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Parametric instabilities in shallow water magnetohydrodynamics of astrophysical plasma in external magnetic field

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ABSTRACT

This article deals with magnetohydrodynamic (MHD) flows of a thin rotating layer of astrophysical plasma in external magnetic field. We use the shallow water approximation to describe thin rotating plasma layer with a free surface in a vertical external magnetic field. The MHD shallow water equations with external vertical magnetic field are revised by supplementing them with the equations that are consequences of the magnetic field divergence-free conditions and reveal the existence of third component of the magnetic field in such approximation providing its relation with the horizontal magnetic field. It is shown that the presence of a vertical magnetic field significantly changes the dynamics of the wave processes in astrophysical plasma compared to the neutral fluid and plasma layer in a toroidal magnetic field. The equations for the nonlinear wave packets interactions are derived using the asymptotic multiscale method. The equations for three magneto-Poincare waves interactions, for three magnetostrophic waves interactions, for the interactions of two magneto-Poincare waves and for one magnetostrophic wave and two magnetostrophic wave and one magneto-Poincare wave interactions are obtained. The existence of parametric decay and parametric amplifications is predicted. We found following four types of parametric decay instabilities: magneto-Poincare wave decays into two magneto-Poincare waves, magnetostrophic wave decays into two magnetostrophic waves, magneto-Poincare wave decays into one magneto-Poincare wave and one magnetostrophic wave, magnetostrophic wave decays into one magnetostrophic wave and one magneto-Poincare wave. Following mechanisms of parametric amplifications are found: parametric amplification of magneto-Poincare waves, parametric amplification of magnetostrophic waves, magneto-Poincare wave amplification in magnetostrophic wave presence and magnetostrophic wave amplification in magneto-Poincare wave presence. The instabilities growth rates are obtained respectively.

1. Introduction

Plasma in various stars and planets is described by magnetohydrodynamics of thin fluid layer with a free surface in the gravity field. As an example we refer to solar tachocline flows (a thin layer inside the sun located above the convection zone) [11,20,21], neutron stars atmosphere dynamics [9,22], accreting matter flows in the neutron stars [12], tidally synchronized exoplanets with magnetoactive atmospheres [8,10,2]. MHD shallow water approximation [6] and quasigeostrophic MHD approximation [6,18,1] are used to describe such flows of astrophysical plasma. The present article is concerned with the study of weakly non-linear waves interactions in shallow water magnetohydrodynamics. We consider MHD flows in plasma layer for which the characteristic horizontal scales significantly exceed the scales of vertical variations. Therefore we assume the layer to be thin and use the shallow water approximation. The MHD equations in shallow water approximation are the alternative for heavy fluid MHD equations in the case where the layer of incompressible inviscid fluid in gravity field with a free surface is considered. The non-inertial reference frame is rotating with the fluid layer. The MHD shallow water equations are obtained by depth-averaging from general incompressible MHD equations. The pressure is assumed to be hydrostatic and layer height is assumed to be much smaller than characteristic horizontal linear scale of plasma layer [6,7,13–15,3,4,23]. The obtained equations play the important role in astrophysical plasmas studies as do the classical shallow water equations in neutral fluid hydrodynamics. It should be noted that in hydrodynamics of neutral fluid in shallow water approximation there are only gravitational Poincare waves and the dispersion relation in shallow water equa-

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tions for neutral fluid exclude Poincare waves interactions [19], because in this case phase matching conditions are not fulfilled. In common shallow water magnetohydrodynamics used in solar tachocline studies toroidal magnetic field exists and it leads to existence of two linear modes: fast magnetogravity waves (similar to Poincare waves in neutral fluid) called magneto-Poincare and slow Alfven waves. Simultaneously the dispersion relation doesn't provide phase matching conditions in weakly non-linear interactions [20,6,7]. In present study we consider MHD shallow water equations in the external vertical magnetic field. Such configuration of magnetic field is typical for the neutron stars [9,2] and for the exoplanets [2]. The external vertical magnetic field significantly modifies MHD shallow water equations and in linear approximation two new fast waves appear: magneto-Poincare mode and magnetostrophic mode [9]. We revised the MHD shallow water equations with external vertical magnetic field supplementing them with the equations that are consequences of the magnetic field divergence-free conditions that are satisfied identically. The new supplementary equations reveal the existence of three components of the magnetic field and provide the expressions for variation of the vertical magnetic field. This is not the case in commonly used shallow water equations without external magnetic fields when magnetic field is inevitably horizontal. In the present work we generalize the linear theory of MHD shallow water flows developed in [9] to the case of finite amplitude waves in weakly non-linear approximation. It is shown that the dispersion relations of linear waves in external vertical magnetic field provide the phase matching conditions needed for non-linear interactions. The interactions of wave packets are investigated in shallow water magnetohydrodynamics of rotating plasma layer. The analysis of the dispersion relations for both modes shows that there are several types of three waves interactions: three magneto-Poincare waves, three magnetostrophic waves and also intermode interactions: two magneto-Poincare waves and magnetostrophic wave, two magnetostrophic waves and magneto-Poincare wave. We use multiscale asymptotic method to describe non-linear interactions [16]. The non-linear equations of wave amplitudes interaction are derived for each case. The analysis of obtained non-linear equations for the three-waves interactions shows that there are two types of instabilities: parametric decay and parametric amplification. We found following four types of parametric decay instabilities: magneto-Poincare wave decays into two magneto-Poincare waves, magnetostrophic wave decays into two magnetostrophic waves, magneto-Poincare wave decays into one magneto-Poincare wave and one magnetostrophic wave, magnetostrophic wave decays into one magnetostrophic wave and one magneto-Poincare wave. The instability growth rates are found. Also following four types of parametric amplification mechanisms were investigated: parametric amplification of magneto-Poincare waves, parametric amplification of magnetostrophic waves, magneto-Poincare wave amplification in magnetostrophic wave presence and magnetostrophic wave amplification in magneto-Poincare wave presence. Increments are found for each type of instability.

