A Survey of Difference Methods for Non-Steady Fluid Dynamics

Robert D. Richtmyer
Courant Institute of Mathematical Sciences,
New York University

Manuscript Completed August 27, 1962

NATIONAL CENTER FOR ATMOSPHERIC RESEARCH
BOULDER, COLORADO

NCAR Technical Notes 63-2

Published by the National Center for Atmospheric Research

Boulder, Colorado

Operated by the University Corporation for Atmospheric Research

a private, non-profit corporation, sponsored by

the National Science Foundation

TABLE OF CONTENTS

ABSTRACT

Several finite-difference approximations to the hyperbolic equations of fluid dynamics are described, and their qualities assessed. They include the schemes normally used and, for comparison, an unstable and a completely stable implicit scheme. Accuracy and stability are discussed. The Lax-Wendroff scheme, for a system of conservation laws, is discussed for problems in one and in two space variables.

1. Introduction

Several finite-difference equation systems that approximate the hyperbolic partial differential equations of fluid dynamics will be described, and some of their qualities assessed. The fluid is assumed to be non-viscous and thermally non-conducting; its thermodynamic properties are described by an equation of state,

$$p = P\ (E,\ V), \tag{1}$$

where p, E, and V are its pressure, internal energy per unit mass, and volume per unit mass. The (local, adiabatic) sound speed c is given by

$$c = c\ (E,\ V) = V\sqrt{P\frac{\partial P}{\partial E} - \frac{\partial P}{\partial V}}\ . \tag{2}$$

Problems will be considered in which the flow depends on either one or two space variables (not necessarily cartesian) and the time t, in which there are no body forces and no heat sources, and in which turbulence is not expected. The partial differential equations are of hyperbolic type and, when taken in conjunction with initial data, boundary conditions, jump conditions and the like on shocks and interfaces, are assumed to determine a well-posed initial value problem.

Typical problems are the calculation of explosions, implosions, non-steady supersonic flight, and the impact of shock waves on rigid bodies. The mathematical methods are rather similar to many of those used in meteorology. See [7].

For problems with one space variable, the most commonly used method is based on the simple difference scheme (equations (4.1). . . (4.5)) given below for the Lagrangean formulation. Shocks have been treated either by shock fitting or by an artificial viscosity ([5], Ch. X).

The Lagrangean method with artificial viscosity has been extended to two space variables, but is much less satisfactory for such problems, because (1) the economics of machine computing requires a somewhat coarse point net when there are three independent variables, which reduces accuracy, (2) after a short time, the Lagrangean point net becomes distorted in space, which further reduces accuracy, and (3) the smearing out of shocks caused by the viscosity is more serious, owing to the coarseness of the net and the generally more complicated shock configuration.

In attempts [6] to derive better methods for multidimensional problems, the following principles have been proposed:

(1) The Eulerian formulation should be used.

(2) Difference equations of second-order accuracy should be used for the smooth part of the flow.

(3) Boundary conditions and jump conditions should be applied by fitting procedures of second-order accuracy on all surfaces of singularity (shocks, interfaces, slip surfaces, contact discontinuities, and boundaries of rarefaction fans).

It is assumed that for problems with piecewise smooth (say, analytic) initial data, the solution is piecewise smooth (analytic) in space-time. Two-dimensional fitting procedures are now under development; so also are methods for detecting the onset of singularities of various kinds during a calculation.

In this report, finite-difference approximations to the partial differential equations for the smooth part of the flow are discussed. The following notation will be used: if $\phi(x, t)$ is any function of one space variable and time appearing in the problem, ϕ_j^n denotes the finite-difference approximation to $\phi(j\Delta x, n\Delta t)$, where Δx and Δt are the dimensions of the unit mesh of the computational point-net in the x, t plane and n and j have integer or half-odd-integer values. Similarly, for a function $\phi(x, y, t)$ of two space variables and time, $\phi_{j,k}^n$ denotes an approximation to $\phi(j\Delta x, k\Delta y, n\Delta t)$.

2. One Space Variable: the Classical Lagrangean Scheme

In a problem with slab symmetry, the Lagrangean variable ξ of a fluid element is the value of a cartesian coordinate of that fluid element in the configuration at time $t = 0$ or in some other reference configuration; the corresponding cartesian coordinate at time t is $x(\xi, t)$; the fluid velocity u and the quantities p, V, E are also functions of ξ and t. The equations are

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial \xi}, \tag{3.1}$$

$$\frac{\partial x}{\partial t} = u, \tag{3.2}$$

$$V = \frac{1}{\rho_0}\frac{\partial x}{\partial \xi}, \tag{3.3}$$

$$\frac{\partial E}{\partial t} = -p\frac{\partial V}{\partial t}, \tag{3.4}$$

and the equation of state (1). Here, $\rho_0 = \rho_0(\xi)$ is the fluid density in the reference configuration. In the following, we shall assume this configuration so chosen that $\rho_0 = $ constant.

