

# Chapter - 1

## Real Numbers

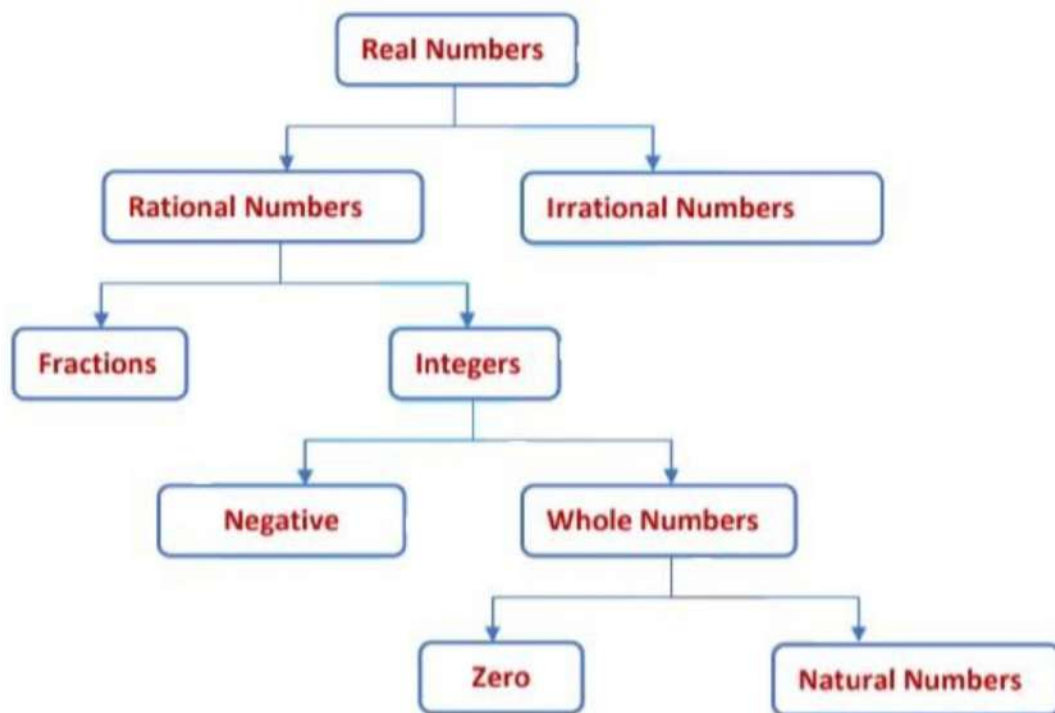
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### Euclids Division Lemma

All rational and irrational numbers are called real numbers. E.g.

3, 4,  $\sqrt{5}$ ,  $\frac{8}{9}$ . The set of real numbers is denoted by R. Real Numbers can be represented on a number line.

Classification of Real Numbers:



**Natural Numbers:** Natural Numbers is a set of counting numbers. They are denoted by N.

$$N = \{ 1, 2, 3, 4, \dots, \infty \}$$

**Whole Numbers:** Whole numbers is a set of natural numbers plus zero.

$$W = \{ 0, 1, 2, 3, \dots, \infty \}$$

**Integers:** Integers is a set of whole numbers and negative of all natural numbers.

$$\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3\}$$

**Rational Numbers:** All the numbers that can be written in the form  $\frac{p}{q}$ , where  $p$  and  $q$  are integers and  $q \neq 0$  are called rational numbers.

E.g.  $\frac{8}{11}, \frac{-3}{17}$

**Irrational Numbers:** All the numbers that cannot be written in the form  $\frac{p}{q}$ , are called irrational numbers. All the non-terminating and

non-repeating decimal numbers are irrational numbers.

E.g.  $\sqrt{5}, \sqrt{3}, \sqrt{5} + \sqrt{3}, \pi$

- Now, consider  $\sqrt{3}$ . When we calculate the value of  $\sqrt{3}$  we get  $\sqrt{3} = 1.7320508075688$  here the decimal number do not repeat and do not terminate. Hence, it is an irrational

number.

- $\sqrt{3}$  is an irrational number, but  $\sqrt{9}$  is not an irrational number because  $\sqrt{9} = 3$ , which is a rational number.

Hence, the square roots of all the numbers do not give an irrational number.

