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Stability of Richtmyer Type Difference Schemes in any Finite Number of Space Variables and Their Comparison with Multistep Strang Schemes

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The stability of generalized Richtmyer two step difference schemes in any finite number of space variables is examined and a sufficient stability condition obtained for each scheme. In certain cases this condition is shown to be optimal C.F.L. The efficiency of these schemes in solving time dependent problems in two and three space variables is examined and the schemes are seen to compare favourably with the corresponding multistep forms of Strang's schemes.

1. Introduction

IN recent years several finite difference schemes have been proposed for the numerical solution of the hyperbolic equation $u_t+F(u)_x+G(u)_y = 0$ in the two Cartesian variables x and y. Lax & Wendroff (1964) proposed a scheme of second order accuracy for the solution of this equation. By considering the linearized form of the difference scheme they derived the local stability condition

$$\lambda(A)\frac{\Delta t}{\Delta x} \leqslant \frac{1}{\sqrt{8}}, \qquad \lambda(B)\frac{\Delta t}{\Delta x} \leqslant \frac{1}{\sqrt{8}}$$

in terms of the eigenvalues of the matrices $A(U)$ and $B(U)$ given by $F(U)_x = A(U)U_x$, $G(U)_y = B(U)U_y$, ignoring the dependence of $A(U)$ and $B(U)$ on U.

Richtmyer (1962) proposed a Runge-Kutta type two step version of this scheme of Lax & Wendroff.

Numerical experiments performed by Burstein (1964) and Morris & Gourlay (1968) suggest that the stability conditions, as given above, for this scheme are conservative. One of the purposes of this paper is to derive an improved stability condition for the Richtmyer two step scheme. The result obtained is a generalization of the result obtained by Richtmyer (1962) for hydrodynamic flow in two space variables and that obtained by Rubin & Preiser (1970) for flow in three space variables. This result is obtained in Section 2 for the general case of any finite number of space variables. Two other Richtmyer type schemes are examined in Section 2 and stability conditions again obtained. In Section 3 it is shown that two of these schemes are optimal Courant Friedricks Lewy. It is also shown that, although the stability of a difference scheme may be written as a fraction of the C.F.L. condition, it is not always relevant to compare schemes on this basis. This point is further emphasized in Section 4 where schemes suitable for the numerical integration of the unsteady flow equation in two and three space variables are examined. Section 5 shows the application of the results of Section 2 to a third order scheme for two space variables.

(2.4) is also a Richtmyer type scheme. The points sequence in m space representing this scheme can be obtained from those representing (2.2) by a rotation of $\pi/4$ in each plane. This scheme will be referred to as the rotated scheme.

To examine the local stability of (2.2), (2.3) and (2.4) put $F_p(U)_x = A_p(U)U_x$ for $p = 1, 2, \ldots, m$, where $A_p(U)$ is the Jacobian of $F_p(U)$. The schemes are linearized by taking the matrices $A_p(U)$ to be constant. These constant matrices will be represented by A_p. For the case when the spatial mesh sizes are equal, i.e. $\Delta x_p = \Delta x$, the amplification matrices of the schemes (2.2), (2.3) and (2.4), obtained by substituting

$$U_0^n \exp\left(i \sum_{p=1}^{m} \tfrac{1}{2}k_p j_p \Delta x\right) \quad \text{for } U_j^n,$$

are respectively

$$G_1 = I-\frac{2}{m}irE_1 \sum_{p=1}^{m} \cos\alpha_p - 2r^2E_1^2, \tag{2.5}$$

$$G_2 = I-\frac{2}{m}irE_2 \prod_{p=1}^{m} \cos\alpha_p - 2r^2E_2^2, \tag{2.6}$$

$$G_3 = I-2irE_3 \prod_{p=1}^{m} \cos\alpha_p - 2r^2E_3^2, \tag{2.7}$$

where $\alpha_p = \frac{1}{2}k_p\Delta x$ for $p = 1, 2, \ldots, m$, $r = \Delta t/\Delta x$, $E_1 = E_2 = \sum_{p=1}^{m} A_p \sin\alpha_p$, $E_3 = \sum_{p=1}^{m} A_p \sin\alpha_p \prod_{\substack{q=1\\q\neq p}}^{m} \cos\alpha_q$.

We now consider the characteristic surfaces associated with (2.1). At time t let $(x_1, x_2, \ldots, x_m)$ be any point in Cartesian m space and $\gamma(x_1, x_2, \ldots, x_m, t) = 0$ any surface, with direction cosines l_p, $p = 1, 2, \ldots, m$. Change the variables from the x's and t to $\beta_1, \beta_2, \ldots, \beta_m, \gamma$, where the β's and γ are functions of the x's and t, chosen so that the transformation is non-singular and the surface is $\gamma = 0$.

(2.1) then becomes

$$\sum_{p=1}^{m} \frac{\partial\beta_p}{\partial t}\cdot\frac{\partial U}{\partial\beta_p}+\frac{\partial\gamma}{\partial t}\cdot\frac{\partial U}{\partial\gamma}+\sum_{p=1}^{m} A_p(U)\left(\sum_{j=1}^{m}\frac{\partial\beta_j}{\partial x_p}\cdot\frac{\partial U}{\partial\beta_j}+\frac{\partial\gamma}{\partial x_p}\cdot\frac{\partial U}{\partial\gamma}\right) = 0,$$

that is

$$\sum_{p=1}^{m}\left(\frac{\partial\beta_p}{\partial t}\frac{\partial U}{\partial\beta_p}+A_p(U)\sum_{j=1}^{m}\frac{\partial\beta_j}{\partial x_p}\frac{\partial U}{\partial\beta_j}\right)+N\frac{\partial U}{\partial\gamma} = 0$$

where

$$N = \frac{\partial\gamma}{\partial t}I+\sum_{p=1}^{m} A_p(U)\frac{\partial\gamma}{\partial x_p}.$$

Let c be a scalar such that $dx_p/dt = cl_p$, for $p = 1, 2, \ldots, m$, on a ray lying in the surface. Then

$$\sum_{p=1}^{m} dx_p l_p = c\,dt$$

and since the surface is $\gamma = 0$,

$$d\gamma = k\left\{c\,dt-\sum_{p=1}^{m} l_p\,dx_p\right\} \tag{2.8}$$

where k is some factor.

From (2.8)

$$\frac{\partial \gamma}{\partial t} = kc, \qquad \frac{\partial \gamma}{\partial x_p} = -kl_p$$

and thus $\gamma = 0$ is a characteristic surface if and only if

$$\det\left(k\left\{cI - \sum_{p=1}^{m} A_p(U)l_p\right\}\right) = 0, \quad \text{or} \quad \det\left(cI - \sum_{p=1}^{m} A_p(U)l_p\right) = 0.$$

This result also holds when $A_p(U)$ is replaced by the constant matrix A_p. Therefore c is an eigenvalue of

$$B = \sum_{p=1}^{m} A_p l_p,$$

where B has linear divisors. The matrices E_1, E_2 and E_3 which occur in the expressions (2.5), (2.6) and (2.7) for the amplification matrices G_1, G_2, G_3 have a form similar to that of B.

Indeed if l_p is chosen so that

$$\sin \alpha_p = l_p\left(\sum_{p=1}^{m} \sin^2 \alpha_p\right)^{\frac{1}{2}}$$

then

$$E_1 = E_2 = \left(\sum_{p=1}^{m} \sin^2 \alpha_p\right)^{\frac{1}{2}} B$$

also has linear divisors and the eigenvalues of these matrices are

$$\lambda(E_1) = \lambda(E_2) = \left(\sum_{p=1}^{m} \sin^2 \alpha_p\right)^{\frac{1}{2}} c.$$

If l_p is chosen so that

$$\sin \alpha_p \prod_{\substack{q=1\\ p\neq q}}^{m} \cos \alpha_q = l_p\left\{\sum_{p=1}^{m} \sin^2 \alpha_p\left(\prod_{\substack{q=1\\ p\neq q}}^{m} \cos^2 \alpha_q\right)\right\}^{\frac{1}{2}}$$

then

$$E_3 = \left\{\sum_{p=1}^{m} \sin^2 \alpha_p\left(\prod_{\substack{q=1\\ q\neq p}}^{m} \cos^2 \alpha_q\right)\right\}^{\frac{1}{2}} B$$

also has linear divisors and the eigenvalues of E_3 are

$$\lambda(E_3) = \left\{\sum_{p=1}^{m} \sin^2 \alpha_p\left(\prod_{\substack{q=1\\ q\neq p}}^{m} \cos^2 \alpha_q\right)\right\}^{\frac{1}{2}} c.$$

If $\xi_s = \lambda(E_s)r$, the matrices G_1, G_2, G_3 given by (2.5), (2.6), (2.7) have an eigenvalue g_s respectively given by

$$g_1 = 1 - \frac{2}{m} i\xi_1 \sum_{p=1}^{m} \cos \alpha_p - 2\xi_1^2, \tag{2.9}$$

$$g_2 = 1 - \frac{2}{m} i\xi_2 \prod_{p=1}^{m} \cos \alpha_p - 2\xi_2^2, \tag{2.10}$$

$$g_3 = 1 - 2i\xi_3 \prod_{p=1}^{m} \cos \alpha_p - 2\xi_3^2. \tag{2.11}$$

Using the stability conditions derived in Section 2, the Richtmyer type schemes in two and three space variables are shown in Section 6 to compare favourably with the corresponding multistep Strang schemes of Morris & Gourlay.

