a. Given this universal relation, create and populate with data the three relation tables for the three entities described in Exercise 29. Underline the primary key in each table.

b. Describe any foreign keys in the relation tables.

c. Consider the form of the Employee IDs. This is probably what kind of key?

32. a. If Mary Black moves from the Accounting Department to the Sales Department, how many tuples must be updated in the universal relation?

b. The three relation tables from Exercise 31 should follow the "one fact, one place" rule (see Exercise 26). With the same change to Mary Black's department, how many tuples must be updated in the database using the three relation tables of Exercise 31?

Exercises 33–36 make use of the three relation tables from Exercise 31.

33. Write an SQL query to give the employee ID, pay dates, and payment amounts for all pay dates with amounts $>$ \$100. Give the result of the query.

34. Write an SQL query to give the contribution ID, pay date, and payment amount for all payments by Mary Black. Give the result of the query.

35. Write an SQL query to give the first and last names and payment amount of all employees who had a payroll deduction on 1/15/2013. Give the result of the query.

36. Write an SQL query to reproduce the universal relation of Exercise 31 from the three relation tables.

SECTION 5.4 FUNCTIONS

In this section we discuss functions, which are really special cases of binary relations from a set S to a set T. This view of a function is a rather sophisticated one, however, and we will work up to it gradually.

Definition

Function is a common enough word even in nontechnical contexts. A newspaper may have an article on how starting salaries for this year's college graduates have increased over those for last year's graduates. The article might say something like, "The salary increase varies depending on the degree program," or, "The salary increase is a function of the degree program." It may illustrate this functional relationship with a graph like Figure 5.11. The graph shows that each degree program has some figure for the salary increase associated with it, that no degree program has more than one figure associated with it, and that both the physical sciences and the liberal arts have the same figure, 1.5%.

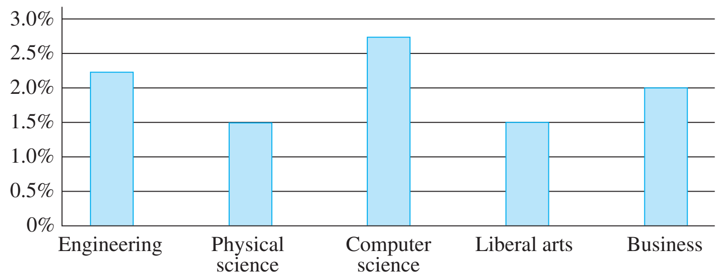


Figure 5.11

Of course, we also use mathematical functions in algebra and calculus. The equation $g(x) = x^3$ expresses a functional relationship between a value for x and the corresponding value that results when the value for x is used in the equation. Thus an x value of 2 has the number $2^3 = 8$ associated with it. (This number is expressed as $g(2) = 8$.) Similarly, $g(1) = 1^3 = 1$, $g(-1) = (-1)^3 = -1$, and so on. For each x value, the corresponding $g(x)$ value is unique. If we were to graph this function on a rectangular coordinate system, the points (2, 8), (1, 1), and $(-1, -1)$ would be points on the graph. If we allow x to take on any real number value, the resulting graph is the continuous curve shown in Figure 5.12.

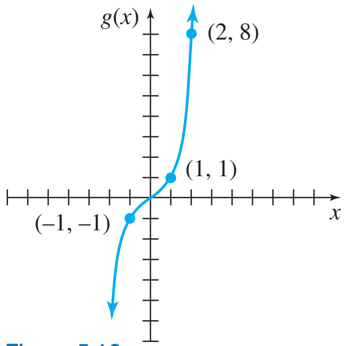


Figure 5.12

The function in the salary increase example could be described as follows. We set the stage by the diagram in Figure 5.13, which indicates that the function always starts with a given degree program and that a particular salary increase is associated with that degree program. The association itself is described by the set of ordered pairs {(engineering, 2. 25%), (physical sciences, 1. 5%), (computer science, 2. 75%), (liberal arts, 1. 5%), (business, 2. 0%)}.

