ECE 310 Fall 2023

Lecture 2

Math preliminaries, complex numbers

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Learning Objectives

After this lecture, you should be able to:

- Explain what defines a complex number, be able to identify the magnitude and phase of a complex number, and visualize them on the complex plane.
- Convert between rectangular and polar forms of complex numbers
- State Euler's identities for complex numbers.
- Interpret and visualize common discrete-time signals including Kronecker delta and unit step.

Recap from previous lecture

In the last lecture, we discussed what defines a signal and described what differentiates analog, continuous-time signals from digital, discrete-time signals. Finally, we provided a brief motivation for why we work with digital signals and the advantages offered by digital signal processing. In this lecture, we will give important math preliminaries as we begin to describe and analyze digital systems and signals.

1 Complex Numbers

1.1 Rectangular form

We may define a complex number as being the sum of a *real* and *imaginary* part. For a complex number $x \in \mathbb{C}$, we have

$$x = a + jb, \tag{1}$$

where $a = \mathrm{Re}\{x\}$ gives the real part and $b = \mathrm{Im}\{x\}$ is the imaginary part. This representation is known as the *rectangular form* of a complex number. We use j to denote the imaginary unit $j = \sqrt{-1}$ (other classes or texts may use i instead of j). We say x is a complex number using the notation $x \in \mathbb{C}$ the same way we would say $y \in \mathbb{R}$ to say y is a real number. We also frequently find it useful to take the *complex conjugate* of a complex number. For complex number $x = a + jb$, we denote its complex conjugate as x^* where

$$x^* = a - jb. \tag{2}$$

Notice that we only need to change the sign of the imaginary part. Moreover, the product of a complex number and its complex conjugate, x^*x, is always a real-valued number since the imaginary parts cancel out! To add or subtract complex numbers, we simply add and subtract their real and imaginary parts separately. To multiply complex numbers, we must multiply each real and imaginary part with the other complex number's real and imaginary part and remember that $j^2 = (\sqrt{-1})^2 = -1$. Dividing two complex numbers in rectangular form is a bit more challenging. Consider, for example $x = 1 + j2$ and $y = 3 - j$ where we would like to compute x/y. We may do so as follows:

$$\begin{aligned} \frac{x}{y} &= \frac{xy^*}{yy*} && (3) \\ &= \frac{(1+j2)(3+j)}{(3-j)(3+j)} && (4) \\ &= \frac{1+j7}{10}. && (5) \end{aligned}$$

Above, we multiplied the top and bottom of the expression by the complex conjugate of the denominator. This gives us a real-valued denominator and thus makes it easy to combine our terms into standard rectangular form.

For real numbers, we may plot them on the one-dimensional real number line. Complex numbers, however, require us to visualize them on the two-dimensional *complex plane*. The complex plane consists of the x-axis for the real part and the y-axis for the imaginary part of our complex numbers. Figure 1 depicts how we represent an arbitrary $x = a + jb$ complex number and its complex conjugate.

1.2 Polar form

We also commonly represent complex numbers in a more compact form given by a *complex exponential*:

$$x = Re^{j\theta}. \tag{6}$$

Equation 6 gives the *polar form* of a complex number.

Referring back to complex arithmetic, multiplication and division are quite straightforward with complex exponentials. Suppose we have complex numbers $x = Re^{j\theta}$ and $y = Se^{j\phi}$. We may compute xy and x/y as follows:

$$\begin{aligned} xy &= \left(Re^{j\theta}\right)\left(Se^{j\phi}\right) && (7) \\ &= RS^{j(\theta+\phi)} && (8) \\ \frac{x}{y} &= \frac{Re^{j\theta}}{Se^{j\phi}} && (9) \\ &= \frac{R}{S}e^{j(\theta-\phi)}. && (10) \end{aligned}$$

Lastly, by the odd symmetry of the arctan function, we can easily write the complex conjugate of a complex number in polar form as $x^* = Re^{-j\theta}$. This representation makes it especially clear how the complex conjugate cancels out the phase or imaginary component of a complex number when we compute x^*x. Figure 1 also depicts the polar representation of complex numbers where θ and R give the angle and length, respectively, of the complex number.