2. Stability Condition

We consider the conservation equation

$$\frac{\partial u}{\partial t}+\sum_{p=1}^{m}\frac{\partial F(u)_p}{\partial x_p}=0 \tag{2.1}$$

in $m(\geqslant 1)$ space variables x_p for $p = 1, 2, \ldots, m$, where u and $F(u)$ are vector quantities. (2.1) is taken to be hyperbolic.

Three two step second order difference schemes are considered for the numerical integration of (2.1).

(1)

$$\begin{aligned} U_{\mathbf{j}}^{n+\frac{1}{2}} &= \frac{1}{m}\sum_{p=1}^{m}\mu_p U_{\mathbf{j}}^n-\frac{1}{2}\sum_{p=1}^{m}\frac{\Delta t}{\Delta x_p}\delta_p F_{p;\mathbf{j}}^n \\ U_{\mathbf{j}}^{n+1} &= U_{\mathbf{j}}^n-\sum_{p=1}^{m}\frac{\Delta t}{\Delta x_p}\delta_p F_{p;\mathbf{j}}^{n+\frac{1}{2}} \end{aligned} \tag{2.2}$$

where the step sizes are taken as Δt, Δx_p for $p = 1, 2, \ldots, m$ and μ_p and δ_p are defined by

$$\mu_p V_{\mathbf{j}} = \tfrac{1}{2}(V_{\mathbf{j}+\frac{1}{2}\mathbf{e}_p}+V_{\mathbf{j}-\frac{1}{2}\mathbf{e}_p}) \quad \text{and} \quad \delta_p V_{\mathbf{j}} = V_{\mathbf{j}+\frac{1}{2}\mathbf{e}_p}-V_{\mathbf{j}-\frac{1}{2}\mathbf{e}_p}$$

with $\mathbf{j} = (j_1, j_2, \ldots, j_m)$ the vector of suffices and $\mathbf{e}_p$ a unit vector in m space with pth entry unity.

(2.2) is a generalization to m variables of the scheme given by Richtmyer (1962).

(2)

$$\begin{aligned} U_{\mathbf{j}}^{n+\frac{1}{2}} &= \nu U_{\mathbf{j}}^n-\frac{1}{2}\sum_{p=1}^{m}\frac{\Delta t}{\Delta x_p}\delta_p F_{p;\mathbf{j}}^n \\ U_{\mathbf{j}}^{n+1} &= U_{\mathbf{j}}^n-\sum_{p=1}^{m}\frac{\Delta t}{\Delta x_p}\delta_p F_{p;\mathbf{j}}^{n+\frac{1}{2}} \end{aligned} \tag{2.3}$$

where

$$\nu V_{\mathbf{j}} = \frac{1}{2^m}\sum_{\mathbf{i}\in\Omega} V_{\mathbf{i}} \quad \text{and} \quad \Omega = \{\mathbf{i}\colon |\mathbf{i}_k-\mathbf{j}_k| = \tfrac{1}{2}\mathbf{e}_k, \qquad k = 1, 2, \ldots, m\}.$$

This scheme differs from the first scheme in the form of the averaging term of the first step. It will be referred to as the modified scheme.

(3)

$$\begin{aligned} U_{\mathbf{j}}^{n+\frac{1}{2}} &= \nu U_{\mathbf{j}}^n-\frac{1}{2}\sum_{p=1}^{m}\frac{\Delta t}{\Delta x_p}\delta_{\nu;p} F_{p;\mathbf{j}}^n \\ U_{\mathbf{j}}^{n+1} &= U_{\mathbf{j}}^n-\sum_{p=1}^{m}\frac{\Delta t}{\Delta x_p}\delta_{\nu;p} F_{p;\mathbf{j}}^n \end{aligned} \tag{2.4}$$

where

$$\delta_{\nu;p} V_{\mathbf{j}} = \frac{1}{2^{m-1}}\Big(\sum_{\mathbf{i}\in\Omega_1} V_{\mathbf{i}}-\sum_{\mathbf{i}\in\Omega_2} V_{\mathbf{i}}\Big),$$

$$\Omega_1 = \{\mathbf{i}\colon |\mathbf{i}_k-\mathbf{j}_k| = \tfrac{1}{2}\mathbf{e}_k, \qquad k = 1, 2, \ldots, m, k \neq p, \mathbf{i}_p = \mathbf{j}_p+\tfrac{1}{2}\}$$

and

$$\Omega_2 = \{\mathbf{i}\colon |\mathbf{i}_k-\mathbf{j}_k| = \tfrac{1}{2}\mathbf{e}_k, \qquad k = 1, 2, \ldots, m, k \neq p, \mathbf{i}_p = \mathbf{j}_p-\tfrac{1}{2}\}.$$

From (2.9) $|g| \leqslant 1$ if

$$\left(c\frac{\Delta t}{\Delta x}\right)^2 \leqslant \frac{m^2-\left(\sum_{p=1}^{m} \cos\alpha_p\right)^2}{m^2\left(\sum_{p=1}^{m} \sin^2\alpha_p\right)}. \tag{2.12}$$

Similarly from (2.10), (2.11) respectively $|g| \leqslant 1$ if

$$\left(c\frac{\Delta t}{\Delta x}\right)^2 \leqslant \frac{m^2-\left(\prod_{p=1}^{m} \cos\alpha_p\right)^2}{m^2\left(\sum_{p=1}^{m} \sin^2\alpha_p\right)}, \tag{2.13}$$

and

$$\left(c\frac{\Delta t}{\Delta x}\right)^2 \leqslant \frac{1-\left(\prod_{p=1}^{m} \cos\alpha_p\right)^2}{\sum_{p=1}^{m} \sin^2\alpha_p\left(\prod_{\substack{q=1\\ q\neq p}}^{m} \cos^2\alpha_q\right)}. \tag{2.14}$$

The minimum value of RHS of (2.12) occurs when $\cos\alpha_1 = \ldots = \cos\alpha_m$ and is $1/m$. For (2.13) the minimum is $1/m$ and occurs when $\alpha_1 = \alpha_2 = \ldots = \alpha_m = \pi/2+n\pi$. The RHS of (2.14) is a minimum when $\alpha_1 = \ldots \alpha_m = n\pi$ and has the value 1. The matrices E_s, $s = 1, 2, 3$, have linear divisors so that G_1, G_2 and G_3 each have a complete set of linearly independent eigenvectors. Hence the sufficient stability conditions for the linearized form of the given difference schemes (2.2), (2.3), (2.4) are respectively

$$c(\alpha_1, \alpha_2, \ldots, \alpha_m)\frac{\Delta t}{\Delta x} \leqslant \frac{1}{\sqrt{m}} \tag{2.15}$$

$$c(\alpha_1, \alpha_2, \ldots, \alpha_m)\frac{\Delta t}{\Delta x} \leqslant \frac{1}{\sqrt{m}} \tag{2.16}$$

and

$$c(\alpha_1, \alpha_2, \ldots, \alpha_m)\frac{\Delta t}{\Delta x} \leqslant 1 \tag{2.17}$$

for all

$$(\alpha_1, \alpha_2, \ldots, \alpha_m).$$

These conditions are also necessary conditions for the schemes (2.2), (2.3) and (2.4).

