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A Method for the Numerical Calculation of Hydrodynamic Shocks

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The equations of hydrodynamics are modified by the inclusion of additional terms which greatly simplify the procedures needed for stepwise numerical solution of the equations in problems involving shocks. The quantitative influence of these terms can be made as small as one wishes by choice of a sufficiently fine mesh for the numerical integrations. A set of difference equations suitable for the numerical work is given, and the condition that must be satisfied to insure their stabilty is derived.

I. INTRODUCTION

In the investigation of phenomena arising in the flow of a compressible fluid, it is frequently desirable to solve the equations of fluid motion by stepwise numerical procedures, but the work is usually severely complicated by the presence of shocks. The shocks manifest themselves mathematically as surfaces on which density, fluid velocity, temperature, entropy and the like have discontinuities; and clearly the partial differential equations governing the motion require boundary conditions connecting the values of these quantities on the two sides of each such surface. The necessary boundary conditions are, of course, supplied by the Rankine-Hugoniot equations, but their application is complicated because the shock surfaces are in motion relative to the network of points in space-time used for the numerical work, and the differential equations and boundary conditions are non-linear. Furthermore, the motion of the surfaces is not known in advance but is governed by the differential equations and boundary conditions themselves. In consequence, the treatment of shocks requires lengthy computations (usually by trial and error) at each step, in time, of the calculation.

We describe here a method for automatic treatment of shocks which avoids the necessity for application of any such boundary conditions. The approximations in it can be rendered as accurate as one wishes, by suitable choice of interval sizes and other parameters occurring in the method. It treats all shocks, correctly and automatically, whenever and wherever they may arise.

The method utilizes the well-known effect on shocks of dissipative mechanisms, such as viscosity and heat conduction.[1] When viscosity is taken into account, for example, the shocks are seen to be smeared out, so that the mathematical surfaces of discontinuity are replaced by thin layers in which pressure, density, temperature, etc. vary rapidly but continuously. Our idea is to introduce (artificial) dissipative terms into the equations so as to give the shocks a thickness comparable to (but preferably somewhat larger than) the spacing of the points of the network. Then the differential equations (more accurately, the corresponding difference equations) may be used for the entire calculation, just as though there were no shocks at all. In the numerical results obtained, the shocks are immediately evident as near-discontinuities that move through the fluid with very nearly the correct speed and across which pressure, temperature, etc. have very nearly the correct jumps.

It will be seen that for the assumed form of dissipation (and, indeed, for many others as well), the Rankine-Hugoniot equations are satisfied, provided the thickness of the shock layers is small in comparison with other physically relevant dimensions of the system. We then consider one way in which the transition from differential to finite-difference equations can be made and we discuss the mathematical stability of these equations. It will be seen that the dissipative terms have the effect of making the stability condition more stringent than the familiar one of Courant, Friedrichs, and Lewy,[2] but not seriously so if the amount of dissipation introduced is only enough to produce a shock thickness comparable with the spatial interval length of the network used.

The method has been applied, so far, only to one-dimensional flows, but appears to be equally suited to the study of more complicated flows; where, indeed, shock calculations by direct application of the Hugoniot equations would ordinarily be prohibitively difficult, even for rapid, automatic computers.

The discussions which follow are primarily intended to give a complete picture of the ideas and mathematical procedures involved. In some places (Chapter VII, also the essential inferences from some of the material of Chapters IV, V) the mathematical discussions are, however, carried through only with a view to give a complete chain of the necessary procedure, but not with all the detail that rigorous proofs in a primarily mathematical paper would require. The reason for doing this was partly desire to avoid inconvenient length, partly that of not wishing to have to select now the precise degree of generality for the

[1] Lord Rayleigh (Proc. Roy. Soc. **A84**, 247 (1910)) and G. I. Taylor (Proc. Roy. Soc. **A84**, 371 (1910)) showed, on the basis of general thermodynamical considerations, that dissipation is necessarily present in shock waves. Later, R. Becker (Zeits. f. Physik **8**, 321 (1922)) gave a detailed discussion of the effects of heat conduction and viscosity. Recently, L. H. Thomas (J. Chem. Phys. **12**, 449 (1944)) has investigated these effects further in terms of the kinetic theory of gases.