In the corresponding finite-difference equations, u carries half-odd-integer superscripts and integer subscripts, while the reverse is true for p, V, E. The equations are

$$\rho_0\left(u_j^{n+\frac{1}{2}} - u_j^{n-\frac{1}{2}}\right) = -\frac{\Delta t}{\Delta \xi}\left(p_{j+\frac{1}{2}}^n - p_{j-\frac{1}{2}}^n\right), \tag{4.1}$$

$$x_j^{n+1} - x_j^n = \Delta t \cdot u_j^{n+\frac{1}{2}}, \tag{4.2}$$

$$V_{j+\frac{1}{2}}^{n+1} = \frac{1}{\rho_0 \Delta \xi}\left(x_{j+1}^{n+1} - x_j^{n+1}\right), \tag{4.3}$$

$$E_{j+\frac{1}{2}}^{n+1} - E_{j+\frac{1}{2}}^n = -\left(p_{j+\frac{1}{2}}^n\right)\cdot\left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n\right) \tag{4.4a}$$

or

$$-\tfrac{1}{2}\left[p_{j+\frac{1}{2}}^n + P\left(E_{j+\frac{1}{2}}^{n+1}, V_{j+\frac{1}{2}}^{n+1}\right)\right]\cdot\left(V_{j+\frac{1}{2}}^{n+1} - V_{j+\frac{1}{2}}^n\right), \tag{4.4b}$$

$$p_{j+\frac{1}{2}}^{n+1} = P\left(E_{j+\frac{1}{2}}^{n+1}, V_{j+\frac{1}{2}}^{n+1}\right). \tag{4.5}$$

Of the two alternatives in (4.4), (4.4a) is slightly easier to use than (4.4b), but less accurate.

If all quantities are known for the relevant values of t up to $n\Delta t$, and if the difference equations are taken in the order written, in each case for all the relevant values of j, each equation contains only one unknown, and the system determines all quantities for the relevant values of t up to $(n+1)\Delta t$, thus completing a cycle.

Note that the new quantities ($u_j^{n+\frac{1}{2}}$ $(j = 1, 2, 3, \ldots)$) obtained from (4.1) are used in (4.2), and so on. A system whose equations are explicit in this sense, if taken in a suitable order, is called effectively explicit. If each equation determines one unknown in terms of old quantities alone, so that the order is immaterial, the system is called (completely) explicit; if the solution of simultaneous equations cannot be avoided, the system is called implicit.

If (4.4b) is used, each equation of the above system is centered, by which is meant that for each there is a point of the x, t plane at which the partial derivatives are approximated to $0(\Delta t^2)$ or $0(\Delta x^2)$ by the difference quotients used.

The system is subject to the well-known stability condition discovered by Courant, Friedrichs, and Lewy in [1], which takes here the form

$$\frac{c\Delta t}{\rho_0 V \Delta\xi} < 1 \tag{5}$$

and says that the distance traveled by a sound wave relative to the fluid must not exceed the distance $x_{j+1} - x_j = \rho_0 V \Delta\xi$ between neighboring net points. Since c and V are functions of ξ and t, the condition (5) should be monitored by the computer throughout the computation.

For the treatment of shocks, interfaces, and boundaries, the reader is referred to [5].

Problems in which the space variable is a radial coordinate in spherical or cylindrical symmetry can be similarly treated; ξ is replaced by the value of the radial coordinate in a reference configuration, and some of the equations are slightly modified.

In the foregoing case of slab symmetry, the system (3.1). . .(3.4) can be modified by replacement of (3.2) and (3.3) by

$$\frac{\partial V}{\partial t} = \frac{1}{\rho_0} \frac{\partial u}{\partial \xi} \tag{6}$$

and by substitution of the right member of this equation for $\partial V/\partial t$ into (3.4). The system then assumes the form of the hyperbolic systems discussed in the next section.

3. Various Schemes for Hyperbolic Systems in One Space Variable

In the Eulerian formulation, where the independent variable is x, the differential equations have additional terms containing the operator $u(\partial/\partial x)$. If the corresponding terms of the difference equations are centered in a simple way, the equations become unstable, and other difference schemes must be considered. That we shall do, by way of preparation for the Eulerian treatment of multi-dimensional problems.

The system of differential equations has the form

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0 \tag{7}$$

where U is a vector whose components, say m in number, are the dependent variables and A is an $m \times m$ matrix, whose elements depend generally on the components of U and on x and t. In this section, to simplify the notation, the equations will be written as though A were a constant matrix.

The system is called hyperbolic if A has all real eigenvalues and m linearly independent eigenvectors. A is not usually a symmetric matrix.

Six schemes are given in Table I. Scheme b., which has been used by J. B. Keller, by P. D. Lax, and by K. O. Friedrichs, suffers from rather severe diffusion effects. Scheme c. has been used by Courant, Isaacson, and Rees [2] and, in a modified form, by R. Lelevier. It presupposes that the equations have been so transformed as to make A diagonal, and only one equation of the system has been written, a being an eigenvalue of A. Scheme d. has been extensively used in meteorological calculations. Scheme e., described by Lax and Wendroff [3], was independently discovered and used by C. Leith in studies of the general circulation of the atmosphere.

The order of magnitude of the truncation error is given in the third column of Table I. It is assumed that the limiting process Δx, $\Delta t \to 0$ is carried out in such a way that the ratio $\Delta x/\Delta t$ remains constant. Hence Δx and Δt are small quantities of the same order and Δ is used to represent either of them. If a sufficiently smooth exact solution of the differential equation is substituted into the difference equation, the two members fail to be equal by an amount of the indicated order of magnitude.

4. Stability

It has long been known from observations that, when instabilities develop in the numerical solution of partial differential equations, they appear as oscillations of rather short wave length and initially small amplitude superimposed on a smooth solution. Therefore, the equations of first variation (i.e., the linearized equations) can be used to predict the growth or decay of the instabilities. Furthermore, they generally appear first in a very small region of space; therefore, if the coefficients of the equation are smooth functions, one can approximate by taking them as constant in this region, and can usually ignore the presence of boundaries. In this discussion, therefore, stability will be taken to mean the stability of the corresponding linearized problems with constant coefficients, and this leads to a local stability condition, usually a limitation on the magnitude of the ratio $\Delta t/\Delta x$; this ratio must be taken small enough to satisfy the limitation for all x, t. For further details, see [5].