In section 2 we provide the shallow water MHD equations in rotating frame on a flat surface. Linear solutions and dispersion curves are provided. The analysis of phase matching conditions is shown the possibility of three-waves interactions. In section 3 we derive the equations for the slowly varying amplitudes of three-waves interactions in shallow water magnetohydrodynamics in the external vertical magnetic field. In section 4 the obtained equations of three-waves interactions are used to analyse physical effects of weakly non-linear interactions of magneto-Poincare and magnetostrophic waves. Parametric decays and parametric amplifications are analysed. The obtained results are formulated in section 5. We revise the shallow water MHD equations in external magnetic field in the appendix.

2. Shallow water approximation. Qualitative analysis of MHD flows of astrophysical plasma

Here, we introduce MHD shallow water equations on a flat plane that describes the flows of thin plasma layer with a free surface in a gravity field in the presence of rotation. Linear solutions of this system are used to qualitatively analyse the dispersion curves for linear waves and to determine the phase matching conditions that make the interwave interaction possible.

The MHD shallow water equations are obtained from the full set of three-dimensional MHD equations. These shallow water equations are obtained for the thin fluid layer with a free surface in the gravitational field which is rotating with the Coriolis parameter f in the external vertical magnetic field B_0 [9]. For magnetic field normalized by the factor $(4\pi\rho)^{-\frac{1}{2}}$, in which ρ is the (assumed constant) density the equations take the form:

$$\frac{\partial h}{\partial t}+\frac{\partial h v_x}{\partial x}+\frac{\partial h v_y}{\partial y}=0 \tag{2.1}$$

$$\frac{\partial (h v_x)}{\partial t}+\frac{\partial (h(v_x^2-B_x^2))}{\partial x}+gh\frac{\partial h}{\partial x}+\frac{\partial (\,h(v_x v_y-B_x B_y))}{\partial y} + B_0 B_x = f h v_y \tag{2.2}$$

$$\frac{\partial (h v_y)}{\partial t}+\frac{\partial (h(v_x v_y-B_x B_y))}{\partial x}+\frac{\partial (h(v_y^2-B_y^2))}{\partial y}+gh\frac{\partial h}{\partial y} + B_0 B_y = -f h v_x \tag{2.3}$$

$$\frac{\partial (h B_x)}{\partial t}+\frac{\partial (h(B_x v_y-B_y v_x))}{\partial y}+B_0 v_x=0 \tag{2.4}$$

$$\frac{\partial (h B_y)}{\partial t}+\frac{\partial (h(B_y v_x-B_x v_y))}{\partial x}+B_0 v_y=0 \tag{2.5}$$

In (2.1)–(2.5) h is free surface vertical coordinate, v_x, v_y are horizontal velocities in shallow water approximation in the xy-plane, B_x, B_y are horizontal components of depth-averaging magnetic fields in shallow water approximation in the x and y directions respectively, B_0 is an external magnetic field which is directed perpendicular to the xy-plane, f is the Coriolis parameter of the flow. The set (2.1)–(2.5) is the result of integrating the three-dimensional MHD equations over z-axis. We assume the complete pressure (hydrodynamic and magnetic) to be hydrostatic [14,13,15,23]. The first equation of the set (2.1)–(2.5) is the result of integrating the continuity equation, the second and the third ones are the equations of conservation of momentum, the fourth and the fifth ones are the equations of magnetic field variations. This closed set of equations, derived in [9], is complete for analysing linear waves and non-linear interactions. In the limit when $B_0=0$ these equations reduce to the common MHD shallow water equations [6]. Indeed, we supplement (2.1)–(2.5) with the equations, that distinguish significantly the MHD shallow water equations with external vertical magnetic field from the equations without vertical magnetic field:

$$\frac{\partial B_z}{\partial t}+B_0\left(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}\right)=0 \tag{2.6}$$

$$\frac{\partial h B_x}{\partial x}+\frac{\partial h B_y}{\partial y}+B_z=0 \tag{2.7}$$

In the traditional derivation of the MHD shallow water equations from the full set of three-dimensional MHD equations in [14,13,23] the vertical component of magnetic field is assumed to be zero. Note that the presence of a vertical magnetic field leads to essential changes of horizontal magnetic field dynamics in shallow water approximation. It should be noted that the horizontal magnetic field is solenoidal in the case without external magnetic field. Indeed it is not the case in the presence of external vertical magnetic field. The vertical variations of magnetic field are nonzero and the divergence-free condition contains vertical component (2.7). Therefore to cover completely magnetic field dynamics it is necessary to consider the equation for the vertical variation of magnetic field (2.6). Thus the magnetic field is principal three-dimensional and each of its components is depending on horizontal coordinates only. The divergence-free condition (2.7) is satisfied identically as a consequence of the equations for magnetic field (2.4), (2.5), (2.6) and is used to set correct initial conditions. The derivations of the system (2.1)–(2.5) and the equations (2.6) and (2.7) are given in appendix.

We linearise the equations (2.1)–(2.5) with respect to the stationary solution $h = \text{const}$, $v_x = v_y = 0$ and $B_x = B_y = 0$. The dispersion equation of the linear waves in (2.1)–(2.5) is following:

$$\omega^5 - \omega^3(ghk^2 + f^2 + 2(\frac{B_0}{h})^2) + (\frac{B_0}{h})^2(ghk^2 + (\frac{B_0}{h})^2)\omega = 0 \quad (2.8)$$

There are two types of waves in this case due to a vertical magnetic field presence. The first type is the generalization of linear Poincare waves in classical shallow water equations for neutral fluid and has following dispersion relations:

$$\omega_{1,3} = \pm\sqrt{\frac{ghk^2}{2} + \frac{f^2}{2} + (\frac{B_0}{h})^2 + \frac{1}{2}\sqrt{ghk^2(ghk^2 + 2f^2) + f^2(f^2 + 4(\frac{B_0}{h})^2)}} \quad (2.9)$$

where sign "+" corresponds to the wave that propagate along vector $\mathbf{k}$, and sign "−" corresponds to the wave that propagate in opposite direction. The obtained linear solutions are called magneto-Poincare waves. The second type of the linear solutions describes magnetostrophic waves that do not have the analogies in neutral fluid. It has following dispersion relations:

$$\omega_{2,4} = \pm\sqrt{\frac{ghk^2}{2} + \frac{f^2}{2} + (\frac{B_0}{h})^2 - \frac{1}{2}\sqrt{ghk^2(ghk^2 + 2f^2) + f^2(f^2 + 4(\frac{B_0}{h})^2)}} \quad (2.10)$$

where sign + corresponds to the wave that propagate along vector $\mathbf{k}$, and sign − corresponds to the wave that propagate in the opposite direction. Subscripts 1, 2 refer to the waves propagating along vector $\mathbf{k}$, and subscripts 3, 4 refer to the waves propagating in the opposite direction.

