Finite F-representation type for homogeneous coordinate rings

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Frobenius pushforwards

- Let R be a local ring over a field of characteristic p.
- We write $F^e : R \to R$, $g \mapsto g^{p^e}$ for the e-th iterate of the Frobenius map.
- Restricting scalars along F^e, we get an R-module $F^e_* R$, called the e-th Frobenius pushforward of R.
- As a set, $F^e_* R$ is just $\{F^e_* r : r \in R\}$, with R-module structure $x \cdot F^e_* r = F^e_*(x^{p^e} r)$.
- In this notation, the Frobenius is an R-linear map $R \to F^e_* R$ determined by $1 \mapsto F^e_* 1$.

We will assume F^e is a finite morphism, i.e., $F^e_* R$ is a finite R-module. This is true for most rings encountered in geometry.

Question

What is the R-module $F^e_* R$? In particular, what are its indecomposable summands and their multiplicities as e varies?

Example and motivations

Example

If $R = \mathbb{F}_p[x_1, \ldots, x_n]$, you can check that $F^e_* R$ is the free R-module with basis $x_1^{a_1} \cdots x_n^{a_n}$ with $0 \leq a_i \leq p^e - 1$. Thus, $F^e_* R = \bigoplus_{i=1}^{p^{en}} R$.

Why do we care about $F^e_* R$? One reason: the summands of R reflect the singularities of $\operatorname{Spec} R$.

Theorem (Kunz)

Let R be local. $F^e_* R$ is a free R-module iff R is regular.

If you care about characteristic-0 singularities, there is an analogy:

char. 0	char. p
log canonical	$F^e_* R$ has a free summand
KLT	$F^e_* R$ has many free summands

which can be made precise.

Finite F-representation type

One question one can ask is, as e varies, how many different isomorphism classes of summands of $F^e_* R$ are there?

Definition

- An R-module M is a Frobenius summand[1] if M is an indecomposable summand of $F^e_* R$ for some $e \geq 1$.
- R has finite F-representation type (FFRT) if there are only finitely many isomorphism classes of Frobenius summands.

(FFRT was introduced originally by Smith–Van den Bergh to study D_R, the ring of differential operators on R.)

We have just seen that polynomial rings have FFRT.

[1] Really, we should assume R is graded or complete to make this definition.

A second example

Consider $R = (\mathbb{Z}/2)[x,y,z]/(x^2 - yz)$. Then F_*R has generators

$$F_* x^i y^j z^k, \quad 0 \le i,j,k \le 1$$

subject to the relations

$$\begin{gathered} xF_*1 = F_*x^2 = F_*(yz), \quad xF_*x = F_*(xyz), \\ xF_*z = zF_*y, \quad xF_*y = yF_*z \\ xF_*(xz) = zF_*(xy), \quad xF_*(xy) = yF_*(xz). \end{gathered}$$

Thus

$$F_*R \cong R \oplus R \oplus \operatorname{coker}\begin{pmatrix} x & z \\ y & x \end{pmatrix} \oplus \operatorname{coker}\begin{pmatrix} x & z \\ y & x \end{pmatrix}.$$

Moreover, one can check that

$$F_*^e R \cong R^{\oplus 2^{2e-1}} \oplus \operatorname{coker}\begin{pmatrix} x & z \\ y & x \end{pmatrix}^{\oplus 2^{2e-1}}.$$

Thus R has FFRT.

Rings with FFRT

Example

If R is a Cohen–Macaulay ring, then the summands of $F^e_* R$ are Cohen–Macaulay. Thus, CM rings with finite CM representation type also have FFRT. Thus:

- Quadric hypersurfaces have FFRT.
- All the rational double point singularities have FFRT.

Example (Smith–Van den Bergh)

If S is a ring with FFRT and $R \subset S$ is a direct summand, and either S is graded or $R \subset S$ is finite, R has FFRT. Thus invariant rings of finite groups and toric rings have FFRT.

Thus, for example, $k[x, y, z]/(x^2 - yz)$ has FFRT in any characteristic.

Rings without FFRT

Not every ring has FFRT:

Example

Let $k = \bar{k}$, and let $R = k[x, y, z]/F_d$ be the homogeneous coordinate ring of a smooth degree-d curve. Then:

d	$X = \operatorname{Proj} R$	$F_*^e R$	FFRT
2	$\mathbb{P}^1$	$R^{\oplus p^{2e}}$	yes
3	elliptic curve	(see board)	no
≥ 4	general type	$\bigoplus^{p^e}$ rank-p^e indecomposable	no

When $d \geq 3$, there are indecomposable summands of arbitrarily large rank as $e \to \infty$, and thus R fails to have FFRT.