[2] Courant, Friedrichs, and Lewy, Math. Ann. **100**, 32 (1928). It is in this important paper that these authors first published their discovery of the conditional stability of the difference-equation integration method for partial differential equations.

validity of the method. It seems preferable to reserve such discussions for later occasions.

The validity of our methods has been tested empirically on various computational applications. These are partly still under analysis, and will be published and discussed elsewhere.

II. THE BASIC EQUATIONS

Consider a one-dimensional fluid motion. Let x be the Lagrangean coordinate, and $X=X(x,t)$ be the Eulerian coordinate. That is, $X(x,t)$ gives the position, at time t, of a fluid element that was initially at position x. Let $\rho_0(x)$ be the initial density, so that V and U, given by

$$V(x,t)=(1/\rho_0(x))(\partial X/\partial x) \tag{1}$$

and

$$U(x,t)=\partial X/\partial t, \tag{2}$$

are the specific volume and fluid velocity, respectively. The equations of motion, of energy, and of continuity are:

$$\rho_0(\partial U/\partial t)=-(\partial/\partial x)(p+q), \tag{3}$$

$$(\partial \mathcal{E}/\partial t)+(p+q)(\partial V/\partial t)=0, \tag{4}$$

and

$$\rho_0(\partial V/\partial t)=(\partial U/\partial x). \tag{5}$$

In these equations, $p=p(x,t)$ is the ordinary (or static) fluid pressure and $\mathcal{E}=\mathcal{E}(x,t)$ is the internal energy per unit mass. A connection between $\mathcal{E}$, p, V is established by an equation of state, which will be assumed, for the purpose of illustration, to have the form

$$\mathcal{E}=(pV)/(\gamma-1) \tag{6}$$

which holds, for example, in the case of a perfect gas. γ is a constant >1. It is supposed that the dissipative mechanism can be represented by the additional term q in the pressure, which is assumed to be negligibly small, except in the neighborhood of the shock.

III. THE EXPRESSION FOR q

The dissipation is introduced for purely mathematical reasons. Therefore q may be taken as any convenient function of p, V, etc. and their derivatives, provided that the following requirements are met:

1. The Eqs. (3), (4), and (5) must possess solutions without discontinuities.
2 The thickness of the shock layers must be everywhere of the same order as the interval length Δx used in the numerical computation, independently of the strength of the shock and of the condition of the material into which it is running.
3. The effect of the terms containing q in (3) and (4) must be negligible outside of the shock layers.
4. The Hugoniot equations must hold when all other dimensions characterizing the flow are large compared to the shock thickness.

We shall show that the expression

$$q=-\frac{(\rho_0 c\Delta x)^2}{V}\frac{\partial V}{\partial t}\cdot\left|\frac{\partial V}{\partial t}\right| \tag{7}$$

meets the requirements. c is a dimensionless constant near unity. The dissipative mechanism is essentially a non-linear viscosity as can be seen more clearly if (7) is written equivalently (for the one-dimensional case) as

$$q=-\frac{(c\Delta x)^2}{V}\frac{\partial U}{\partial x}\cdot\left|\frac{\partial U}{\partial x}\right| \tag{8}$$

by use of (5). To show that the expression (7) meets the stated requirements, we consider a steady-state shock.

IV. STEADY-STATE PLANE SHOCK

Imagine a long pipe containing a fluid initially in equilibrium (thermally and mechanically), into which a piston is pushing from one end with constant speed, as shown in Fig. 1. In the absence of dissipation the specific volume, V, and the fluid velocity, U, are, at a given instant, as shown by the solid curves in the graphs, whereas in the presence of dissipation they are as shown by the dashed curves. In either case, the shock is steady, at least approximately, after it has gone to a sufficiently great distance from the initiating piston. Then U, V, $\mathcal{E}$, etc. depend on x and t only through the combination

$$w=x-st, \tag{9}$$

where s is the speed of the shock relative to the original, or Lagrangean, coordinates. We suppose that ρ_0 and Δx are constants (independent of x).