- The value of the famous irrational number  $\pi$  is 3.145926..... Again the decimal number is neither repeating nor terminating.

**Euclid's Division Lemma**

A lemma is a proven statement used for proving another statement.

Euclid's Division Lemma states that any positive integer can be divided by another positive integer in such a way that it leaves a remainder which is smaller than and this is

also known as the long division process.

According to Euclid's Division Lemma, if  $a$  and  $b$  are two positive

integers, there exist unique integers  $q$  and  $r$  which satisfies the condition,  $a = bq + r$ , where  $0 \leq r < b$

Here,  $a \rightarrow$  Dividend

$b \rightarrow$  Divisor

$q \rightarrow$  Quotient

$r \rightarrow$  Remainder

Dividend = (Divisor Quotient) Remainder.

Let us consider a pair of integers 46 and 7 If we divide 46 by 7,

$$\begin{array}{r} 6 \\ 7 \overline{) 46} \\ \underline{- 42} \\ 4 \end{array}$$

Here, 7 is the divisor, 46 is the dividend, 6 is the quotient and 4 is the remainder. Clearly, the remainder, that is 4, is smaller than the divisor, 7.

This can also be written as,

Example : Show that every positive even integer is of the form  $2q$ , and that every positive odd integer is of the form  $2q + 1$ , where  $q$  is some integer.

Let  $b$  be any positive integer and  $b = 2$

According to Euclid's division lemma there exist  $q$  and  $r$  such that

$a = bq + r$ , where  $0 \leq r < b$ .  $\rightarrow$  (i)

Here,  $b = 2$

Putting  $b = 2$  in Eq. (i) we get,

$a = 2q + r$ , where  $q$  is any integer

Now,  $0 \leq r < 2$

Therefore, the possible remainders are 0 and 1.

When  $r = 0$ ,  $a = 2q + 0$

$$a = 2q$$

When  $r = 1$ ,  $a = 2q + 1$

If  $a = 2q$ , then is an even integer.

If  $a = 2q + 1$ , then is an odd integer.

Let us consider an even positive integer, 6 and an odd positive integer, 9.

$$\text{Then, } 6 = 2 \times 3$$

$$\text{and } 9 = 2 \times 4 + 1$$

Clearly, the positive even integer is of the form and odd positive integer is of the form  $2q + 1$ .

Example: Show that any positive odd integer is of the form  $6q + 1$ ,

or  $6q + 3$  or  $6q + 5$ , where  $q$  is any integer.

Let  $a$  be any odd positive integer and  $b = 6$ .

According to Euclid's division lemma there exist  $q$  and  $r$  such that

$$a = bq + r, \text{ where } 0 \leq r < b \rightarrow (i)$$

Putting  $b = 6$  in Eq.(i) we get,

$$a = 6q + r, \text{ where } q \text{ is any integer and } 0 \leq r < 6$$

$$\text{As } 0 \leq r < 6, \therefore r = 0, 1, 2, 3, 4, 5$$

$$\text{When } r = 0, a = 6q + 0$$

$$r = 1, a = 6q + 1$$

$$r = 2, a = 6q + 2$$

$$r = 3, a = 6q + 3$$

$$r = 4, a = 6q + 4$$

$$r = 5, a = 6q + 5$$

Assuming  $q=2$ , then  $a=6q=6 \times 2=12$ , an even integer

$$a=6q+2=6 \times 2+2=14, \text{ an even integer}$$

$$a=6q+4=6 \times 2+4=16. \text{ an even integer.}$$

Since,  $a$  is an odd integer  $a \neq 6q$ ,  $a \neq 6q+2$ ,  $a \neq 6q+4$

Any positive odd integer is of the form  $6q+1, 6q+3, 6q+5$

Now, the value of  $q$  can be any integer. Let us assume that the value of  $q$  is 2.

$$\text{Then, } 6q + 1 = 6 \times 2 + 1 = 13, \text{ positive odd integer}$$

$$6q+3=6 \times 2+3=12+3=15, \text{ positive odd integer}$$

$$6q+5 = 6 \times 2 + 5 = 12 + 5 = 17, \text{ positive odd integer}$$

Example: Show that the square of any positive odd integer is of the form  $4q + 1$  for some integer  $q$ .

