Numer. Math. 20, 350—355 (1973)

On Two Step Lax-Wendroff Methods in Several Dimensions

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Received January 11, 1971

Abstract. A version of Richtmyer's two step Lax-Wendroff scheme for solving hyperbolic systems in conservation form, is considered. This version uses only the nearest points, has second order accuracy at every time cycle and allows a time step which is larger by a factor of $\sqrt{d}$ than Richtmyer's, where d is the number of spatial dimensions. The scheme appears to be competitive with the optimal stability schemes proposed by Strang and carried out by Gourlay and Morris.

Introduction

The advantages of solving two dimensional hyperbolic systems in conservation form by the use of Richtmyer's two step method [6] over the original Lax-Wendroff scheme [5], are—simplicity and larger time steps. Indeed, these advantages hold even more for three dimensional cases (see [8]).

Let the hyperbolic system of equations be

$$W_t = [F(W)]_x + [G(W)]_y \tag{1}$$

with given initial conditions $W(0, x, y)$. The simplicity mentioned above is that the two step method does not use the Jacobian matrices

$$A = F_W; \qquad B = G_W \tag{2}$$

in contrast to the Lax-Wendroff original scheme. Instead of taking the two step scheme as proposed by Richtmyer we would like to consider the following version, mentioned already by Burstein [1], and in [13],

$$\begin{aligned} W^{n+1/2}_{j+1/2,m+1/2} &= \widehat{W}^n_{j+1/2,m+1/2} + \frac{\lambda}{2}\,[\widetilde{F}^n_{j+1,m+1/2} - \widetilde{F}^n_{j,m+1/2} + \widetilde{G}^n_{j+1/2,m+1} - \widetilde{G}^n_{j+1/2,m}] \\ W^{n+1}_{j,m} &= W^n_{j,m} + \lambda\,[\widetilde{F}^{n+1/2}_{j+1/2,m} - \widetilde{F}^{n+1/2}_{j-1/2,m} + \widetilde{G}^{n+1/2}_{j,m+1/2} - \widetilde{G}^{n+1/2}_{j,m-1/2}] \end{aligned} \tag{3}$$

where

$$\lambda = \frac{\Delta t}{\Delta x} = \frac{\Delta t}{\Delta y}; \qquad W^n_{j,m} = W(t_n, x_j, y_m)$$

and

$$\widehat{W}^n_{j+1/2,m+1/2} = [W^n_{j+1,m+1} + W^n_{j+1,m} + W^n_{j,m+1} + W^n_{j,m}]/4.$$

$$\widetilde{F}^n_{j+1,m+1/2} = F\left((W^n_{j+1,m+1} + W^n_{j+1,m})/2\right);$$

$$\widetilde{F}^{n+1/2}_{j+1/2,m} = F\left((W^{n+1/2}_{j+1/2,m+1/2} + W^{n+1/2}_{j+1/2,m-1/2})/2\right)$$

etc. It should be noted that for example,

$$\{\tilde F^{n+1/2}_{j+1/2,m} - \tilde F^{n+1/2}_{j-1/2,m}\} - \{(F^{n+1/2}_{j+1/2,m+1/2} + F^{n+1/2}_{j+1/2,m-1/2})/2 - (F^{n+1/2}_{j-1/2,m+1/2} - F^{n+1/2}_{j-1/2,m-1/2})/2\}$$
$$= O[(\Delta x)^3]$$

This is important because simplicity of such schemes, as suggested by Strang [6], is estimated by the number of calculations of F and G per time step. This number is 8 in our scheme (3), and clearly the results produced at *each* time step are of second order of accuracy.