It is convenient to define

$$M=\rho_0 s \tag{10}$$

—in a co-moving coordinate system, M is the mass crossing unit area in unit time—whereupon Eqs. (3)–(5) become:

$$M(dU/dw)=(d/dw)(p+q), \tag{11}$$

$$(d\mathcal{E}/dw)+(p+q)(dV/dw)=0, \tag{12}$$

and

$$-M(dV/dw)=dU/dw. \tag{13}$$

Then, (11) and (13) give:

$$-M^2(dV/dw)=(d/dw)(p+q); \tag{14}$$

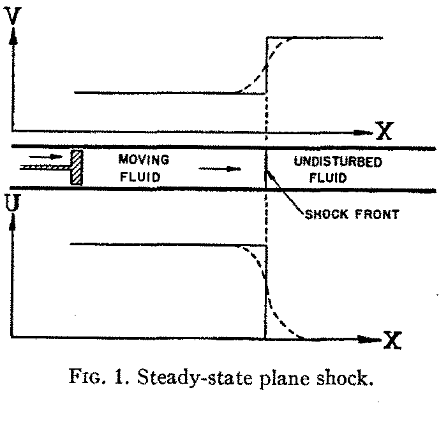

FIG. 1. Steady-state plane shock.

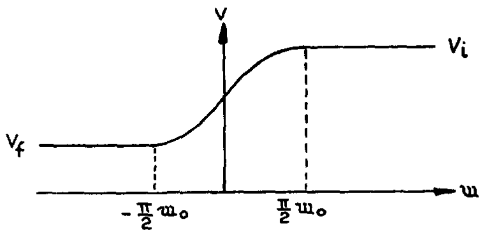

FIG. 2. Specific volume in steady shock.

and (12) and (14) give

$$\frac{d\mathcal{E}}{dw}+\frac{d}{dw}[(p+q)V]+M^2V\frac{dV}{dw}=0. \qquad (15)$$

The solutions of (13), (14), (15) are:

$$MV+U=C_1 \qquad (16)$$

$$M^2V+p+q=C_2 \qquad (17)$$

$$\mathcal{E}+(p+q)V+\tfrac{1}{2}M^2V^2=C_3, \qquad (18)$$

where C_1, C_2, and C_3 are constants.

Let the initial and final values be denoted by:

$$\text{As } w\rightarrow\infty;\quad V\rightarrow V_i,\quad p\rightarrow p_i,\quad \mathcal{E}\rightarrow\mathcal{E}_i,\quad q\rightarrow 0; \qquad (19)$$

$$\text{As } w\rightarrow-\infty;\quad V\rightarrow V_f,\quad p\rightarrow p_f,\quad \mathcal{E}\rightarrow\mathcal{E}_f,\quad q\rightarrow 0. \qquad (20)$$

Then (17) gives:

$$M^2(V_i-V_f)=P_f-P_i; \qquad (21)$$

and (18) and (21) together give:

$$\mathcal{E}_f-\mathcal{E}_i=\tfrac{1}{2}(P_i+P_f)(V_i-V_f). \qquad (22)$$

(21) and (22) are the equations of Hugoniot and are seen to be independent of the amount and form of the dissipation, provided that $q\rightarrow 0$ as $w\rightarrow\pm\infty$. The physical reason for this is that the Hugoniot equations are direct consequences of the conservation laws of mass, momentum and energy, and the form of dissipation assumed is such as top reserve the over-all conservation of these quantities. These laws require that in a shock a certain amount of mechanical energy be converted irreversibly into heat. In the steady state, the motion adjusts itself, in the shock layer, until precisely that amount of work is done against the pressure q according to Eq. (3) and is converted into heat according to Eq. (4).