3. Courant Friedricks Lewy Condition

We consider the solution of the equation

$$\frac{\partial u}{\partial t}+\sum_{p=1}^{m}\frac{\partial F(u)_p}{\partial x_p} = 0 \quad \text{at } (P, t_0)$$

where P is any point in m space and t_0 any time. Let $D(P, t_0)$ denote the set of points on the initial hyperplane $t = 0$ where the values of the initial data influence the values of the solutions of the given equation at (P, t_0) and $D_h(P, t_0)$ the corresponding set for the difference equation approximating the given differential equation. It was observed by Courant, Friedricks & Lewy (1928) that if a difference scheme is convergent for all smooth initial data, then as $h \to 0$ for any P and t_0, the set $D(P, t_0)$ must be contained in the set of limit points of $D_h(P, t_0)$. We consider $D_h(P, t_0)$. The value of the quantity to be evaluated at the nodal point (P, t_0) at a time $(n+1)\Delta t$, say, depends upon its value at a set of nodal points at the previous time level $n\Delta t$. The value at each of these nodes in turn depends on its value at a set of nodes at the time level $(n-1)\Delta t$, the domain of dependence of (P, t_0) at the level being the union of these sets. If the process is extended back to the initial hyperplane we obtain a region in space and its boundary defines the set of limit points of $D_h(P, t_0)$. Thus $D_h(P, t_0)$ depends upon the space increments Δx_p, $p = 1, 2, \ldots, m$ and the time increment Δt. Because of homogeneity, P may be taken as the origin and t_0 as unity in determining $D_h(P, t_0)$ and $D(P, t_0)$ in the case of constant coefficients. In this case $D_h(P, t_0)$ is obtained as above by a single projection back to the initial hyperplane. $D(P, t_0)$ is not so easy to determine. However its support function

$$h_D(l_1, \ldots, l_m) = \lambda_{\max}\left(\sum_{p=1}^{m} A_p l_p\right) \quad \text{(see Lax, 1958)}$$

can be obtained since the right hand side of this equality is the maximum speed of propagation c in the direction $(l_1, \ldots, l_m)$. The Courant Friedricks Lewy observation, given above, reduces to the condition that the convex hull of $D_h(P, t_0)$ must include the convex hull of $D(P, t_0)$ and this is the case if and only if the support function of $D(P, t_0)$ never exceeds that of $D_h(P, t_0)$.

The support function $h_s(l_1, l_2, \ldots, l_m)$ of $D_h(P, t_0)$ is the "distance" from the "centre" to a limit point of the set. It is therefore proportional to $\Delta x/\Delta t$ and we can write

$$h_s(l_1, l_2, \ldots, l_m) = \frac{\Delta x}{\Delta t}\sigma(l_1, l_2, \ldots, l_m),$$

where $\sigma(l_1, l_2, \ldots l_m)$ is independent of Δx and Δt. Thus, for all l, the CFL condition is

$$h_D(l_1, \ldots, l_m) \leqslant h_s(l_1, \ldots, l_m). \tag{3.1}$$

Lax & Wendroff (1962) have shown that convergence and stability are equivalent so that the CFL condition gives a necessary condition for stability.

We now consider the difference scheme (2.2). $D_h(P, t_0)$ will be an m dimensional double "pyramid" with diagonals $2\Delta x_p$, $p = 1, 2, \ldots, m$. When the spatial mesh sizes all equal Δx, the domain becomes a square, with vertices at $(\pm 1, 0)$, $(0, \pm 1)$ in the two space variable case and a regular m dimensional double "pyramid" with vertices at the unit points on the axes in the general case.

Both stability conditions (2.15) and (3.1) imply upper limits for $\Delta t/\Delta x$ for this scheme.

In the case of (2.15) this condition is

$$\frac{\Delta t}{\Delta x} \leqslant \min_{\text{all } l}\left\{\frac{1}{\sqrt{m}\,.\,h_D(l)}\right\} = r_1 \text{ (say)} \tag{3.2}$$

and for (3.1) it is

$$\frac{\Delta t}{\Delta x} \leqslant \min_{\text{all } l}\left\{\frac{\sigma(l)}{h_D(l)}\right\} = r_2 \text{ (say)}. \tag{3.3}$$

For the generalized Richtmyer scheme (2.2) $1/\sqrt{m} \leqslant \sigma(l) \leqslant 1$. If we write $q = r_1/r_2$

then q is a measure for a particular difference equation of how close the sufficient condition for stability (2.15) is to the necessary condition (3.1) so that the positive number q cannot exceed one. Indeed $q = 1$ gives an optimal scheme.

We will now determine the maximum and minimum values of q for the scheme (2.2). The minimum value of q occurs when $D(P, t_0)$ touches $D_h(P, t_0)$ at a vertex so that $\sigma(l) = 1$ therefore $q_{\min} = 1/\sqrt{m}$. Such conditions are uncommon in practice since they would necessitate $D(P, t_0)$ having a sharp convex corner, at least in the limit, and the hypercone from the next time level to $D(P, t_0)$ being steeply skewed. Figure 1 illustrates these conditions in the case of two space variables. The maximum value of q occurs when $D(P, t_0)$ touches $D_h(P, t_0)$ at the centroid of a face, or the mid point of a side in the two variable case, so that $\sigma(l) = 1/\sqrt{m}$ and therefore $q_{\max} = 1$. Conditions such as these are attainable since many problems in mechanics and fluid dynamics involve flow in an isotropic medium and in such cases $D(P, t_0)$, referred to a Cartesian system, is often circular in two space dimensions and a hypersphere in a general finite dimensional space touching $D_h(P, t_0)$ as indicated above.

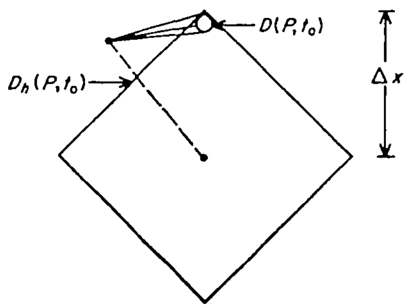


FIG. 1.

Hence for the generalized Richtmyer scheme (2.2), $1/\sqrt{m} \leqslant q \leqslant 1$. Since $q_{\max} = 1$, the sufficient and necessary conditions given respectively by (3.2) and (3.3) are equal so that under certain conditions the linearized form of the generalized Richtmyer scheme (2.2) is optimal CFL. This implies that the stability condition (2.15) cannot be improved upon for this difference scheme. The difference schemes (2.3) and (2.4) can be similarly treated. When the spatial mesh sizes are equal $D_h(P, t_0)$ for the rotated Richtmyer scheme is a hypercube with the 2^m vertices at points whose co-ordinates are all either $+1$ or -1. The sufficient and necessary stability conditions again imply upper limits for $\Delta t/\Delta x$.

The sufficient condition (2.17) gives

$$\frac{\Delta t}{\Delta x} \leqslant \min_{\text{all } l}\left\{\frac{1}{h_D(l)}\right\} = r_1$$

and the necessary condition (3.1) gives

$$\frac{\Delta t}{\Delta x} \leqslant \min_{\text{all } l}\left\{\frac{\sigma(l)}{h_D(l)}\right\} = r_2.$$

$1 \leqslant \sigma(l) \leqslant \sqrt{m}$ and $q_{\max}$ occurs when $D(P, t_0)$ touches $D_h(P, t_0)$ at the centroid of a face so that $\sigma(l) = 1$ and $q_{\max}$ is unity. Similarly $q_{\min} = 1/\sqrt{m}$ occurs for coincidence at a corner and corresponds to $\sigma(l) = \sqrt{m}$. q_{max} being unity means that the sufficient and necessary stability conditions can be equal. Thus scheme (2.4) can be optimal so that the stability condition (2.17) for the rotated scheme cannot be improved upon. Similarly the modified scheme (2.3), for which $D_h(P, t_0)$ is a hypercube with its edges removed, yields $q_{max} = 1/\sqrt{m}$ so that the modified scheme is not optimal.

