ECE 310 Fall 2023

Lecture 4

Convolution and impulse response

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Learning Objectives

After this lecture, you should be able to:

- Define the impulse response of a discrete-time system
- Show how the response of LTI systems to input signals can be fully described by their impulse response.
- Define the convolution between two discrete-time signals and key properties.
- Be able to compute the convolution between two finite-length discrete-time signals by hand.

Recap from previous lecture

We defined important properties of discrete-time systems in the previous lecture. These properties included linearity, time-invariance, causality, and stability. The former two properties will be particularly important for this lecture as we motivate discrete-time convolution using the impulse response of LTI systems.

1 Impulse response

The *impulse response* of a discrete-time system T is the response of that system to a unit impulse or Kronecker delta, $\delta[n]$:

$$h[n] = T(\delta[n]). \tag{1}$$

This definition is quite straightforward: the impulse response is the system's response to an impulse! Before moving on, let's consider the impulse response of a couple simple systems. First, consider our difference system from Lecture 3: $y_1[n] = x[n] - x[n-1]$. Following Eqn. 1, we see that

$$h_1[n] = \delta[n] - \delta[n-1]. \tag{2}$$

For our second system, let T be a *median filter* of width three. Thus,

$$y_2[n] = \text{median}\left(x[n], x[n-1], x[n-2]\right), \tag{3}$$

where the median of three numbers is the middle number after sorting the three values. Since there is only one non-zero value in $\delta[n]$, our second impulse response $h_2[n]$ is given by

$$h_2[n] = 0. \tag{4}$$

We will return to these two impulse responses later in this lecture. Now, let's consider how we can utilize the impulse response of discrete-time systems; specifically, LTI systems.

1.1 Signals as linear combination of impulses

Any discrete-time signal can be represented as a sum of scaled and shifted unit impulses. Consider, for example, $x[n] = \sin\left(\frac{\pi}{4}n\right)(u[n] - u[n-8])$. This is one period of a sinusoid with period 8. We can also represent $x[n]$ as

$$x[n] = \frac{\sqrt{2}}{2}\delta[n-1] + \delta[n-2] + \frac{\sqrt{2}}{2}\delta[n-3] - \frac{\sqrt{2}}{2}\delta[n-5] - \delta[n-6] - \frac{\sqrt{2}}{2}\delta[n-7]. \tag{5}$$

Here, we take each sample of $x[n]$ and express it as a Kronecker delta centered at $n = k$ with height $x[k]$. In general, we can then write any $x[n]$ as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]. \tag{6}$$

1.2 System response for LTI systems

We now know how to define the impulse response of a discrete-time system and how to decompose discrete-time signals as a summation of scaled and shifted unit impulses. We can combine these two concepts to elegantly describe the output of a particular class of systems to any input signal: LTI systems!

Suppose we have an LTI system T with impulse response $h[n]$ that receives an input $x[n]$. Using Eqns. 1 and 6, our output $y[n]$ will be

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k], \ \forall n. \tag{7}$$

Equation 7 is known as the *convolution* between $x[n]$ and $h[n]$. It is important to reinforce that this is only possible for LTI systems! Just because we have the impulse response of a system does not mean we know its response to scaled unit impulses (homogeneity), the summation of unit impulses (additivity), or shifted unit impulses. We need a linear system (satisfying superposition) to guarantee the first two pieces while we need time-invariance to guarantee the last piece.

This brings us to a relationship of fundamental importance. A system's output can only be defined by a convolution *if and only if* it is LTI. Moreover, a system is fully defined by its impulse response *if and only if* it is LTI. "Fully defined" means we know the system's output for any given input signal. We cannot use the impulse response and decomposition of discrete-time signals as impulses to obtain a system output if the system is not LTI. Therefore, we cannot obtain its output using convolution for any input signal. These three statements form an *if and only if* relationship and thus are all equivalent! We will not prove both directions of these three statements, but Fig. 1 is important to keep in mind.

To close the loop on this point, let's return to our two impulse response from before: $h_1[n]$ and $h_2[n]$. Our difference system given by $h_1[n]$ tells us we can obtain any output by taking our signal $x[n]$ and subtracting the same signal delayed by one sample, $x[n-1]$. We can also accomplish this same operation via the convolution sum in Eqn. 7 and $h[n] = \delta[n] - \delta[n-1]$ because we already proved this system is LTI in Lecture 3! The median filter, on the other hand, is clearly not fully described by its impulse response. We have $h_2[n] = 0$, which may suggest the median filter would give a zero output to any input signal. But this is obviously not true since we use the median in signal processing and statistical operations all the time. The underlying issue is that the median is a non-linear system. The non-linearity breaks our nice relationships given in Fig. 1 and prevents us from using the impulse response or convolution when computing the response of a median filter.