To investigate the shape of the shock, we first look for solutions satisfying

$$(\partial V/\partial t)\leq 0 \quad\text{or}\quad (dV/dw)\geq 0. \qquad (23)$$

This is normally the case for a shock moving to the right. Then (7) can be written:

$$qV=+(Mc\Delta x)^2(dV/dw)^2. \qquad (24)$$

From (17) and (18)

$$\mathcal{E}-\tfrac{1}{2}M^2V^2=C_3-C_2V, \qquad (25)$$

so that by (6),

$$pV=[(\gamma-1)/2]M^2V^2+C_4V+C_5, \qquad (26)$$

whereupon (17) gives:

$$qV=C_2-[(\gamma+1)/2]M^2V^2-C_4V-C_5. \qquad (27)$$

The right member of (27) is a quadratic in V that vanishes for $V=V_i$ and for $V=V_f$, so that, clearly,

$$(Mc\Delta x)^2\left(\frac{dV}{dw}\right)^2=qV=\frac{\gamma+1}{2}M^2(V_i-V)(V-V_f). \qquad (28)$$

(Equation (28) can also be obtained from (27) by direct use of the Hugoniot equations and the equations that fix C_2, C_4, C_5.)

To solve (28), put:

$$\psi=V-\frac{V_i+V_f}{2},\quad \psi_0=\frac{V_i-V_f}{2},\quad \varphi=\frac{\psi}{\psi_0}, \qquad (29)$$

so that

$$c\Delta x\frac{d\psi}{dw}=[(\gamma+1)/2]^{\frac{1}{2}}(\psi_0^2-\psi^2)^{\frac{1}{2}}, \qquad (30)$$

or

$$c\Delta x\frac{d\varphi}{dw}=[(\gamma+1)/2]^{\frac{1}{2}}(1-\varphi^2)^{\frac{1}{2}}. \qquad (31)$$

Therefore,

$$w=[2/(\gamma+1)]^{\frac{1}{2}}c\Delta x\int\frac{d\varphi}{(1-\varphi^2)^{\frac{1}{2}}}=w_0\arcsin\varphi \qquad (32)$$

where

$$w_0=[2/(\gamma+1)]^{\frac{1}{2}}c\Delta x. \qquad (33)$$

Finally,

$$\psi=V-\frac{V_i+V_f}{2}=\frac{V_i-V_f}{2}\sin\frac{w}{w_0}. \qquad (34)$$

Because of our initial assumption (23) that $dV/dw\geq 0$, we can use only a half wave of the solution (34), but this half wave can be pieced together with two other particular solutions,

$$V\equiv V_i \quad\text{and}\quad V\equiv V_f \qquad (35)$$

(they satisfy Eq. (28)), to make the composite continuous solution depicted in Fig. 2. w_0 is a measure of the thickness of the shock, and is of order Δx, provided c is of order unity, independently of the strength of the shock and conditions ahead of it, by Eq. (33). Throughout most of the system q is negligible in comparison with the ordinary pressure, p, because of the factor $(\Delta x)^2$ in (7); but in the shock layer q becomes comparable with p because of the abnormally large value of $\partial V/\partial t$ there.

Expression (7) thus meets all requirements.

It may be noted that if we had looked instead for solutions having $dV/dt>0$, no solution would have been found for which all quantities are continuous and bounded, because in that case the opposite sign would occur in (28), leading to the hyperbolic instead of ordinary sine function. Therefore negative shocks do not exist in the steady state, for our equations. Negative shocks do not exist in the physical world, either, so that our expression (7) is satisfactory from this point of view.

V. STABILITY OF THE DIFFERENTIAL EQUATIONS

Suppose that at some instant there is superposed on a desired solution $U(x, t)$, $V(x, t)$, etc., a small perturbation δU, δV, etc. It is of interest to know whether the perturbation grows with increasing time. To investigate this we replace U, V, etc. by $U+\delta U$, $V+\delta V$, etc. in (3), (4), (5), (8), after first rewriting (4), by use of (6), in the form:

$$[\gamma p+(\gamma-1)q](\partial V/\partial t)+V(\partial p/\partial t)=0. \tag{36}$$