In solving a particular partial differential equation problem several difference schemes, for which sufficient stability conditions are known in terms of a bound for $\Delta t/\Delta x$ are usually available to effect a numerical integration of the given equation. It is therefore possible to determine a number q, $0 < q \leqslant 1$, for each of these schemes and this quantity appears to be a possible criterion for comparing the stability of these difference schemes. It will now be illustrated that the comparison of schemes on this basis can lead to results which may not be relevant. Consider the solution of a hyperbolic partial differential equation in two space variables so that $D(P, t_0)$ is fixed irrespective of the difference scheme used. Two schemes (a) and (b) will be considered. $D_h(P, t_0)$ for these schemes is shown in Fig. 2(a): that for scheme (a) by the solid line and that for scheme (b) by the dashed line. Let the sufficient stability condition, $\Delta t/\Delta x \leqslant r_s$ say, be the same for each scheme. In the situation indicated by Fig. 2(a), $D_h(P, t_0)$ differs for these schemes but the necessary condition

$$\frac{\Delta t}{\Delta x} \leqslant r_N = \min_{\text{all } l} \left\{ \frac{\sigma(l)}{h_D(l)} \right\}$$

is the same for each domain, that for (a) using the $\sigma(l)_{min}$ and that for (b) using the $\sigma(l)_{max}$ appropriate to the scheme. Therefore q will be the same for each scheme, being q_{max} for (a) and q_{min} for scheme (b). If the physical situation changes so that $D(P, t_0)$ is displaced, the necessary conditions will now differ.

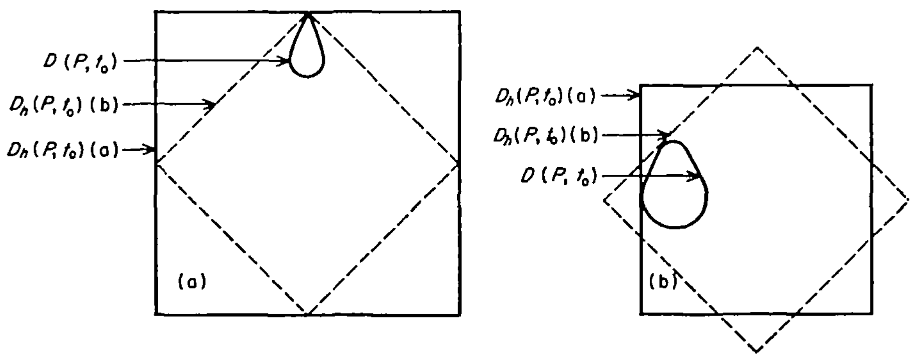


FIG. 2.

In the situation shown in Fig. 2(b), where the solid line again represents $D_h(P, t_0)$ for scheme (a) and the dashed line for scheme (b), let the sufficient conditions be identical for the schemes. In this case q will be different for schemes under consideration with $q(b) > q(a)$. If stability were the only criterion for comparing schemes and this comparison was made in terms of the number q the above results would indicate that in the situation shown in Fig. 2(a) both schemes would be equally efficient and that scheme (b) would be the more efficient in the situation shown in Fig. 2(b), one scheme being regarded as more efficient than another if, for a given accuracy, computation over a given space is performed more quickly. Hence the relative efficiency of the schemes for the solution of a given equation would alter with variation of q. Therefore comparison of schemes using the number q as a criterion can only be used when the variation in q is identical for the schemes under consideration, this requiring $D_h(P, t_0)$ to be similar for the schemes. If this condition is not satisfied, comparison can only be made in terms of the sufficient stability conditions with the CFL condition providing an upper bound for the sufficient conditions, although the use of the q comparison with non similar $D_h(P, t_0)$ may lead to correct comparisons in some specific cases as indicated above. On the basis of the sufficient stability conditions the schemes (a) and (b) would be equally efficient. The incorrectness of using the number q to compare the stability of various difference schemes will be further illustrated in Section 4 by considering specific difference schemes. Details of the two and three dimensional forms of the schemes discussed in Sections 2 and 3 will be included in this section for the sake of completeness.

4. Application of Previous Results to Specific Schemes

We consider six difference schemes for the numerical integration of

$$u_t+F(u)_x+G(u)_y = 0, \tag{4.1}$$

which is (2.1) with $m = 2$ and five schemes for the numerical integration of

$$u_t+F(u)_x+G(u)_y+H(u)_z = 0, \tag{4.2}$$

which is (2.1) with $m = 3$. In each case necessary and sufficient stability conditions for the linearized schemes will be given when the spatial mesh sizes are equal. The schemes for the integration of equation (4.1) will be considered first.

(i) Richtmyer Scheme

$$\begin{aligned} U^{n+\frac{1}{2}}_{j,k} &= \tfrac{1}{4}(U^n_{j+\frac{1}{2},k}+U^n_{j-\frac{1}{2},k}+U^n_{j,k+\frac{1}{2}}+U^n_{j,k-\frac{1}{2}})- \\ &\qquad \frac{\Delta t}{2\Delta x}(F^n_{j+\frac{1}{2},k}-F^n_{j-\frac{1}{2},k})-\frac{\Delta t}{2\Delta y}(G^n_{j,k+\frac{1}{2}}-G^n_{j,k-\frac{1}{2}}) \\ U^{n+1}_{j,k} &= U^n_{j,k}-\frac{\Delta t}{\Delta x}(F^{n+\frac{1}{2}}_{j+\frac{1}{2},k}-F^{n+\frac{1}{2}}_{j-\frac{1}{2},k})-\frac{\Delta t}{\Delta y}(G^{n+\frac{1}{2}}_{j,k+\frac{1}{2}}-G^{n+\frac{1}{2}}_{j,k-\frac{1}{2}}). \end{aligned} \tag{4.3}$$

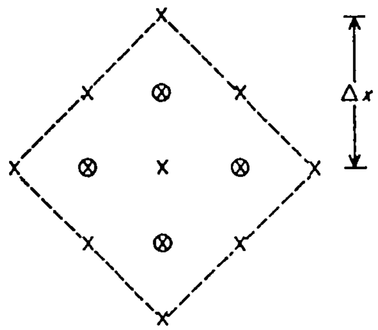
This scheme, which is (2.2) with $m = 2$, $F_1(U) = F(U)$, $F_2(U) = G(U)$, is a nine point scheme for which $D_h(P, t_0)$, shown in Fig. 3, is a square.

The tightest CFL condition, corresponding to r_2 giving q_{max}, is $c\Delta t/\Delta x \leqslant 1/\sqrt{2}$. q can be unity since (2.15) gives the sufficient stability condition as

$$\frac{\Delta t}{\Delta x}c \leqslant \frac{1}{\sqrt{2}}. \tag{4.4}$$

(ii) Modified Richtmyer Scheme

$$\begin{aligned} U^{n+\frac{1}{2}}_{j,k} &= \tfrac{1}{4}(U^n_{j+\frac{1}{2},k+\frac{1}{2}}+U^n_{j-\frac{1}{2},k+\frac{1}{2}}+U^n_{j+\frac{1}{2},k-\frac{1}{2}}+U^n_{j-\frac{1}{2},k-\frac{1}{2}})- \\ &\qquad \frac{\Delta t}{2\Delta x}(F^n_{j+\frac{1}{2},k}-F^n_{j-\frac{1}{2},k})-\frac{\Delta t}{2\Delta y}(G^n_{j,k+\frac{1}{2}}-G^n_{j,k-\frac{1}{2}}) \\ U^{n+1}_{j,k} &= U^n_{j,k}-\frac{\Delta t}{\Delta x}(F^{n+\frac{1}{2}}_{j+\frac{1}{2},k}-F^{n+\frac{1}{2}}_{j-\frac{1}{2},k})-\frac{\Delta t}{\Delta y}(G^{n+\frac{1}{2}}_{j,k+\frac{1}{2}}-G^{n+\frac{1}{2}}_{j,k-\frac{1}{2}}). \end{aligned} \tag{4.5}$$



⊗ Intermediate plane points

FIG. 3.

This is (2.3) with $m = 2$, $F_1(U) = F(U)$, $F_2(U) = G(U)$. The scheme, for which $D_h(P, t_0)$ is shown in Fig. 4, is a 21 point scheme. The tightest CFL condition is $c\Delta t/\Delta x \leqslant 1$. The scheme is not optimal and $q_{max} = 1/\sqrt{2}$ since (2.16) gives the sufficient condition

$$c\frac{\Delta t}{\Delta x} \leqslant \frac{1}{\sqrt{2}}. \tag{4.6}$$

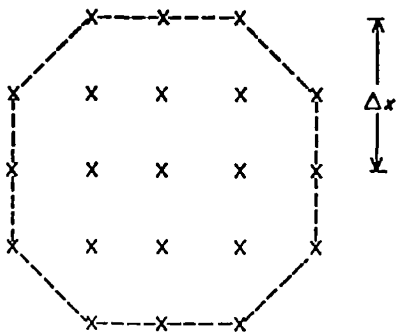


FIG. 4.