Let  $a$  be any positive integer

According to Euclid's division lemma there exist  $q$  and  $r$  such that

$$a = bq + r, \text{ where } 0 \leq r < b$$

Now,  $a$  is of the form  $2m + 1$ , as  $a$  is a positive odd integer.

$$\text{When } a=2m+1$$

$$a^2 = (2m + 1)^2$$

$$a^2 = 4m^2 + 4m + 1$$

$$a^2 = 4(m^2 + m) + 1 = 4q + 1$$

$$a^2 = 4q + 1, \text{ where } q = (m^2 + m)$$



Hence, for some integer , the square of any odd integer is of the form  $4m+1$

Let us assume that the value of the odd positive integer is 3

$$\text{Square of } 3 = 3^2 = 9$$

$$9 = 2 \times 4 + 1$$

$$\text{Here, } q = 2$$

Clearly, square of odd integer 3, that is 9, can be written in the form  $4q+1$

Example: Show that the cube of any positive integer is of the form  $9m$  ,  $9m+1$  or  $9m+8$ , for some integer  $m$  .

Let  $a$  be any positive integer such that it is of the form

$$3q, 3q+1, 3q+2$$

$$\text{When } a=3q$$

$$a^3 = (3q)^3 = 27q^3 = 9(3q^3) = 9m, \text{ where } m = 3q^3$$

$$\text{When } a = 3q+1$$

$$a^3 = (3q+1)^3 = 27q^3 + 27q^2 + 9q + 1$$

$$\therefore (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$a^3 = 9q(3q^2 + 3q + 1) + 1$$

$$a^3 = 9m + 1, \text{ where } m = q(3q^2 + 3q + 1)$$

$$\text{When } a=3q+2$$

$$a^3 = (3q+2)^3 = 27q^3 + 54q^2 + 36q + 8$$

$$a^3 = 9q(3q^2 + 6q + 4) + 8$$

$$a^3 = 9m + 8, \text{ where } m = q(3q^2 + 6q + 4)$$

Hence  $a^3$  is of the form  $9m$ ,  $9m+1$  or  $9m+8$

Let us consider positive integers 3,4 and 5

Cube of 3 =  $3^3 = 27 = 9 \times 3$

Cube of 4 =  $4^3 = 64 = 9 \times 7 + 1$

Cube of 5 =  $5^3 = 125 = 9 \times 13 + 8$

So, the cube of 3 is of the form  $9m$

The cube of 4 is of the form  $9m + 1$

The cube of 5 is of the form  $9m + 8$

Euclid's Division Algorithm

An algorithm is a series of well-defined steps which gives a procedure to solve a type of problem.

Euclid's division Algorithm is a technique used to compute the Highest Common Factor (HCF) of two given positive integers. HCF of two positive integers and is the largest

positive integer that divides both and .

To obtain the HCF of two positive integers and , where , we use Euclid's division Algorithm.

Step 1 : Apply Euclid's division Lemma, to a and b to find whole numbers q and r such that ,  $a = bq + r$ ,  $0 \leq r < b$

Step 2 : If  $r = 0$ , b is the HCF of a and b . If  $r \neq 0$  then apply division lemma to b and r .

Step 3: Continue the process till the remainder is zero. The divisor at this stage will be the required HCF.

Example: Use Euclid's division algorithm, to find the HCF of 870 and 225.

Given numbers are 870 and 225.

Here,  $870 > 225$  . By using Euclid's Division Lemma,  $a = bq + r$ ,  $0 \leq r < b$  ,

we get,

$$870 = 225 \times 3 + 195$$

Now,  $r = 195 \neq 0$ .

We consider new divisor 225 and new remainder 195 and apply the division lemma to get,

$$225 = 195 \times 1 + 30$$

Again  $r = 30 \neq 0$ .

We consider new divisor 195 and new remainder 30 and apply the division lemma to get,.

$$195 = 30 \times 6 + 15$$

Again  $r = 15 \neq 0$ .

We consider new divisor 30 and new remainder 15 and apply the division lemma to get,

$$30 = 15 \times 2 + 0$$

Now, the remainder is equal to zero, we stop our procedure, the divisor at this stage is 15. Hence the HCF of 870 and 225 is 15.

Example : Use Euclid's division algorithm, to find the HCF of 176 and 38220.

Given numbers are 176 and 38220.