We obtain in this way the equations of first variation:

$$\rho_0(\partial/\partial t)\delta U=-(\partial/\partial x)(\delta p+\delta q), \tag{37}$$

$$\left.\begin{aligned}\frac{\partial V}{\partial t}[\gamma\delta p+(\gamma-1)\delta q]+[\gamma p+(\gamma-1)q]\frac{\partial}{\partial t}\delta V\\ +V\frac{\partial}{\partial t}\delta p+\frac{\partial p}{\partial t}\delta V=0,\end{aligned}\right\} \tag{38}$$

$$\delta q=\frac{(c\Delta x)^2}{V^2}\frac{\partial U}{\partial x}\cdot\left|\frac{\partial U}{\partial x}\right|\delta V-2\frac{(c\Delta x)^2}{V}\cdot\left|\frac{\partial U}{\partial x}\right|\frac{\partial}{\partial x}\delta U, \tag{39}$$

$$\rho_0\frac{\partial}{\partial t}\delta V=\frac{\partial}{\partial x}\delta U. \tag{40}$$

In writing the last term of (39) we have assumed that the perturbation is not large enough to alter the sign of $\partial U/\partial x$. (37) to (40) are a set of simultaneous linear differential equations for δU, δV, δp, δq. Their coefficients depend on the desired solution U, V, p, q, and are thought of as smoothly varying functions of x, t. We shall be concerned with rapidly varying perturbations. We therefore treat the coefficients of (37) to (40) as constants in a small region and look for solutions having the form:

$$\delta U=\delta U_0e^{ikx+\alpha t},\quad \delta V=\delta V_0e^{ikx+\alpha t},\ \text{etc.}, \tag{41}$$

where δU_0, δV_0, δp_0, δq_0, k, and α are constant and k is real. Substitution of (41) into (37) to (40) leads to four simultaneous homogeneous linear equations in δU_0, δV_0, δp_0, δq_0. The vanishing of the determinant of these equations establishes an equation connecting α and k. By solving this equation, for given k, and examining the real part of α, we can determine whether a given Fourier component (41) of the perturbation grows with increasing time. This program is readily carried out, but we omit the details in the interest of brevity. The determinantal equation is:

$$\begin{aligned}&(\alpha\rho_0)^2\gamma\frac{\partial V}{\partial t}+2\frac{\rho_0\alpha}{V}\frac{\partial V}{\partial t}(kc\Delta x)^2\left|\frac{\partial U}{\partial x}\right|\\ &-\frac{1}{V^2}\frac{\partial V}{\partial t}(kc\Delta x)^2\frac{\partial U}{\partial x}\cdot\left|\frac{\partial U}{\partial x}\right|+k^2\alpha[\gamma p+(\gamma-1)q]\\ &+\rho_0^2\alpha^3V+2\rho_0\alpha^2(kc\Delta x)^2\left|\frac{\partial U}{\partial x}\right|\\ &-\frac{\alpha}{V}(kc\Delta x)^2\frac{\partial U}{\partial x}\cdot\left|\frac{\partial U}{\partial x}\right|+k^2\frac{\partial p}{\partial t}=0.\end{aligned} \tag{42}$$

If we restrict our attention to Fourier components with very large k, only certain terms of (42) need be retained, the others being negligible either by viture of being of lower order in k or by virtue of being of lower order in α. Two cases are distinguished: in shock regions we retain all terms in (42); in normal regions we drop the dissipative terms (those containing Δx). In the two regions, the dominant terms, in the sense explained, give: shock regions:

$$\alpha=-2\frac{(kc\Delta x)^2}{\rho_0V}\left|\frac{\partial U}{\partial x}\right| \tag{43}$$

normal regions:

$$\alpha^2=-\frac{k^2}{\rho_0V}\frac{\gamma p}{\rho_0}. \tag{44}$$

It is seen that small disturbances are damped out in the shock layers but propagate without either growth or decay in normal regions. This corresponds to physical reality, so our expression (7) is satisfactory also as regards stability of the resulting differential equations.

We can furthermore identify the terms in the equations of variation that lead to the dominant terms in (42). They give:

shock regions:

$$\frac{\partial}{\partial t}\delta U\approx\sigma\frac{\partial^2}{\partial x^2}\delta U, \tag{45}$$

where

$$\sigma=\frac{2(c\Delta x)^2}{V\rho_0}\left|\frac{\partial U}{\partial x}\right|;$$

normal regions:

$$\frac{\partial^2}{\partial t^2}\delta U\approx s_0^2\frac{\partial^2}{\partial x^2}\delta U, \tag{46}$$

where

$$s_0^2=\frac{\gamma p}{\rho_0^2V}.$$

Thus our system has the character of a diffusion equation in the shock layers and of a sound equation elsewhere.