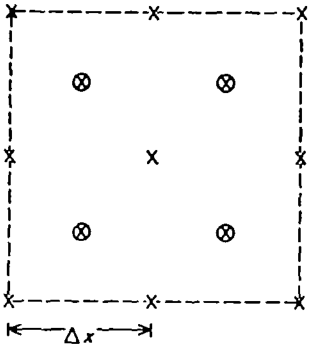
(iii) Rotated Richtmyer Scheme

$$\begin{aligned} U^{n+\frac{1}{2}}_{j,k} &= \tfrac{1}{4}(U^n_{j+\frac{1}{2},k+\frac{1}{2}}+U^n_{j-\frac{1}{2},k+\frac{1}{2}}+U^n_{j+\frac{1}{2},k-\frac{1}{2}}+U^n_{j-\frac{1}{2},k-\frac{1}{2}})-\frac{\Delta t}{4\Delta x}(F^n_{j+\frac{1}{2},k+\frac{1}{2}}+F^n_{j+\frac{1}{2},k-\frac{1}{2}}- \\ &\quad F^n_{j-\frac{1}{2},k+\frac{1}{2}}-F^n_{j-\frac{1}{2},k-\frac{1}{2}})-\frac{\Delta t}{4\Delta y}(G^n_{j+\frac{1}{2},k+\frac{1}{2}}+G^n_{j-\frac{1}{2},k+\frac{1}{2}}-G^n_{j+\frac{1}{2},k-\frac{1}{2}}-G^n_{j-\frac{1}{2},k-\frac{1}{2}}) \\ U^{n+1}_{j,k} &= U^n_{j,k}-\frac{\Delta t}{2\Delta x}(F^{n+\frac{1}{2}}_{j+\frac{1}{2},k+\frac{1}{2}}+F^{n+\frac{1}{2}}_{j+\frac{1}{2},k-\frac{1}{2}}-F^{n+\frac{1}{2}}_{j-\frac{1}{2},k+\frac{1}{2}}-F^{n+\frac{1}{2}}_{j-\frac{1}{2},k-\frac{1}{2}})- \\ &\quad \frac{\Delta t}{2\Delta y}(G^{n+\frac{1}{2}}_{j+\frac{1}{2},k+\frac{1}{2}}+G^{n+\frac{1}{2}}_{j-\frac{1}{2},k+\frac{1}{2}}-G^{n+\frac{1}{2}}_{j+\frac{1}{2},k-\frac{1}{2}}-G^{n+\frac{1}{2}}_{j-\frac{1}{2},k-\frac{1}{2}}). \end{aligned} \tag{4.7}$$

(4.7) is (2.4) with $m = 2$, $F_1(U) = F(U)$, $F_2(U) = G(U)$. $D_k(P, t_0)$ for this nine point scheme is a square.

The tightest CFL condition is $c\Delta t/\Delta x \leqslant 1$ so that q can be unity since the sufficient stability condition is given by (2.17) as

$$\frac{\Delta t}{\Delta x}c \leqslant 1. \tag{4.8}$$



⊗ Intermediate plane points

FIG. 5.

(*iv*) *Singleton Scheme*

Singleton (1968) proposed the scheme

$$\begin{aligned}
U^{n+\frac{1}{2}}_{j+\frac{1}{2},k} &= \tfrac{1}{2}(U^n_{j,k}+U^n_{j+1,k})-\frac{\Delta t}{2\Delta x}(F^n_{j+1,k}-F^n_{j,k})-\\
&\qquad \frac{\Delta t}{8\Delta y}(G^n_{j,k+1}+G^n_{j+1,k+1}-G^n_{j,k-1}-G^n_{j+1,k-1})\\
U^{n+\frac{1}{2}}_{j,k+\frac{1}{2}} &= \tfrac{1}{2}(U^n_{j,k}+U^n_{j,k+1})-\frac{\Delta t}{2\Delta y}(G^n_{j,k+1}-G^n_{j,k})-\\
&\qquad \frac{\Delta t}{8\Delta x}(F^n_{j+1,k}+F^n_{j+1,k+1}-F^n_{j-1,k}-F^n_{j-1,k+1})\\
U^{n+1}_{j,k} &= U^n_{j,k}-\frac{\Delta t}{\Delta x}(F^{n+\frac{1}{2}}_{j+\frac{1}{2},k}-F^{n+\frac{1}{2}}_{j-\frac{1}{2},k})-\frac{\Delta t}{\Delta y}(G^{n+\frac{1}{2}}_{j,k+\frac{1}{2}}-G^{n+\frac{1}{2}}_{j,k-\frac{1}{2}})
\end{aligned} \tag{4.9}$$

This scheme has the amplification matrix of the Lax-Wendroff single step method. Lax & Wendroff (1964) showed the sufficient stability condition for (4.9) to be

$$\lambda(A \text{ or } B)\frac{\Delta t}{\Delta x} \leqslant \frac{1}{\sqrt{8}} \tag{4.10}$$

and the CFL condition to be $\lambda(A \text{ or } B)\ \Delta t/\Delta x \leqslant 1$, with the symbols as defined in Section 1. The scheme is not optimal since $q = 1/\sqrt{8}$.

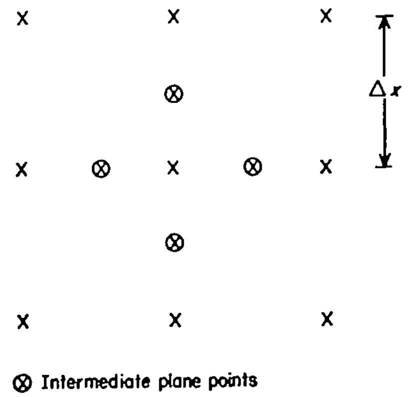


FIG. 6.

(v) *Strang Schemes*

Strang (1963, 1968) proposed schemes in any finite number of space variables using one dimensional Lax-Wendroff operators. We consider the two space variable schemes. The first of these is

$$U^{n+1} = \tfrac{1}{2}(L_x L_y + L_y L_x)U^n$$

where L_x, L_y are one dimensional Lax-Wendroff operators in the x and y directions respectively. Morris & Gourlay (1968) formulated this as a multistep method.

$$\begin{aligned}
V_{(1)}^{n+1} &= \mu_y U^n - \tfrac{1}{2} r\delta_y G^n & W_{(1)}^{n+1} &= \mu_x U^n - \tfrac{1}{2} r\delta_x F^n \\
V_{(2)}^{n+1} &= U^n - r\delta_y G_{(1)}^{n+1} & W_{(2)}^{n+1} &= U^n - r\delta_x F_{(1)}^{n+1} \\
V_{(3)}^{n+1} &= \mu_x V_{(2)}^{n+1} - \tfrac{1}{2} r\delta_x F_{(2)}^{n+1} & W_{(3)}^{n+1} &= \mu_y W_{(2)}^{n+1} - \tfrac{1}{2} r\delta_y G_{(2)}^{n+1} \\
V_{(4)}^{n+1} &= V_{(2)}^{n+1} - r\delta_x F_{(3)}^{n+1} & W_{(4)}^{n+1} &= W_{(2)}^{n+1} - r\delta_y G_{(3)}^{n+1} \\
U_{j,k}^{n+1} &= \tfrac{1}{2}(V_{(4)}^{n+1} + W_{(4)}^{n+1})
\end{aligned} \tag{4.11}$$

where $r = \Delta t/\Delta x$, $\mu_x Z = \tfrac{1}{2}(Z_{j+\frac{1}{2},k} + Z_{j-\frac{1}{2},k})$, $\delta_x Z = Z_{j+\frac{1}{2},k} - Z_{j-\frac{1}{2},k}$ with $\mu_y Z$, $\delta_y Z$ corresponding operators in the y direction.

