Chapter 3

Discretisation of shallow water equations

3.1 Staggered grids

For the discretisation of the two-dimensional shallow water equations, see sections 3.2, and 3.3 there have been various grid layouts suggested for the distribution of discrete point for water elevation, and the horizontal velocity components (see *Arakawa and Lamb* [1977] and figure 3.1). These have been denoted by A-, B-, C-, D-, and E-grid. Since the E-grid is basically a rotated B-grid, it is often neglected in numerical analyses (see e.g. *Beckers and Deleersnijder* [1993]).

A-GRID

B-GRID

C-GRID
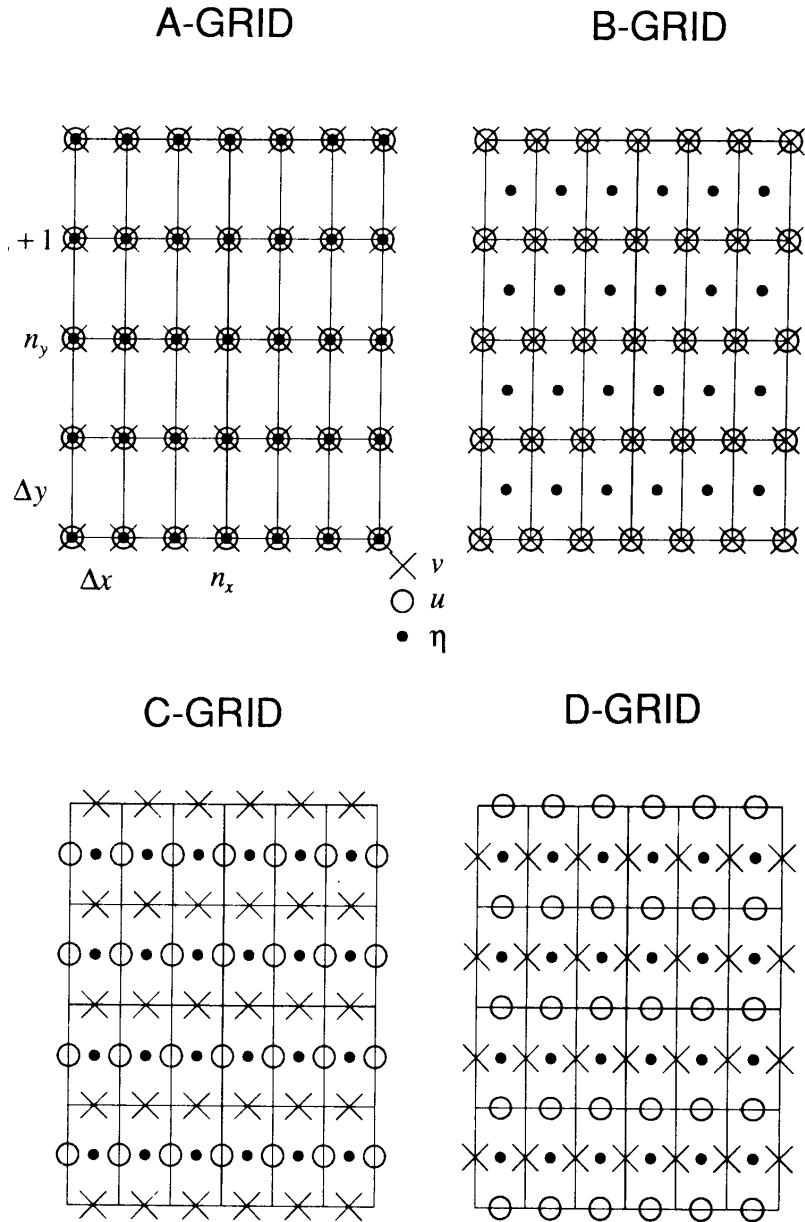
D-GRID

Figure 3.1: Design of the Arakawa A-, B-, C-, and D-grids, from *Beckers and Deleersnijder* [1993], with $\bullet$ denoting the location of the surface elevation η, and $\circ$ and $\times$ denoting the locations of the horizontal velocity components u and v, respectively.