VI. FINITE DIFFERENCE EQUATIONS

There are many systems of finite difference equations equivalent to the differential equations, but we shall restrict the discussion to one of the simplest. Certain other systems are much superior from the point of view of stability but are more complicated from the point of view of numerical solution. Systems of the latter type will be discussed elsewhere.

Let the points of a rectangular network with spacings Δx and Δt be denoted by x_l, $t^n (l=0, 1, 2, \cdots, L; n=0, 1, 2, \cdots)$. We shall also have occasion to deal with intermediate points, having coordinates $x_{l+\frac{1}{2}} \stackrel{\text{def.}}{=} \frac{1}{2}(x_{l+1}+x_l)$ etc. To facilitate the writing, we introduce abbreviations such as:

$$V_{l+\frac{1}{2}}{}^n = V(x_{l+\frac{1}{2}}, t^n) \text{ etc.} \tag{47}$$

The difference equations corresponding to (3), (5), (8)' and (36) are:

$$\rho_0 \frac{U_l{}^{n+\frac{1}{2}} - U_l{}^{n-\frac{1}{2}}}{\Delta t} = -\frac{p_{l+\frac{1}{2}}{}^n + q_{l+\frac{1}{2}}{}^{n-\frac{1}{2}} - p_{l-\frac{1}{2}}{}^n - q_{l-\frac{1}{2}}{}^{n-\frac{1}{2}}}{\Delta x}, \tag{48}$$

$$\rho_0 \frac{V_{l+\frac{1}{2}}{}^{n+1} - V_{l+\frac{1}{2}}{}^n}{\Delta t} = \frac{U_{l+1}{}^{n+\frac{1}{2}} - U_l{}^{n+\frac{1}{2}}}{\Delta x}, \tag{49}$$

$$q_{l+\frac{1}{2}}{}^{n+\frac{1}{2}} = -\frac{2(c\Delta x)^2}{V_{l+\frac{1}{2}}{}^n + V_{l+\frac{1}{2}}{}^{n+1}} \frac{(U_{l+1}{}^{n+\frac{1}{2}} - U_l{}^{n+\frac{1}{2}}) \cdot |U_{l+1}{}^{n+\frac{1}{2}} - U_l{}^{n+\frac{1}{2}}|}{(\Delta x)^2}, \tag{50}$$

and

$$\left[\gamma \frac{p_{l+\frac{1}{2}}{}^{n+1} + p_{l+\frac{1}{2}}{}^n}{2} + (\gamma - 1) q_{l+\frac{1}{2}}{}^{n+\frac{1}{2}}\right] \frac{V_{l+\frac{1}{2}}{}^{n+1} - V_{l+\frac{1}{2}}{}^n}{\Delta t} + \frac{V_{l+\frac{1}{2}}{}^{n+1} + V_{l+\frac{1}{2}}{}^n}{2} \frac{p_{l+\frac{1}{2}}{}^{n+1} - p_{l+\frac{1}{2}}{}^n}{\Delta t} = 0. \tag{51}$$

These equations are correct to second order of small quantities Δx and Δt, except for the terms containing q in (48) which are only correct to the first order; but these terms are negligible except in the shock layers and are physically artificial in any case.

For numerical solution, suppose that all quantities are known for superscript n or less. Compute $U_l{}^{n+\frac{1}{2}}$ from (48) for each l; compute $V_{l+\frac{1}{2}}{}^{n+1}$ from (49) for each l; compute $q_{l+\frac{1}{2}}{}^{n+\frac{1}{2}}$ from (50) for each l; compute $p_{l+\frac{1}{2}}{}^{n+\frac{1}{2}}$ from (51) for each l; this completes a cycle. Boundary conditions are needed, an example being (rigid walls at ends of a tube): $U_0{}^{n+\frac{1}{2}} \equiv 0$, $U_L{}^{n+\frac{1}{2}} \equiv 0$.