The second of these is

$$U^{n+1} = L_{x/2} L_y L_{x/2} U^n$$

where $L_{x/2}$ is a one dimensional operator over a half time span. The multistep formulation of this scheme is given by Morris & Gourlay (1970).

$$\begin{aligned}
V_{(1)}^{n+1} &= \mu_x U^n - \tfrac{1}{4} r\delta_x F^n \\
V_{(2)}^{n+1} &= U^n - \tfrac{1}{2} r\delta_x F_{(1)}^{n+1} \\
V_{(3)}^{n+1} &= \mu_y V_{(2)}^{n+1} - \tfrac{1}{2} r\delta_y G_{(2)}^{n+1} \\
V_{(4)}^{n+1} &= V_{(2)}^{n+1} - r\delta_y G_{(3)}^{n+1} \\
V_{(5)}^{n+1} &= \mu_x V_{(4)}^{n+1} - \tfrac{1}{4} r\delta_x F_{(4)}^{n+1} \\
V_{(6)}^{n+1} &= V_{(4)}^{n+1} - \tfrac{1}{2} r\delta_x F_{(5)}^{n+1}, \qquad U_{j,k}^{n+1} = V_{(6)}^{n+1}.
\end{aligned} \tag{4.12}$$

Strang (1963) showed that scheme (4.11), for which $D_h(P, t_0)$ is shown in Fig. 7, has a sufficient stability condition

$$\lambda(A \text{ or } B)\frac{\Delta t}{\Delta x} \leqslant 1. \tag{4.13}$$

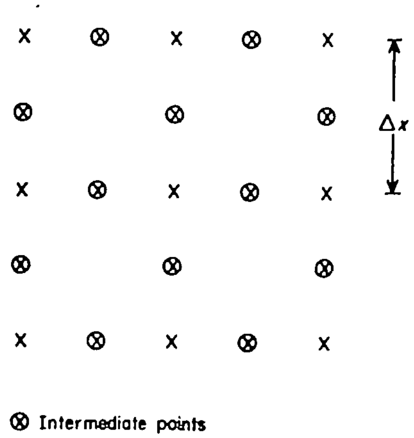


FIG. 7.

which is identical to the CFL condition so that $q = 1$ for the scheme. Strang (1968) also showed that scheme (4.12), for which $D_h(P, t_0)$ is shown in Fig. 8, has a sufficient stability condition

$$\lambda(A)\frac{\Delta t}{\Delta x} \leqslant 2, \qquad \lambda(B)\frac{\Delta t}{\Delta x} \leqslant 1 \tag{4.14}$$

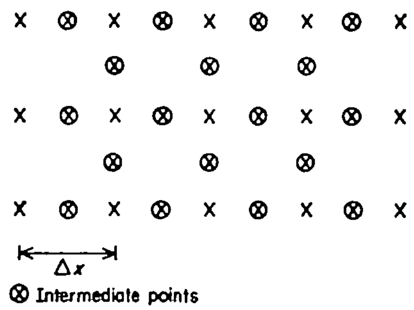


FIG. 8.

which is identical to the CFL condition so that again $q = 1$. In practice the $L_{x/2}$ at the end of one cycle and the $L_{x/2}$ at the beginning of the next are compounded to L_x so that the scheme involves the successive application of L_x and L_y with the half time operators being used only at the initial and printout levels.

The schemes for the numerical integration of (4.2) will now be considered.

(*vi*) *Richtmyer Scheme*

$$U^{n+\frac{1}{2}}_{j,k,l} = \tfrac{1}{6}(U^n_{j+\frac{1}{2},k,l}+U^n_{j-\frac{1}{2},k,l}+U^n_{j,k+\frac{1}{2},l}+U^n_{j,k-\frac{1}{2},l}+U^n_{j,k,l+\frac{1}{2}}+U^n_{j,k,l-\frac{1}{2}})-\frac{\Delta t}{2\Delta x}(F^n_{j+\frac{1}{2},k,l}-F^n_{j-\frac{1}{2},k,l})-\frac{\Delta t}{2\Delta y}(G^n_{j,k+\frac{1}{2},l}-G^n_{j,k-\frac{1}{2},l})-\frac{\Delta t}{2\Delta z}(H^n_{j,k,l+\frac{1}{2}}-H^n_{j,k,l-\frac{1}{2}}). \tag{4.16}$$

$$U^{n+1}_{j,k,l} = U^n_{j,k,l}-\frac{\Delta t}{\Delta x}(F^{n+\frac{1}{2}}_{j+\frac{1}{2},k,l}-F^{n+\frac{1}{2}}_{j-\frac{1}{2},k,l})-\frac{\Delta t}{\Delta y}(G^{n+\frac{1}{2}}_{j,k+\frac{1}{2},l}-G^{n+\frac{1}{2}}_{j,k-\frac{1}{2},l})-\frac{\Delta t}{\Delta z}(H^{n+\frac{1}{2}}_{j,k,l+\frac{1}{2}}-H^{n+\frac{1}{2}}_{j,k,l-\frac{1}{2}}).$$

$D_h(P, t_0)$ is a double pyramid for this scheme, one of the octants being shown in Fig. 9. $OA = \Delta x/\sqrt{3}$ so that the tightest CFL condition is $c\Delta t/\Delta x \leqslant 1/\sqrt{3}$. Since (2.15) gives the sufficient stability condition as

$$\frac{\Delta t}{\Delta x}c \leqslant \frac{1}{\sqrt{3}} \tag{4.17}$$

q again can be unity for this scheme.

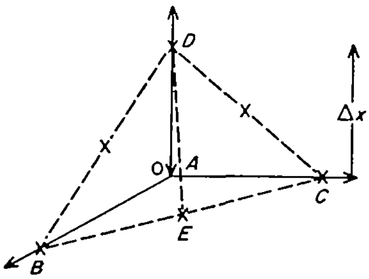


FIG. 9.

(*vii*) *Modified Richtmyer Scheme*

$$U^{n+\frac{1}{2}}_{j,k,l} = \tfrac{1}{8}(U^n_{j+\frac{1}{2},k+\frac{1}{2},l+\frac{1}{2}}+U^n_{j+\frac{1}{2},k+\frac{1}{2},l-\frac{1}{2}}+U^n_{j+\frac{1}{2},k-\frac{1}{2},l+\frac{1}{2}}+U^n_{j+\frac{1}{2},k-\frac{1}{2},l-\frac{1}{2}}+U^n_{j-\frac{1}{2},k+\frac{1}{2},l+\frac{1}{2}}+U^n_{j-\frac{1}{2},k+\frac{1}{2},l-\frac{1}{2}}+U^n_{j-\frac{1}{2},k-\frac{1}{2},l+\frac{1}{2}}+U^n_{j-\frac{1}{2},k-\frac{1}{2},l-\frac{1}{2}})-\frac{\Delta t}{2\Delta x}(F^n_{j+\frac{1}{2},k,l}-F^n_{j-\frac{1}{2},k,l})-\frac{\Delta t}{2\Delta y}(G^n_{j,k+\frac{1}{2},l}-G^n_{j,k-\frac{1}{2},l})-\frac{\Delta t}{2\Delta z}(H^n_{j,k,l+\frac{1}{2}}-H^n_{j,k,l-\frac{1}{2}}). \tag{4.18}$$

$$U^{n+1}_{j,k,l} = U^n_{j,k,l}-\frac{\Delta t}{\Delta x}(F^{n+\frac{1}{2}}_{j+\frac{1}{2},k,l}-F^{n+\frac{1}{2}}_{j-\frac{1}{2},k,l})-\frac{\Delta t}{\Delta y}(G^{n+\frac{1}{2}}_{j,k+\frac{1}{2},l}-G^{n+\frac{1}{2}}_{j,k-\frac{1}{2},l})-\frac{\Delta t}{\Delta z}(H^{n+\frac{1}{2}}_{j,k,l+\frac{1}{2}}-H^{n+\frac{1}{2}}_{j,k,l-\frac{1}{2}}).$$

One of the octants of $D_k(P, t_0)$ is shown in Fig. 10. From this the tightest CFL condition is $c\Delta t/\Delta x \leqslant 1$. Since the sufficient stability condition, as given by (2.16), is

$$\frac{\Delta t}{\Delta x}c \leqslant \frac{1}{\sqrt{3}} \tag{4.19}$$

the scheme is not optimal since $q_{\max} = 1/\sqrt{3}$.

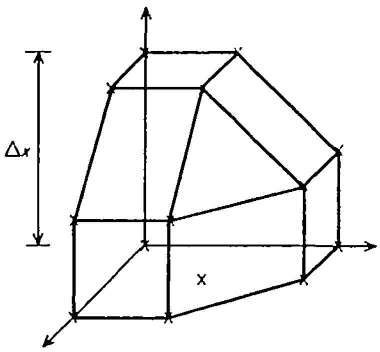


FIG. 10.

