Singularities

University of Michigan

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Any errors or inaccuracies are likely introduced by myself, Devlin Mallory.

Resolutions of singularities (September 20, Matt Stevenson)

The setting today is varieties over a field k of characteristic 0.

Definition. A resolution of a variety X over k is a proper birational morphism $f : X' \to X$, with X' smooth (equivalently nonsingular, or regular, etc.) over k. We often ask for more: we'd like to control the open set $U \subset X$ over which f is an isomorphism, and we'd like for $f^{-1}(X - U)$ to have "nice" geometry. Moreover, we'd like to have a resolution of X and some extra data, which we'll discuss later.

Example. If X is dimension 1, then the normalization $X' \to X$ is a resolution of singularities: X' is normal, hence regular in codimension 1, so smooth, and $X' \to X$ is proper.

This, of course, has no hope of working in higher dimension.

Remark (The common approach)**.** Many approaches share the same general strategy:

(1) embed $X \subset M$, where M is something we understand (for example, M smooth).

(2) blow up the "worst" singularities of X inside M.

(3) Look at the strict transform X' of X; if this is smooth we're done, and if not repeat.

This is often called an "embedded" resolution of singularities.

Example (cuspidal cubic)**.** Let $X = V(y^2 - x^3) \subset \mathbb{A}^2$ (we know of course that the normalization resolves the singularity, but this remains a good example). Step (1) is complete, so now we blow up $\mathbb{A}^2$ at the origin, obtaining $\operatorname{Proj} k[x, y][u, v]/(xv - yu)$. This is covered by two charts: in one, we have that $f^{-1}(y^2 - x^3) = x^2(v^2 - x)$; thus the strict transform is just the parabola $v^2 - x$, so this gives a resolution of singularities.

Example (D_4-singularity)**.** Let $X = V(x^2 + y^3 + z^3) \subset \mathbb{A}^3$; this is singular at the origin, so we blow up at the origin, obtaining the map

$$\frac{\operatorname{Proj}(k[x, y, z][u, v, w]}{I_2\begin{pmatrix} x & y & z \\ u & v & w \end{pmatrix}} \to X$$

In the chart $v \neq 0$ the equation $x^2 + y^3 + z^3$ pulls back to $(uy)^2 + y^3 + (wy)^3 = y^2(u^2 + y(1 + w^3))$. Then the strict transform X' is locally in this chart given by $u^2 + y(1 + w^3)$. But this is still singular when $u = y = 1 + w^3 = 0$, giving three singular points. Now, restrict further to $D(v) \cap D(1 - w + w^2)$ and blow up at $u = v = 0$ and $w = -1$, giving

$$\frac{\operatorname{Proj}(k[u, y, w][r, s, t]}{I_2\begin{pmatrix} u & y & w+1 \\ r & s & t \end{pmatrix}};$$

checking the chart $t \neq 0$ we see the pullback $g^{-1}(\{u^2 + y(1+w^3) = 0\})$ is defined by $r^2(w+1)^2 + s(w+1)(1+w^3) = (w+1)^2(r^2 + s(1-w+w^2))$, so the new strict transform is $r^2 + s(1-w+w^2)$, which is smooth. The other two singular points blow up similarly, so we've found our resolution.

Example (Whitney umbrella)**.** Let $X = V(x^2 - y^2z) \subset \mathbb{A}^3$; this is singular along the line $V(x,y)$, and is even more singular at the origin. If we try to blow up the very bad point at the origin, we get that the equation pulls back to (locally on the chart $w \neq 0$) $(uz)^2 - (vz)^2z = z^2(u^2 - v^2z)$. That is, we've reproduced the exact same singularity! If instead we blow up the line, this does resolve the singularity.

This hints that the general problem of constructing resolutions is quite difficult. Thankfully, we have the following general theorem, which gives us a "strong resolution":

Theorem (Hironaka 1964)**.** *There exists a projective birational isomorphism $f : X' \to X$, X' smooth, such that:*

(1) f is an isomorphism over the smooth locus of X.

(2) $f^{-1}(X_{\mathrm{sing}})$ is a very mildly singular (more precisely, snc) codimension-1 subset.

(3) f is a composition of blowups along smooth centers.

By snc, we mean the following:

Definition. If X is smooth over k, then an effective divisor $D \subset X$ is snc (or has simple normal crossings) if

(1) each irreducible component D_i of D is smooth, and

(2) the intersections of the D_i are transverse, i.e., at all $p \in D$ we can choose $D = V(x_1 \cdots x_r)$ where $x_1, \cdots x_d \in \mathcal{O}_{X,p}$ are part of a regular system of parameters.

(1) If $X = \mathbb{A}^n$ then $D = V(x_1 \cdots x_r)$ is snc for $r \leq n$.

(2) If X is a smooth toric variety and D is the toric boundary then D is snc.

Definition. D has normal crossings (or is nc) if D is étale-locally snc.