3.2 Stability of schemes for irrotational flows

The linearised shallow water equations for irrotational flow are of the following form:

$$\partial_{\tilde{t}}\tilde{\eta} = -H\partial_{\tilde{x}}\tilde{u} - H\partial_{\tilde{y}}\tilde{v}$$

$$\partial_{\tilde{t}}\tilde{u} = -g\partial_{\tilde{x}}\tilde{\eta} \qquad\qquad (3.1)$$

$$\partial_{\tilde{t}}\tilde{v} = -g\partial_{\tilde{y}}\tilde{\eta}$$

with the surface elevation $\tilde{\eta}$, the eastward velocity, $\tilde{u}$, the northward velocity, $\tilde{v}$, the time $\tilde{t}$, the spatial coordinates, $\tilde{x}$ and $\tilde{y}$, and the average water depth H and the gravitational acceleration, g.

In order make this system of equations non-dimensional, we introduce a length scale $\tilde{L}$, a time scale $\tilde{T}$, a velocity scale $\tilde{U}$, and an elevation scale $\tilde{E}$, such that we define non-dimensional variables as

$$t = \frac{\tilde{t}}{\tilde{T}}; \quad x = \frac{\tilde{x}}{\tilde{L}}; \quad y = \frac{\tilde{y}}{\tilde{L}};$$

$$\eta = \frac{\tilde{\eta}}{\tilde{E}}; \quad u = \frac{\tilde{u}}{\tilde{U}}; \quad v = \frac{\tilde{v}}{\tilde{U}}. \qquad\qquad (3.2)$$

After inserting (3.2) into (3.1) and defining the relations

$$\tilde{L}^2 = gH\tilde{T}^2 \qquad\qquad (3.3)$$

and

$$\tilde{E}^2 = \frac{H\tilde{U}^2}{g}, \qquad\qquad (3.4)$$

we obtain the non-dimensionalised irrotational shallow water equations (see *Beckers and Deleersnijder* [1993]):

$$\partial_t\eta = -\partial_x u - \partial_y v$$

$$\partial_t u = -\partial_x\eta \qquad\qquad (3.5)$$

$$\partial_t v = -\partial_y\eta$$

A typical explicit numerical discretisation on a C-grid is of the following form (see *Beckers and Deleersnijder* [1993]):

$$\frac{\eta_{n_x,n_y}^{n_t+1} - \eta_{n_x,n_y}^{n_t}}{\Delta t} = -\frac{u_{n_x+1/2,n_y}^{n_t} - u_{n_x-1/2,n_y}^{n_t}}{\Delta x} - \frac{v_{n_x,n_y+1/2}^{n_t} - v_{n_x,n_y-1/2}^{n_t}}{\Delta y}$$

$$\frac{u_{n_x+1/2,n_y}^{n_t+1} - u_{n_x+1/2,n_y}^{n_t}}{\Delta t} = -\frac{\eta_{n_x+1,n_y}^{n_t+1} - \eta_{n_x,n_y}^{n_t+1}}{\Delta x}$$

$$\frac{v_{n_x,n_y+1/2}^{n_t+1} - v_{n_x,n_y+1/2}^{n_t}}{\Delta t} = -\frac{\eta_{n_x,n_y+1}^{n_t+1} - \eta_{n_x,n_y}^{n_t+1}}{\Delta y}$$

(3.6)

Expressed as discrete Fourier modes, the numerical solution may be written as:

$$\begin{pmatrix} \eta_{n_x,n_y}^{n_t} \\ u_{n_x+1/2,n_y}^{n_t} \\ v_{n_x,n_y+1/2}^{n_t} \end{pmatrix} = \begin{pmatrix} E^{n_t} \exp\left(i\left(k_x n_x \Delta x + k_y n_y \Delta y\right)\right) \\ U^{n_t} \exp\left(i\left(k_x(n_x+1/2)\Delta x + k_y n_y \Delta y\right)\right) \\ V^{n_t} \exp\left(i\left(k_x n_x \Delta x + k_y(n_y+1/2)\Delta y\right)\right) \end{pmatrix} \quad (3.7)$$