In nonrotating case where $f = 0$ we have:

$$\omega^2 = \begin{cases} ghk^2 + (\frac{B_0}{h})^2 \\ (\frac{B_0}{h})^2 \end{cases} \quad (2.11)$$

The first mode in (2.11) is the magnetogravity wave, where the restoring force is a combination of the gravity acceleration and magnetic tension. The second mode in (2.11) is non-dispersive and represents the Alfven wave, and its restoring force is magnetic tension. In our case of the rotating system magnetic tension tries to balance the Coriolis force. To emphasize this balance this mode is called magnetostrophic [9].

The general solution is the sum of the four linear waves.

The general form of the dispersion curves in case when $\omega > 0$ is shown in Fig. 1. The upper curve displays magneto-Poincare mode, the bottom curve displays magnetostrophic mode. The frequency ω is plotted on the vertical axis, wave number k_x is plotted on the horizontal axis, when the wave vector $\mathbf{k}$ is directed along the x-axis. Dispersion relation $\omega(\mathbf{k})$ defines the dispersion surface $\omega(k_x, k_y)$. This surface is cylindrically symmetric because the dispersion relation is symmetric with respect to k_x and k_y. For $\omega < 0$ (solutions ω_3 and ω_4) dispersion surface is symmetric with respect to the $\omega = 0$ plane.

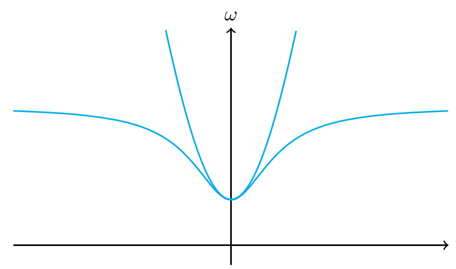

Fig. 1. Dispersion curves. The upper one is Poincare mode, the bottom one is magnetostrophic mode.

To assess the possibility of waves interactions there is a need to analyse the dispersion relations and determine the dispersion curves asymptotic for both modes. The phase matching conditions necessary for interactions of waves with different wave vectors and different frequencies in general case have the form $\omega(\mathbf{k_1}) + \omega(\mathbf{k_2}) = \omega(\mathbf{k_3}), \mathbf{k_1} + \mathbf{k_2} = \mathbf{k_3}$. Let us imagine these conditions as the curves on the plane (k_x, ω). Then the first term specifies a point $(k_1, \omega(k_1))$ which is on the first dispersion curve. The second term specifies a point $(k_2, \omega(k_2))$ which is on the second dispersion curve. The phase matching conditions are satisfied when the second dispersion curve shifted by $(k_1, \omega(k_1))$ crosses the first one in a point $(k_3, \omega(k_3))$, it implies the existence of three wave interactions. Below we qualitatively analyse the possibility of three wave interactions for magneto-Poincare and magnetostrophic waves.

The phase matching conditions for magneto-Poincare have the form $\omega_1(\mathbf{k}_1) + \omega_1(\mathbf{k}_2) = \omega_1(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ where the subscript 1 corresponds to the forward magneto-Poincare wave. At small frequencies the dispersion surfaces for the magneto-Poincare mode on the k_x, k_y-plane are convex. Fig. 2.a shows that the curve $\omega_1(k_x)$ and the curve $\omega_1(k_x - k_{x1}) + \omega_1(k_{x1})$ (the second term in the phase matching condition) may intersect and therefore the surface $\omega_1(\mathbf{k})$ and the surface $\omega_1(\mathbf{k} - \mathbf{k}_1) + \omega_1(\mathbf{k}_1)$ may intersect along a curve. The intersection points present to magneto-Poincare wave with frequency equal to the frequencies sum and the wave vector is equal to the wave vectors sum of two waves. Thus the phase matching conditions are satisfied for three magneto-Poincare waves, so their nonlinear interaction is allowed.

The phase matching conditions for three magnetostrophic waves take the form $\omega_2(\mathbf{k}_1) + \omega_2(\mathbf{k}_2) = \omega_2(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. The subscript 2 corresponds to the magnetostrophic wave. To understand whether these conditions are realized let us turn to the Fig. 2.b. The bottom curve $\omega_2(k_x)$ represents the first term in the phase matching condition. The upper curve $\omega_2(k_x - k_{x1}) + \omega_2(k_{x1})$ represents the second term. The intersection of curves causes the intersection of dispersion surfaces. Thus there are such wave vectors $\mathbf{k}_1, \mathbf{k}_2$ that the phase matching condition is satisfied.

In the case of interaction of one magnetostrophic wave and one magneto-Poincare wave magnetostrophic wave with following conditions can turn: $\omega_2(\mathbf{k}_1) + \omega_1(\mathbf{k}_2) = \omega_2(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ (Fig. 2.c) and magneto-Poincare wave can turn and its phase matching condition is: $\omega_1(\mathbf{k}_1) + \omega_2(\mathbf{k}_2) = \omega_1(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ (Fig. 2.d). In both cases the intersection of dispersion surfaces means that there are such $\mathbf{k}_1$ and $\mathbf{k}_2$ that will satisfy the phase matching conditions. Thus such three-wave interactions exist.