Here,  $38220 > 176$ . By using Euclid's Division Lemma,  $a = bq + r$ ,  $0 \leq r < b$ , we get,

$$38220 = 176 \times 217 + 28$$

Now,  $r = 28 \neq 0$ .

We consider new divisor 176 and new remainder 28 and apply the division lemma to get,

$$176 = 28 \times 6 + 8$$



Again  $r = 8 \neq 0$ .

We consider new divisor 28 and new remainder 8 and apply the division lemma to get,

$$28 = 8 \times 3 + 4$$

$$r = 4 \neq 0,$$

We consider new divisor 8 and new remainder 4 and apply the division lemma to get,

$$8 = 4 \times 2 + 0$$

Now, the remainder is equal to zero, we stop our procedure, the divisor at this stage is 4. Hence the HCF of 176 and 38220 is 4.

Example: Find the HCF of 180, 252 and 324 by using Euclid's division Lemma.

Given numbers are 180, 252 and 324.

Now,  $324 > 252 > 180$

We will first find the HCF of 324 and 252

By using Euclid's Division Lemma,  $a = bq + r$ ,  $0 \leq r < b$ , where we get,

$$324 = 252 \times 1 + 72$$

Now,  $r = 72 \neq 0$ .

We consider now new divisor 252 and new remainder 72 and apply the division lemma to get,

$$252 = 72 \times 3 + 36$$

Again  $r = 36 \neq 0$ .

We consider new divisor 72 and new remainder 36 and apply the division lemma to get,

$$72 = 36 \times 2 + 0.$$

The remainder is equal to zero, the divisor at this stage is 36. Hence the HCF of 324 and 252 is 36.

Now, we will use Euclid's Division Lemma for 180 and 36 to get,

$$180 = 36 \times 5 + 0$$

The HCF of 180 and 36 is 36.

Hence the HCF of 324, 252 and 180 is 36.

Example : Two tanks contain 504 L and 735 L of milk, respectively. Find the maximum capacity of a container which can measure the milk of either tank in exact number of

times.

The capacities of the two tanks are 504L and 735L.

The maximum capacity of a container will be the HCF of 504 and 735.

$$735 > 504$$

By using Euclid's Division Lemma we get,

$$735 = 504 \times 1 + 231$$

Now,  $r = 231 \neq 0$ .

We will consider new divisor 504 and new remainder 231 and apply Euclid's Division Lemma to get,

$$504 = 231 \times 2 + 42$$

Again,  $r = 42 \neq 0$ .

We now consider new divisor 231 and new remainder 42, and apply Euclid's Division Lemma to get,

$$231 = 42 \times 5 + 21$$

$$r = 21 \neq 0.$$

We now consider new divisor 42 and new remainder 21, and apply Euclid's Division Lemma to get,

$$42 = 21 \times 2 + 0$$

The remainder is equal to zero, the divisor at this stage is 21. Hence the HCF of 504 and 735 is 21.

Therefore, the maximum capacity of the required container is 21L.

Example : Three pieces of timber 42 m, 49 m and 56 m long have to be divided into planks of the same length. What is the greatest possible length of each plank?

The greatest possible length of the plank will be the HCF of 42m, 49 m and 56 m.

Now,  $56 > 49 > 42$

We will first find the HCF of 56 and 49.

By using Euclid's Division Lemma we get,

$$56 = 49 \times 1 + 7$$

$$\text{Now, } r = 7 \neq 0.$$

We will consider new divisor 49 and new remainder 7 and apply Euclid's Division Lemma to get,

$$49 = 7 \times 7 + 0$$

The remainder is equal to zero, the divisor at this stage is 7. Hence the HCF of 56 and 49 is 7.

Now, using Euclid's Division Lemma for 42 and 7 we get,

$$42 = 7 \times 6 + 0$$

As the remainder is 0, the divisor in this case is the HCF of 42 and 7.

Therefore, the HCF of 42 and 7 is 7.

The HCF of and is 7.

Hence the greatest possible length of the plank is 7m

### **Fundamental Theorem of Arithmetic**

#### **Composite Number**

Composite Numbers are those numbers that have at least one factor other than 1 and the number itself.

Consider a number, 10. Now the factors of 10 are 1, 2, 5 and 10. So it is a composite number.

