{aligned} \Longrightarrow \int_{D^k} \left(\frac{\partial u_h^k}{\partial t}\psi_n - a u_h^k \frac{d\psi_n}{dx} \right) dx &= -[a u_h^k \psi_n]_{x_l^k}^{x_r^k} \\ &= -\int_{\partial D^k} \left(\hat{n} \cdot a u_h^k\right) \psi_n dx, \quad 1 \le n \le N_p. \end{aligned}$$

- Lack of conditions on the local solution and test functions.
  $\longrightarrow$ the solution at the interfaces between elements is multiply defined and we will need to choose which solution, or combination of solution, is correct!
- We refer this solution as $(au_h)^*$ known as the numerical flux.

Semidiscrete scheme :

$$\int_{D^k} \left(\frac{\partial u_h^k}{\partial t}\psi_n - a u_h^k \frac{d\psi_n}{dx} \right) dx = -\int_{\partial D^k} \left(\hat{n} \cdot (au_h)^*\right) \psi_n dx, \quad 1 \le n \le N_p \qquad (1)$$

We recover a total of $K \times N_p$ equations for the same number of unknowns.
Integration by parts in (1) one more time yields

$$\int_{D^k} R_h(x,t)\psi_n dx = \int_{\partial D^k} \hat{n} \cdot (a u_h^k - (au_h)^*)\psi_n dx, \quad 1 \le n \le N_p.$$

: strong form

As in the equivalent weak form, it is the right-hand-side that is responsible for recovering the global solution from the local solutions and imposing boundary conditions.
$\longrightarrow$ Key role played by the numerical flux $(au_h)^*$.

[Example 2.1] Modal expansion :

$$x \in D^k,\ u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\phi_n(x) \quad : \text{ local solution}$$

$$\hat{M}^k \frac{d}{dt}\hat{u}_h^k - (\hat{S}^k)^T a\hat{u}_h^k = -(au_h)^*\psi(x_r^k) + (au_h)^*\psi(x_l^k), \tag{2}$$

where $\hat{M}_{ij}^k = (\psi_i, \psi_j)_{D^k}$, $\hat{S}_{ij}^k = \left(\psi_i, \frac{d\psi_j}{dx}\right)_{D^k}$, $\hat{u}_h^k = [\hat{u}_1^k, \cdots, \hat{u}_{N_p}^k]^T$,$\psi = [\psi_1(x), \cdots, \psi_{N_p}(x)]^T$.
: Classical discontinuous Galerking finite element method
: most widely used, but certainly not a unique choice

[Example 2.2] Nodal expansion :

$$x \in D^k,\ u_h^k(x,t) = \sum_{i=1}^{N_p} u_h^k(x_i^k,t)l_i^k(x),$$

where N_p nodal values, $u_h^k(x_i^k,t)$ are the unknowns.
Now choose N_p nodes, y_i^k.
Since the weak form does not allow a space of nonsmooth test function, we consider the strong form

$$M^k \frac{du_h^k}{dt} + S^k a u_h^k = (au_h^k - (au_h)^*)l^k(x_r^k) - (au_h^k - (au_h)^*)l^k(x_l^k).$$

[Remark] : The specification of the numerical flux $(au_h)^*$ is most naturally related to the dynamics of the PDE being solved. At the left end and of the local domain D^k, this numerical flux should be a function of $[au_h^{k-1}(x_r^{k-1}), au_h^k(x_l^k)]$, while the right end depends on $[au_h^k(x_r^k), au_h^{k+1}(x_l^{k+1})]$.
$(au_h)^* \longrightarrow$ the flux one would wish to know at the interface
$\longrightarrow$ at a physical boundary, a reasonable, but not unique.

