1. Impulse response

Impulse response

The impulse response of a discrete-time system T is the response of that system to a unit impulse, $\delta[n]$:

$$h[n] = T(\delta[n])$$

1.1 Signals as linear combination of impulses

Any DT signal can be represented as a sum of scaled and shifted unit impulses.

ex.

$$x[n] = sin(\frac{\pi}{4}n)(u[n] - u[n-8])$$

It's obvious to see that when $n \geq 8$, $x[n] = 0$. So we just have to write down every value of $x[n]$ when $n \leq 7$.

In this way, we can respresent $x[n]$ as

$$x[n] = \frac{\sqrt{2}}{2}\delta[n-1] + \delta[n-2] + \frac{\sqrt{2}}{2}\delta[n-3] - \frac{\sqrt{2}}{2}\delta[n-5] - \delta[n-6] - \frac{\sqrt{2}}{2}\delta[n-7]$$

In general we can the then write any $x[n]$ as

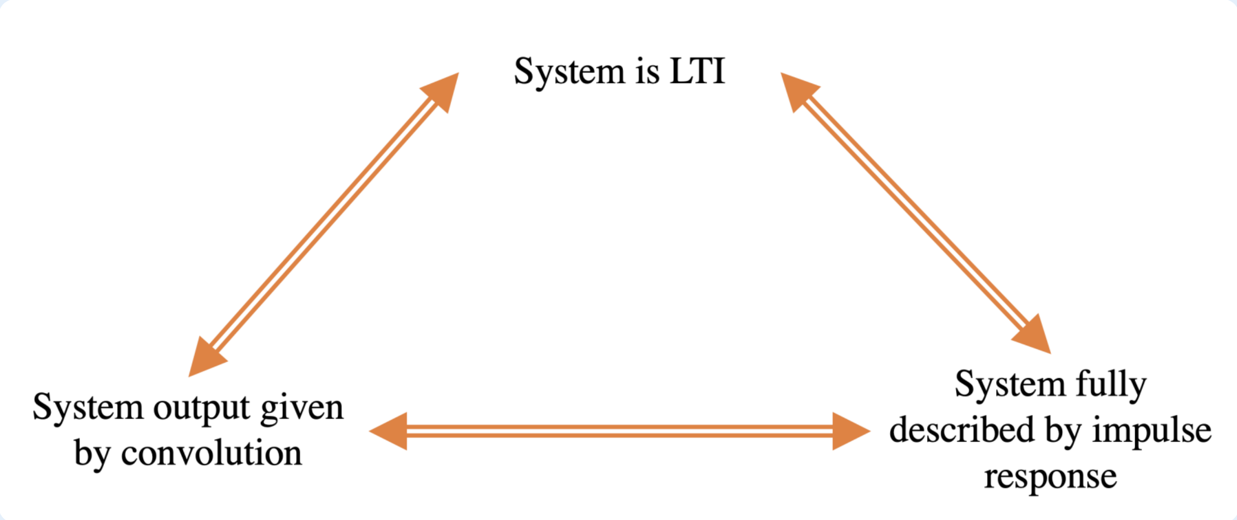
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

1.2 System response for LTI systems

Suppose we have an LTI system T with impulse response $h[n]$ that receives an input $x[n]$, our output $y[n]$ will be

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \forall n.$$

This eqn is known as *the convolution* between $x[n]$ and $h[n]$.

> ✎ **Relationships of fundamental importance**
>
> A system's output can only be defined by a convolution iff it is LTI.
> A system is fully defined by its impulse response iff it is LTI. (Fully defined means we know the system's output for any given input signal)
>
> 
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2. Convolution

The convolution between two DT signals $x[n]$ and $h[n]$:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n].$$

Notice that the convolution operator is *commutative*: the order of convolution doesn't matter.

2.1 Properties of convolution and impulse response

- Commutativity.

$$x[n] * h_1[n] * h_2[n] = x[n] * h_2[n] * h_1[n] = h_1[n] * h_2[n] * x[n]$$

- Associativity.

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n])$$

- Distributive property.

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- Identity.

$$x[n] * \delta[n] = x[n]$$

- Causality.
  An LTI system with impulse response $h[n]$ is *causal* if

$$h[n] = 0 for\, n < 0$$

- Stability.
  An LTI system with impulse response $h[n]$ is BIBO stabel if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- Start\End points and length.
  For
  Signal $x[n]$ of length N, starting at index n_s, and ending at index n_e.
  Signal $h[n]$ of length M, starting at index m_s, and ending at index m_e.
  If

$$y[n] = x[n] * h[n]$$

  Then

$$y_s = n_s + m_s$$

$$y_e = n_e + m_e$$

$$Y = N + M - 1$$