#### **Prime Number**

Prime Numbers are those numbers which have exactly two factors, 1 and the number itself.

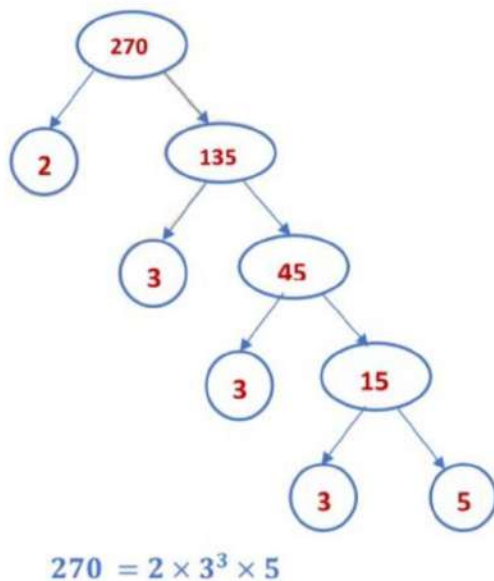
Let us take one more number 23. Now the factors of 23 are 1 and 23. That means it has two factors 1 and the number itself, so it is called a prime number.

Fundamental Theorem of Arithmetic states that every composite number can be expressed as a product of primes, and this factorization is unique, apart from the order in

which the prime factors occur.

Any composite number can be written as a product of primes in one way only as long as we are not particular about the order in which the primes occur.

Let us consider an example here, we will use a tree diagram to show the factors of 270.



$$270 = 5 \times 3^3 \times 2$$

Here, in the prime factorization of 270, the prime numbers appearing in both the cases are the same only the order in which they appear are different.

Therefore, the prime factorization of 270 is unique except for the order in which the primes occur.

Example: Check whether  $15^n$  can end with digit zero for any natural number  $n$ .

If a number ends with the digit 0, then it is divisible by both 2 and 5.

But prime factors of 15 are 3 and 5.

$$15^n = (3 \times 5)^n = 3^n \times 5^n$$

Here, the prime factorization of  $15^n$  contains only 5 but not 2.

The uniqueness of the Fundamental Theorem of Arithmetic guarantees that there are no other primes in the prime factorization of  $15^n$ .

Therefore,  $15^n$  cannot end with digit zero for any natural number  $n$ .

Example: Explain, why  $(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) + 5$  and  $(3 \times 5 \times 13 \times 46) + 23$  is a composite number?



Composite numbers are those numbers that have at least one factor other than 1 and the number itself. E.g.4, 6, 9

$$\text{i) } (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) + 5 = 5040 + 5 = 5045$$

$$5045 = 5 \times 1009$$

As the factors of 5045 are 5 and 1009, it is a composite number.

$$\text{ii) } (3 \times 5 \times 13 \times 46) + 23 = 8970 + 23 = 8993$$

$$8993 = 17 \times 23 \times 23 \times 2$$

As the factors of 8993 are 17, 23 and 2, it is a composite number.

#### HCF and LCM by Prime Factorisation Method

In this method, we first express the given numbers as a product of prime factors separately. Then, HCF is the product of the smaller power of each common prime factor in the numbers and LCM is the product of the greatest power of each prime factor involved in the numbers.

For any two positive integers a and b,

$$\text{HCF}(a, b) \times \text{LCM}(a, b) = a \times b$$

Example: Find the LCM and HCF of 120 and 144 by the fundamental theorem of arithmetic.

$$120 = 2^3 \times 3 \times 5$$

$$144 = 2^4 \times 3^2$$

Now, HCF is the product of the smallest power of each common prime factor in the numbers

Common Prime Factors	Smallest Power of Prime Factor
2	$2^3$
3	$3^1$

$$\text{HCF}(120, 144) = 2^3 \times 3 = 8 \times 3 = 24$$

Prime Factors of 120 and 144	Greatest Power of Prime Factor
2	$2^4$
3	$3^2$
5	$5^1$

LCM is the product of the greatest power of each prime factor involved in the numbers.

$$\text{LCM}(120, 144) = 2^4 \times 3^2 \times 5 = 16 \times 9 \times 5 = 720$$

Example: If two positive integers  $p$  and  $q$  can be expressed as  $p = ab^2$  and  $q = a^3b$ , where  $a, b$  are prime numbers, find the  $\text{LCM}(p, q)$ .