[Example 2.3]

$$\int_\Omega \left(\frac{\partial u}{\partial t} + \frac{\partial (au)}{\partial x} = 0 \right) u \, dx$$
$$\Longrightarrow \frac{d}{dt}\|u\|_\Omega^2 = -a(u^2(R) - u^2(L)) \ : \ \text{energy method}$$

Consider a nodal approach :

$$x \in D^k, \quad u_h^k(x,t) = \sum_{i=1}^{N_p} u_h^k(x_i^k, t) l_i^k(x).$$

Strong form :

$$M^k \frac{du_h^k}{dt} + S^k(au_h^k) = [l^k(x)(au_h^k - (au_h)^*)]_{x_l^k}^{x_r^k}, \tag{3}$$

where $M_{ij}^k = (l_i^k, l_j^k)_{D^k}$, $S_{ij}^k = \left(l_i^k, \frac{dl_i^k}{dx}\right)_{D^k}$.
Notice that

- $u_h^T M^k u_h = \int_{D^k} \sum_{i=1}^{N_p} u_h^k(x_i^k) l_i^k(x) \sum_{j=1}^{N_p} u_h^k(x_j^k) l_j^k(x) dx = \|u_h^k\|_{D^k}^2$
-

$$\begin{aligned} u_h^T S^k u_h &= \int_{D^k} \sum_{i=1}^{N_p} u_h^k(x_i^k) l_i^k(x) \sum_{j=1}^{N_p} u_h^k(x_j^k) \frac{dl_j^k(x)}{dx} dx = \int_{D^k} (u_h^k)(u_h^k)' dx \\ &= \int_{D^k} \frac{1}{2} \left((u_h^k)^2 \right)' dx = \frac{1}{2} [(u_h^k)^2]_{x_l^k}^{x_r^k} \end{aligned}$$

Therefore, ${u_h^k}^T(3)$ and the above two calculations yield

$$\frac{d}{dt}\|u_h^k\|_{D^k}^2 = -a[(u_h^k)^2]_{x_l^k}^{x_r^k} + 2[u_h^k(au_h^k - (au_h)^*)]_{x_l^k}^{x_r^k}. \tag{4}$$

For stability, we must, as for the original equation, require that

$$\sum_{k=1}^{K} \frac{d}{dt}\|u_h^k\|_{D^k}^2 = \frac{d}{dt}\|u_h\|_\Omega^2 \le 0.$$

At every interface, it is enough to control

$$\hat{n}^- \cdot (au_h^2(x^-) - 2u_h(x^-)(au_h)^*(x^-)) + \hat{n}^+ \cdot (au_h^2(x^+) - 2u_h(x^+)(au)^*(x^+)) \le 0.$$

Let us consider a numerical flux like

$$f^* = (au)^* = \{\{au\}\} + |a|\frac{1-\alpha}{2}[[u]].$$

At any internal interface, this gives a contribution to the energy integral as

$$-|a|(1-\alpha)[[u_h]]^2$$

which is nonpositive, provided $0 \le \alpha \le 1$.

Assume first that $a > 0$. Then taking

$$f_L = 0, \;\; f_R = au_h(x_r^k)$$

as numerical boundary fluxes, reflect a Dirichlet boundary condition of $u(x_l^1) = 0$ and a purely internal choice at the outflow, as i reasonable.
In this case, we recover the global energy condition from (4) by summing over all elements

$$\frac{d}{dt}\|u_h\|_\Omega^2 = -|a|(1-\alpha)\sum_{k=1}^{K-1}[[u_h^k(x_r^k)]]^2 - (1-\alpha)a(u_h^1(x_l^1))^2 - a(u_h^K(x_r^K))^2.$$

For $0 \le \alpha \le 1$, we have global stability.

Alternative choice :

$$f_L = -au_h^1(x_l^1), \;\; f_R = au_h^K(x_r^K).$$

$$\frac{d}{dt}\|u_h\|_\Omega^2 = -|a|(1-\alpha)\sum_{k=1}^{K-1}[[u_h^k(x_r^k)]]^2 - a(u_h^1(x_l^1))^2 - a(u_h^K(x_r^K))^2.$$

: stable for $0 \le \alpha \le 1$

- $\alpha = 1$: numerical flux = simply the average of two solutions
  $\Longrightarrow$ central flux
- $\alpha = 0$: flux always takes information from where it is coming
  $\Longrightarrow$ upwind flux

(Here, we have not talked about whether the above choices are good or bad or even convergent!)

- Properties of DG
  - The solutions are piecewise smooth, often polynomial but discontinuous between elements
  - Boundary conditions and interface continuity are enforced only weakly
  - All operators are local
  - The schemes are well suited to various order and element sizes, as all information exchange is across the interfaces only