Stability Analysis

The amplification matrix of the linearized version of (3) turns out to be

$$P = I + i\lambda\left[A \sin\xi \frac{1+\cos\eta}{2} + B \sin\eta \frac{1+\cos\xi}{2}\right]$$
$$-\lambda/2\,[A\sqrt{(1-\cos\xi)(1+\cos\eta)} + B\sqrt{(1+\cos\xi)(1-\cos\eta)}]^2 \tag{4}$$

where ξ and η are the dual variables after Fourier transformation. Now, if we denote

$$M = \sqrt{(1-\cos\xi)(1+\cos\eta)}\,A + \sqrt{(1+\cos\xi)(1-\cos\eta)}\,B \tag{5}$$

then P takes the form

$$P = I + \frac{i}{2}\lambda\sqrt{(1+\cos\xi)(1+\cos\eta)}\,M - \frac{1}{2}\lambda^2 M^2 \tag{6}$$

and by the spectral mapping theorem we have

$$p = 1 + \frac{i}{2}\lambda\sqrt{(1+\cos\xi)(1+\cos\eta)}\,m - \frac{1}{2}\lambda^2 m^2 \tag{7}$$

where p and m are the eigenvalues of P and M respectively. In order to meet the Von-Neumann condition, the inequality

$$|p|^2 = (1-\tfrac{1}{2}\lambda^2 m^2)^2 + \tfrac{1}{4}\lambda^2 m^2 (1+\cos\xi)(1+\cos\eta) \leqq 1, \tag{8}$$

must be true for all values of ξ and η such that $|\xi|, |\eta| \leqq \pi$.

If we denote

$$\alpha = 1 - \sin^2\frac{\xi}{2}$$

$$\beta = 1 - \sin^2\frac{\eta}{2}$$

we obtain the inequality

$$\lambda^2 m^2 \leqq 4(1-\alpha\beta) \tag{9}$$

which must be fulfilled for every $0 \leqq \alpha, \beta \leqq 1$.

Let us now consider the case where $AB = BA$, then we know that

$$m_i = 2\sqrt{\beta(1-\alpha)}\cdot a_i + 2\sqrt{\alpha(1-\beta)}\,b_i \tag{10}$$

where m_i, a_i, and b_i are the i-th eigenvalues of M, A, and B respectively. Now (9) takes the form

$$\lambda^2 \leqq \frac{1-\alpha\beta}{a_i^2\beta(1-\alpha)+b_i^2\alpha(1-\beta)+2a_i b_i\sqrt{\alpha\beta(1-\alpha)(1-\beta)}} \tag{11}$$

for all $0 \leqq \alpha$, $\beta \leqq 1$. In this range for α and β it easy to verify that the minimal value of the right-hand side is $1/a_i^2+b_i^2$, and therefore the stability criterion is

$$\lambda \leqq \frac{1}{\operatorname*{Max}_i \sqrt{a_i^2+b_i^2}} \tag{12}$$

while Richtmyer's condition in this case turns out to be

$$\lambda \leqq \frac{1}{\operatorname*{Max}_i \sqrt{2a_i^2+2b_i^2}}.$$

In the case where A and B are symmetric (12) takes the form

$$\lambda \leqq [\varrho(A^2+B^2)]^{-1/2} \tag{12a}$$

where $\varrho(A^2+B^2)$ is the spectral radius of A^2+B^2. In view of the Lax-Wendroff stability theorem ([5], p. 384) condition (12a) is sufficient for stability in the symmetric case.

Let us now take the hydrodynamic equations where $AB \neq BA$. Although these matrices are simultaneously symmetrizable [10], it is better to follow Richtmyer [6] and consider the matrices A' and B' after a simplifying transformation. As in [6] we have:

$$A' = \begin{pmatrix} u & \varrho & 0 & 0 \\ 0 & u & 0 & 1/\varrho \\ 0 & 0 & u & 0 \\ 0 & \varrho c^2 & 0 & u \end{pmatrix}; \quad B' = \begin{pmatrix} v & 0 & \varrho & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & 1/\varrho \\ 0 & 0 & \varrho c^2 & v \end{pmatrix} \tag{13}$$

where

u = the velocity component in the x direction.

v = the velocity component in the y direction.