Inserting (3.7) into (3.6) gives:

$$E^{n_t+1} - E^{n_t} = -\Delta t 2i \left(\frac{U^{n_t}}{\Delta x}\sin(1/2 k_x \Delta x) + \frac{V^{n_t}}{\Delta y}\sin(1/2 k_y \Delta y)\right)$$

$$U^{n_t+1} - U^{n_t} = -\frac{\Delta t}{\Delta x} 2i E^{n_t+1} \sin(1/2 k_x \Delta x) \quad (3.8)$$

$$V^{n_t+1} - V^{n_t} = -\frac{\Delta t}{\Delta y} 2i E^{n_t+1} \sin(1/2 k_y \Delta y),$$

which may be converted to

$$\begin{pmatrix} E^{n_t+1} \\ U^{n_t+1} \\ V^{n_t+1} \end{pmatrix} = A \begin{pmatrix} E^{n_t} \\ U^{n_t} \\ V^{n_t} \end{pmatrix} \quad (3.9)$$

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with the amplification matrix

$$
A = \begin{pmatrix}
1 & -2i\frac{\Delta t}{\Delta x}\sin(a_x) & -2i\frac{\Delta t}{\Delta y}\sin(a_y) \\[2mm]
-2i\frac{\Delta t}{\Delta x}\sin(a_x) & \left(1 - 4\frac{\Delta t^2}{\Delta x^2}\sin^2(a_x)\right) & -4\frac{\Delta t^2}{\Delta x\Delta y}\sin(a_x)\sin(a_y) \\[2mm]
-2i\frac{\Delta t}{\Delta x}\sin(a_y) & -4\frac{\Delta t^2}{\Delta x\Delta y}\sin(a_x)\sin(a_y) & \left(1 - 4\frac{\Delta t^2}{\Delta y^2}\sin^2(a_y)\right)
\end{pmatrix}
\tag{3.10}
$$

with $a_x = 1/2k_x\Delta x$ and $a_y = 1/2k_y\Delta y$. According to the von Neumann stability criterium, the norm of the matrix A must not be larger than unity. The norm of the matrix is equal to the largest module of the eigenvalues.

To demonstrate how a stability criterion is derived we assume one-dimensional flow only with $V^{n_t} = V^{n_t+1} = 0$, such that the matrix A reduces to

$$
A = \begin{pmatrix} 1 & -ia \\ -ia & \left(1 - a^2\right) \end{pmatrix}
\tag{3.11}
$$

with

$$
a = 2\frac{\Delta t}{\Delta x}\sin(1/2k_x\Delta x).
\tag{3.12}
$$

This the eigenvalues $\lambda_{1,2}$ are calculated by means of the following equation:

$$
(1 - \lambda)(1 - a^2 - \lambda) + a^2 = 0
\tag{3.13}
$$

which may be transformed to

$$
\lambda^2 - 2b\lambda + 1 = 0
\tag{3.14}
$$

with

$$
b = 1 - \frac{a^2}{2}
\tag{3.15}
$$

Eq. (3.14) has the solutions

$$
\lambda_{1,2} = b \pm \sqrt{b^2 - 1}.
\tag{3.16}
$$

For the case of $b^2 \leq 1$ we obtain

$$
\lambda_{1,2} = b \pm i\sqrt{1 - b^2},
\tag{3.17}
$$

such that

$$
|\lambda_{1,2}| = |b \pm i\sqrt{1 - b^2}| = \sqrt{b^2 + (1 - b^2)} = 1.
\tag{3.18}
$$

For the case of $b^2 > 1$, we obtain for $b > 1$

$$
|\lambda_1| = |b + \sqrt{b^2 - 1}| > 1.
\tag{3.19}
$$

and for $b < -1$

$$
|\lambda_2| = |b - \sqrt{b^2 - 1}| > 1.
\tag{3.20}
$$

Thus, the scheme will be instable for $b^2 > 1$, and $-1 \leq b \leq 1$ is a necessary stability condition. With $b^2 \leq 1$ we have $a^2 \leq 4$ and thus

$$4\frac{\Delta t^2}{\Delta x^2}\sin^2(1/2k_x\Delta x) \leq 4. \tag{3.21}$$

A sufficient condition for this is

$$\frac{\Delta t^2}{\Delta x^2} \leq 1. \tag{3.22}$$

Converting this non-dimensional criterion to dimensional form by means of (3.3), we obtain

$$\frac{\Delta \tilde{t}^2}{\Delta \tilde{x}^2} \leq \frac{\tilde{T}^2}{\tilde{L}^2} = \frac{1}{gH} \quad \Rightarrow \quad \frac{\Delta \tilde{t}}{\Delta \tilde{x}}\sqrt{gH} \leq 1, \tag{3.23}$$

which means that in the one-dimensional case, a shallow water surface wave (the phase speed of which is $\sqrt{gH}$) must not travel further than the distance of one grid box width. For the full two-dimensional case, the stability criterion reads

$$\Delta \tilde{t}\sqrt{\frac{gH}{\Delta \tilde{x}^2} + \frac{gH}{\Delta \tilde{y}^2}} \leq 1, \tag{3.24}$$

or, for the equidistant case,

$$\frac{\Delta \tilde{t}}{\Delta \tilde{x}}\sqrt{gH} \leq \sqrt{\frac{1}{2}}. \tag{3.25}$$