(This example requires nontrivial theorems by X. Sun and others about stability of Frobenius pushforwards.)

Properties of rings with FFRT

FFRT has strong consequences for other algebraic properties of R. For example:

- *Local cohomology*: If R has FFRT, its iterated local cohomology modules have finitely many associated primes
- *Differential operators*: If R is strongly F-regular and has FFRT, then the ring of differential operators D_R is a simple algebra.
- Let R be a local ring of characteristic p with FFRT. Then there exists a finitely generated Cohen–Macaulay R-module M.

(The first is due in various forms to Takagi–Takahashi, Hochster–Núñez-Betancourt, Dao–Quy, and Quinlan-Gallego. The second is Smith–Van den Bergh, and the last is Yao.)

Is FFRT rare?

- These examples suggest that perhaps FFRT is rare.
- Few examples are known either way, particularly in higher dimensions.
- The case of homogeneous rings of curves suggests that perhaps coordinate rings of non-Fano varieties do not have FFRT.
- (Which is not to say that coordinate rings of Fano varieties have FFRT!)

Today we'll show that many coordinate rings of non-Fano varieties (e.g., high-degree complete intersections) do not have FFRT.

The main theorem

Theorem (-)

Let X be a smooth variety over a perfect field k of characteristic p from the following list:

- *A non-Fano smooth complete intersection.*
- *A non-unirational K3 surface.*
- *A non-uniruled Calabi–Yau variety with $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, \dots, \dim X - 1$.*

Let L be a very ample line bundle on X. Then the homogeneous coordinate ring $R = S(X, L)$ of X does not have FFRT.

Remark

The proof is by contradiction, and tells us nothing about the Frobenius summands that occur, simply that there infinitely many. It would be very nice to know $F^e_* R$ or even just $F^e_* \mathcal{O}_X$.

An example

Just to make things more down-to-earth:

Example

For *any* characteristic $p > 0$ and any $d \geq 5$ not divisible by p, the ring

$$\frac{k[x_0, x_1, x_2, x_3, x_4]}{x_0^d + x_1^d + x_2^d + x_3^d + x_4^d}$$

does not have FFRT.

(This is the homogeneous coordinate ring of a smooth threefold in $\mathbb{P}^4$, which is Calabi–Yau if $d = 5$ and general-type if $d \geq 6$.)

Sketch of the proof

There are three main ingredients in the proof:

1. Use a theorem of Takagi–Takahashi to show that FFRT implies the existence of negative-degree differential operators on R.
2. Show that existence of such differential operators gives nonzero global sections of divided powers of the tangent sheaf T_X.
3. Show that no such global sections can exist, using nonpositivity of T_X and stability results from algebraic geometry.

A brief intro to differential operators

Write $D_R = D_{R/k}$ for the ring of k-linear differential operators on R. If R is of finite type over a perfect field of characteristic p, D_R is $\bigcup_e \operatorname{End}_{R^{p^e}}(R)$; you can think of this as the definition of D_R.

Example

Let $R = \mathbb{F}_p[x]$. Then $D_{R/k}$ is generated as an R-algebra by the differential operators

$$1, \partial/\partial_x, \frac{1}{p!}(\partial/\partial_x)^p, \frac{1}{p^2!}(\partial/\partial_x)^{p^2}, \frac{1}{p^3!}(\partial/\partial_x)^{p^3}, \ldots .$$

Note that $R[1/x]$ is generated by $1/x$ over $D_{R/k}$: it suffices to obtain $1/x^{p^e+1}$ for any e, and a simple calculation shows that

$$\frac{1}{p^e!}(\partial/\partial_x)^{p^e} \cdot \frac{1}{x} = \frac{1}{x^{p^e+1}}$$

A criteria for FFRT via differential operators

A theorem of Álvarez–Montaner, Blickle, and Lyubeznik says that if $R = \mathbb{F}_p[x_1, \dots, x_n]$, then *any* localization $R[1/f]$ is generated over D_R by $1/f$.

(Note this is false in characteristic 0, and its failure to be true is measured by the Bernstein–Sato polynomial.)

This was generalized to the FFRT setting as follows:

Theorem (Takagi–Takahashi 2008)

Let R be a graded domain of finite type over $R_0 = k$ a perfect field. If R has finite F-representation type, then for any $f \in R$, $R[1/f]$ is generated by $1/f$ as a $D_{R/k}$-module.

Thus, if we can show that $R[1/f]$ is not generated by $1/f$ as a D_R-module, for some f, then R cannot have FFRT.