ϱ = the density per unit volume.

c = the adiabatic sound speed.

We now define

$$r = \sqrt{\sin^2\frac{\xi}{2}\cos^2\frac{\eta}{2} + \sin^2\frac{\eta}{2}\cos^2\frac{\xi}{2}}$$

$$\cos\phi = \sin\frac{\xi}{2}\cos\frac{\eta}{2}\Big/ r \tag{14}$$

$$\sin\phi = \sin\frac{\eta}{2}\cos\frac{\xi}{2}\Big/ r$$

$$u' = u\cos\phi + v\sin\phi,$$

u' being the component of the velocity vector V in the direction ϕ. Using these notations the matrix

$$M' = r \cdot \begin{pmatrix} u' & \varrho \cos\phi & \varrho \sin\phi & 0 \\ 0 & u' & 0 & \cos\phi/\varrho \\ 0 & 0 & u' & \sin\phi/\varrho \\ 0 & \varrho c^2 \cos\phi & \varrho c^2 \sin\phi & u' \end{pmatrix} \tag{15}$$

is obtained having the eigenvalues:

$$m = m' = r \cdot \begin{bmatrix} u' \\ u' \\ u' + c \\ u' - c \end{bmatrix}. \tag{16}$$

With these results, the inequality (8) takes the form;

$$|p|^2 = 1 - 4\lambda^2 m^2 \left\{ 1 - \alpha\beta - \lambda^2 r^2 \cdot \begin{bmatrix} u' \\ u' \\ u' + c \\ u' - c \end{bmatrix}^2 \right\} \leqq 1 \tag{17}$$

which by (14) leads to

$$\lambda^2 (|V| + c)^2 \leqq \frac{1-\alpha\beta}{r^2} = \frac{1-\alpha\beta}{\alpha+\beta-2\alpha\beta}; \qquad (0 \leqq \alpha; \beta \leqq 1). \tag{18}$$

Finally, since the right hand side has the minimal value of 1, we obtain the condition

$$\lambda \leqq \frac{1}{|V| + c}. \tag{19}$$

This condition can be written as $|V| + c \leqq \Delta x/\Delta t$ expressing the Courant-Friedrichs-Lewy requirement that the mesh speed $\Delta x/\Delta t$ will not be less than the largest physical speed $|V| + c$. In this sense (19) is a *CFL* condition and has the factor $\sqrt{2}$ in its favor when compared to Richtmyer's result [6]. A similar condition is given in [12], in Appendix (b).

Condition (19) is sufficient for linear stability because Richtmyer's sufficiency proof (see [7] page 364) is valid in the present case too (except for the fact that ϕ is not the same as his θ but this makes no difference in the proof).

It should be noted that the amplification matrix here is equal to I only when $\xi = \eta = 0$ or $\xi = \eta = \pi$, while in Richtmyer's case this is so also for $\xi = 0$, $\eta = \pi$ and $\xi = \pi$, $\eta = 0$.

Three Dimensions

If three dimensional conservation laws are considered, namely

$$W_t = [F(W)]_x + [G(W)]_y + [H(W)]_z \tag{20}$$

then the corresponding two step scheme should be,

Step 1.

$$\begin{aligned}
W^{n+1/2}_{j+1/2,m+1/2,k+1/2} = \widehat{W}^n_{j+1/2,m+1/2,k+1/2} &+ \frac{\lambda}{2}\,[\widetilde{F}^n_{j+1,m+1/2,k+1/2} - \widetilde{F}^n_{j,m+1/2,k+1/2} \\
&+ \widetilde{G}^n_{j+1/2,m+1,k+1/2} - \widetilde{G}^n_{j+1/2,m,k+1/2} \\
&+ \widetilde{H}^n_{j+1/2,m+1/2,k+1} - \widetilde{H}^n_{j+1/2,m+1/2,k}]
\end{aligned} \tag{21}$$

Step 2.