Here, the surface wave must not travel further than the distance between a u-point and a v-point, which is $\sqrt{\frac{1}{2}}\Delta \tilde{x}$.

3.3 Stability of schemes for rotational flows

When earth rotation is considered, then the dimensionless shallow water equations (3.26) are of the following form

$$\begin{aligned}
\partial_t \eta &= -\partial_x u - \partial_y v \\
\partial_t u - fv &= -\partial_x \eta \\
\partial_t v + fu &= -\partial_y \eta
\end{aligned} \tag{3.26}$$

with the Coriolis parameter f, which has been nondimensionalised by multiplication with $\tilde{T}$ from (3.2).

The stability criteria for central difference approximations for the C-grid will be shown for a scheme which is an extension of (3.6):

$$\frac{\eta_{n_x,n_y}^{n_t+1} - \eta_{n_x,n_y}^{n_t}}{\Delta t} = -\frac{u_{n_x+1/2,n_y}^{n_t} - u_{n_x-1/2,n_y}^{n_t}}{\Delta x} - \frac{v_{n_x,n_y+1/2}^{n_t} - v_{n_x,n_y-1/2}^{n_t}}{\Delta y}$$

$$\frac{u_{n_x+1/2,n_y}^{n_t+1} - u_{n_x+1/2,n_y}^{n_t}}{\Delta t} = -\frac{\eta_{n_x+1,n_y}^{n_t+1} - \eta_{n_x,n_y}^{n_t+1}}{\Delta x} + f\,(\bar{v}^{xy})_{n_x+1/2,n_y}^{n_t+s}$$

$$\frac{v_{n_x,n_y+1/2}^{n_t+1} - v_{n_x,n_y+1/2}^{n_t}}{\Delta t} = -\frac{\eta_{n_x,n_y+1}^{n_t+1} - \eta_{n_x,n_y}^{n_t+1}}{\Delta y} - f\,(\bar{u}^{xy})_{n_x,n_y+1/2}^{n_t+1-s}\,,$$

$$(3.27)$$

with

$$s = \begin{cases} 0 & \text{for even } n_t, \\[2mm] 1 & \text{for odd } n_t. \end{cases} \tag{3.28}$$

The terms $\bar{u}^{xy}$ and $\bar{u}^{xy}$ denote spatial averaging to the location to which the discrete equation is located. After having transferred the discrete C-grid solution adequately to the A-, the B- and the D-grid and with $2\theta_x = k_x$ and $2\theta_y = k_y$ and

$$\phi = f\Delta t; \quad c_x = \frac{\Delta t}{\Delta x}\sqrt{gH}; \quad c_y = \frac{\Delta t}{\Delta y}\sqrt{gH}; \tag{3.29}$$

$$\alpha = \begin{cases} 1 & \text{for A- and B-grids,} \\[2mm] |\cos\theta_x \cos\theta_y| & \text{for C- and D-grids} \end{cases} \tag{3.30}$$

the stability of all these schemes is as shown in figure 3.2.

3.4 Implicit schemes

Neglecting rotation again and discretising the set of equations (3.5), an efficient way to increase numerical stability is to apply implicit schemes for the free surface. the two known methods are the two-dimensional one-step schemes (see section 3.4.1) and the alternating directional split schemes (see section 3.4.2).

3.4.1 Two-dimensional implicit schemes

For simplicity, the two-dimensional implicit schemes (see, e.g., *Backhaus* [1985]; *Burchard* [1995]) are explained here with the example of a one-dimensional implicit scheme. The extension to two-dimensional schemes is then straightforward.