For the linearized problems, the growth or decay of small disturbances can be determined by Fourier expansion. If a typical Fourier term,

$$U = U_0 e^{ikx} , \tag{8}$$

where U_0 is a constant vector, is substituted into the difference equations for U_j^n, with $x = j\Delta x$, it is found that U_j^{n+1} is of the same form but with U_0 replaced by GU_0, where the matrix G, whose elements depend on k, Δt, and Δx, is called the amplification matrix.

Similarly, if a Fourier component of an exact solution is equal to (8) at time $t = n\Delta t$, it will have the same form at $t = (n+1)\Delta t$ but with U_0

replaced by ΓU_0, where

$$\Gamma = e^{ikA\Delta t} . \tag{9}$$

The eigenvalues of Γ all lie on the unit circle in the complex plane, corresponding to the fact that for a hyperbolic system waves neither grow nor decay. For any reasonable finite difference scheme, the eigenvalues of G lie very nearly on the unit circle for $k \ll 1/\Delta x$ (wave lengths long compared with Δx); then, if the flow being computed is smooth and Δx is small, the components of shorter wave lengths are of no importance, and all we ask of them is that they not grow exponentially, in the numerical solution, for if they do they will swamp the desired components.

This suggests, as a condition for stability, that the eigenvalues of the amplification matrix G should not exceed 1 in absolute value, and this requirement is the essential content of the <u>von Neumann necessary condition</u> for stability for a hyperbolic system. (The slightly more generous condition that the eigenvalues should not exceed $1 + O(\Delta t)$ must be used for problems in which there is a mechanism permitting a growth of the true solution; see [5], Ch. IV.) As will be indicated below, the von Neumann condition is also sufficient for stability of the schemes listed in Table I for the fluid-dynamical problems.

The equation for the amplification matrix is given in Table II, for the schemes listed in Table I. For stability, we require that the eigenvalues of G lie in the closed unit disk of the complex plane for all real values of $\alpha = k\Delta x$. In each case G is of the form $f(A)$, where $f(\cdot)$ is a rational algebraic function; its eigenvalues are given by $f(a_1), \ldots, f(a_m)$, where $a_1, \ldots, a_m$ are the eigenvalues of A. Each eigenvalue of G traces out a curve C in the complex plane, as α varies; typical curves of this kind are shown in figure 1, for the six schemes. Whether such a curve lies in the unit disk depends in general on the value of $a(\Delta t/\Delta x)$, where a is the corresponding eigenvalue of A; the condition is given in the last column of Table II.

5. Sufficiency of the Stability Condition

It is shown in [5] that the von Neumann condition is also sufficient for stability in particular if G is a normal matrix. (A normal matrix is one that commutes with its Hermitian conjugate; it can always be written in the form $C+iD$, where C and D are Hermitian (self-conjugate) matrices; necessary and sufficient for reducibility of a matrix M to diagonal form by a unitary transformation U^*MU is that M be normal.) Our matrices $G=f(A)$ will be normal if A, which is a real matrix, is symmetric. Although this is not usually the case for the equations of fluid dynamics, the following argument can be used, for hyperbolic problems in one space variable:

Since A has real eigenvalues and a complete set of eigenvectors, it can be reduced to real diagonal form by a similarity transformation

$$A \to \tilde{A} = PAP^{-1} . \tag{10.1}$$

Therefore, if we take new dependent variables $v_1, \ldots, v_m$ given by

$$V = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = P \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} = PU , \tag{10.2}$$

the differential equations take the form

$$\frac{\partial V}{\partial t} = \tilde{A} \frac{\partial V}{\partial x} ; \tag{11}$$

similarly, the difference equations take the same forms as those discussed above, but with V in place of U and $\tilde{A}$ in place of A; since $\tilde{A}$ is diagonal, hence symmetric, it is now seen that the von Neumann condition is sufficient for stability in these problems.

More precisely, the concept of stability depends on the choice of a norm in the function space of the dependent variables, and the use of Fourier analysis implies an L_2 or root-mean-square type of norm. Clearly, we have been somewhat arbitrary, since the norm is not in general invariant under the transformation (11), because P is not in general a unitary matrix; that is, the norm given by

$$\|V\|^2 = \int (|v_1|^2 + \ldots + |v_m|^2)\, dx \tag{12.1}$$

is not the same as that given by

$$\|U\|^2 = \int (|u_1|^2 + \ldots + |u_m|^2)\, dx \,. \tag{12.2}$$

However, the new norm (12.1) is precisely the one appropriate for stability analysis, since the differential equation (11) has the property that, for its solutions, this norm is constant in time; that is, with respect to this choice of norm, a small disturbance neither grows nor decays in a true solution.

For fluid dynamics, this is equivalent to writing the equations in characteristic form

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p - c^2\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\rho = 0 \tag{13.1}$$

$$\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)p + \rho c\left(\frac{\partial}{\partial t} + (u+c)\frac{\partial}{\partial x}\right)u = 0 \tag{13.2}$$

$$\left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)p - \rho c\left(\frac{\partial}{\partial t} + (u-c)\frac{\partial}{\partial x}\right)u = 0 \tag{13.3}$$

and then linearizing by calling the coefficients u_0, c_0^2, etc. instead of u, c^2, etc. Then we have

$$V = \begin{pmatrix} p + c_0^2 p \\ p + \rho_0 c_0 u \\ p - \rho_0 c_0 u \end{pmatrix} ,$$

$$A = \begin{pmatrix} u_0 & 0 & 0 \\ 0 & u_0+c & 0 \\ 0 & 0 & u_0-c \end{pmatrix} .$$

The largest eigenvalue of A has the magnitude $|u_0| + c$. For schemes b., c., d., and e. of Table I the stability condition is therefore

$$(|u_0| + c) \frac{\Delta t}{\Delta x} < 1 \quad . \tag{14}$$

(The possibility of equality in (14) is not regarded as of practical importance, since numerical calculations are always performed with a finite number of binary or decimal places.)