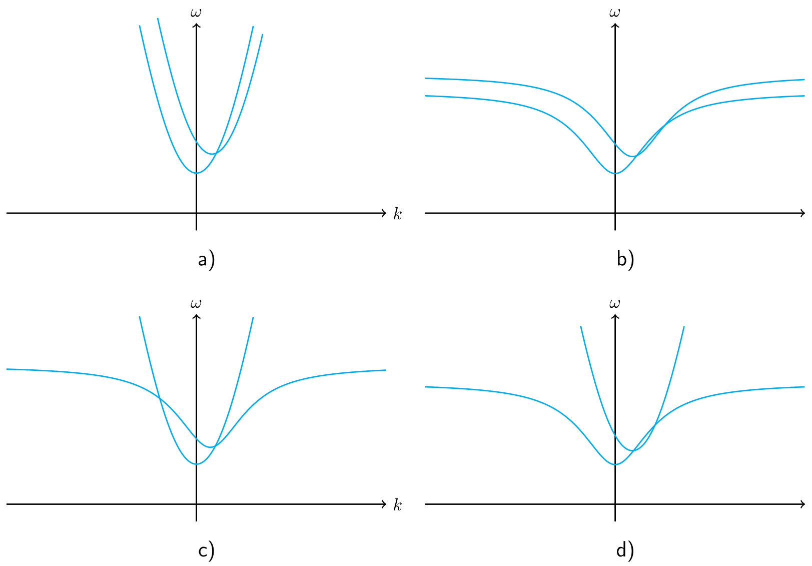

Fig. 2. The intersection of the dispersion curves. a – three magneto-Poincare waves interaction, b – three magnetostrophic waves interaction, c – two magneto-Poincare and one magnetostrophic wave interaction, d – two magnetostrophic waves and one magneto-Poincare wave interaction.

3. Three-wave interactions in shallow water magnetohydrodynamics

This section is devoted to the weakly nonlinear wave interactions in shallow water magnetohydrodynamics in the external vertical magnetic field. To study non-linear effects we use multiscale asymptotic method [16,17]. The solution $\mathbf{u} = (h, v_x, v_y, B_x, B_y)$ of original equations (2.1)–(2.5) is represented as a series in the small parameter ϵ:

$$\mathbf{u} = \epsilon \mathbf{u}_0 + \epsilon^2 \mathbf{u}_1 + \dots \tag{3.1}$$

where $\mathbf{u}_0$ is linear solution of initial system (2.1)–(2.5), $\mathbf{u}_1$ is a term describing quadratic non-linearity effect. Keeping the terms proportional to ϵ^2, we obtain a set of linear inhomogeneous equations for $\mathbf{u}_1$ that contains the secular terms that result in asymptotic solutions growing linearly with time. In this case, the condition $\epsilon \mathbf{u}_1 << \mathbf{u}_0$ is violated on large scales. Therefore, to obtain the non-linear correction, we introduce the dependence of the amplitudes of linear waves on the "slow" time and large linear scales, that ensure the eliminating of secular terms on the corresponding scales. To implement such a procedure, we introduce "fast" (T_0, X_0, Y_0) and "slow" (T_1, X_1, Y_1) variables for the variables (t, x, y) according to:

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial T_0} + \frac{\partial}{\partial T_1} + \dots \tag{3.2}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial X_0} + \frac{\partial}{\partial X_1} + \dots \tag{3.3}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial Y_0} + \frac{\partial}{\partial Y_1} + \dots \tag{3.4}$$

In this way, we obtain the compatibility conditions on the slow amplitudes under which the secular terms are eliminate and $\mathbf{u}_0(T_1, X_1, Y_1)$ is defined. As a consequence we obtain the complete set of equations for waves interactions in shallow water MHD in the external vertical magnetic field.

Considering three magneto-Poincare waves interaction with amplitudes α, β and γ ($\mathbf{l}$ is an eigenvector):

$$\begin{aligned}\mathbf{u}_0 &= \alpha \mathbf{l} \exp(i(\omega_1(\mathbf{k}_1)t - k_{1x}x - k_{1y}y)) + \\ &+ \beta \mathbf{l} \exp(i(\omega_1(\mathbf{k}_2)t - k_{2x}x - k_{2y}y)) + \\ &+ \gamma \mathbf{l} \exp(i(\omega_1(\mathbf{k}_3)t - k_{3x}x - k_{3y}y))\end{aligned} \tag{3.5}$$

with the phase matching condition $\omega_1(\mathbf{k}_1) + \omega_1(\mathbf{k}_2) = \omega_1(\mathbf{k}_3)$ and $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ we obtain amplitude equations:

$$\begin{aligned}&a\frac{\partial \gamma}{\partial T} + b^1 \frac{\partial \gamma}{\partial X} + b^2 \frac{\partial \gamma}{\partial Y} \\ &\quad = -i(k(\omega_1(\mathbf{k}_1) + \omega_1(\mathbf{k}_2)) + l^1(k_{x1} + k_{x2}) \\ &\qquad + l^2(k_{y1} + k_{y2}) + 2m)\alpha\beta\end{aligned} \tag{3.6}$$

$$\begin{aligned}&a\frac{\partial \alpha}{\partial T} + b^1 \frac{\partial \alpha}{\partial X} + b^2 \frac{\partial \alpha}{\partial Y} \\ &\quad = -i(k(\omega_1(\mathbf{k}_3) - \omega_1(\mathbf{k}_2)) + l^1(k_{x3} - k_{x2}) \\ &\qquad + l^2(k_{y3} - k_{y2}) + 2m)\gamma\beta\end{aligned} \tag{3.7}$$

$$\begin{aligned}&a\frac{\partial \beta}{\partial T} + b^1 \frac{\partial \beta}{\partial X} + b^2 \frac{\partial \beta}{\partial Y} \\ &\quad = -i(k(\omega_1(\mathbf{k}_3) - \omega_1(\mathbf{k}_1)) + l^1(k_{x3} - k_{x1}) \\ &\qquad + l^2(k_{y3} - k_{y1}) + 2m)\gamma\alpha\end{aligned} \tag{3.8}$$

where the coefficients a, b^1, b^2, k, l^1, l^2, m are constant and are determined by the initial conditions (f, H, g, B_z).

