Dual Descent and Air Resistance

Devin Crowley

1 Introduction

One of the most archetypal problems in the field of physics is modeling trajectories. This is widely applicable to a host of military concerns, including the prediction and potential adjustment of projectile weapon trajectories, originating either from within Earth's atmosphere or from orbit. The models involved range considerably in their intricacy, but ultimately the best approach is to use the assumptions and simplifications most appropriate to the situation. In this paper we will be numerically modeling and analyzing the vertical descent of two identical objects, A and B, in the presence of air resistance. A is dropped from rest, but B is given an initial lateral velocity. Our objective is to determine which object will land first, and how the relevant variables affect the time difference between their descents.

2 Model

We have made several assumptions to simplify our analysis.

- There is no spin to account for
- The density of the air is constant over the altitude range
- The force of gravity is constant over the altitude range
- The object's shape has no asymmetry that would affect its trajectory
- The quadratic air resistance term dominates the linear term

We have chosen to model this situation using Euler's Method. These are the relevant equations of motion:

$$\frac{d^2x}{dt^2} = -\frac{B_2 v v_x}{m},\ \frac{d^2y}{dt^2} = -g - \frac{B_2 v v_y}{m}, \tag{1}$$

where B_2 is the drag coefficient for quadratic drag, m is the object's mass, v is its velocity, and g is the acceleration due to gravity. However, to model this numerically using Euler's Method, we have reduced these 2nd order differential equations into a set of 1st order differential equations.

$$\frac{dx}{dt} = v_x,\ \frac{dy}{dt} = v_y \tag{2}$$

$$\frac{dv_x}{dt} = -\frac{B_2 v v_x}{m},\ \frac{dv_y}{dt} = -g - \frac{B_2 v v_y}{m} \tag{3}$$

We have discretized these equations into the following form, from which we directly implemented our code.

$$v_{x,i+1} = v_{x,i} - \frac{B_2 v v_{x,i}}{m}\Delta t \tag{4}$$

$$v_{y,i+1} = v_{y,i} - g\Delta t - \frac{B_2 v v_{y,i}}{m}\Delta t \tag{5}$$

$$y_{i+1} = y_i + v_{y,i}\Delta t \tag{6}$$

These equations describe the evolution of both the x and y components of the velocity for these objects, and their height. We will use initial values for these quantities at the time of release $(t = 0)$ and use equations 4-6 to evolve them with time in increments of Δt. Each iteration of this procedure corresponds to an additional step forward in time of Δt.

To delve further into this problem we can analyze the contributions of some key variables on the objects' descent times. Specifically, we will investigate the effects of object B's horizontal launch speed, the initial height of the drop h, the objects' mass m, and the drag coefficient B_2.

One simplification we can make is reducing m and B_2 to a single parameter $C = \frac{B_2}{m}$. B_2 and m are both strictly positive and only appear in our equations of motion as a ratio, $(\frac{B_2}{m})$. Therefore, if we vary just one when we perform our analysis, we could equivalently have (inversely) varied the other instead, so keeping these variables separate adds no new information or depth to our investigation.

In figures 3, 4, and 5, we have plotted the difference between the descent times of objects A and B as functions of the initial height, the initial lateral velocity, and the drag coefficient C. Each plot varies one of these quantities while keeping the other two parameters constant to isolate each variable's effect on the difference in descent times.

3 Results

3.1 Overall Trajectory

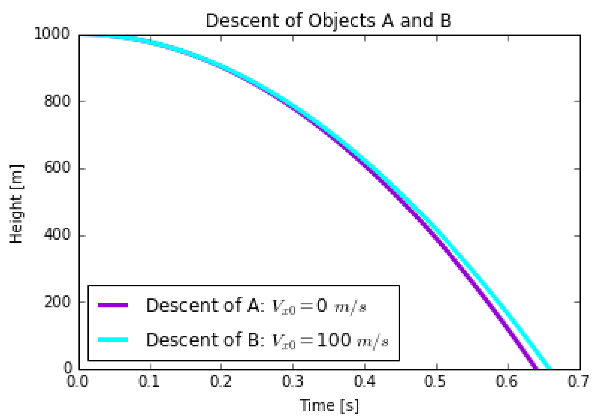

Figure 1: Vertical trajectories with $C = 0.003\ m^{-1}$ and a time step of $\Delta t = 0.005s$

We have found that object A, which was dropped from rest, landed before object B, as shown in figure 1. We could have predicted this by examining the last term of our 2nd equation of motion (3), for the y-component of the velocity. This term has a factor of v, the overall velocity. As we know,

$$v = \sqrt{v_x^2 + v_y^2}.$$

From here it is trivial that the object with a lateral component to its velocity will experience greater vertical air resistance and therefore accelerate more slowly; object A lands before object B.