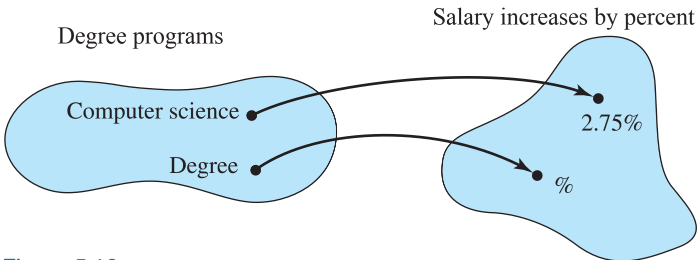


Figure 5.13

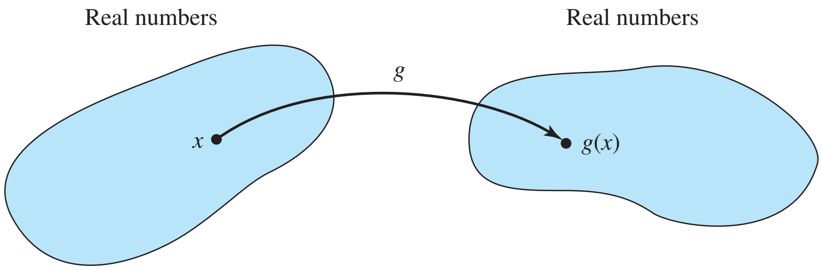


Figure 5.14

For the algebraic example $g(x) = x^3$, Figure 5.14 shows that the function always starts with a given real number and associates a second real number with it.

The association itself is described by $\{(x, g(x)) | g(x) = x^3\}$, or simply $g(x) = x^3$. This set includes (2, 8), (1, 1), $(-1, -1)$, but because it is an infinite set, we cannot list all its members; we have to describe them.

From the above examples, we can conclude that there are three parts to a function: (1) a set of starting values, (2) a set from which associated values come, and (3) the association itself. The set of starting values is called the *domain* of the function, and the set from which associated values come is called the *codomain* of the function. Thus both the domain and codomain represent pools from which values may be chosen. (This usage is consistent with our use of the word *domain* when discussing predicate wffs in Section 1.2. There the domain of an interpretation is a pool of values that variables can assume and to which constant symbols may be assigned. Similarly, the domain D_i of an attribute A_i in a database relation, discussed in Section 5.3, is a pool of potential values for the attribute.)

The picture for an arbitrary function f is shown in Figure 5.15. Here f is a function from S to T, symbolized $f: S \to T$. S is the domain and T is the codomain. The association itself is a set of ordered pairs, each of the form (s, t) where $s \in S$, $t \in T$, and t is the value from T that the function associates with the value s from S; $t = f(s)$. Hence, the association is a subset of $S \times T$ (a binary relation from S to T). But the important property of this relation is that every member of S must have one and only one T value associated with it, so every $s \in S$ will appear exactly once as the first component of an (s, t) pair. (This property does not prevent a given T value from appearing more than once.)

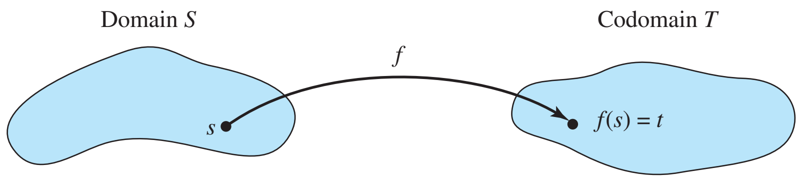


Figure 5.15

We are now ready for the formal definition of a function.

DEFINITIONS **TERMINOLOGY FOR FUNCTIONS**

Let S and T be sets. A **function** (**mapping**) f from S to T, $f: S \to T$, is a subset of $S \times T$ where each member of S appears exactly once as the first component of an ordered pair. S is the **domain** and T is the **codomain** of the function. If (s, t) belongs to the function, then t is denoted by $f(s)$; t is the **image** of s under f, s is a **preimage** of t under f, and f is said to map s to t. For $A \subseteq S$, $f(A)$ denotes $\{f(a) | a \in A\}$.