In equations (13.1), (13.2), and (13.3), the Eulerian formulation has been used; a similar argument can be made for the Lagrangean formulation and leads to the stability condition (5).

6. The Two-Step Lax-Wendroff Scheme for a System of Conservation Laws

Of the three second-order-accuracy schemes in Table I, the Lax-Wendroff scheme e. seems to be the most suitable for extension to the case of two space variables, because, first, the damping of short-wave-length disturbances has a slight smoothing effect, which is beneficial, for second order accuracy depends on smoothness of the functions, and, second, the implicit scheme furthermore requires relaxation (or equivalent) solution of simultaneous equations in two space variables. This section contains a further discussion, still for one space variable, of scheme e.

So far, the Lax-Wendroff scheme has been described only for linear equations with constant coefficients. In general, we shall understand it to be a scheme in which U_j^{n+1} is expressed in terms of U_{j-1}^n, U_j^n, and U_{j+1}^n by a formula of the type

$$U_{j+1}^n = U_j^n + \Delta t \left\langle \frac{\partial U}{\partial t} \right\rangle_j^n + \frac{(\Delta t)^2}{2} \left\langle \frac{\partial^2 U}{\partial t^2} \right\rangle_j^n \tag{15}$$

where $\langle\cdot\rangle_j^n$ denotes a difference approximation to $\partial U/\partial t$ or to $\partial^2 U_2/\partial t$, obtained by use of the differential equation and so chosen that the error term of (15) is $0(\Delta^3)$. (Lax and Wendroff also constructed similar schemes of still higher accuracy, but these have apparently not been used, so far.)

By the two-step Lax-Wendroff scheme we mean a scheme in which schemes b. and d. are used for alternate time cycles, as follows:

$$1)\quad U_j^{n+1} = \tfrac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{\Delta t}{2\Delta x} A\left(U_{j+1}^n - U_{j-1}^n\right), \tag{16.1}$$

$$2)\quad U_j^{n+2} = U_j^n - \frac{\Delta t}{\Delta x} A\left(U_{j+1}^{n+1} - U_{j-1}^{n+1}\right) . \tag{16.2}$$

If A is constant, this reduces, upon substitution of (1) into (2), to

$$U_j^{n+2} = U_j^n - \frac{\Delta t}{2\Delta x} A\cdot\left(U_{j+2}^n - U_{j-2}^n\right) + \tfrac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2\cdot\left(U_{j+2}^n - 2U_j^n + U_{j-2}^n\right) \tag{16.3}$$

which is exactly the Lax-Wendroff scheme for a net with spacings $2\Delta x$ and $2\Delta t$. If A is variable but is properly centered in (16.2), e.g., by averaging the function A(U) between points $(j+1)\Delta x$ and $(j-1)\Delta x$, the over-all scheme has second-order accuracy. (The intermediate quantities at time $(n+1)\Delta t$ have only first-order accuracy.)

If the equations are written in conservation-law form

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{17}$$

where F(U) is a vector-valued function of the vector U, the two-step Lax-Wendroff scheme takes the simple form

$$U_j^{n+1} = \tfrac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(F_{j+1}^n - F_{j-1}^n\right), \tag{18.1}$$

$$U_j^{n+2} = U_j^n - \frac{\Delta t}{\Delta x}\left(F_{j+1}^{n+1} - F_{j-1}^{n+1}\right), \tag{18.2}$$

where F_j^n stands for $F(U_j^n)$, etc.

This form has the additional advantage that the finite-difference analogue of the quantity $\int U\,dx$ is exactly related to the difference analogue of the integrated flux $\int F(U)\,dt$. That is, the finite-difference analogue of the conservation law

$$\int_{x_1}^{x_2} U\,dx \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} F(U)\,dt \Bigg|_{x_1}^{x_2} = 0 \tag{19}$$

is exactly satisfied, since summation of (18.2) over all netpoints between x_1 and x_2 yields a telescoping sum of the quantities F_j^n.

7. The Two-Step Lax-Wendroff Scheme in Two Space Variables

If x and y are cartesian coordinates, the Eulerian equations of fluid dynamics can be written in conservation-law form

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} = 0, \tag{20}$$

where

$$U = \begin{pmatrix} \rho \\ m \\ n \\ e \end{pmatrix}, \quad F(U) = \begin{pmatrix} m \\ \frac{m^2}{\rho} + p \\ \frac{mn}{\rho} \\ (e+p)\frac{m}{\rho} \end{pmatrix}, \quad G(U) = \begin{pmatrix} n \\ \frac{mn}{\rho} \\ \frac{n^2}{\rho} + p \\ (e+p)\frac{n}{\rho} \end{pmatrix}. \tag{21}$$

Here, m and n are the x and y components of momentum per unit volume, and e is the total energy per unit volume. In terms of previous variables,

$$m = \rho u, \quad n = \rho v, \quad e = \rho\left(E + \tfrac{1}{2}(u^2 + v^2)\right); \tag{22}$$

in (21), p is to be regarded as a function of ρ, m, n, e obtained from the equation of state:

$$p = P(E, V) = P\left(\frac{e}{\rho} - \frac{m^2 + n^2}{2\rho^2}, \frac{1}{\rho}\right). \tag{23}$$

If the differentiations in (20) are expanded by the chain rule, the equations take the form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} + B(U)\frac{\partial U}{\partial y} = 0; \tag{24}$$

the matrices A and B so obtained are rather complicated.