Our plots for the time difference as functions of initial height y_0, initial lateral velocity v_{x0}, and the drag coefficient C were much more computationally intensive than our program to generate the trajectory plot, figure 1. To save on computation time we chose a larger value for Δt to increment by. However, these less precise plots still show the trends we are investigating.

3.1.1 Numerical Accuracy

Figure 1 shows the basic results of our simulation: the trajectories of objects A and B for a single set of parameter values, with a time step of $\Delta t = 0.005s$. In this paper you will encounter other plots of the difference in descent times between these two objects as we vary several parameters, but each point on these plots is the result of a single run of this basic trajectory simulation. Therefore, we can demonstrate the numerical accuracy of them all by showing the convergence of our trajectory plot for a representative set of parameter values, as we decrease our temporal step size Δt. Figure 2 shows the same trajectories for two smaller step sizes, $\Delta t = 0.002$ and $\Delta t = 0.001$. As you can see, they do not change much, and experience diminishing changes with decreased Δt values, so we are confident that we are close to the true solution.

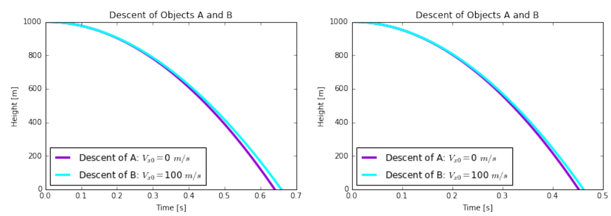

Figure 2: Vertical trajectories with $C = 0.003\ m^{-1}$, respectively using time steps of $\Delta t = 0.002s$ and $\Delta t = 0.001s$

3.2 Initial Height

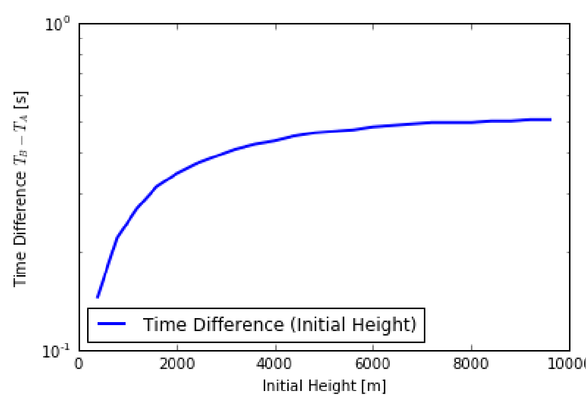

Figure 3: The difference in descent times as a function of the initial height, with $C = 0.05\ m^{-1}$ and $v_{x0} = 100\ m/s$

Figure 3 shows the dependence on the initial height. It is monotonically increasing, but ultimately approaches an asymptote. Interpreting this graph is quite straightforward. The further the two objects fall, the longer the difference in vertical air resistance has to separate their trajectories. However, they eventually reach terminal velocity, object B gradually loses its lateral velocity, and the two objects come to fall at the same rate. Further increases in the initial height have vanishingly small impacts on the time difference; technically they do have some effect, but only insofar as the two objects approach terminal velocity asymptotically, never exactly reaching it.