An observation

In general, it is not easy to say when $R[1/f]$ is generated by some negative power of f.

But there is one case where it's easy:

Lemma

If R is graded, $\deg f > 0$, and D_R contains no differential operators of negative degree, then $R[1/f]$ is not generated over D_R by $1/f$.

(In fact, $R[1/f]$ can't be finitely generated over D_R at all.)

The proof is immediate once one verifies that the action of D_R on $R[1/f]$ respects the $\mathbb{Z}$-grading: $R[1/f]$ has elements of arbitrarily negative degree, so to generate by $1/f$ over D_R you have to have differential operators that decrease degree.

Negative-degree operators in characteristic p

If a graded ring R is strongly F-regular, it must have differential operators of negative degree.

It's less clear what to expect outside of the F-regular setting. We prove the following technical result:

Theorem

Let R be a Gorenstein graded ring of finite type over $R_0 = k$. Let $X = \operatorname{Proj} R$, with very ample line bundle $L = \mathcal{O}_X(1)$. If

$$H^0((\operatorname{Sym}^m \Omega_X)^\vee \otimes L^{-1}) = 0 \qquad (*)$$

for all m, then R has no differential operators of negative degree.

The sheaf $(\operatorname{Sym}^m \Omega_X)^\vee$ is equal to the m-th *divided* power of $T_X = \Omega_X^\vee$, so if T_X is suitably "nonpositive" then $(*)$ might hold.

Nonpositivity and stability of T_X

We need two properties in order to show $(*)$ holds:

1. (easy) Nonpositivity of T_X (i.e., of $c_1(T_X)$).
2. (harder) Strong semistability of T_X (equivalently, of Ω_X).

Why these properties?

- Nonpositivity of T_X says that $(\mathrm{Sym}^m \Omega_X)^\vee \otimes L^{-1}$ is also nonpositive, and thus "numerically" should have no sections.
- If a vector bundle E is nonpositive and semistable then $H^0(E \otimes L^{-1}) = 0$.
- We want to apply this to for each $E = (\mathrm{Sym}^m \Omega_X)^\vee$, and thus need all $\mathrm{Sym}^m \Omega_X$ to be semistable.
- In characteristic 0, semistability of $\mathrm{Sym}^m E$ follows from semistability of E, but not in positive characteristic.

Results on strong semistability of Ω_X

The semistability of all $\operatorname{Sym}^m \Omega_X$ follows from *strong* semistability of Ω_X:

Definition

A sheaf E is strongly semistable if $(F^e)^* E$ is semistable for any e.

If E is strongly semistable, it is a consequence that $\operatorname{Sym}^m E$ is as well for any m.

Algebraic geometers have studied the strong semistability of Ω_X:

- For non-Fano complete intersections in $\mathbb{P}^N$, work of Noma guarantees strong semistability of Ω_X.
- For Calabi–Yau varieties/K3 surfaces, work of Langer guarantees strong semistability of Ω_X if X is not uniruled/unirational.

Thus, we obtain the cases of the main theorem.

Strong semistability of Ω_X will fail for X a uniruled/unirational Calabi–Yau/K3; however, probably such examples still lack FFRT.

Fano varieties

What about Fano varieties? We've seen that quadrics and toric varieties have FFRT. These are not the only such exexamples:

Example (RŠVdB)

If R is the coordinate ring of the Grassmannian $\mathrm{Gr}(2, n)$ and $p \geq \max\{3, n-2\}$, then R has FFRT.

(This requires hard representation theory!)
Another example:

Theorem (- 2022)

Let R be the homogeneous coordinate of a quintic del Pezzo surface. Then R has FFRT.

This was the first non-toric Fano surface for which FFRT is known.

An open question

It still remains to answer whether the coordinate ring of *any* smooth Fano variety has FFRT. I suspect the answer is negative:

Conjecture

Let X be a general Fano hypersurface in $\mathbb{P}^N$ of degree $3 \leq d \leq N$. Then

$$F_*\mathcal{O}_X = \mathcal{O}_X \oplus \mathcal{O}_X(-1)^{a_1} \oplus \cdots \oplus \mathcal{O}_X(n-d)^{a_{n-d}} \oplus M,$$

and $F_*^e M$ is indecomposable for every $e \geq 0$. In particular, the coordinate ring of X does not have FFRT.

We know that the line bundle summands above do occur, and these are the only line bundle summands possible, so the content is that the "remainder" M is indecomposable, as are its pushforwards.

The simplest case of the conjecture is cubic surfaces in $\mathbb{P}^3$; even for these, we don't know if FFRT holds or not!