Example. The nodal cubic $V(y^2 - x^2(x+1))$ is nc but not snc, as can be seen by taking the étale cover $\operatorname{Spec} k[x,y,z]/(z^2 - (x+1)) \to \mathbb{A}^2$, since when we pullback D to this cover it decomposes as $(y - xz)(y + xz)$.

We can, in fact, ask more from resolutions of singularities: we can resolve not only X, but X plus some "extra" data. There are two choices of this data, essentially equivalent: we can fix D a Weil divisor on X or $a \subset \mathcal{O}_X$ an ideal sheaf.

Definition. A log resolution of (X, D) or (X, a) is a proper birational morphism $f : X' \to X$, with X' smooth, such that

(1) $\mathrm{Exc}(f)$ is a divisor (i.e., pure codimension 1).

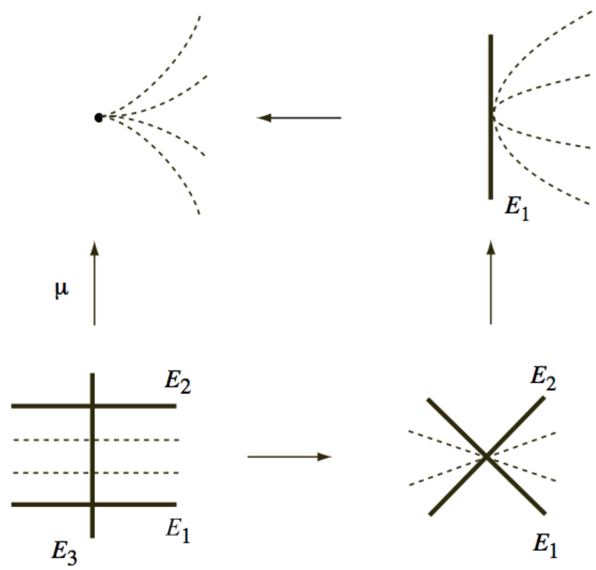
(2a) For (X, D) we demand that $f^*D + \mathrm{Exc}(f)$ is snc[1].

(2b) For (X, a) we demand that $a\mathcal{O}_{X'} = \mathcal{O}_{X'}(-F)$ for F an effective divisor with $F + \mathrm{Exc}(f)$ snc.

Theorem (Hironaka)**.** *Log resolutions exist.*

[1]When D is not Cartier, we ask instead that the support $f^{-1}D \cup \mathrm{Exc}(f)$ is snc.

Example. Consider $D = V(y^2 - x^3) \subset X = \mathbb{A}^2$ (equivalently, this gives a log resolution of $(X, a = (y^2, x^3)))$. We begin by blowing up at the origin; in one chart (where $y = xv$) we get $f_1^{-1}(D) = x^2(v^2 - x) = 0$. Each component is smooth, but they intersect in a length-2 subscheme, i.e., this certainly isn't snc. So, we blow up at the intersection of $V(x)$ and $V(v^2 - x)$. In the interesting chart we have $(f_1 \circ f_2)^{-1}(D) = V(v^3 r^2 (v - r))$. This is three lines meeting at the origin, which is not snc, so we blow up one more time, finally obtaining an snc divisor. Thus we've finally computed a log resolution of (X, D). The following illustration of this case is taken from Rob Lazarsfeld's "Positivity in Algebraic Geometry II", where the dotted lines depict general k-linear combinations of x^3 and y^2.



Remark. Note that $\mathrm{Exc}(f)$ need not be a divisor for a birational morphism $X' \to X$! For example, take $X = V(xy - uv) \subset \mathbb{A}^4$, and blow up the plane $V(u, v)$ and consider the strict transform X' of X. Then we have X' is smooth, but the exceptional locus of the blowup is a $\mathbb{P}^1$, and thus of codimension 2.

We do at least have the following useful fact:

Theorem. *If X is normal and $\mathbb{Q}$-factorial (i.e., every Weil divisor has a Cartier multiple) and X' is quasiprojective and smooth, then the exceptional locus of any birational map $f : X' \to X$ is a divisor.*

So, we have resolutions; so, how do we use them to analyze singularities? Classically, given a variety X over k, for $x \in X$ we can attach invariants to valuations (or valuation rings) in $k(X)/k$ centered at x. Hironaka's theorem thus gives a geometric reformulation of the study of these valuations.

Theorem (Zariski)**.** *For all k and all X over k, and v a valuation on $k(X)/k$ centered on X, the following are equivalent:*

(1) v is discrete (and Abhyankar).

(2) There exists $Y \to X$ proper birational with Y normal and $E \subset Y$ a prime divisor such that $v = c \cdot \mathrm{ord}_E$ for some $c > 0$.

Of course, in characteristic 0 by resolution of singularities we may take Y smooth by resolving the normal variety Y and considering the strict transform of E.

Example. If $X = \mathbb{A}^2_k$, then we can define a valuation

$$v\left(\sum_{i,j} a_{ij} x^i y^j\right) = \min_{a_{ij} \neq 0} i + j;$$

this is the same as taking the order of vanishing along the exceptional divisor of the blowup at the origin.