The one-dimensional idealisation of the non-dimensional and non-rotational system of equations (3.5) is of the following form:

Grid	Inertia oscillations	Pure gravity waves		Inertia-gravity waves	
		$c_x = c = c_y$	$c_x \neq c_y$	$c_x = c = c_y$	$c_x \neq c_y$
A	$\phi^2 \leqslant 1$	$c^2 \leqslant 2$	$c_x^2 + c_y^2 \leqslant 4$	$c^2 \leqslant 2\,\dfrac{2-|\phi|-\phi^2}{4-\phi^2}$	$c_x^2 + c_y^2 \leqslant \dfrac{2-\phi^2-|\phi|}{1-\phi^2\sin^2\beta\cos^2\beta}\sqrt{\phi^2+(1-\phi^2)\sin^2 2\beta}$ where $\sin^2\beta = \dfrac{c_y^2}{c_x^2+c_y^2}$
B	$\phi^2 \leqslant 1$	$c^2 \leqslant 1$	$c_x^2, c_y^2 \leqslant 1$	$c^2 \leqslant \dfrac{1-\phi^2}{2}$	$c_x^2, c_y^2 \leqslant \dfrac{1-\phi^2}{2}$
C	$\phi^2 \leqslant 1$	$c^2 \leqslant \frac{1}{2}$	$c_x^2 + c_y^2 \leqslant 1$	$\phi^2 \leqslant 1$ and $c^2 \leqslant \frac{1}{4}$	$\phi^2 \leqslant 1$ and $c_x^2 + c_y^2 \leqslant \frac{1}{2}$
D	$\phi^2 \leqslant 1$	$c^2 \leqslant \frac{27}{8}$	$c_x^2, c_y^2 \leqslant 4$ and $\dfrac{(c_x^2+c_y^2)^3}{27c_x^2c_y^2} \leqslant 1$ if $c_x^2+c_y^2 \geqslant 6$	$c^2 \leqslant \dfrac{1}{2}\min_{0\leqslant\alpha\leqslant 1}\left[\dfrac{1-\alpha|\phi|}{\alpha^2(1-\alpha)(2-\alpha|\phi|)}\right]$	Unknown in general $c_x^2, c_y^2 \leqslant \frac{1}{2}$ if $\phi^2=1$ $c_x^2 \leqslant \frac{1}{2}(1+\sqrt{1-\phi^2})^2$ if $c_y=0$

Figure 3.2: Stability criteria for inertia-gravity waves, from *Beckers and Deleersnijder* [1993]. For the meaning of c_x, c_y, ϕ, and α, see equations (3.29) and (3.30).

$$\partial_t \eta = -\partial_x u$$
$$\partial_t u = -\partial_x \eta$$
(3.31)

A semi-implicit discretisation is of the following form:

$$\frac{\eta_i^{n+1} - \eta_i^n}{\Delta t} = -\sigma \frac{u_{i+1/2}^{n+1} - u_{i-1/2}^{n+1}}{\Delta x} - (1-\sigma)\frac{u_{i+1/2}^n - u_{i-1/2}^n}{\Delta x}$$

$$\frac{u_{i+1/2}^{n+1} - u_{i+1/2}^n}{\Delta t} = -\sigma \frac{\eta_{i+1}^{n+1} - \eta_i^{n+1}}{\Delta x} - (1-\sigma)\frac{\eta_{i+1}^n - \eta_i^n}{\Delta x}$$
(3.32)

with $0 \leq \sigma \leq 1$. For $\sigma = 0$, we have a fully explicit scheme, for $\sigma = 1/2$, we have a semi-implicit, and for $\sigma = 1$ we have a fully implicit scheme.

For solving this implicit scheme, we first write down the discretisation for $u_{i-1/2}^{n+1}$:

$$\frac{u_{i-1/2}^{n+1} - u_{i-1/2}^n}{\Delta t} = -\sigma \frac{\eta_i^{n+1} - \eta_{i-1}^{n+1}}{\Delta x} - (1-\sigma)\frac{\eta_i^n - \eta_{i-1}^n}{\Delta x},$$
(3.33)

and then we formally insert the solutions for $u_{i+1/2}^{n+1}$ and $u_{i-1/2}^{n+1}$ into the discrete

equation for η_i^{n+1}:

$$\eta_{i+1}^{n+1}\left(-\sigma^2\frac{\Delta t^2}{\Delta x^2}\right) + \eta_i^{n+1}\left(1 + 2\sigma^2\frac{\Delta t^2}{\Delta x^2}\right) + \eta_{i-1}^{n+1}\left(-\sigma^2\frac{\Delta t^2}{\Delta x^2}\right)$$

$$= \eta_{i+1}^n\left(\sigma(1-\sigma)\frac{\Delta t^2}{\Delta x^2}\right) + \eta_i^n\left(1 - 2\sigma(1-\sigma)\frac{\Delta t^2}{\Delta x^2}\right) + \eta_{i-1}^n\left(\sigma(1-\sigma)\frac{\Delta t^2}{\Delta x^2}\right)$$

$$- \frac{\Delta t}{\Delta x}\left(u_{i+1/2}^n - u_{i-1/2}^n\right)$$

$$(3.34)$$

This is a linear system of equations with a so-called tri-diagonal matrix, which is diagonally dominant, i.e. the sum of the elements outside the diagonal is smaller than the diagonal itself. For such simple linear systems of equations, usually simplified Gaussian solvers are used.

3.4.2 ADI schemes

In order to avoid the solution of large two-dimensional implicit linear systems of equations for the surface elevation, a method called *alternating directions implicit* (ADI) has been suggested by *Abbott* [1979]. These schemes calculate subsequently first one horizontal direction and then the other. An example for the discretisation of the linear shallow water equations without Earth rotation, (3.5) is given here:

$$\frac{\eta_{n_x,n_y}^{n_t+1/2} - \eta_{n_x,n_y}^{n_t}}{\Delta t} = -\frac{u_{n_x+1/2,n_y}^{n_t+1} - u_{n_x-1/2,n_y}^{n_t+1}}{\Delta x}$$

$$\frac{u_{n_x+1/2,n_y}^{n_t+1} - u_{n_x+1/2,n_y}^{n_t}}{\Delta t} = -\frac{\eta_{n_x+1,n_y}^{n_t+1/2} - \eta_{n_x,n_y}^{n_t+1/2}}{\Delta x}$$

$$(3.35)$$

$$\frac{\eta_{n_x,n_y}^{n_t+1} - \eta_{n_x,n_y}^{n_t+1/2}}{\Delta t} = -\frac{v_{n_x,n_y+1/2}^{n_t+1} - v_{n_x,n_y-1/2}^{n_t+1}}{\Delta y}$$

$$\frac{v_{n_x,n_y+1/2}^{n_t+1} - v_{n_x,n_y+1/2}^{n_t}}{\Delta t} = -\frac{\eta_{n_x,n_y+1}^{n_t+1} - \eta_{n_x,n_y}^{n_t+1}}{\Delta y}$$

$$(3.36)$$

Each of the equations (3.35) and (3.36) is equivalent to the fully implicit (for $\sigma = 1$) version of the one-dimensional scheme (3.32). Both, (3.35) and (3.36) are solved subsequently, with alternating sequence (in order to obtain symmetry between the two directions). If the η equations in (3.35) and (3.36) are added, the contribution from the intermediate surface elevation $\eta_{n_x,n_y}^{n_t+1/2}$ drops out such that it is obvious that (3.35) and (3.36) provide a consistent discretisation of (3.5).

Nowadays, three-dimensional models are either discretised using a two-dimensional implicit scheme for the surface elevation, as shown by means of a one-dimensional

example in section 3.4. Alternatively, an explicit discretisation of the surface elevation is often used according to (3.5). For this, a comparably short time step is needed, in order to fulfil the stability criterium (3.24). Since the stability criterium for the advection of momentum and tracers (e.g., temperature and salinity) is typically much weaker due to the small external Froude number (current speed much smaller than the phase speed of long surface waves), split schemes are used, with many small times steps for the surface elevation per time step for the equations for velocity and tracers.