(*viii*) *Rotated Richtmyer Scheme*

$$\begin{aligned}
U^{n+\frac{1}{2}}_{j,k,l} = {} & \tfrac{1}{8}(U^n_{j+\frac12,k+\frac12,l+\frac12}+U^n_{j+\frac12,k+\frac12,l-\frac12}+U^n_{j+\frac12,k-\frac12,l+\frac12}+U^n_{j+\frac12,k-\frac12,l-\frac12}+ \\
& U^n_{j-\frac12,k+\frac12,l+\frac12}+U^n_{j-\frac12,k+\frac12,l-\frac12}+U^n_{j-\frac12,k-\frac12,l+\frac12}+U^n_{j-\frac12,k-\frac12,l-\frac12})- \\
& \frac{\Delta t}{8\Delta x}(F^n_{j+\frac12,k+\frac12,l+\frac12}+F^n_{j+\frac12,k-\frac12,l+\frac12}+F^n_{j+\frac12,k+\frac12,l-\frac12}+F^n_{j+\frac12,k-\frac12,l-\frac12}- \\
& F^n_{j-\frac12,k+\frac12,l+\frac12}-F^n_{j-\frac12,k+\frac12,l-\frac12}-F^n_{j-\frac12,k-\frac12,l+\frac12}-F^n_{j-\frac12,k-\frac12,l-\frac12})- \\
& \frac{\Delta t}{8\Delta y}(G^n_{j+\frac12,k+\frac12,l+\frac12}+G^n_{j+\frac12,k+\frac12,l-\frac12}+G^n_{j-\frac12,k+\frac12,l+\frac12}+G^n_{j-\frac12,k+\frac12,l-\frac12}- \\
& G^n_{j+\frac12,k-\frac12,l+\frac12}-G^n_{j+\frac12,k-\frac12,l-\frac12}-G^n_{j-\frac12,k-\frac12,l+\frac12}-G^n_{j-\frac12,k-\frac12,l-\frac12})- \\
& \frac{\Delta t}{8\Delta z}(H^n_{j+\frac12,k+\frac12,l+\frac12}+H^n_{j+\frac12,k-\frac12,l+\frac12}+H^n_{j-\frac12,k+\frac12,l+\frac12}+H^n_{j-\frac12,k-\frac12,l+\frac12}- \\
& H^n_{j+\frac12,k+\frac12,l-\frac12}-H^n_{j+\frac12,k-\frac12,l-\frac12}-H^n_{j-\frac12,k+\frac12,l-\frac12}-H^n_{j-\frac12,k-\frac12,l-\frac12}). \\
U^{n+1}_{j,k,l} = {} & U^n_{j,k,l}-\frac{\Delta t}{4\Delta x}(F^{n+\frac12}_{j+\frac12,k+\frac12,l+\frac12}+F^{n+\frac12}_{j+\frac12,k+\frac12,l-\frac12}+F^{n+\frac12}_{j+\frac12,k-\frac12,l+\frac12}+ \\
& F^{n+\frac12}_{j+\frac12,k-\frac12,l-\frac12}-F^{n+\frac12}_{j-\frac12,k+\frac12,l+\frac12}-F^{n+\frac12}_{j-\frac12,k+\frac12,l-\frac12}-F^{n+\frac12}_{j-\frac12,k-\frac12,l+\frac12}- \\
& F^{n+\frac12}_{j-\frac12,k-\frac12,l-\frac12})-\frac{\Delta t}{4\Delta y}(G^{n+\frac12}_{j+\frac12,k+\frac12,l+\frac12}+G^{n+\frac12}_{j+\frac12,k+\frac12,l-\frac12}+G^{n+\frac12}_{j-\frac12,k+\frac12,l+\frac12}+ \\
& G^{n+\frac12}_{j-\frac12,k+\frac12,l-\frac12}-G^{n+\frac12}_{j+\frac12,k-\frac12,l+\frac12}-G^{n+\frac12}_{j+\frac12,k-\frac12,l-\frac12}-G^{n+\frac12}_{j-\frac12,k-\frac12,l+\frac12}- \\
& G^{n+\frac12}_{j-\frac12,k-\frac12,l-\frac12})-\frac{\Delta t}{4\Delta z}(H^{n+\frac12}_{j+\frac12,k+\frac12,l+\frac12}+H^{n+\frac12}_{j+\frac12,k-\frac12,l+\frac12}+H^{n+\frac12}_{j-\frac12,k+\frac12,l+\frac12}+ \\
& H^{n+\frac12}_{j-\frac12,k-\frac12,l+\frac12}-H^{n+\frac12}_{j+\frac12,k+\frac12,l-\frac12}-H^{n+\frac12}_{j+\frac12,k-\frac12,l-\frac12}-H^{n+\frac12}_{j-\frac12,k+\frac12,l-\frac12}- \\
& H^{n+\frac12}_{j-\frac12,k-\frac12,l-\frac12}).
\end{aligned} \tag{4.20}$$

For (4.17) $D_h(P, t_0)$ is a cube of side $2\Delta x$ so that the tightest CFL condition is $c\Delta t/\Delta x \leqslant 1$. (2.17) gives the sufficient stability condition as

$$\frac{\Delta t}{\Delta x}c \leqslant 1. \tag{4.21}$$

Therefore q again can be unity.

(*ix*) *Strang schemes*

The three space variable Strang schemes were formulated as step methods by Gourlay, McGuire & Morris (1971).

The three space variable form of the Strang I scheme is

$$U^{n+1} = \tfrac{1}{2}(L_x L_y L_z + L_z L_y L_x)U^n \tag{4.22}$$

where L_z is the one dimensional Lax-Wendroff operator in the Z direction and L_x, L_y are a simple extension of that previously given.

The format of the multistep formulation is a simple extension of the format of the two variable case.

The three space variable form of the Strang II scheme is

$$U^{n+1} = L_{x/2}L_{y/2}L_z L_{y/2}L_{x/2}U^n \tag{4.23}$$

where $L_{x/2}$ and $L_{y/2}$ represent operators over a half time span. The multistep formulation has a format similar to the two dimensional scheme. For computational efficiency this scheme is again compounded to give

$$U^{n+\tau} = L_{x/2}L_{y/2}L_z L_{y/2}L_x L_{y/2}L_z L_{y/2}L_x \dots L_{x/2}U^n. \tag{4.24}$$

The multistep formulation of this scheme is based on the cycle $L_x L_{y/2} L_z L_{y/2}$ and its format is similar to that of the two space variable case. $D_h(P, t_0)$ for (4.22) and (4.24) is a cube of side $2\Delta x$. Strang (1963, 1968) has shown that both the sufficient stability condition and the CFL condition for (4.22) and (4.24) are given by

$$\frac{\Delta t}{\Delta x}c \leqslant 1 \tag{4.25}$$

so that $q = 1$ for each of these schemes.

(*x*) *Comparison of Schemes*

We first consider the schemes for the numerical integration of (4.1) and compare them on the basis of the stability results of this section. Although the Richtmyer, Rotated Richtmyer & Strang schemes have $q = 1$ the results of Section 3 show that these schemes are not necessarily equivalent. For the Rotated Richtmyer and Strang schemes $D_h(P, t_0)$ is similar so that the q comparison leads to the result of equivalence obtained by comparing the sufficient stability conditions (4.8) and (4.13). $D_h(P, t_0)$ for the Richtmyer scheme differs from those mentioned above so that comparison is only possible on the basis of the sufficient stability condition. Examination of (4.4), (4.8), and (4.12) shows that both the Rotated scheme and the Strang schemes are $\sqrt{2}$ times more efficient than the Richtmyer scheme. Because of $D_h(P, t_0)$ the modified scheme can only be compared in terms of (4.6). The Richtmyer scheme and the modified scheme are therefore equivalent. By virtue of their $D_h(P, t_0)$'s the Strang and Singleton schemes can be compared in terms of q and this comparison leads to the correct result, obtained by comparing (4.10) and (4.13), that the Strang scheme is $\sqrt{8}$ times more efficient than the Singleton scheme. (4.4) and (4.10) show the Richtmyer scheme to be twice as efficient as the Singleton scheme.

Similar results are obtained when the schemes given for the numerical integration of (4.2) are compared. Basing the comparisons on the sufficient stability conditions (4.17), (4.19), (4.21), (4.25), it is found that the Strang and Rotated Richtmyer schemes are equivalent, the Richtmyer and Modified schemes are equivalent and that any of the former is $\sqrt{3}$ times more efficient than either of the latter. Because of the similarity of $D_k(P, t_0)$ the q comparison would also lead to correct results for the Strang and Rotated Richtmyer schemes.