The original Eulerian equations are also of the form (24), with U, A, B replaced by

$$U' = \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix}, \quad A'(U') = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & 0 & 1/\rho \\ 0 & 0 & u & 0 \\ 0 & \rho c^2 & 0 & u \end{pmatrix},$$

$$B'(U') = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\rho \\ 0 & 0 & \rho c^2 & v \end{pmatrix}; \tag{25}$$

these matrices are not normal and they do not commute.

The two-step Lax-Wendroff scheme for (20) is

$$1)\quad U^{n+1}_{j\,k} = \tfrac{1}{4}\left(U^n_{j+1\,k} + U^n_{j-1\,k} + U^n_{j\,k+1} + U^n_{j\,k-1}\right) - \frac{\Delta t}{2\Delta x}\left(F^n_{j+1\,k} - F^n_{j-1\,k}\right) - \frac{\Delta t}{2\Delta y}\left(G^n_{j\,k+1} - G^n_{j\,k-1}\right), \tag{26.1}$$

$$2)\quad U^{n+2}_{j\,k} = U^n_{j\,k} - \frac{\Delta t}{\Delta x}\left(F^{n+1}_{j+1\,k} - F^{n+1}_{j-1\,k}\right) - \frac{\Delta t}{\Delta y}\left(G^{n+1}_{j\,k+1} - G^{n+1}_{j\,k-1}\right), \tag{26.2}$$

where F^n_{jk} is an abbreviation for $F(U^n_{jk})$, etc.

Equations (26.1) and (26.2) give no coupling between net points (in space-time) with even values of $n+j+k$ and those with odd values; half of the net points can therefore be omitted, if desired.

(Substitution of (26.1) into (26.2) for the case $F(U)=AU$ gives an equation for U^{n+2} in terms of U^n and its first and second partial differences with respect to x and y analogous to (16.3); a nine-point cluster is used, as in Figure 2; $\partial/\partial x$ is evaluated four times, using the point pairs labeled 1, 2, 3, 4 and averaging, $\partial^2/\partial x^2$ is evaluated using the points labeled A, and $\partial^2/\partial x \partial y$ is evaluated using the points labeled B.)

As in one space variable, these equations have error $0(\Delta^3)$, and the finite-difference analogues of the integrated conservation laws hold exactly. I shall show that the von Neumann stability condition, for the case $\Delta y = \Delta x$, is satisfied if

$$(|\vec{v}| + c) \frac{\Delta t}{\Delta x} < \frac{1}{\sqrt{2}} , \qquad (27)$$

where $|\vec{v}| = \sqrt{u^2+v^2}$; presumably (27) is also sufficient for stability but I have not yet worked out the proof.

The amplification matrix of the combined system (26.1) and (26.2), obtained by substituting $U_0 \exp(i\mathcal{K}_x j\Delta x + i\mathcal{K}_y k\Delta y)$ for U^n_{jk}, is

$$G = I - 2i(\cos\alpha + \cos\beta)\left[\frac{\Delta t}{\Delta x}(A \sin\alpha + B \sin\beta)\right] + 2\left[\frac{\Delta t}{\Delta x}(A \sin\alpha + B \sin\beta)\right]^2 , \qquad (28)$$

where $\alpha = \mathcal{K}_x \Delta x$ and $\beta = \mathcal{K}_y \Delta y$.

For the stability discussion, we shall take the linearized equations in the original Eulerian form and thus use the matrices A' and B' given by (25), which can be obtained from A, B by a similarity transformation like (10.2). As pointed out in section 5, this similarity transformation implies a change of the norm adopted for the function space of the dependent variables; however, the von Neumann condition remains unchanged, since the eigenvalues of G are invariant under the similarity transformation.

We call

$$C = A \sin\alpha + B \sin\beta$$

and let θ be the angle between the x axis and a direction whose direction cosines are proportional to $\sin\alpha$ and $\sin\beta$, i.e.,

$$\cos\theta = \frac{\sin\alpha}{\sqrt{\sin^2\alpha + \sin^2\beta}} .$$

Then, from (25),

$$C = \sqrt{\sin^2\alpha + \sin^2\beta}\begin{pmatrix} u' & \rho\cos\theta & \rho\sin\theta & 0 \\ 0 & u' & 0 & (1/\rho)\cos\theta \\ 0 & 0 & u' & (1/\rho)\sin\theta \\ 0 & \rho c^2\cos\theta & \rho c^2\sin\theta & u' \end{pmatrix}, \tag{29}$$

where $u' = u\cos\theta + v\sin\theta$ is the component of $\vec{v}$ in the direction θ. The eigenvalues of C are

$$\lambda = \sqrt{\sin^2\alpha + \sin^2\beta}\begin{Bmatrix} u' \\ u' \\ u' + c \\ u' - c \end{Bmatrix}, \tag{30}$$

and if we call $\mu = (\Delta t/\Delta x)\lambda$, the eigenvalues of G are

$$g = 1 - i\mu(\cos\alpha + \cos\beta) - 2\mu^2 . \tag{31}$$

Hence,

$$
\begin{aligned}
|g|^2 &= (1 - 2\mu^2)^2 + \mu^2(\cos\alpha + \cos\beta)^2 \\
&\leqq (1 - 2\mu^2)^2 + 2\mu^2(\cos^2\alpha + \cos^2\beta) \\
&= 1 - 2\mu^2\left[(\sin^2\alpha + \sin^2\beta) - 2\mu^2\right] \\
&= 1 - 2\mu^2(\sin^2\alpha + \sin^2\beta)\left[1 - 2\left(\frac{\Delta t}{\Delta x}\right)^2 \begin{Bmatrix} u' \\ u' \\ u' + c \\ u' - c \end{Bmatrix}^2\right].
\end{aligned}
$$

