$$
\begin{cases}
\dfrac{\partial h}{\partial t}+\dfrac{\partial (hv_x)}{\partial x}+\dfrac{\partial (hv_y)}{\partial y}=0\\
\dfrac{\partial (hv_x)}{\partial t}+\dfrac{\partial\left(h\left(v_x^2-B_x^2\right)+\frac{1}{2}gh^2\right)}{\partial x}+\dfrac{\partial\left(h(v_xv_y-B_xB_y)\right)}{\partial y}+\left(B_0B_x-hv_y\left(f_0+\dfrac{\partial f}{\partial y}y\right)\right)=-h\nu\left(\dfrac{\partial^2 v_x}{\partial x^2}+\dfrac{\partial^2 v_x}{\partial y^2}\right)\\
\dfrac{\partial (hv_y)}{\partial t}+\dfrac{\partial\left(h(v_xv_y-B_xB_y)\right)}{\partial x}+\dfrac{\partial\left(h\left(v_y^2-B_y^2\right)+\frac{1}{2}gh^2\right)}{\partial y}+\left(B_0B_y+hv_x\left(f_0+\dfrac{\partial f}{\partial y}y\right)\right)=-h\nu\left(\dfrac{\partial^2 v_y}{\partial x^2}+\dfrac{\partial^2 v_y}{\partial y^2}\right)\\
\dfrac{\partial (hB_x)}{\partial t}+\dfrac{\partial\left(h(v_yB_x-v_xB_y)\right)}{\partial y}-B_0v_x=0\\
\dfrac{\partial (hB_y)}{\partial t}+\dfrac{\partial\left(h(v_xB_y-v_yB_x)\right)}{\partial x}-B_0v_y=0
\end{cases}
\tag{1}
$$

$$
\frac{\partial H}{\partial t}+\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}+F+\nu V=0 \tag{2}
$$

$$
\begin{cases}
H=\begin{pmatrix}h & hv_x & hv_y & B_x & B_y\end{pmatrix}^T\\
X=\begin{pmatrix}hv_x & h\left(v_x^2-B_x^2\right)+\dfrac{1}{2}gh^2 & h\left(v_xv_y-B_xB_y\right) & 0 & h\left(v_xB_y-v_yB_x\right)\end{pmatrix}^T\\
Y=\begin{pmatrix}hv_y & h\left(v_xv_y-B_xB_y\right) & h\left(v_y^2-B_y^2\right)+\dfrac{1}{2}gh^2 & h\left(v_yB_x-v_xB_y\right) & 0\end{pmatrix}^T\\
F=\begin{pmatrix}0 & B_0B_x-hv_y\left(f_0+\dfrac{\partial f}{\partial y}y\right) & B_0B_y+hv_x\left(f_0+\dfrac{\partial f}{\partial y}y\right) & -B_0v_x & -B_0v_y\end{pmatrix}^T\\
V=\begin{pmatrix}0 & -h\left(\dfrac{\partial^2 v_x}{\partial x^2}+\dfrac{\partial^2 v_x}{\partial y^2}\right) & -h\left(\dfrac{\partial^2 v_y}{\partial x^2}+\dfrac{\partial^2 v_y}{\partial y^2}\right) & -h\left(\dfrac{\partial^2 B_x}{\partial x^2}+\dfrac{\partial^2 B_x}{\partial y^2}\right) & -h\left(\dfrac{\partial^2 B_y}{\partial x^2}+\dfrac{\partial^2 B_y}{\partial y^2}\right)\end{pmatrix}^T
\end{cases}
\tag{3}
$$

$$
\begin{cases}
H_{i-\frac{1}{2},j}^{n+\frac{1}{2}}=\dfrac{H_{i,j}^n+H_{i-2,j}^n+H_{i-1,j+1}^n+H_{i-1,j-1}^n}{4}-\dfrac{\Delta t}{2}\left(\dfrac{X_{i,j}^n-X_{i-2,j}^n}{\Delta x}+\dfrac{Y_{i-1,j+1}^n-Y_{i-1,j-1}^n}{\Delta y}+2F_{i-1,j}^n+2\nu V_{i-1,j}^n\right)\\
H_{i+\frac{1}{2},j}^{n+\frac{1}{2}}=\dfrac{H_{i+2,j}^n+H_{i,j}^n+H_{i+1,j+1}^n+H_{i+1,j-1}^n}{4}-\dfrac{\Delta t}{2}\left(\dfrac{X_{i+2,j}^n-X_{i,j}^n}{\Delta x}+\dfrac{Y_{i+1,j+1}^n-Y_{i+1,j-1}^n}{\Delta y}+2F_{i+1,j}^n+2\nu V_{i+1,j}^n\right)\\
H_{i,j-\frac{1}{2}}^{n+\frac{1}{2}}=\dfrac{H_{i+1,j-1}^n+H_{i-1,j-1}^n+H_{i,j}^n+H_{i,j-2}^n}{4}-\dfrac{\Delta t}{2}\left(\dfrac{X_{i+1,j-1}^n-X_{i-1,j-1}^n}{\Delta x}+\dfrac{Y_{i,j}^n-Y_{i,j-2}^n}{\Delta y}+2F_{i,j-1}^n+2\nu V_{i,j-1}^n\right)\\
H_{i,j+\frac{1}{2}}^{n+\frac{1}{2}}=\dfrac{H_{i+1,j+1}^n+H_{i-1,j+1}^n+H_{i,j+2}^n+H_{i,j}^n}{4}-\dfrac{\Delta t}{2}\left(\dfrac{X_{i+1,j+1}^n-X_{i-1,j+1}^n}{\Delta x}+\dfrac{Y_{i,j+2}^n-Y_{i,j}^n}{\Delta y}+2F_{i,j+1}^n+2\nu V_{i,j+1}^n\right)
\end{cases}
\tag{4}
$$

$$
H_{i,j}^{n+1}=H_{i,j}^n-\frac{\Delta t}{2}\left(\frac{X_{i+\frac{1}{2},j}^{n+\frac{1}{2}}-X_{i-\frac{1}{2},j}^{n+\frac{1}{2}}}{\Delta x}+\frac{Y_{i,j+\frac{1}{2}}^{n+\frac{1}{2}}-Y_{i,j-\frac{1}{2}}^{n+\frac{1}{2}}}{\Delta y}+2F_{i,j}^n+2\nu V_{i,j}^n\right) \tag{5}
$$