Given:  $p = ab^2$  and  $q = a^3b$

Prime Factors of $p$ and $q$	Greatest Power of Prime Factor
$a$	$a^3$
$b$	$b^2$

LCM is the product of the greatest power of each prime factor involved in the numbers.

$$\text{LCM}(p, q) = a^3 \times b^2 = a^3b^2$$

Example: Write the HCF and LCM of the smallest odd composite number and the smallest odd prime number.

Smallest odd composite number is 9 and the smallest odd prime number is 3.

$$9 = 3^2$$

$$3 = 3^1$$

Now, the smallest power of the common prime factor is  $3^1$ .

$$\text{HCF}(9, 3) = 3$$

The greatest power of the common prime factor is  $3^2$ .

$$\text{LCM}(9, 3) = 3^2 = 9$$

Example: If  $\text{HCF}(253, 440) = 11$  and  $\text{LCM}(253, 440) = 253 \times R$ . Find the value of R.

We know that,

$$\text{HCF}(a, b) \times \text{LCM}(a, b) = a \times b$$

$$\therefore \text{HCF}(253, 440) \times \text{LCM}(253, 440) = 253 \times 440$$

$$11 \times 253 \times R = 253 \times 440$$

$$R = \frac{253 \times 440}{253 \times 11}$$

$$R = 40$$

Example: Ravi and Shikha drive around a circular sports field. Ravi takes 16 min to complete one round, while Shikha completes the round in 20 min. If both start at the same point, at the same time and go in the same direction, then after how much time will they meet at the starting point?

Time taken by Ravi to drive one round of the circular field = 16 min  
Time taken by Shikha to drive one round of the circular field = 20 min

The time after which they will again meet at the starting point will be equal to the LCM of 16 min and 20 min.

$$16 = 2^4$$

$$20 = 2^2 \times 5$$

$$\text{LCM}(16, 20) = 2^4 \times 5 = 16 \times 5 = 80$$

Therefore, Ravi and Shikha will meet again at the starting point after 80 min.

### Revisiting Irrational Numbers

Revisiting Irrational Numbers

Irrational numbers are those numbers which cannot be written in the form  $\frac{p}{q}$ , where p and q are integers and  $q \neq 0$ . E.g.  $\sqrt{2}, \sqrt{3}, \sqrt{15}$

The square roots of all the numbers do not give an irrational number.

For example,  $\sqrt{2}$  is an irrational number but  $\sqrt{4} = 2$ , which is rational.

Therefore square roots of all prime numbers are irrational.

If  $p$  is a prime number then  $\sqrt{p}$  is an irrational number.

Theorem 1:

If a prime number  $p$  divides  $a^2$ , then  $p$  divides  $a$ , where  $a$  is a positive integer.

Proof:

Every positive integer can be expressed as the product of primes.

Let  $a = p_1 p_2 p_3 \dots p_n$  where  $p_1 p_2 p_3 \dots p_n$  are all the prime numbers of  $a$ .

$$a^2 = (p_1 p_2 p_3 \dots p_n)$$

$$a^2 = (p_1 p_2 p_3 \dots p_n)(p_1 p_2 p_3 \dots p_n)$$

$$a^2 = (p_1^2 p_2^2 p_3^2 \dots p_n^2)$$

It is given that  $p$  divides  $a^2$ . According to the Fundamental Theorem of Arithmetic, we can say that  $p$  is one of the prime factors of  $a^2$ .

According to the Fundamental Theorem of Arithmetic, the prime factorization of a natural number is unique.

Now the only prime factors of  $a^2$  are  $p_1 p_2 p_3 \dots p_n$ . Therefore,  $p$  is one of  $p_1 p_2 p_3 \dots p_n$ .

So  $p$  is also a factor of  $a$

If  $p$  divides  $a^2$  then  $p$  also divides  $a$ .

Let us consider a positive integer 12.

Now, the factors of 12 are 2, 2 and 3.

On squaring 12 we get,

$$12^2 = 144$$



The factors of 144 are 2, 2, 2, 2, 3 and 3.

If one of the factors, let's say 3 divides 144. Then this factor, 3 will also divide 144, as it is one of the factors of 144 also.

Example: Prove that  $\sqrt{2}$  is an irrational number.