5. Schemes of Order of Accuracy Greater than the Second

In Section 2 the stability of certain difference schemes in any finite number of space variables was examined. In each scheme the amplification matrix was a polynomial function of a single matrix, this form being guaranteed by ensuring that derivative approximations are formulated in the same manner at each step of the method. We now illustrate that the stability analysis of Section 2 can be extended to schemes of accuracy greater than the second provided that the amplification matrix is a polynomial function of a single matrix.

We consider the difference scheme

$$U^{n+1} = L_3 U^n \tag{5.1}$$

where $L_3 = I + r(A\delta_x + B\delta_y) + \frac{1}{2}r^2(A\delta_x + B\delta_y)^2 + \frac{1}{6}r^3(A\delta_x + B\delta_y)^3$ is the Dunn difference operator (see Burstein & Mirin, 1969) approximating the differential operator

$$S_3 = \frac{4}{3}\left\{\frac{S_1(A,B)+S_1(B,A)}{2}\right\} - \frac{1}{3}S_2(A,B),$$

S_1 and S_2 being Strang operators (see Strang, 1968).

The amplification matrix for this linearized scheme, obtained by substituting $U_0^n \exp\{i(k_x j + k_y k)\Delta x\}$ for $U_{j,k}^n$ and α, β respectively for $k_x\Delta x$ and $k_y\Delta x$, is

$$G = I + 2r(A\sin\alpha + B\sin\beta) - \frac{1}{2}r^2(A\sin\alpha + B\sin\beta)^2 - \frac{1}{6}ir^3(A\sin\alpha + B\sin\beta)^3. \tag{5.2}$$

The analysis of Section 2 shows that $c(\sin^2\alpha + \sin^2\beta)^{\frac{1}{2}}$ is an eigenvalue of the matrix $A\sin\alpha + B\sin\beta$.

The amplification matrix (5.2) thus has an eigenvalue g given by

$$g = 1 + i\mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{6}i\mu^3$$

where $\mu = rc(\sin^2\alpha + \sin^2\beta)^{\frac{1}{2}}$.

$$|g| \leqslant 1 \quad \text{if} \quad \mu^2 \leqslant 3.$$

Hence

$$\left(c\frac{\Delta t}{\Delta x}\right)^2 \leqslant \frac{3}{\sin^2\alpha + \sin^2\beta}. \tag{5.3}$$

The minimum value of the right hand side of (5.3) occurs when $\sin^2\alpha = \sin^2\beta = 1$ and has the value 1·5.

The sufficient stability condition for the scheme is therefore

$$c\frac{\Delta t}{\Delta x} \leqslant 1{\cdot}5^{\frac{1}{2}}$$

so that the scheme is not optimal.

The method can obviously be extended to higher order schemes.

6. Comparison of Schemes in Two and Three Space Variables

We consider the six difference schemes in two space variables given in Section 4 and seek some measure of their relative efficiencies. In discussing the efficiency we will investigate the combined effect of the stability condition, the number of arithmetic operations required to advance the solution net and the local truncation error. In each of the schemes under consideration the principal part T of the local truncation error can be expressed in the form

$$\mathbf{T} = \Delta t^3 \mathbf{C} + \Delta t\, \Delta x^2 \mathbf{D}, \tag{6.1}$$

the stability condition in the form

$$\frac{\Delta t}{\Delta x} \leqslant k \tag{6.2}$$

and the number N of arithmetical operations in the form

$$N = \frac{M}{\Delta t\, \Delta x^2}, \tag{6.3}$$

where **T**, **C**, **D** are vector quantities and k, M scalars which depend on the choice of scheme. In assessing the arithmetical requirements necessary to advance the solution net some authors use the number of vector calculations as their criterion. This criterion can be misleading, being detrimental to Richtmyer type schemes in many situations, since the evaluation of $F(U)$ and $G(U)$ at the same point generally involves fewer arithmetical operations than the evaluation of $F(U)$ at one point and $G(U)$ at another. Computational requirements based on the total arithmetical count overcome this but only allow comparison on a problem basis. Index 1 in Table 1 is based on the total arithmetical count and index 2 on the number of vector calculations. It should also be noted that in assessing the number of arithmetical operations required to advance the solution net with the Richtmyer scheme use is made of the fact that mesh points having even values of $j+k+n$ are not coupled to points having odd values of $j+k+n$ so that half of the points may be omitted. Using (6.1), (6.2) and (6.3) it is easy to show that each of the schemes under consideration is most efficient computationally when the stability condition attains its maximum value.

We compare the given schemes on a general basis and also for the problem of the unsteady flow of a polytropic gas with the computational requirement taken as the total count and consider a fixed number of arithmetic operations in each case. Δt and Δx are obtained from (6.2) and (6.3) and substitution into (6.1) yields estimates of **T** to be used for comparison purposes. Unfortunately, although the expressions for **T** contain similar terms, a direct comparison is not possible.

Table 1 gives an index for comparing schemes. This index, which is such that the smaller the figure given the better the scheme, is based only upon the stability condition and the number of arithmetical operations required.

In this table Strang I refers to the 1968 scheme of Morris & Gourlay and Strang II to their 1970 compounded scheme. If other factors are equal, truncation produces errors of approximately the same magnitude for the Singleton and Richtmyer schemes and an error for the Modified scheme which certainly is not smaller than these. This result, together with the results of Table 1 shows that the Modified scheme can be eliminated and that the Singleton scheme only compares with the Richtmyer scheme when the number of vector evaluations is taken as the arithmetical count. Truncation produces errors, other factors being equal, of approximately the same magnitude for the Strang schemes and the Rotated Richtmyer scheme and each of these appears to be slightly greater than that for the Richtmyer scheme. These results, together with Table 1, imply that there is little to choose between the Richtmyer and Strang I scheme and that neither of these schemes can compare with the Strang II or Rotated Richtmyer schemes. The choice therefore appears to be between the latter schemes, the schemes being equally efficient if the arithmetic count is taken as the number of vector evaluations. However there appears to be a slight bias in favour of the Rotated Richtmyer scheme when the vectors $\mathbf{F}(u)$ and $\mathbf{G}(u)$ have elements in common.

TABLE 1

Scheme	Singleton	Richt.	Mod.-Richt.	Rot. Richt.	Strang I	Strang II
Index 1	3·5	2·5	5·0	1	2·4	1·15
Index 2	2·8	2·8	5·6	1	2·0	1

In the above analysis it is assumed that if estimates are given for the solution of two of the difference schemes, the one with the smaller error is the one closest to the solution of the differential equation. If this assumption is not made, the error estimate must be made in terms of the normalized error (see Strang, 1968). This depends upon the principal error function which can only be obtained in the form of an integral for the schemes under consideration.

We now consider the schemes in three space variables given in Section 4. Although the truncation error associated with the Rotated scheme is probably larger than that associated with the Richtmyer scheme the stability condition and arithmetical requirements of the former outweigh this disadvantage so that the Richtmyer scheme can be eliminated. The modified scheme can similarly be eliminated. As observed by Morris & Gourlay (1970), the Strang I scheme can be eliminated in favour of the Strang II scheme. We compare therefore the Strang II scheme and the Rotated Richtmyer scheme for which the stability conditions are equal. However the latter requires only 75% of the number of vector evaluations required by the former (indeed if $m \geqslant 2$ is the number of space dimensions, m_1 the vector evaluations for the Rotated Richtmyer scheme and m_2 the evaluations for the Strang II scheme then $m_1/m_2 = 2m/4(m-1) \leqslant 1$) and this makes no allowance for the vector quantities having elements in common. Although the Rotated Richtmyer scheme may have a slightly larger truncation error than the Strang II scheme this is insufficient to counteract the advantage of the Rotated Richtmyer scheme's smaller arithmetical requirements.

7. Conclusions

When a difference scheme is to be considered it is essential that the scheme be stable. It has been shown how to determine the stability condition for certain types of schemes in any number of space variables and for any order of accuracy. It has also been shown that, although an index can be obtained for any scheme by expressing its stability as a fraction of the CFL condition, meaningful comparison of schemes on this basis is not always possible. The Rotated Richtmyer schemes in two and three space variables have been shown to be probably the most efficient of the difference schemes considered and it is likely that this advantage will persist as the number of space variables is increased.

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