The largest possible value of the quantities in the curly bracket is $|\vec{v}| + c$; hence $|g|^2 \leqq 1$ if

$$\frac{\Delta t}{\Delta x}(|\vec{v}| + c) < \frac{1}{\sqrt{2}} \quad .$$

8. On Non-Linear Instability

In this section, the implications for the foregoing difference methods of the phenomenon of non-linear instability described by Phillips [4], are briefly discussed. Although a rigorous theory of this phenomenon is lacking, it seems clear that a distinction must be made between the usual problems of ordinary fluid dynamics and the problems of meteorology. In the first, turbulence plays no role, either because the flow is stable or because the time is too short for the physical instabilities to develop into turbulence. For these problems, the flow is piecewise smooth; it is conjectured that if shocks and other singularities are properly handled, and if a difference scheme, like c. or e., that has positive stability (in the sense that short-wave disturbances are damped out) is used, non-linear instabilities will not occur. This conclusion is borne out by experience. The example given below suggests that even the metastable leap-frog scheme d. is stable against small non-linear disturbances if the ordinary stability condition is met; however, it is precisely because of the doubtful nature of any general conclusion of this kind that the Lax-Wendroff scheme e. is preferred over d.

(It may be mentioned in passing that the short-wave oscillations often observed when shocks are treated by the artificial-viscosity method can be interpreted in terms of the ideas used in discussing non-linear instability: energy is transferred from longer to shorter waves; this energy ought to be converted into heat at the shocks, but if the dissipation by the viscosity term is inadequate, some of the energy accumulates at wave lengths of the order of $2\Delta x$, since shorter waves cannot be represented in the calculation.)

As an example, almost identical with that of Phillips but for a hyperbolic problem, the very simple equation

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial (u)^2}{\partial x} \tag{32}$$

is considered, together with the leap-frog difference equation

$$u_j^{n+1} - u_j^{n-1} = r\left[\left(u_{j+1}^{n}\right)^2 - \left(u_{j-1}^{n}\right)^2\right], \tag{33}$$

where $r = \Delta t/2\Delta x$. An exact solution of (33) having the form

$$u_j^n = C^n \cos\frac{\pi}{2}j + S^n \sin\frac{\pi}{2}j + U^n \cos\pi j + V \tag{34}$$

can be found; it contains wave numbers $(\pi/2)\Delta x$, $\pi\Delta x$, and 0. (Phillips omitted the constant term corresponding to V in (34), but we retain it because we are interested in cases in which the smooth part of the flow predominates, in a sense described below.) Upon substituting (34) into (33), we obtain recurrence relations for the coefficients:

$$C^{n+1} - C^{n-1} = -4rS^n(U^n - V), \tag{35.1}$$

$$S^{n+1} - S^{n-1} = -4rC^n(U^n + V), \tag{35.2}$$

$$U^{n+1} - U^{n-1} = 0. \tag{35.3}$$

According to (35.3), U^n takes on constant values, say A and B, on alternate cycles, and then the other two equations give

$$C^{n+2} - 2C^n + C^{n-2} = 16\, r^2\, (A+V)(B-V)C^n \tag{36}$$

and a similar equation for S^n. The solution of (36) is bounded, as n increases (in fact $|C^n|$ is constant) if the coefficient of C^n on the right lies between -4 and 0. The usual stability condition,

$$\frac{\Delta t}{\Delta x} \max |u| < 1,$$

implies

$$2r \left[|V| + \max\{|A|, |B|\} \right] < 1.$$

Therefore, even for the leap-frog scheme, this solution remains bounded if the constant term dominates, in the sense that

$$|A| \text{ and } |B| < |V|. \tag{37}$$

However, if the disturbance is out of the linear range, in the sense that (37) is violated, and if $A+V$ and $B-V$ have the same sign, then, as observed by Phillips, (36) always has exponentially growing solutions whose rate of growth cannot be decreased (is in fact increased) by decreasing Δt for given r.

In the problems of meteorology, one apparently cannot assume that the short-wave disturbances are small, even at the beginning of a calculation. Phillips found [4] that the non-linear instabilities could be avoided, in a typical numerical prediction of the general circulation of the atmosphere, by systematically filtering out all wave lengths between $2\Delta x$ and $4\Delta x$ at intervals of 2 hours of real time (every 6 cycles). However, Phillips was using a variant of the leap-frog scheme, and the question arises whether the automatic filtering of the Lax-Wendroff scheme, which attenuates the short waves, might accomplish the same result. Clearly this question cannot be answered without a detailed investigation, but a slight indication may be had by computing the linear attenuation factor $|g|^6$ for 6 cycles of a Lax-Wendroff calculation in a simple case. In Figure 3, $|g|^6$ is plotted against wave number k for a hyperbolic problem in one space variable in which Δt is one-half the maximum allowed value. Comparison with the filtering of Phillips (dashed line) suggests that, at least in some problems, the positive linear stability of the Lax-Wendroff scheme should overcome non-linear instability.

It should be noted that when the fluid-dynamical conservation-law equations are treated by the Lax-Wendroff scheme, the wave energy that is removed from the short waves by the automatic filtering is not removed from the system (since energy is conserved) but is converted into other forms of energy, as it would be by a true dissipation.

August 27, 1962

REFERENCES

[1.] Courant, R., K. O. Friedrichs, and H. Lewy: "Über die partiellen differenzengleichungen der mathematischen physik," Mathematische Annalen, 100, 32, 1928.

[2.] Courant, R., E. Isaacson, and M. Rees: "On the solution of non-linear hyperbolic differential equations by finite differences," Communications on Pure and Applied Mathematics, 5, 243, 1952.