1.3 Magnitude and phase of complex numbers

Two important quantities for any complex number are the *magnitude* and *phase*. There are multiple ways to describe the magnitude and phase of any complex number. Let $x = a + jb = Re^{j\theta}$ be a complex number given in both rectangular and polar forms. The magnitude of x may be denoted as

$$\text{Magnitude}(x) = |x| = \sqrt{a^2 + b^2} = \sqrt{xx^*} = R. \tag{11}$$

All of the above forms are equivalent. We may similarly represent the phase of x as

$$\text{Phase}(x) = \angle x = \theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right), & a \geq 0 \\ \tan^{-1}\left(\frac{b}{a}\right) + \pi, & a < 0 \end{cases}. \tag{12}$$

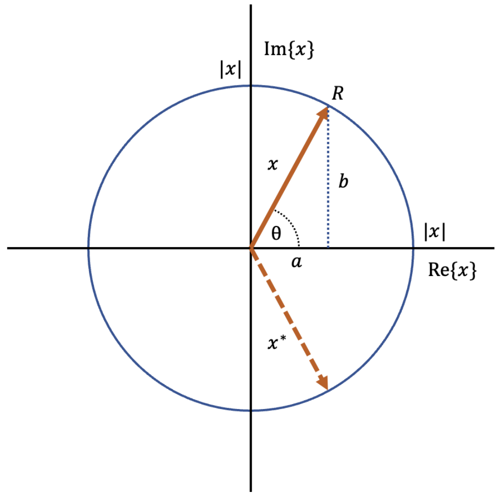


Figure 1: Example complex number on the complex plane.

Using the above relationships, we can easily convert from rectangular form to polar form by finding the magnitude and phase of a complex number and plugging those quantities into the polar form. To convert from polar to rectangular we will need to invoke Euler's formula.

Lastly, we should point out a couple subtle points regarding complex numbers, in particular for the polar form. First, note that real numbers can only have phase of zero or $\pm\pi$. Looking at Fig. 1, we see that numbers lying on the real line of the complex plane only occur at angles of 0 or $\pm\pi$. Furthermore, we should note the *periodicity* of complex numbers. We will discuss periodic signals in much more detail later in the course, but as a simple example consider $x = e^{j\frac{\pi}{3}}$ and $y = e^{j\frac{7\pi}{3}}$. While x and y may appear to be different complex numbers, they in fact point to the same point on the complex plane. This is because they have the same magnitude ($R = 1$) and they share the same *principal angle*. We define the principal angle as the equivalent angle of any complex number that lies between $[-\pi, \pi]$ or $[0, 2\pi]$ (conventions vary). For any angle outside of the principal angle range, we simply add or subtract 2π until we lie in this range. Concretely, we may express this relation as follows:

$$Re^{j\theta} = Re^{j(\theta_p + 2\pi k)}. \tag{13}$$

Above, k is any integer, i.e. $k \in \mathbb{Z}$, and θ_p gives the principal angle of a complex number. If θ already lies in the principal angle range, $\theta = \theta_p$ already. In the previous example, the principal angle was $\theta_p = \frac{\pi}{3}$.

1.4 Euler's identities

Euler's formula for $x = Re^{j\theta}$ is given by:

$$x = R\left(\cos\theta + j\sin\theta\right). \tag{14}$$

Thus, Eqn. 14 gives us the conversion from polar to rectangular form of complex numbers. We may also use Euler's formula to derive the highly important Euler's identities. For simplicity, let's set $R = 1$. We begin with the identity for cosine.

$$\frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{\cos(\theta) + j\sin(\theta) + \cos(-\theta) + j\sin(-\theta)}{2} \tag{15}$$

$$\stackrel{(a)}{=} \frac{2\cos\theta}{2} \tag{16}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \tag{17}$$

Next, we can derive the identity for sine.

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \frac{\cos(\theta) + j\sin(\theta) - \cos(-\theta) - j\sin(-\theta)}{2j} \tag{18}$$

$$\stackrel{(b)}{=} \frac{2j\sin\theta}{2j} \tag{19}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{20}$$

Above, (a) and (b) both follow from the even and odd symmetry of cos and sin, respectively, i.e. $\cos(\theta) = \cos(-\theta)$ and $\sin(\theta) = -\sin(-\theta)$. The identities given in Eqns. 17 and 20 will be of great importance throughout this course.

2 Common discrete-time signal representations

We have several common discrete-time signals that will be used throughout this course. Before going through these signals, let's establish a bit of useful notation for writing out any arbitrary signal. Take $x[n]$ as our example signal that is length 7 as follows:

$$x[n] = \begin{bmatrix} 0 & 4 & -2 & 0 & 3 & 1 & 0 \end{bmatrix} = \{0, 4, -2, 0, 3, 1\}.$$

We have written $x[n]$ as a sequence of seven values, but we do not know which samples correspond to each n index! We typically choose to identify the $n = 0$ sample clearly in our signal in some manner. Common choices including bold text, arrow underneath $n = 0$, or underlining:

$$x[n] = \{0, \mathbf{4}, -2, 0, 3, 0\}$$

$$x[n] = \{0, \underset{\uparrow}{4}, -2, 0, 3, 0\}$$

$$x[n] = \{0, \underline{4}, -2, 0, 3, 0\}$$

In each of these examples, we have $x[-1] = 0$, $x[0] = 4$, $x[1] = -2$, and so on.