3.3 Initial Lateral Velocity

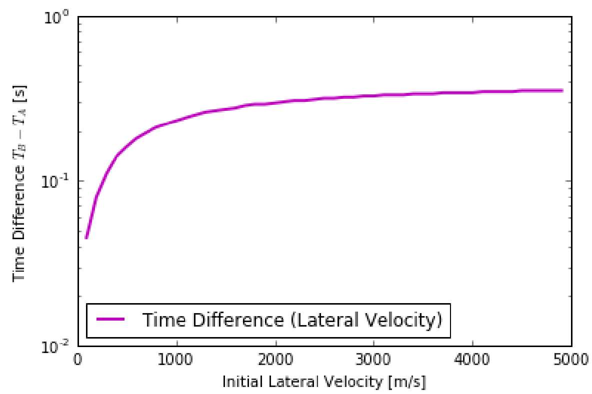

Figure 4: The difference in descent times as a function of the initial lateral velocity, with $C = 0.003\ m^{-1}$ and $y_0 = 1000\ m$

Figure 4 shows the dependence on the initial lateral velocity of object B. It is also monotonically increasing, eventually approaching an asymptote. It is fairly intuitive that the graph is positively sloped, since the initial lateral velocity is the only difference between the two objects, so as it increases the difference between their descent times increases as well. More specifically, the greater object B's initial lateral velocity, the greater the difference between their vertical drag forces, and therefore descent times. However, the drag force is proportional to v^2, so as v_{x0} is further increased, it will diminish more and more quickly due to the horizontal drag force. Eventually, even very large initial velocities will almost immediately reduce to lesser velocities before the objects gain an appreciable downward velocity and are able to separate due to the disparity in their vertical drags. That is the mechanism illustrated by figure 4.

3.4 Drag Coefficient

Figure 5 shows the dependence on the drag coefficient C. Unlike the dependence on initial height and lateral velocity, it increases at first, but eventually goes down again. The interpretation of this graph is similar to that of figure 4, for the dependence on v_{x0}. It is positively sloped initially because the drag force on object B is greater than

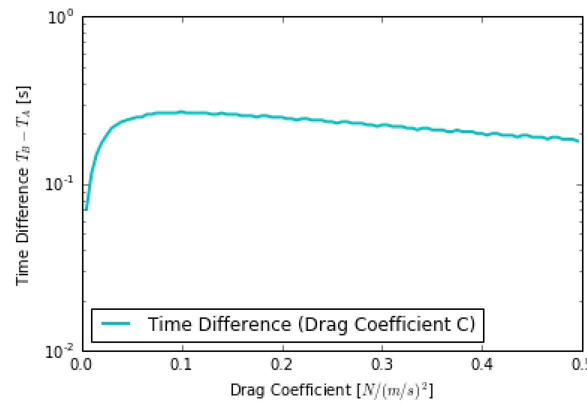

Figure 5: The difference in descent times as a function of the drag coefficient, with $v_{x0} = 100\ m/s$ and $y_0 = 1000\ m$

the drag force on object A due to B's lateral velocity. Increasing the magnitude of both drag forces by increasing C makes this difference greater, especially at the top of the drop when object B's lateral velocity is largest. However, if the drag coefficient is made sufficiently large, then object B's lateral velocity will approach zero very quickly: before the two objects' vertical velocities become significant enough for their vertical drag forces to be appreciable, and for the difference in those forces to have an appreciable effect.

To recap, when C is made large enough, object B's initial lateral velocity quickly vanishes; further increasing C makes v_x vanish even more quickly such that the two objects are not given the opportunity to separate before their velocities converge. Essentially, a sufficiently powerful drag reduces the difference between the objects' initial velocities quickly enough for it to have only a very small effect. For this reason, we can infer that this graph will eventually project down and approach zero.

4 Conclusion

We have analyzed the descent of two identical objects released from the same height, one from rest and the other with an initial horizontal velocity. Our aim was to explore the difference in descent times between the two objects and how it was affected by changing the initial height, the initial lateral velocity of one object, and the drag coefficient of air resistance. In order to focus on these parameters, and to simplify our approach, we made several assumptions in our model. We neglected spin, attenuation of atmospheric density and weight with altitude, the linear drag term, and any possible asymmetry in the shape of the objects. Within certain parameter regimes these are good assumptions, particularly in light of the purpose of this exploration.

We found some numerical difficulties with certain versions of figure 5, in which our algorithm became unstable for larger values of the drag coefficient C. However, we were able to identify trends, figure out the involved physics, and support them with this plot. We were able to do this for all three parameters in question, y_0, v_{x0}, and C.