Considering the case when three magnetostrophic waves satisfy the phase matching condition $\omega_2(\mathbf{k}_1) + \omega_2(\mathbf{k}_2) = \omega_2(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ we obtain the following system describing slow amplitudes χ, ϕ, ψ of three magnetostrophic waves:

$$\begin{aligned}&a\frac{\partial \chi}{\partial T} + b^1 \frac{\partial \chi}{\partial X} + b^2 \frac{\partial \chi}{\partial Y} \\ &\quad = -i(k(\omega_2(\mathbf{k}_1) + \omega_2(\mathbf{k}_2)) + l^1(k_{x1} + k_{x2}) \\ &\qquad + l^2(k_{y1} + k_{y2}) + 2m)\phi\psi\end{aligned} \tag{3.9}$$

$$\begin{aligned}&a\frac{\partial \phi}{\partial T} + b^1 \frac{\partial \phi}{\partial X} + b^2 \frac{\partial \phi}{\partial Y} \\ &\quad = -i(k(\omega_2(\mathbf{k}_3) - \omega_2(\mathbf{k}_2)) + l^1(k_{x3} - k_{x2}) \\ &\qquad + l^2(k_{y3} - k_{y2}) + 2m)\chi\psi\end{aligned} \tag{3.10}$$

$$\begin{aligned}&a\frac{\partial \psi}{\partial T} + b^1 \frac{\partial \psi}{\partial X} + b^2 \frac{\partial \psi}{\partial Y} \\ &\quad = -i(k(\omega_2(\mathbf{k}_3) - \omega_2(\mathbf{k}_1)) + l^1(k_{x3} - k_{x1}) \\ &\qquad + l^2(k_{y3} - k_{y1}) + 2m)\chi\phi\end{aligned} \tag{3.11}$$

In case when two magneto-Poincare waves and one magnetostrophic wave satisfy the phase matching condition $\omega_1(\mathbf{k}_1)+\omega_2(\mathbf{k}_2)=\omega_1(\mathbf{k}_3)$, $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3$ we have:

$$a\frac{\partial\beta}{\partial T}+b^1\frac{\partial\beta}{\partial X}+b^2\frac{\partial\beta}{\partial Y}= \\ =-i(k(\omega_2(\mathbf{k_1})+\omega_1(\mathbf{k_2}))+l^1(k_{x1}+k_{x2}) \\ +l^2(k_{y1}+k_{y2})+2m)\alpha\phi \tag{3.12}$$

$$a\frac{\partial\phi}{\partial T}+b^1\frac{\partial\phi}{\partial X}+b^2\frac{\partial\phi}{\partial Y}= \\ =-i(k(\omega_1(\mathbf{k_3})-\omega_1(\mathbf{k_2}))+l^1(k_{x3}-k_{x2}) \\ +l^2(k_{y3}-k_{y2})+2m)\beta\alpha \tag{3.13}$$

$$a\frac{\partial\alpha}{\partial T}+b^1\frac{\partial\alpha}{\partial X}+b^2\frac{\partial\alpha}{\partial Y}= \\ =-i(k(\omega_1(\mathbf{k_3})-\omega_2(\mathbf{k_1}))+l^1(k_{x3}-k_{x1}) \\ +l^2(k_{y3}-k_{y1})+2m)\beta\phi \tag{3.14}$$

When two magnetostrophic waves and one magneto-Poincare wave interact the phase matching conditions $\omega_2(\mathbf{k}_1)+\omega_1(\mathbf{k}_2)=\omega_2(\mathbf{k}_3)$, $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3$ are satisfied and we obtain:

$$a\frac{\partial\psi}{\partial T}+b^1\frac{\partial\psi}{\partial X}+b^2\frac{\partial\psi}{\partial Y}= \\ =-i(k(\omega_1(\mathbf{k_1})+\omega_2(\mathbf{k_2}))+l^1(k_{x1}+k_{x2}) \\ +l^2(k_{y1}+k_{y2})+2m)\alpha\phi \tag{3.15}$$

$$a\frac{\partial\alpha}{\partial T}+b^1\frac{\partial\alpha}{\partial X}+b^2\frac{\partial\alpha}{\partial Y}= \\ =-i(k(\omega_2(\mathbf{k_3})-\omega_2(\mathbf{k_2}))+l^1(k_{x3}-k_{x2}) \\ +l^2(k_{y3}-k_{y2})+2m)\psi\phi \tag{3.16}$$

$$a\frac{\partial\phi}{\partial T}+b^1\frac{\partial\phi}{\partial X}+b^2\frac{\partial\phi}{\partial Y}= \\ =-i(k(\omega_2(\mathbf{k_3})-\omega_1(\mathbf{k_1}))+l^1(k_{x3}-k_{x1}) \\ +l^2(k_{y3}-k_{y1})+2m)\psi\alpha \tag{3.17}$$

Below we use the obtained nonlinear three-wave interactions equations for a qualitative analysis of wave transformation in shallow water magnetohydrodynamics of astrophysical plasma in weakly nonlinear approximation [5].

4. Parametric instabilities of magneto-Poincare and magnetostrophic waves

4.1. Decay instabilities

We are interested in the initial stage of decay of magneto-Poincare or magnetostrophic waves. Therefore we are able to linearise three waves equations by neglecting the changes of initial wave due to inverse interactions. Thus we assume the amplitude of the initial wave to be constant when the nonlinear interactions rise.

Let us consider three magneto-Poincare waves interaction. Assuming the amplitude of one wave is much higher than the amplitudes of other waves $\gamma=\gamma_0>>\alpha,\beta$ in the equations (3.6)–(3.8) we obtain the linear differential equations with the arguments (α,β). The obtained solution displays amplitude growth with the following increment:

$$\Gamma=\frac{|f^*|\gamma_0}{a} \tag{4.1}$$

where

$$f^*=-i(k(\omega_1(\mathbf{k_3})-\omega_1(\frac{\mathbf{k_3}}{2}))+l^1(\frac{k_{x3}}{2})+l^2(\frac{k_{y3}}{2})+2m) \tag{4.2}$$

Thus, parent magneto-Poincare wave with a frequency ω_3 and the wave vector $\mathbf{k_3}$ decays into two magneto-Poincare waves with frequencies $\frac{\omega_3}{2}$ and wave vectors $\frac{\mathbf{k_3}}{2}$ in magnetohydrodynamics of astrophysical plasma in external magnetic field.

Let us set the initial conditions in the equations (3.9)–(3.11) for three magnetostrophic waves interaction as the amplitude of one wave is much higher than the amplitudes of other waves $\chi=\chi_0>>\phi,\psi$ we obtain amplitude growth with the following increment:

$$\Gamma=\frac{f^*\chi_0}{a} \tag{4.3}$$

where

$$f^*=-i(k(\omega_2(\mathbf{k_3})-\omega_2(\frac{\mathbf{k_3}}{2}))+l^1(\frac{k_{x3}}{2})+l^2(\frac{k_{y3}}{2})+2m) \tag{4.4}$$

Hence, magnetostrophic wave with frequency ω_3 and wave vector $\mathbf{k_3}$ decays into two magnetostrophic waves with frequencies $\frac{\omega_3}{2}$ and wave vectors $\frac{\mathbf{k_3}}{2}$ in shallow water magnetohydrodynamics of astrophysical plasma.