Kronecker delta. The *Kronecker delta*, or *unit impulse*, takes a value of one at $n = 0$ and is zero everywhere else:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}. \tag{21}$$

We can scale our Kronecker delta or shift it to any index n. For example, $x[n] = 3\delta[n - 10]$ will take value 3 at $n = 10$ and be zero everywhere else. In fact, we can represent any of our digital signals as simply being the summation of scaled and shifted Kronecker deltas. This of course would be quite tedious, so let's move on to more descriptive signals!

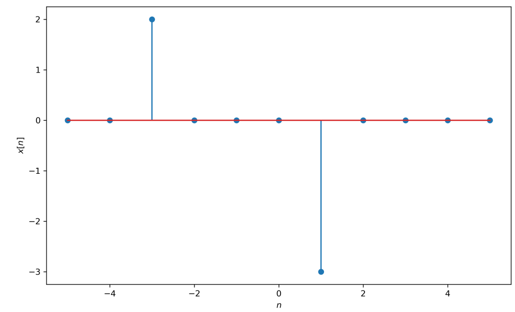
Unit step. The *unit step function* takes value one starting at $n = 0$ and keeps this value for all $n > 0$:

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}. \tag{22}$$

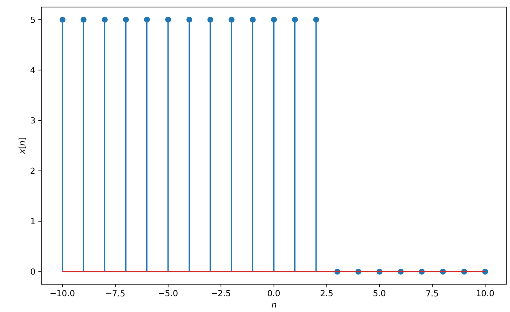
Like the Kronecker delta, we can scale or shift the unit step function or even flip it like $x[n] = 4u[3 - n]$. This signal would have value 4 for all $n \leq 3$ and value 0 for all $n > 3$.

Sinusoids. We have seen sinusoids before in plenty of courses, but let's establish our notation for each part of a sine or cosine signal. We can describe an arbitrary sinusoid as follows:

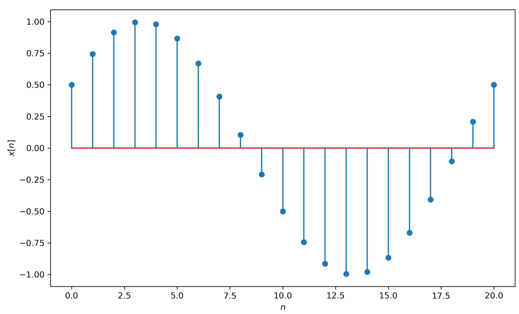
$$x[n] = A\sin(\omega_0 n + \theta), \ -\infty < n < \infty. \tag{23}$$



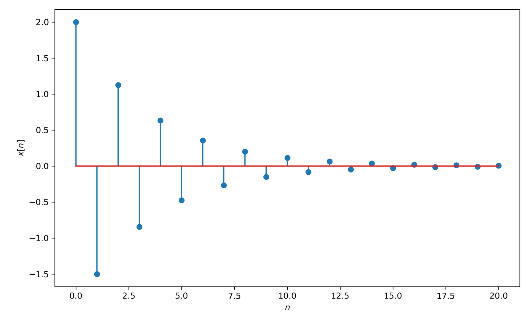
(a) $x[n] = 2\delta[n+3] - 3\delta[n-1]$



(b) $x[n] = 5u[2-n]$



(c) $x[n] = \cos\left(\frac{\pi}{10}n - \frac{\pi}{3}\right)$



(d) $x[n] = 2\left(-\frac{3}{4}\right)^n$

Figure 2: Example signals for (a) Kronecker delta, (b) unit step function, (c) sinusoids, and (d) exponentials.

Above, A gives the *amplitude*, ω_0 denotes the *radial frequency* (in radians/sample), and θ is the phase. All of these components are real-valued.

Exponentials. Finally, let's set our notation for exponentials:

$$x[n] = Ba^n, \ \infty < n < \infty. \tag{24}$$

Here, both B and a can be real-valued or complex-valued. Thus, exponentials are flexible to represent geometric sequences or complex exponentials within the same notation.

The above list of examples is of course not exhaustive, but gives us a good foundation as we begin this course. Figure 2 provides examples of each of the above signals.