2 Convolution

Now that we have derived the convolution operation, we can provide a more complete definition and intuition, summary of related properties, and demonstrate how to compute the convolution between two signals. Building on Eqn. 7, we can define the convolution between two discrete-time signals $x[n]$ and $h[n]$ as

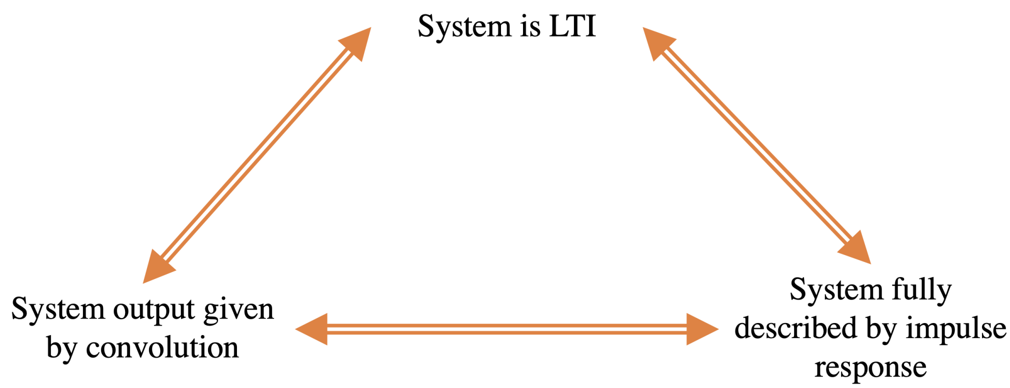

Figure 1: The "if and only if" relationship between LTI systems, convolution, and impulse response.

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n]. \tag{8}$$

Here, we denote the convolution operation with the $*$ symbol. Furthermore, we see that the convolution operator is *commutative*: the order of convolution does not matter. This can be shown by a simple change of variables in the convolution sum, e.g. $m = n - k$. In words, when we *convolve* (don't say convolute!) two signals, we are following these steps:

1. Flip one of our signals, e.g. $h[-k]$.
2. Shift by n samples, e.g. $h[-(k-n)] \equiv h[n-k]$.
3. Multiply the two signals element-wise, e.g. $z_n[k] = x[k]h[n-k]$.
4. Sum up our product of these two signals to obtain the output at index n, e.g. $y[n] = \sum_{k=-\infty}^{\infty} z_n[k]$.

We follow Steps (2)–(4) for all values of n in order to obtain our output signal $y[n]$. Note that we add the n subscript to $z_n[k]$ to make clear that the intermediate $z_n[k]$ signal depends on both n and k.

2.1 Properties of convolution and impulse response

Below we list some helpful properties for the convolution operator and the impulse response of LTI systems:

Commutativity. We have already discussed convolution being commutative, but we can further extend this to applying multiple LTI systems to an input signal $x[n]$:

$$x[n] * h_1[n] * h_2[n] = x[n] * h_2[n] * h_1[n] = h_1[n] * h_2[n] * x[n] \ \cdots \tag{9}$$

Associativity.

$$(x[n] * h_1[n]) * h_2[n] = x[n] * (h_1[n] * h_2[n]). \tag{10}$$

Distributive property.

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]. \tag{11}$$

Identity.

$$x[n] * \delta[n] = x[n]. \tag{12}$$

Causality. An LTI system with impulse response $h[n]$ is causal if

$$h[n] = 0,\ n < 0. \tag{13}$$

Intuitively, this means the impulse response will never reach into the future when we flip and shift to perform convolution.

Stability. An LTI system with impulse response $h[n]$ is BIBO stable if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{14}$$

Lecture exercise: Prove an LTI system with impulse response $h[n]$ is BIBO stable if Eqn. 14 holds.

Start point, end point, length. For signal $x[n]$ of length N, starting at index n_s, and ending at index n_e and signal $h[n]$ of length M, starting at m_s, and ending at index m_e, let $y[n] = x[n] * h[n]$. The starting index k_s, ending index k_e, and length K are given by:

$$k_s = n_s + m_s \tag{15}$$

$$k_e = n_e + m_e \tag{16}$$

$$K = N + M - 1. \tag{17}$$

2.2 Computing the convolution of two signals

We will conclude this lecture with a few different perspectives on how to compute the convolution between two signals by hand.