$$\begin{aligned}
W^{n+1}_{j,m,k} = W^n_{j,m,k} + \lambda\,[\widetilde{F}^{n+1/2}_{j+1/2,m,k} &- \widetilde{F}^{n+1/2}_{j-1/2,m,k} \\
+ \widetilde{G}^{n+1/2}_{j,m+1/2,k} &- \widetilde{G}^{n+1/2}_{j,m-1/2,k} \\
+ \widetilde{H}^{n+1/2}_{j,m,k+1/2} &- \widetilde{H}^{n+1/2}_{j,m,k-1/2}
\end{aligned} \tag{22}$$

where:

$$\begin{aligned}
\widehat{W}^n_{j+1/2,m+1/2,k+1/2} = [W^n_{j+1,m+1,k+1} &+ W^n_{j+1,m+1,k} + W^n_{j+1,m,k+1} \\
&+ W^n_{j,m+1,k+1} + W^n_{j+1,m,k} + W^n_{j,m+1,k} \\
&+ W^n_{j,m,k+1} + W^n_{j,m,k}]/8,
\end{aligned}$$

$$\widetilde{F}^n_{j+1,m+1/2,k+1/2} = F\left((W^n_{j+1,m+1,k+1} + W^n_{j+1,m+1,k} + W^n_{j+1,m,k+1} + W^n_{j+1,m,k})/4\right),$$

$$\begin{aligned}
\widetilde{F}^{n+1/2}_{j+1/2,m,k} = \widetilde{F}\left((W^{n+1/2}_{j+1/2,m+1/2,k+1/2} \right. &+ W^{n+1/2}_{j+1/2,m+1/2,k-1/2} \\
&\left. + W^{n+1/2}_{j+1/2,m-1/2,k+1/2} + W^{n+1/2}_{j+1/2,m-1/2,k-1/2})/4\right)
\end{aligned}$$

etc.

Here, 12 calculations of the vector functions F and G are needed for each time cycle, except for the averaging. The linear stability analysis is similar to the two dimensional case; for the hydrodynamical equations, for example, the necessary and sufficient condition, is again

$$\frac{\Delta x}{\Delta t} \geqq |V| + c \tag{24}$$

having a factor of $\sqrt{3}$ in its favor when compared to the Richmyer-like scheme (see [8] and [2]).

It is interesting to note that the Lax-Wendroff method, as in [5] can easily be extended to d spatial dimensions, having the stability condition

$$\lambda\sigma \leqq \frac{1}{d\sqrt{d}} \tag{25}$$

where σ is the largest of the spectral radii of the matrices involved. The proof is for the symmetric case, and follows the proof in [5]. If one adds to the d-dimensional Lax-Wendroff scheme, the stabilizing term

$$-(\Delta x)^4 \lambda^2/8 \sum_{i\neq j} (A_i^2 + A_j^2)\, W_{x_i x_i x_j x_j} \tag{26}$$

then the stability criteria becomes

$$\lambda\sigma \leqq \frac{1}{d}\,. \tag{27}$$

The case with $d = 2$ is given in [5].

A detailed comparison between the different schemes producing second order results at *every* Δt, is presented in [11]. In any case, if we examine the optimized Lax-Wendroff schemes suggested by Strang [9] and formulated by Gourlay and Morris [3, 4], we observe that although a larger time step is allowed there, more calculations of F and G are needed.

In d spatial dimension the multistep Strang schemes, known as the optimized Lax-Wendroff methods, allow in general a time step which is larger by a factor of $\sqrt{d}$ than the stable Δt in scheme (3). On the other hand the functions F and G must be calculated a number of times which is larger than needed in (3) by a factor of $(2d-1)/d$ or 2, depending on which of Strang's schemes is used [4]. For d being 2 or 3 scheme (3), therefore, appears to be competitive with the optimized Lax-Wendroff methods, as in demonstrated in [11].

Acknowledgment. Special thanks are due to Dr. David Gottlieb for many helpful discussions.

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