Considering two magneto-Poincare and one magnetostrophic wave interaction the initial conditions in the equations (3.12)–(3.14) are selected as the amplitude of one magneto-Poincare wave is much higher than the amplitudes of other waves $\beta=\beta_0>>\phi,\alpha$. We obtain the linear system of differential equations with the argument (ϕ,α). The obtained solution displays amplitude growth with the following increment:

$$\Gamma=\frac{\sqrt{|f_2^*||f_3^*|}\beta_0}{a} \tag{4.5}$$

where

$$f_2^*=-i(k(\omega_1(\mathbf{k_3})-\omega_1(\mathbf{k_3}-\mathbf{k_1^0}))+l^1k_{x1}^0)+l^2k_{y1}^0)+2m) \tag{4.6}$$

$$f_3^*=-i(k(\omega_1(\mathbf{k_3})-\omega_2(\mathbf{k_1^0})) \\ +l^1(k_{x3}-k_{x1}^0)+l^2(k_{y3}-k_{y1}^0)+2m) \tag{4.7}$$

Therefore, magneto-Poincare wave with a frequency ω_3 and wave vector $\mathbf{k_3}$ decays into magnetostrophic wave with a frequency ω_1^0 and wave vector $\mathbf{k_1^0}$ and magneto-Poincare wave with frequency $\omega_3-\omega_1^0$ and wave vector $\mathbf{k_3}-\mathbf{k_1^0}$ where $\mathbf{k_1^0}$ is determined from the constants in free surface MHD flows in shallow water approximation.

Finally, let two magnetostrophic waves and one magneto-Poincare wave interact and satisfy the phase matching condition $\omega_2(\mathbf{k_1})+\omega_1(\mathbf{k_2})=\omega_2(\mathbf{k_1}+\mathbf{k_2})$. Considering the initial conditions in the equations (3.15)–(3.17) when the amplitude of one magnetostrophic wave is much higher than the amplitudes of other waves $\psi=\psi_0>>\phi,\alpha$ we obtain similarly amplitude growth with the following increment:

$$\Gamma=\frac{\sqrt{|f_2^*||f_3^*|}\psi_0}{a} \tag{4.8}$$

where

$$f_2^*=-i(k(\omega_2(\mathbf{k_3})-\omega_2(\mathbf{k_3}-\mathbf{k_1^0})) \\ +l^1(k_{x1}^0)+l^2(k_{y1}^0)+2m) \tag{4.9}$$

$$f_3^*=-i(k(\omega_2(\mathbf{k_3})-\omega_1(\mathbf{k_1^0})) \\ +l^1(k_{x3}-k_{x1}^0)+l^2(k_{y3}-k_{y1}^0)+2m) \tag{4.10}$$

Here we showed that magnetostrophic wave with a frequency ω_3 and wave vector $\mathbf{k_3}$ decays into magneto-Poincare wave with frequency ω_1^0 and wave vector $\mathbf{k_1^0}$ and magnetostrophic wave with frequency $\omega_3 - \omega_1^0$ and wave vector $\mathbf{k_3} - \mathbf{k_1^0}$ where $\mathbf{k_1^0}$ is determined from the constants in shallow water magnetohydrodynamics of astrophysical plasma in the external magnetic field.

4.2. Parametric amplification

The decay instability corresponds to the energy transfer from one wave into two waves, the reverse process of parametric amplification corresponds to the energy transfer from two waves to the third wave. Thus the processes are similar and provide the energy transfer in opposite directions. Below we study the initial stage of parametric amplification when two parent waves give rise to the third one. As it is done in section 4.1 we neglect the inverse action of the rising wave to the parent waves. Therefore this allows us to linearise the nonlinear three wave equations.

Let us consider three magneto-Poincare waves interaction which satisfy the phase matching conditions $\omega_1(\mathbf{k}_1) + \omega_1(\mathbf{k}_2) = \omega_1(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. The initial conditions in equations (3.6)–(3.8) are selected such as the amplitude of the third wave is much smaller than the amplitudes of two parent waves $\gamma << \alpha_0, \beta_0$. These conditions lead to the linear differential equation with the argument γ. The obtained solution displays amplitude growth with the following increment:

$$\Gamma = \frac{|f_1|\alpha_0\beta_0}{a} \tag{4.11}$$

where

$$f_1 = -i(k(\omega_1(\mathbf{k_1}) + \omega_1(\mathbf{k_2})) + l^1(k_{x1} + k_{x2}) + l^2(k_{y1} + k_{y2}) + 2m) \tag{4.12}$$

Thus, we obtained that rising magneto-Poincare wave has frequency $\omega_3 = \omega_1 + \omega_2$ and wave vector $\mathbf{k_3} = \mathbf{k_1} + \mathbf{k_2}$ in magnetohydrodynamics of astrophysical plasma in external magnetic field.

When considering three magnetostrophic wave interaction which satisfy the phase matching conditions $\omega_2(\mathbf{k}_1) + \omega_2(\mathbf{k}_2) = \omega_2(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$ the initial condition in equations (3.9)–(3.11) is that the amplitude of one wave is much smaller than the amplitudes of two other waves $\chi << \psi_0, \phi_0$. These conditions lead to the linear differential equation with the argument χ. The obtained solution displays amplitude growth with the following increment:

$$\Gamma = \frac{|f_1|\psi_0\phi_0}{a} \tag{4.13}$$

where

$$f_1 = -i(k(\omega_2(\mathbf{k_1}) + \omega_2(\mathbf{k_2})) + l^1(k_{x1} + k_{x2}) + l^2(k_{y1} + k_{y2}) + 2m) \tag{4.14}$$

Hence, rising magnetostrophic wave has frequency $\omega_3 = \omega_1 + \omega_2$ and wave vector $\mathbf{k_3} = \mathbf{k_1} + \mathbf{k_2}$ in shallow water MHD of astrophysical plasma in the external magnetic field.