Let us assume that  $\sqrt{2}$  is rational. Since it is a rational number it can be expressed in the form  $\frac{a}{b}$ , where a and b are integers and  $b \neq 0$ .

Now, a and b have no common factor other than 1.

$\sqrt{2} = \frac{a}{b}$ , where a and b are coprime

On squaring both sides, we get

$$2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

Therefore, 2 divides  $a^2$ .

We know that when 2 divides  $a^2$  then 2 divides a also.

We can write  $a = 2m$ , where m is an integer.

Putting  $a = 2m$  in  $2b^2 = a^2$

$$2b^2 = (2m)^2$$

$$2b^2 = 4m^2$$

$$b^2 = 2m^2$$

Again if 2 divides  $b^2$ , then 2 divides b also.

Therefore, 2 is a common factor of a and b.

But this contradicts the fact that a, b have no common factor other than 1.

So, we conclude that  $\sqrt{2}$  is irrational.



Example: Show that  $3\sqrt{2}$  is an irrational number.

Let us assume that  $3\sqrt{2}$  is rational.

Then  $3\sqrt{2}$  can be expressed in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b \neq 0$ .  
Now,  $a$  and  $b$  have no common factor other than 1.

$$3\sqrt{2} = \frac{a}{b}, \text{ where } a \text{ and } b \text{ are coprime integers}$$

$$\sqrt{2} = \frac{a}{3b}$$

Since, 3,  $a$  and  $b$  are integers,  $\frac{a}{3b}$  is rational.

Now, a rational number cannot be equal to an irrational number, that is  $\sqrt{2}$

So, we conclude that  $3\sqrt{2}$  is irrational.

Example: Show that  $5 - \sqrt{3}$  is irrational.

Let us assume that  $5 - \sqrt{3}$  is rational. As it is a rational number, it can be

expressed in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers and  $b \neq 0$ . Now,  $a$  and  $b$  have no common factor other than 1.

$$5 - \sqrt{3} = \frac{a}{b}, \text{ where } a \text{ and } b \text{ are coprime}$$

$$5 - \frac{a}{b} = \sqrt{3}$$

$$\frac{5b - a}{b} = \sqrt{3}$$

As, 5,  $a$ , and  $b$  are integers,  $\frac{5b - a}{b}$  is rational.

Now,  $\sqrt{3}$  is an irrational number and cannot be equal to a rational number.

So, we conclude that  $5 - \sqrt{3}$  is irrational.

Example: Prove that  $2\sqrt{3} + \sqrt{5}$  is an irrational number.

Let us assume that  $2\sqrt{3} + \sqrt{5}$  is rational. As it is a rational number, it can be expressed in the form  $\frac{a}{b}$  where a and b are integers and  $b \neq 0$ . Now, a and b have no common factor other than 1.

$$2\sqrt{3} + \sqrt{5} = \frac{a}{b}, \text{ where a and b are coprime}$$

$$2\sqrt{3} = \frac{a}{b} - \sqrt{5}$$

On squaring both sides, we get

$$12 = \left(\frac{a}{b} - \sqrt{5}\right)^2$$

$$12 = \frac{a^2}{b^2} - 2\sqrt{5}\frac{a}{b} + 5$$

$$\frac{a^2}{b^2} - 2\sqrt{5}\frac{a}{b} = 7$$

$$\frac{a^2}{b^2} - 7 = 2\sqrt{5}\frac{a}{b}$$

$$\frac{a^2 - 7b^2}{2b^2} = \sqrt{5}\frac{a}{b}$$

$$\sqrt{5} = \frac{a^2 - 7b^2}{2ab}$$

Since, 2, 7, a and b are integers  $\frac{a^2 - 7b^2}{2ab}$  is rational

A rational number cannot be equal to an irrational number.

So, our assumption is wrong. Hence,  $2\sqrt{3} + \sqrt{5}$  is irrational.

### Revisiting rational numbers and their decimal expansion

Revisiting Rational Numbers

Rational Numbers and their Decimal Expansion

The decimal expansion of every rational number is either terminating or non-terminating repeating.

### Terminating Decimal Expansion

The number which terminates after a finite number of steps in the process of division is called terminating decimal expansion. E.g. 6.25, 1.14

### Non-Terminating Decimal Expansion

The number which does not terminate in the process of division is called non-terminating decimal expansion.