VII. STABILITY OF THE DIFFERENCE EQUATIONS

Equations (48) to (51), being only approximations to the differential equations, cannot be expected to give all features of the solution with precision. If U, V are thought of as being expanded in Fourier series with coefficients depending on t, the long-wave-length components are accurately given by (48) to (51), provided Δx and Δt are sufficiently small, but the components whose wave-lengths are of order Δx are always falsified somewhat. This falsification is harmless, provided that all physically relevant components are treated accurately; this requires not only that Δx and Δt be small but also that the physically insignificant components with wave-lengths of order Δx remain small during the entire calculation. It may happen not only that the short-wave-length components are falsified but also that their amplitudes increase with increasing n, in spite of the stability of the *differential* equations. This increase is generally exponential and, if it occurs, quickly makes gibberish of the entire calculation.

The avoidance of such catastrophes, when partial differential equations are approximated by difference equations, has been the subject of study by various investigators, beginning with the fundamental paper of Courant, Friedrichs, and Lewy referred to in reference 1. We shall give a somewhat heuristic discussion of the stability questions met in the present problem.

We again suppose a small perturbation δU, δV, etc., superposed on a smooth solution, and consider the equations of variation of (48) to (51). According to the analysis of Part V, the dominant terms are expected to be (see Eqs. (45) and (46)):

shock regions:

$$\frac{\delta U_l{}^{n+\frac{1}{2}} - \delta U_l{}^{n-\frac{1}{2}}}{\Delta t} \approx \sigma \frac{\delta U_{l+1}{}^{n-\frac{1}{2}} - 2\delta U_l{}^{n-\frac{1}{2}} + \delta U_{l-1}{}^{n-\frac{1}{2}}}{(\Delta x)^2}; \tag{52}$$

normal regions:

$$\frac{\delta U_l{}^{n+\frac{1}{2}} - 2\delta U_l{}^{n-\frac{1}{2}} + \delta U_l{}^{n-\frac{3}{2}}}{(\Delta t)^2} \approx s_0{}^2 \frac{\delta U_{l+1}{}^{n-\frac{1}{2}} - 2\delta U_l{}^{n-\frac{1}{2}} + \delta U_{l-1}{}^{n-\frac{1}{2}}}{(\Delta x)^2}; \tag{53}$$

and this can be verified by writing out the difference equations of variation in detail.

As before, we consider perturbations of the form:

$$\delta U = \delta U_0 e^{ikx+\alpha t}, \text{ etc.,} \tag{54}$$

so that:

$$\delta U_l{}^{n+\frac{1}{2}} = \delta U_0 \zeta^l \xi^{n+\frac{1}{2}}, \tag{55}$$

where:

$$\zeta = e^{ik\Delta x}, \quad \xi = e^{\alpha \Delta t}. \tag{56}$$

For stability we require that $|\xi| \leq 1$ for all real k.

We consider normal regions first. Substitution of (55) into (53) and cancellation of common factors gives:

$$\xi - 2 + \frac{1}{\xi} = 2\left(\frac{s_0 \Delta t}{\Delta x}\right)^2 (\cos k\Delta x - 1). \tag{57}$$

If we call

$$L=\frac{s_0\Delta t}{\Delta x}, \tag{58}$$

the solution of (57) is

$$\xi_{1,2}=b\pm(b^2-1)^{\frac{1}{2}} \tag{59}$$

where

$$b=1-L^2(1-\cos k\Delta x). \tag{60}$$

According to (60), b is always <1, so there are two cases:

$$-1<b<1,\quad |\xi_1|=|\xi_2|=1;\quad \text{stability}; \tag{61}$$

$$b<-1,\quad |\xi_1|<1<|\xi_2|;\quad \text{instability}. \tag{62}$$

From Eq. (60) it is seen that (61) will hold for all k if and only if

$$L\leq 1. \tag{63}$$

This is the familiar condition for stability of hydrodynamical equations of the form (53).