[3.] Lax, P. D., and B. Wendroff: "Systems of conservation laws," Communications on Pure and Applied Mathematics, 13, 217, 1960.

[4.] Phillips, N. A.: "An example of non-linear computational instability," The Atmosphere and the Sea in Motion, Bert Bolin, editor, Rockefeller Institute Press, New York, 501, 1959.

[5.] Richtmyer, R. D.: Difference Methods for Initial Value Problems, Interscience Tracts in Pure and Applied Mathematics, No. 4, Wiley (Interscience), New York, 1957.

[6.] Richtmyer, R. D.: Progress Report on the Mach Reflection Calculation, Computing and Applied Mathematics Center, Courant Institute of Mathematical Sciences, New York University, 1961.

[7.] Thompson, P. D.: Numerical Weather Analysis and Prediction, Macmillan, New York, 1961.

TABLE I

Scheme	Designation	Difference scheme	Truncation error
a.	Unstable	$U_j^{n+1} = U_j^n - \frac{\Delta t}{2\Delta x} A\left(U_{j+1}^n - U_{j-1}^n\right)$	$0(\Delta^2)$
b.	Diffusing	$U_j^{n+1} = \frac{1}{2}\left(U_{j+1}^n + U_{j-1}^n\right) - \frac{\Delta t}{2\Delta x} A\left(U_{j+1}^n - U_{j-1}^n\right)$	same
c.	Upstream differencing	$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\begin{cases} u_{j+1}^n - u_j^n & \text{if } a > 0 \\ u_j^n - u_{j-1}^n & \text{if } a < 0 \end{cases}$	same
d.	Leap frog	$U_j^{n+1} = U_j^{n-1} - \frac{\Delta t}{\Delta x} A\left(U_{j+1}^n - U_{j-1}^n\right)$	$0(\Delta^3)$
e.	Lax-Wendroff	$U_j^{n+1} = U_j^n - \frac{\Delta t}{2\Delta x} A\left(U_{j+1}^n - U_{j-1}^n\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2 A^2\left(U_{j+1}^n - 2U_j^n + U_{j-1}^n\right)$	same
f.	Implicit	$U_j^{n+1} = U_j^n - \frac{\Delta t}{4\Delta x} A\left(\left(U_{j+1}^n + U_{j+1}^{n+1}\right) - \left(U_{j-1}^n + U_{j-1}^{n+1}\right)\right)$	same

TABLE II

Scheme	Amplification matrix	When stable:
a.	$G = I - i\frac{\Delta t}{\Delta x} A \sin\alpha$	never
b.	$G = I\cos\alpha - i\frac{\Delta t}{\Delta x} A \sin\alpha$	if $\left\vert\frac{a\Delta t}{\Delta x}\right\vert < 1$ for all eigenvalues a of A
c.	$g = 1 - a\frac{\Delta t}{\Delta x}\left\{\begin{matrix} e^{i\alpha} - 1 & \text{if } a < 0 \\ 1 - e^{i\alpha} & \text{if } a > 0 \end{matrix}\right\}$	same
d.	$G = G^{-1} - 2i\frac{\Delta t}{\Delta x} A \sin\alpha$	same
e.	$G = I - i\frac{\Delta t}{\Delta x} A \sin\alpha - \left(\frac{\Delta t}{\Delta x}\right)^2 A^2(1 - \cos\alpha)$	same
f.	$\left(I + \frac{i}{2}\frac{\Delta t}{\Delta x} A \sin\alpha\right) G = I - \frac{i}{2}\frac{\Delta t}{\Delta x} A \sin\alpha$	always