3.5 Strategies for 3D models

The possible strategies how to discretise three-dimensional free-surface models is demonstrated here for a simplified two-dimensional (vertical) problem:

$$\partial_t u + \partial_x(uu) + \partial_z(uw) - \partial_z(A_v \partial_z u) = -g\partial_x \eta, \tag{3.37}$$

$$\partial_x u + \partial_z w = 0, \tag{3.38}$$

$$\partial_t c + \partial_x(cu) + \partial_z(cw) = 0, \tag{3.39}$$

where c is a passively drifted non-reactive marker tracer not subject to vertical mixing. This system of equations is solved with the following dynamic boundary conditions

$$-A_v \partial_z u = \tau_b, \quad \text{for } z = -H,$$

$$-A_v \partial_z u = \tau_s, \quad \text{for } z = \eta, \tag{3.40}$$

and kinematic boundary conditions

$$w = -u\partial_x H, \qquad \text{for } z = -H,$$

$$w = \partial_t \eta + u\partial_x \eta, \qquad \text{for } z = \eta. \tag{3.41}$$

With the Leibniz rule

$$\partial_t \int_{a(t)}^{b(t)} f(t,z)\, \mathrm{d}z = \int_{a(t)}^{b(t)} \partial_t f(t,z)\, \mathrm{d}z + \partial_t b(t) f(t,b(t)) - \partial_t a(t) f(t,a(t)), \tag{3.42}$$

and the kinematic boundary conditions (3.41), vertical integration of (3.38) from $z = -H$ to $z = \eta$ gives the surface elevation equation

$$\partial_t \eta = -\partial_x U \tag{3.43}$$

with the vertically integrated horizontal velocity component

$$U = \int_{-H}^{\eta} u\, \mathrm{d}z. \tag{3.44}$$

In the following, one possible strategy is presented to solve the equations (3.37) - (3.39) numerically. This concept is equivalent to the solution strategy of the General Estuarine Transport Model (GETM, www.getm.eu, *Burchard and Bolding* [2002]), which is a coastal ocean model with bottom-following coordinates. In GETM, most of the terms are solved explicitly in time, except for the vertical diffusion terms which are solved semi-implicitly. In this case, there are at least three different constraints which have to be respected to obtain a sufficiently accurate numerical solution of the system of equations (3.37) - (3.39):

1. The dynamic equations (3.37) and (3.39) require that the time step fulfils the CFL criterium $\Delta t \leq \Delta x / u_{\max}$ with $u_{\max}$ denoting the maximum horizontal velocity value during the entire simulation (see section 2.2.3).

2. The explicit solution of the free-surface equation requires that the CFL criterium for the phase velocity of shallow water waves is fulfilled: $\Delta t \leq \Delta x / \sqrt{gH_{max}}$, where $H_{\max}$ is the maximum water depth. Note that the criterium for fully three-dimensional model simulations is even more critical, see equation (3.23).

3. The tracer equation (3.39) must also be fulfilled for a tracer of constant concentration $c = 1$ everywhere, which shows that the continuity equation (3.38) must also be fulfilled numerically. This has the consequence that the volume conservation must be numerically exact, such that the surface elevation equation (3.43) and the continuity equation (3.38) must be numerically consistent and that the definition of the vertically integrated transport (3.44) must be numerically exact.

In order to solve for the surface elevation, the dynamic equation for the vertically integrated transport U needs to be solved. To do so, the u equation in (3.37) has to be integrated vertically:

$$\int_{-H}^{\eta} \partial_t u \, \mathrm{d}z + \int_{-H}^{\eta} \partial_x(uu) \, \mathrm{d}z$$

$$+u(\eta)w(\eta) - u(-H)w(-H) + \tau_s - \tau_b = -g(\eta + H)\partial_x \eta. \tag{3.45}$$

With the Leibniz rule (3.42) the first two integrals in (3.45) can be reformulated as follows:

$$\int_{-H}^{\eta} \partial_t u \, \mathrm{d}z = \partial_t U - \partial_t \eta u(\eta),$$

$$\int_{-H}^{\eta} \partial_x(uu) \, \mathrm{d}z = \partial_x \int_{-H}^{\eta} (uu) \, \mathrm{d}z - \partial_x \eta u(\eta)u(\eta) - \partial_x H u(-H)u(-H),$$

$$\tag{3.46}$$

such that after consideration of the kinematic boundary conditions (3.41), the momentum equation (3.45) is of the following form:

$$\partial_t U + \partial_x \int_{-H}^{\eta} uu \, \mathrm{d}z + \tau_s - \tau_b = -g(\eta + H)\partial_x \eta. \tag{3.47}$$

The non-linear momentum advection term can be approximated as

$$\int_{-H}^{\eta} (uu) \, \mathrm{d}z \approx \frac{U^2}{\eta + H}, \tag{3.48}$$