If we consider two magneto-Poincare waves and one magnetostrophic wave interaction which satisfy the phase matching condition $\omega_1(\mathbf{k_1}) + \omega_2(\mathbf{k_2}) = \omega_1(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. The initial condition in equations (3.12)–(3.14) is that the amplitude of one wave is much smaller than the amplitudes of two other waves $\beta << \alpha_0, \phi_0$. This condition leads to the linear differential equation with the argument β. The obtained solution displays amplitude growth with the following increment:

$$\Gamma = \frac{|f_1|\alpha_0\phi_0}{a} \tag{4.15}$$

where

$$f_1 = -i(k(\omega_2(\mathbf{k_1}) + \omega_1(\mathbf{k_2})) + l^1(k_{x1} + k_{x2}) + l^2(k_{y1} + k_{y2}) + 2m) \tag{4.16}$$

Therefore, magneto-Poincare wave is the sum of magnetostrophic wave with frequency and wave vector ω_1 and $\mathbf{k_1}$ and magneto-Poincare wave with frequency and wave vector ω_2 and $\mathbf{k_2}$. The frequency of the originated wave is $\omega_3 = \omega_1 + \omega_2$, its wave vector is $\mathbf{k_3} = \mathbf{k_1} + \mathbf{k_2}$ in free surface flows in shallow water magnetohydrodynamics of astrophysical plasma.

In the last case we deal with the three-wave interaction of two magnetostrophic waves and one magneto-Poincare wave which satisfy the phase matching condition $\omega_2(\mathbf{k_1}) + \omega_1(\mathbf{k_2}) = \omega_2(\mathbf{k}_3)$, $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3$. The initial condition in equations (3.15)–(3.17) is that the amplitude of one wave is much smaller than the amplitudes of two other waves $\psi << \alpha_0, \phi_0$. This condition causes the linear differential equation with the argument ψ. The obtained solution displays amplitude growth with the following increment:

$$\Gamma = \frac{|f_1|\phi_0\alpha_0}{a} \tag{4.17}$$

where

$$f_1 = -i(k(\omega_1(\mathbf{k_1}) + \omega_2(\mathbf{k_2})) + l^1(k_{x1} + k_{x2}) + l^2(k_{y1} + k_{y2}) + 2m) \tag{4.18}$$

Therefore, amplified magnetostrophic wave is the sum of magneto-Poincare wave with frequency ω_1 and wave vector $\mathbf{k_1}$ and magnetostrophic wave with frequency ω_2 and wave vector $\mathbf{k_2}$. The resulting wave frequency is $\omega_3 = \omega_1 + \omega_2$ and its wave vector is $\mathbf{k_3} = \mathbf{k_1} + \mathbf{k_2}$.

5. Conclusion

The weakly nonlinear theory of wave interactions in large scale flows of rotating plasma layer with a free surface in gravity field on a flat surface in vertical magnetic field is developed. We use MHD shallow water approximation to obtain the interaction equations for the magneto-Poincare and magnetostrophic waves. The analysis of shallow water approximation in the external magnetic field showed the principal difference in waves dynamics compared to those without external magnetic field. Considering the dispersion surfaces of investigated waves we found that such condition can be satisfied for following configurations: three magneto-Poincare waves (Fig. 2.a), three magnetostrophic waves (Fig. 2.b), and also two magnetostrophic waves and one magneto-Poincare wave (Fig. 2.d) and two magneto-Poincare waves and one magnetostrophic wave (Fig. 2.c). We obtain weakly nonlinear solution by asymptotic multiscale method. For each type of interactions the nonlinear equations are obtained. It is shown that there are decay instabilities and parametric amplifications in weakly nonlinear approximation. It is found that there are following types of decay instabilities: magneto-Poincare wave decays into two magneto-Poincare waves with the increment (4.1), magnetostrophic wave decays into two magnetostrophic waves with the increment (4.3), magneto-Poincare wave decays into one magneto-Poincare wave and one magnetostrophic wave with the increment (4.5), magnetostrophic wave decays into one magnetostrophic wave and one magneto-Poincare wave with the increment (4.8). Also there are four types of parametric amplifications: parametric amplification of the magneto-Poincare waves with instability growth rate (4.11), parametric amplification of magnetostrophic waves with instability growth rate (4.13), and also parametric amplification of magneto-Poincare wave in magneto-Poincare wave field with instability growth rate (4.15) and amplification of magnetostrophic wave in magneto-Poincare wave field with instability growth rate (4.17). The work is supported by the program 7 of Russian Academy of

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Appendix A. Shallow water approximation in external magnetic field

Here we derive the MHD shallow water equations taking special attention to the divergence-free condition, which must be inevitably satisfied exactly in shallow water approximation in a presence of the external vertical magnetic field. We use depth-averaging of initial three-dimensional MHD equations as done in [9]. Let us denote the normalized by the factor $(4\pi\rho)^{-\frac{1}{2}}$, in which ρ is the (assumed constant) density components of magnetic field as B_1, B_2, B_3 and the velocity components as v_1, v_2, v_3, $\tilde{p}$ is a pressure. The incompressible MHD equations in a gravity field in a rotating frame of reference with Coriolis parameter f are following:

$$\partial_t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} + \partial_x \begin{pmatrix} v_1^2 - B_1^2 + \tilde{p} \\ v_1 v_2 - B_1 B_2 \\ v_1 v_3 - B_1 B_3 \\ 0 \\ v_1 B_2 - v_2 B_1 \\ v_1 B_3 - v_3 B_1 \end{pmatrix} + \partial_y \begin{pmatrix} v_1 v_2 - B_1 B_2 \\ v_2^2 - B_2^2 + \tilde{p} \\ v_2 v_3 - B_2 B_3 \\ v_2 B_1 - v_1 B_2 \\ 0 \\ v_2 B_3 - v_3 B_2 \end{pmatrix} + \partial_z \begin{pmatrix} v_1 v_3 - B_1 B_3 \\ v_2 v_3 - B_2 B_3 \\ v_3^2 - B_3^2 + \tilde{p} \\ v_3 B_1 - v_1 B_3 \\ v_3 B_2 - v_2 B_3 \\ 0 \end{pmatrix} = \begin{pmatrix} f v_2 \\ f v_1 \\ -g \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{A.1}$$