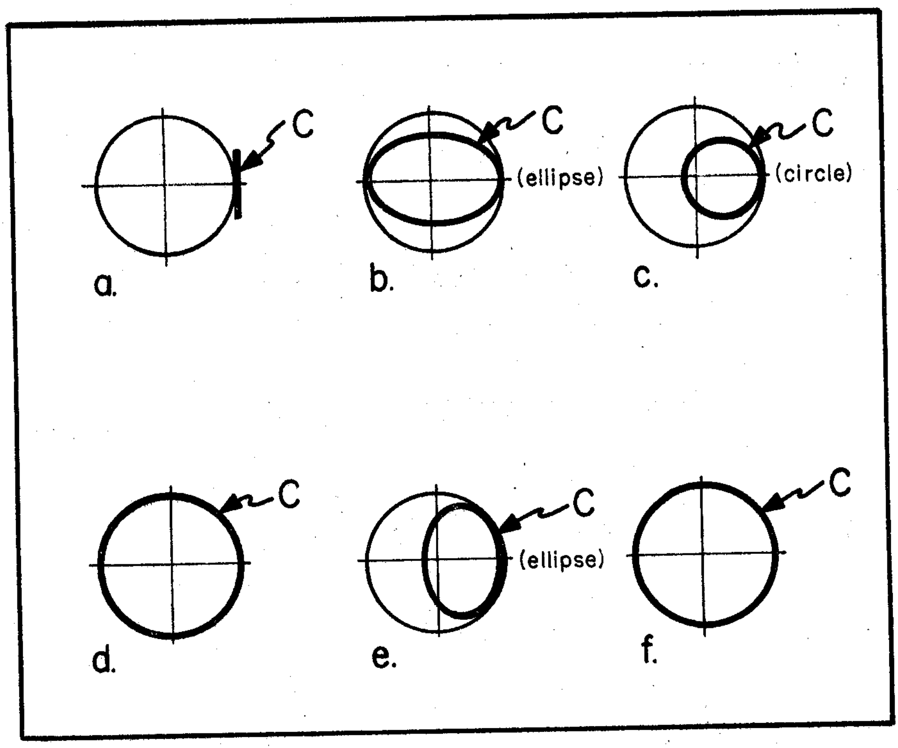

Figure 1: Amplification Values

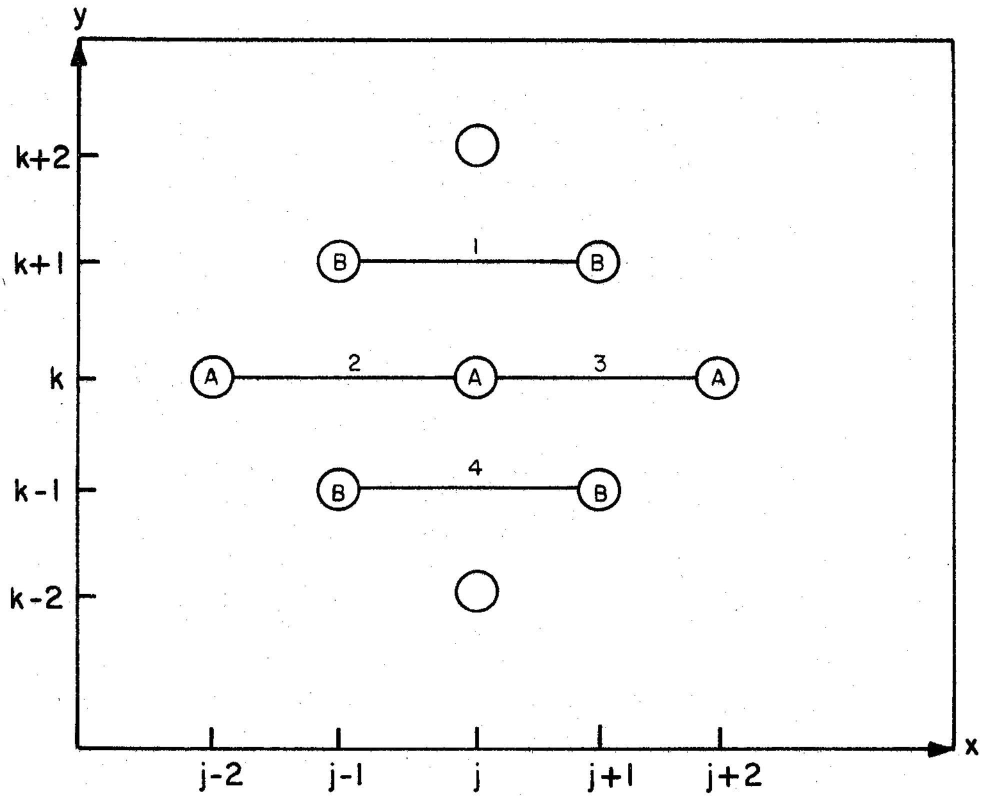

Figure 2: Net Point Cluster for Expression of $U_{j\ k}^{n+2}$ in Terms of U^n

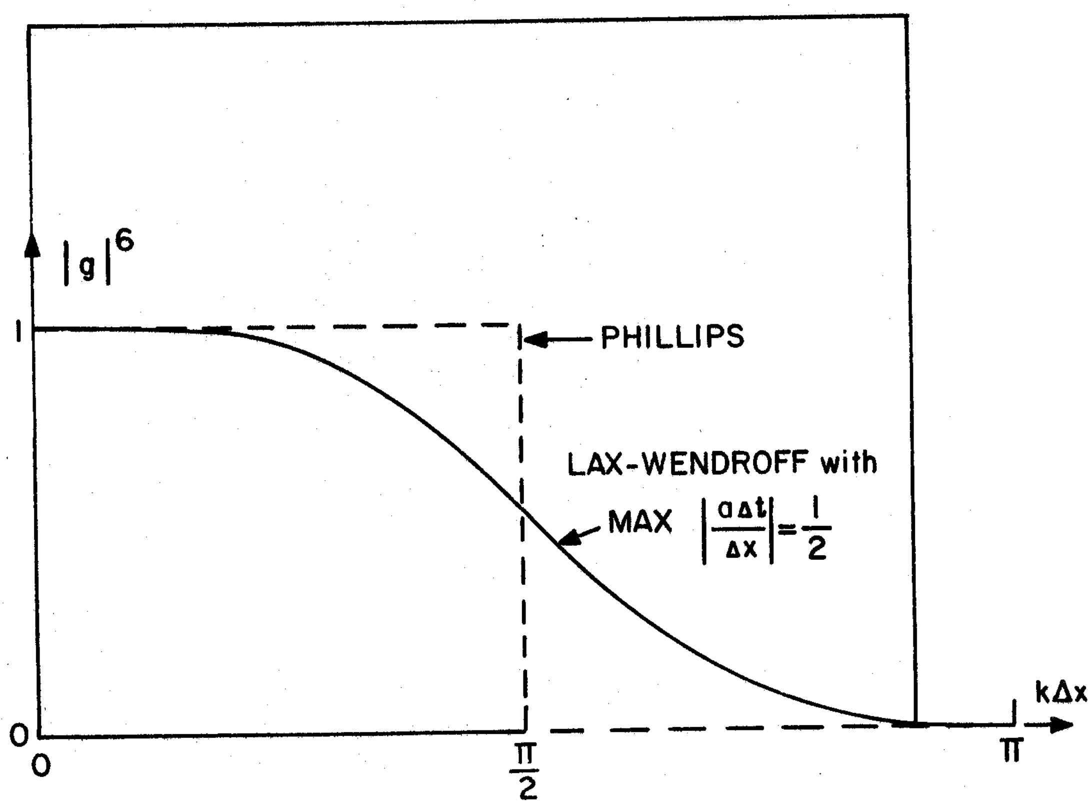

Figure 3: Filtering after Six Cycles