A similar treatment of Eq. (52) for the shock region yields directly:

$$\xi-1=2\frac{\sigma\Delta t}{(\Delta x)^2}(\cos k\Delta x-1), \tag{64}$$

and the stability condition is clearly that

$$\frac{2\sigma\Delta t}{(\Delta x)^2}\leq 1 \quad\text{or}\quad \Delta t\leq\frac{(\Delta x)^2}{2\sigma}. \tag{65}$$

To interpret this result, we calculate σ according to (45) for the steady shock discussed in Part IV. From (33), (34);

$$\left|\frac{\partial U}{\partial x}\right|=\rho_0\left|\frac{\partial V}{\partial t}\right|=\rho_0 s\left|\frac{\partial V}{\partial w}\right|$$

$$=\rho_0 s\frac{V_i-V_f}{2c\Delta x}((\gamma+1)/2)^{\frac{1}{2}}\cos\frac{w}{w_0}. \tag{66}$$

Therefore

$$\sigma=sc\Delta x\frac{V_i-V_f}{V}((\gamma+1)/2)^{\frac{1}{2}}\cos\frac{w}{w_0}, \tag{67}$$

and the stability condition (65) takes the form:

$$\Delta t\leq\frac{V\Delta x}{2sc(V_i-V_f)}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\frac{1}{\cos(w/w_0)}, \tag{68}$$

or,

$$\Delta t\leq\frac{\Delta x}{4sc}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\frac{(\eta+1)/(\eta-1)+\sin(w/w_0)}{\cos(w/w_0)}, \tag{69}$$

by further use of (34), where

$$\eta=V_i/V_f. \tag{70}$$

The quantity η is a measure of the shock strength. Equation (69) shows that different parts of the shock layer (i.e., different values of w) impose different restrictions on Δt. The effective restriction is obtained by replacing the right member of (69) by its minimum value for $-(\pi/2)w_0\leq w\leq(\pi/2)w_0$. The minimum of the last factor in (69) is found to be $2(\eta)^{\frac{1}{2}}/(\eta-1)$, and the stability condition is

$$\Delta t\leq\frac{\Delta x}{2sc}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\frac{\eta^{\frac{1}{2}}}{\eta-1}. \tag{71}$$

For practical application, it is convenient to express (71) in terms of the quantity L appearing in the normal hydrodynamic stability condition (63). By elimination of p_i from the Hugoniot Eqs. (21), (22), and (8), the speed of the shock is found to be

$$\left.\begin{aligned}s=\frac{M}{\rho_0}&=\frac{1}{\rho_0}\left(\frac{2}{(\gamma+1)\eta-(\gamma-1)}\right)^{\frac{1}{2}}\left(\gamma\frac{p_f}{V_f}\right)^{\frac{1}{2}}\\&=\left(\frac{2}{(\gamma+1)\eta-(\gamma-1)}\right)^{\frac{1}{2}}s_{0f},\end{aligned}\right\} \tag{72}$$

where s_{0f} is the speed of sound, relative to Lagrangean coordinates, in the material behind the shock. By combining (71) and (72) and use of (58), the stability condition becomes:

$$L_f=\frac{s_{0f}\Delta t}{\Delta x}\leq\frac{1}{2c}\frac{\{[\eta-(\gamma-1)/(\gamma+1)]\eta\}^{\frac{1}{2}}}{\eta-1}. \tag{73}$$

Lastly, the shock strength, and hence η, is generally unknown until the calculation has been performed; it is therefore advisable to replace the right member of (73) by its minimum with respect to η. η can vary in the range $1\leq\eta\leq(\gamma+1)/(\gamma-1)$ (the latter value corresponding to an infinitely strong shock) and the minimum of (73) is attained at the upper end of this range, where the last factor in (73) has the value $\gamma^{\frac{1}{2}}$. Our final, sufficient condition for stability reads:

$$L_f\leq\gamma^{\frac{1}{2}}/2c. \tag{74}$$

This condition has been found to insure stability in test calculations, whereas a serious violation of it leads to trouble. The choice $c=1$ has been found to yield good results in practice for the representation of shocks, in which case the stability condition is at worst slightly more severe than the one that must be observed, anyway (compare Eq. (63)), to insure stability of the motion behind the shock.