2.2.1 Shift-and-overlap method

The shift-and-overlap or "table" method is the visualization of the procedure we described in Section 2. For each value of n, we will shift one of our two signals, multiply the overlap between the signals, and sum the result. Let our two signals be:

$$x[n] = \{\underset{\uparrow}{3}, 1, 0, 3, 1, 1\}$$

$$h[n] = \{2, \underset{\uparrow}{-1}, 1\}.$$

We will perform the flipping and shifting on $h[n]$ since it is the shorter sequence. From our start point, end point, and length properties, we know that our first non-zero index will be $n = -1$, the end index will be $n = 6$, and the length will be 8. Thus, our answer would be

k	-2	-1	0	1	2	3	4	5	
$x[k]$			3	1	0	3	1	1	
$y[-1] = \sum_k x[k]h[-1-k]$	1	-1	2						$= 2 \cdot 3 = 6$
$y[0] = \sum_k x[k]h[-k]$		1	-1	2					$= -1 \cdot 3 + 2 \cdot 1 = -1$
$y[1] = \sum_k x[k]h[1-k]$			1	-1	2				$= 1 \cdot 3 + (-1) \cdot 1 + 2 * 0 = 2$
$y[2] = \sum_k x[k]h[2-k]$				1	-1	2			$= 1 \cdot 1 + (-1) \cdot 0 + 2 \cdot 3 = 7$
$y[3] = \sum_k x[k]h[3-k]$					1	-1	2		$= 1 \cdot 0 + (-1) \cdot 3 + 2 \cdot 1 = -1$
$y[4] = \sum_k x[k]h[4-k]$						1	-1	2	$= 1 \cdot 3 + (-1) \cdot 1 + 2 \cdot 1 = 4$
$y[5] = \sum_k x[k]h[5-k]$							1	-1	$= 1 \cdot 1 + (-1) \cdot 1 = 0$
$y[6] = \sum_k x[k]h[6-k]$								1	$= 1 \cdot 1 = 1$

$$y[n] = \begin{bmatrix} 6 & \underline{-1} & 2 & 7 & -1 & 4 & 0 & 1 \end{bmatrix}.$$

Note that we assume there are zeros on either side of $x[k]$ when we have partial overlap between $x[k]$ and $h[n-k]$. This is known as *zero extension.*

2.2.2 Matrix-vector multiplication

Similar to the table method, we have the matrix-vector multiplication method. As the name suggests, we will encapsulate the computation performed by the table method into the multiplication of a matrix and a vector. We can accomplish this by constructing an *operator matrix* from one of our signals. Let's use the same signals $x[n]$ and $h[n]$ from Section 2.2.1. The operator matrix of $h[n]$ would be given by

$$\mathbf{H} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{18}$$

and our result will be given by

$$y = \mathbf{H}x. \tag{19}$$

Note that we use boldface text to denote matrices. The structure of the matrix $\mathbf{H}$ is special with its constant diagonals and known as a *Toeplitz* matrix. We see that the flipping and shifting of $h[n]$ is expressed in each row of $\mathbf{H}$ as we shift the reversed sequence $h[-k]$ one index with each successive row. Finally, our output will be given by

$$y = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 3 \\ 1 \\ 1 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ -1 \\ 2 \\ 7 \\ -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}}_{y}. \tag{20}$$

or

$$y = \underbrace{\begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 3 \\ 3 & 0 & 1 \\ 1 & 3 & 0 \\ 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{X}} \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}_{h} = \underbrace{\begin{bmatrix} 6 \\ -1 \\ 2 \\ 7 \\ -1 \\ 4 \\ 0 \\ 1 \end{bmatrix}}_{y} \tag{21}$$

We can associate each index of n to y using our start point property starting with $n = -1$ for the first non-zero sample.

2.2.3 Analytical sum

Our last method is to directly use the definition of the convolution sum. This method can be fairly challenging but works best when our signals are very long or even infinitely long. Let our two signals be given by

$$\begin{aligned} x[n] &= u[n] \\ h[n] &= \left(-\frac{3}{4}\right)^n u[n]. \end{aligned}$$

We can then compute our output $y[n] = x[n] * h[n]$ as follows:

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} \left(-\frac{3}{4}\right)^k u[k]u[n-k] \\
&\overset{(a)}{=} \sum_{k=0}^{n} \left(-\frac{3}{4}\right)^k \\
&\overset{(b)}{=} \frac{1-\left(-\frac{3}{4}\right)^{n+1}}{\frac{7}{4}} \\
&= \frac{4}{7} - \frac{4}{7}\left(-\frac{3}{4}\right)^{n+1}.
\end{aligned}$$

Above, step (a) follows from the limits of where $u[k]$ and $u[n-k]$ are both non-zero (0 to n) and step (b) is from the finite summation of a geometric sequence:

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}. \tag{22}$$