There are two types of non-terminating decimal expansions

#### a) Non-terminating repeating Decimal Expansion

The number which does not terminate but repeats the particular number again and again in the process of division is said to be non-terminating repeating decimal.

E.g. 0.333333

#### b) Non-terminating Non-repeating Decimal Expansion

The number which neither terminates nor repeats the particular number in the process of division is said to be non-terminating repeating decimal. E.g. 1.03303033

### Theorem 1:

Let  $x$  be a rational number whose decimal expansion terminates.

Then  $x$  can be expressed in the form  $\frac{p}{q}$ , where  $p$  and  $q$  are coprime and the prime factorization of  $q$  is of the form  $2^m 5^n$ , where  $m, n$  are non-negative integers.

### Theorem 2:

Let  $x = \frac{p}{q}$  be a rational number, such that the prime factorization of  $q$  is

of the form  $2^m 5^n$ , where  $m, n$  are non-negative integers, then  $\frac{p}{q}$  has a terminating decimal.

$$\frac{23}{80} = \frac{23}{2^4 \times 5}$$

Multiplying both numerator and denominator by  $5^3$

$$= \frac{23 \times 5^3}{2^4 \times 5 \times 5^3} = \frac{23 \times 5^3}{2^4 \times 5^4}$$

$$\frac{2875}{(2 \times 5)^4} = \frac{2875}{(10)^4} = \frac{2875}{10000} = 0.2875$$

Theorem 3:

Let  $x = \frac{p}{q}$  be a rational number, such that the prime factorization of  $q$  is not of the form  $2^m 5^n$ , where  $m, n$  are non-negative integers, then  $x$  has decimal expansion which is non-terminating repeating.

Example:

Without actually performing the long division, state whether  $\frac{543}{225}$  has a terminating decimal expansion or non-terminating recurring decimal expansion.

$$\frac{543}{225} = \frac{181}{75} = \frac{181}{3 \times 5^2}$$

Since the factors of the denominator  $(3 \cdot 5^2)$  is not of the form  $2^m 5^n$ .

So,  $\frac{543}{225}$  is non-terminating recurring decimal expansion.

Example:

The decimal expansion of the rational number  $\frac{53}{2^4 \times 5^3}$  will terminate after how many places of decimal.

$$\frac{53}{2^4 \times 5^3}$$

Now, multiplying both numerator and denominator by 5

$$\begin{aligned}\frac{23 \times 5}{2^4 \times 5^3 \times 5} &= \frac{265}{2^4 \times 5^4} \\ &= \frac{265}{(2 \times 5)^4} = \frac{265}{10^4} = \frac{265}{10000} = 0.0265\end{aligned}$$

The decimal expansion of the rational number  $\frac{53}{2^4 \times 5^3}$  will terminate after 4 places of decimal.

Example: If  $\frac{299}{4000} = \frac{299}{2^m 5^n}$ , then find the value of m and n, where m and n are non-negative integers. Hence, write its decimal expansion without actual division.

$$\frac{299}{4000} = \frac{299}{2^m 5^n}$$

$$\frac{299}{4000} = \frac{299}{2^5 5^3}$$

$$\frac{299}{2^m 5^n} = \frac{299}{2^5 \times 5^3}$$

$$m = 5 \text{ and } n = 3$$

Now,  $\frac{299}{4000}$  can be expressed as  $\frac{299}{2^5 \times 5^3}$

We have,  $\frac{299}{2^5 \times 5^3}$

Multiplying both numerator and denominator by  $5^2$

$$\frac{299 \times 5^2}{2^5 \times 5^3 \times 5^2} = \frac{7475}{2^5 \times 5^5}$$



$$= \frac{7475}{(2 \times 5)^5} = \frac{7475}{10^5} = \frac{7475}{100000} = 0.07475$$

Example:

What can you say about the prime factorization of the denominators of the following rational:

i) 42.123456789

ii)  $\overline{32.56789}$

i) Since 42.123456789 has a terminating decimal expansion. So, its denominator is of the form  $2^m 5^n$ , where m and n are non-negative integers.

ii) Since  $\overline{32.5678}$  has non-terminating decimal expansion. So, its denominator has factors other than 2 or 5.