Discrete-time systems

*DT = discrete-time

Discrete time system

Any computational process that maps a discrete-time input, $x[n]$, to a discrete-time output, $y[n]$.
We can notate a system as follows:

$$x[n] \overset{T}{\mapsto} y[n]$$

$$y[n] = T(x[n]).$$

*T denotes out DT system

ex.

$$y[n] = x[n] - x[n-1]$$

$$y[n] = HalfCenterCrop(x[n])$$

This system takes an input image $x[n]$ and crops down to the center of the image so that the output $y[n]$ is half aas tall and half as wide.

One important point: **discrete-time systems are not just filters** (digital fliters are just one example of discrete-time systems)

Linearity, time-invariance, causality, stability

Linearity

条件

A discrete-time system is *linear* **iff** it satisfies the following **two** conditions:

$$T(ax[n]) = aT(x[n])(1)$$

$$T(x_1[n] + x_2[n]) = T(x_1[n]) + T(x_2[n])(2)$$

And we can combine them into one condition known as *superposition*:

$$T(ax_1[n] + bx_2[n]) = aT(x_1[n]) + bT(x_2[n]), a, b \in R$$

The condition Eqn. 1 is known as *homogeneity*, which says that a scaled input, $ax[n]$, should produce an output that is the same as if we passed the original input, $x[n]$, to the system and scaled the output by a.

The condition Eqn. 2 is known as *additivity*.

证明：

Exercise 1: Suppose we have the system given by $y[n] = x[n] - x[n-1]$ (this is our difference system from Eqn. 3). Is this system linear?.

We can check for linearity immediately by trying to prove superposition. Let $z[n] = ax_1[n] + bx_2[n]$:

$$\begin{aligned}
T(ax_1[n] + bx_2[n]) &\stackrel{?}{=} aT(x_1[n]) + bT(x_2[n]) \\
T(ax_1[n] + bx_2[n]) &= T(z[n]) \\
&= z[n] - z[n-1] \\
&= (ax_1[n] + bx_2[n]) - (ax_1[n-1] + bx_2[n-1]) \\
&= a(x_1[n] - x_1[n-1]) + b(x_2[n] - x_2[n-1]) \\
&= aT(x_1[n]) + bT(x_2[n]). \checkmark
\end{aligned}$$

Thus, the system is **linear**.

证伪：

Exercise 2: Let our system T be given by $y[n] = (x[n])^p,\ p > 0$. Is this system linear?

We can try to prove linearity via superposition like we did in the previous exercise. However, let's start with just trying to prove homogeneity:

$$\begin{aligned}
T(ax[n]) &\stackrel{?}{=} aT(x[n]) \\
T(ax[n]) &= (ax[n])^p \\
&= a^p(x[n])^p \\
&= a^pT(x[n]) \\
&\neq aT(x[n])\ ✗
\end{aligned}$$

Thus, the system is **non-linear**.

Time-invariance

条件

We say a DT system is *time-invariant* or *shift-invariant* if:

$$y[n - n_0] = T(x[n - n_0])$$

or

$$T(x)[n - n_0] = T(x[n - n_0])$$

This means a shift by n_0 in our input yields the same shift in output.

证明/证伪:

Exercise 4: Is $y[n] = x[n] - x[n-1]$ time-invariant?

$$\begin{aligned}
y[n-n_0] &\stackrel{?}{=} T(x[n-n_0]) \\
y[n-n_0] &= x[(n-n_0)] - x[(n-n_0)-1] \\
&= x[n-n_0] - x[n-n_0-1] \\
T(x[n-n_0]) &= x[n-n_0] - x[n-1-n_0] \\
&= x[n-n_0] - x[n-n_0-1].
\end{aligned}$$

The two results are equivalent here, thus the system is **time-invariant**. Notice how we group the indices within each $x[n]$ term differently for the output and input sides of the relation to make clear whether the shift occurs before or after the system T acts on the signal.

Exercise 5: Is $y[n] = nx[3n]$ time-invariant?

$$\begin{aligned}
y[n-n_0] &\stackrel{?}{=} T(x[n-n_0]) \\
y[n-n_0] &= (n-n_0)x[3(n-n_0)] \\
T(x[n-n_0]) &= nx[3n-n_0].
\end{aligned}$$

These two results are not equal, thus the system is **time-varying**. Note how we only replace the n in front of the $x[n]$ with $n - n_0$ for the shifted output. The shifted input only adds a shift by n_0 inside of the actual input signal's argument.

Casuality

条件

A system is *causal* if the output doesn't depend on future samples.

Stability

条件

A digital system T is *bounded-input bounded-output(BIBO)* stable if

$$\text{For any } x[n] \text{ such that } |x[n]| < \beta,\ \forall n,\ |T(x[n])| < \alpha,\ \forall\ n,\ \text{where } 0 \le \alpha < \infty, 0 \le \beta < \infty. \tag{11}$$

证明：

Exercise 7: Suppose we are given the system $y[n] = x^{10}[n] + e^{x[n]}$. Is this system BIBO stable?

By the definition of BIBO stability, we may say that our input $|x[n]| < \beta < \infty$ for all n. This means that

$$\begin{aligned} |y[n]| &= |x^{10}[n] + e^{x[n]}| \\ &\leq |x[n]|^{10} + e^{|x[n]|} \\ &< \beta^{10} + e^{\beta} \\ &< \infty, \ \forall n. \ \checkmark \end{aligned}$$

Thus, our system is **BIBO stable** where $|y[n]|$ is guaranteed to be less than $\alpha = \beta^{10} + e^{\beta}$.