(note that this approximation is exact for a vertically constant velocity profile) such that in the following modification of (3.47),

$$\partial_t U + \partial_x \left(\frac{U^2}{\eta + H} \right) + \left[\partial_x \int_{-H}^{\eta} uu \, \mathrm{d}z - \partial_x \left(\frac{U^2}{\eta + H} \right) \right] + \tau_s - \tau_b =$$
$$-g(\eta + H)\partial_x \eta, \tag{3.49}$$

the term in the square brackets can be assumed to be small. When neglecting the term in the square brackets, the surface elevation equation (3.43) and the vertically integrated momentum equation (3.49) can be discretised explicitly, as shown in section 3.2 for the C-grid. The stability criterium for this system, (3.23), requires a relatively small time step, which is often referred to as the micro time step. The original, vertically resolving momentum equation (3.37), will be calculated at a longer time step (the so-called macro time step), for which the stability criterium is typically dictated by the CFL criterium for the momentum advection,

$$\Delta t \leq \frac{\Delta x}{u_{\max}}, \tag{3.50}$$

with the maximum flow velocity, $u_{\max}$. To properly couple the vertically integrated mode consisting of equations (3.43) and (3.49) with the vertically resolved mode consisting of equation (3.37), the vertically integrated mode is iterated a number of times (typically 10-30) until the next time step for the vertically resolved is calculated once.

Since the neglect of the square term in (3.49) would lead to an inconsistent discretisation of the system of equations (3.37) - (3.38), and on the other hand u in (3.49) is only known at the longer macro time steps, one could keep the value of the square brackets constant during each cycle of micro time steps.

To make the vertically resolved velocity profile u consistent with the vertically integrated velocity U, u needs to be shifted after each macro time step in a way that its vertical integral is equal to U. This is necessary to ensure that the tracer equation (3.39) is consistently solved. Inserting for example the constant value $c = 1$ into (3.39) shows that the reproduction of the continuity equation (3.38) is a necessary condition for this. The barotropic mode consisting of the vertically integrated momentum equation (3.49) and the surface elevation equation (3.43) can either be solved implicitly or explicitly. For both
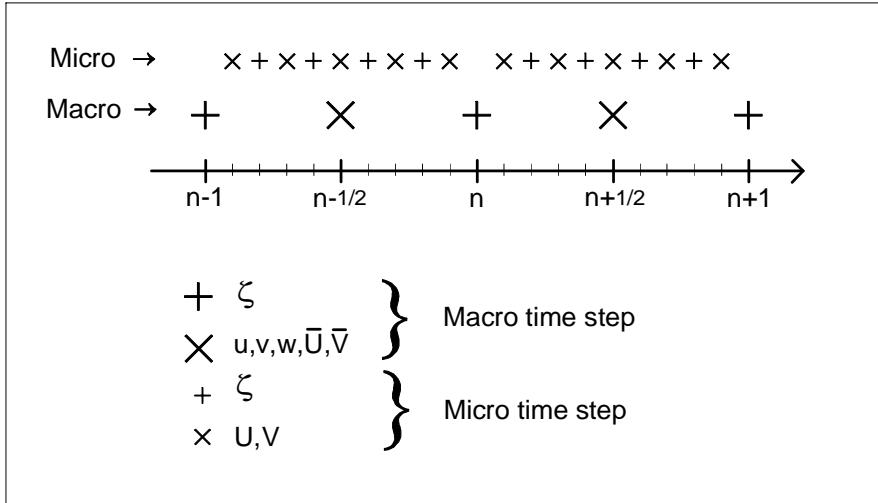
Figure 3.3: Sketch explaining the organisation of the time stepping in GETM. This figure has been taken from *Burchard and Bolding* [2002].

strategies, a consistent coupling to the discretisation of the vertically resolved velocity equation in (3.37), the so-called baroclinic mode, is necessary.

Today, most ocean models use such a mode split technique where a split factor N between the surface elevation calculation and the vertically resolved equations for velocity and tracers is applied. As an example for the time stepping within the mode splitting concept, figure 3.3 shows the time stepping principle of the General Estuarine Transport Model, GETM, see *Burchard and Bolding* [2002] and www.getm.eu, where barotropic mode is calculated with explicit discretisation N times (micro time steps), before the baroclinic mode is calculated once (macro time step).