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0 \tag{A.2}$$

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0 \tag{A.3}$$

The equations (A.1) are the equations for velocity and magnetic field, the equation (A.2) is the continuity equation and (A.3) is the divergence-free condition for magnetic field, that is satisfied identically and used to set correct initial conditions. We supplement these equations with the boundary conditions for a fluid layer in a vertical magnetic field B_0 in the form:

$$v_3|_{z=0} = 0 \tag{A.4}$$

$$v_3|_{z=h} = \frac{\partial h}{\partial t} + v_1|_{z=h}\frac{\partial h}{\partial x} + v_2|_{z=h}\frac{\partial h}{\partial y} \tag{A.5}$$

$$B_3|_{z=0} = B_0 \tag{A.6}$$

$$B_3|_{z=h} = B_0 + B_1|_{z=h}\frac{\partial h}{\partial x} + B_2|_{z=h}\frac{\partial h}{\partial y} \tag{A.7}$$

The pressure (the sum of hydrodynamic and magnetic) is assumed to be hydrostatic:

$$\partial_z(\tilde{p}) = \partial_z\left(p + \frac{B^2}{2}\right) = -g \tag{A.8}$$

To derive shallow water equations, the system (A.1) is integrated along the vertical axis. According to (A.8) we neglect the vertical accelerations, and the equation for the vertical velocity turns into an identity. Using the boundary conditions (A.4)–(A.7) we obtain from (A.1):

$$\frac{\partial}{\partial t}\int_0^h v_1 dz + \frac{\partial}{\partial x}\int_0^h v_1^2 dz - \frac{\partial}{\partial x}\int_0^h B_1^2 dz + gh\frac{\partial h}{\partial x} + \frac{\partial}{\partial y}\int_0^h v_1 v_2 dz - \frac{\partial}{\partial y}\int_0^h B_1 B_2 dz + B_0 B_1 = f v_2 h \tag{A.9}$$

$$\frac{\partial}{\partial t}\int_0^h v_2 dz + \frac{\partial}{\partial x}\int_0^h v_1 v_2 dz - \frac{\partial}{\partial x}\int_0^h B_1 B_2 dz + gh\frac{\partial h}{\partial y} + \frac{\partial}{\partial y}\int_0^h v_2^2 dz - \frac{\partial}{\partial y}\int_0^h B_2^2 dz + B_0 B_2 = -f v_1 h \tag{A.10}$$

$$\frac{\partial}{\partial t}\int_0^h B_1 dz + \frac{\partial}{\partial x}\int_0^h v_2 B_1 dz - \frac{\partial}{\partial y}\int_0^h v_1 B_2 dz + B_0 v_1 = 0 \tag{A.11}$$

$$\frac{\partial}{\partial t}\int_0^h B_2 dz + \frac{\partial}{\partial x}\int_0^h v_1 B_2 dz - \frac{\partial}{\partial y}\int_0^h v_2 B_1 dz + B_0 v_2 = 0 \tag{A.12}$$

$$\frac{\partial}{\partial t}\int_0^h B_3 dz + \frac{\partial}{\partial x}\int_0^h B_3 v_1 dz + \frac{\partial}{\partial y}\int_0^h v_2 B_3 dz = 0 \tag{A.13}$$

The equation (A.13) plays an important role in understanding the shallow water approximation in the external magnetic field since the variation of the vertical magnetic field is principal for satisfying the depth-averaging divergence-free condition of the magnetic field. Depth-averaging of the continuity equation (A.2) leads to:

$$\frac{\partial}{\partial x}\int_0^h v_1 dz + \frac{\partial}{\partial y}\int_0^h v_2 dz + \frac{\partial h}{\partial t} = 0, \tag{A.14}$$

and the depth-averaging divergence-free condition (A.3) of the magnetic field takes the form:

$$\frac{\partial}{\partial x}\int_0^h B_1 dz + \frac{\partial}{\partial y}\int_0^h B_2 dz + B_3 = 0 \tag{A.15}$$

The equations (A.13) and (A.15) are principal in MHD shallow water approximation in the external magnetic field not only as the technical details to obtain correct consequence of the divergence-free condition but also shows the existence of z-component of the magnetic field expressions for that are separated from the shallow water equations [9]. Let us introduce the height-averaged velocities and magnetic fields:

$$v_{x,y} = \frac{1}{h}\int_0^h v_{1,2} dz \tag{A.16}$$

$$B_{x,y,z} = \frac{1}{h}\int_0^h B_{1,2,3} dz \tag{A.17}$$

Using the expressions (A.16), (A.17) we obtain MHD shallow water equations in external vertical magnetic field:

$$\frac{\partial h}{\partial t}+\frac{\partial h v_x}{\partial x}+\frac{\partial h v_y}{\partial y}=0 \tag{A.18}$$

$$\frac{\partial (h v_x)}{\partial t}+\frac{\partial (h(v_x^2-B_x^2))}{\partial x}+gh\frac{\partial h}{\partial x}+\frac{\partial (\,h(v_x v_y-B_x B_y))}{\partial y}+B_0B_x=fhv_y \tag{A.19}$$

$$\frac{\partial (h v_y)}{\partial t}+\frac{\partial (h(v_x v_y-B_x B_y))}{\partial x}+\frac{\partial (h(v_y^2-B_y^2))}{\partial y}+gh\frac{\partial h}{\partial y}+B_0B_y=-fhv_x \tag{A.20}$$

$$\frac{\partial (h B_x)}{\partial t}+\frac{\partial (h(B_x v_y-B_y v_x))}{\partial y}+B_0v_x=0 \tag{A.21}$$

$$\frac{\partial (h B_y)}{\partial t}+\frac{\partial (h(B_y v_x-B_x v_y))}{\partial x}+B_0v_y=0 \tag{A.22}$$

and supplementary equations for depth-averaged z-component of the magnetic field and depth-averaged divergence-free condition:

$$\frac{\partial B_z}{\partial t}+B_0(\frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y})=0 \tag{A.23}$$

$$\frac{\partial hB_x}{\partial x}+\frac{\partial hB_y}{\partial y}+B_z=0 \tag{A.24}$$

The equation (A.18) follows from (A.14), equations (A.19)–(A.23) are derived from the equations for velocity and magnetic field variations (A.9)–(A.13), the equation (A.24) follows from (A.15).

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