A function from S to T is a subset of $S \times T$ with certain restrictions on the ordered pairs it contains. That is why we spoke of a function as a special kind of binary relation. By the definition of a function, a binary relation that is one-to-many (or many-to-many) cannot be a function. Also, each member of S must be used as a first component.

We have talked a lot about values from the sets S and T, but as our example of salary increases shows, these values are not necessarily numbers, nor is the association itself necessarily described by an equation.

PRACTICE 23 Which of the following formulas are functions from the domain to the codomain indicated? For those that are not, why not?

a. $f: S \to T$ where $S = T = \{1, 2, 3\}$, $f = \{(1, 1), (2, 3), (3, 1), (2, 1)\}$
b. $g: \mathbb{Z} \to \mathbb{N}$ where g is defined by $g(x) = |x|$ (the absolute value of x)
c. $h: \mathbb{N} \to \mathbb{N}$ where h is defined by $h(x) = x - 4$
d. $f: S \to T$ where S is the set of all people in your hometown, T is the set of all automobiles, and f associates with each person the automobile that person owns
e. $g: S \to T$ where $S = \{2013, 2014, 2015, 2016\}$, T = {\$20,000, \$30,000, \$40,000, \$50,000, \$60,000}, and g is defined by the graph in Figure 5.16.
f. $h: S \to T$ where S is the set of all quadratic polynomials in x with integer coefficients, $T = \mathbb{Z}$, and h is defined by $h(ax^2 + bx + c) = b + c$
g. f: $\mathbb{R} \to \mathbb{R}$ where f is defined by $f(x) = 4x - 1$
h. $g: \mathbb{N} \to \mathbb{N}$ where g is defined by

$$g(x) = \begin{cases} x + 3 & \text{if } x \geq 5 \\ x & \text{if } x \leq 5 \end{cases}$$

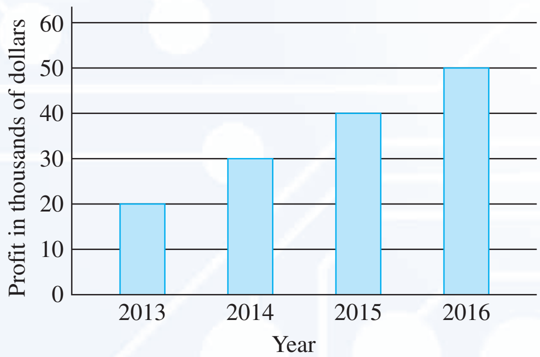


Figure 5.16 Profits of the American Earthworm Corp. ■

PRACTICE 24 Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$.

a. What is the image of -4?
b. What are the preimages of 9? ■

EXAMPLE 27 When we studied recursive definitions in Section 3.1, we talked about sequences, where a sequence S was written as

$$S(1), S(2), S(3), \ldots$$

Changing the notation to

$$f(1), f(2), f(3), \ldots$$

we see that a sequence is nothing but a list of functional values for a function f whose domain is the positive integers, and this is how a sequence is often defined.

Indeed, the algorithms we gave for computing the values in such sequences were pseudocode that computes the function.

Also in Section 3.1, we talked about recursive operations such as a^n where a is a fixed nonzero real number and $n \geq 0$. This is also simply a function $f(n) = a^n$ whose domain is $\mathbb{N}$.

The definition of a function includes functions of more than one variable. We can have a function $f: S_1 \times S_2 \times \cdots \times S_n \to T$ that associates with each ordered n-tuple of elements $(s_1, s_2, \ldots, s_n)$, $s_i \in S_i$, a unique element of T.

EXAMPLE 28 $f: \mathbb{Z} \times \mathbb{N} \times \{1, 2\} \to \mathbb{Z}$ is given by $f(x, y, z) = x^y + z$. Then $f(-4, 3, 1) = (-4)^3 + 1 = -64 + 1 = -63$.

EXAMPLE 29 In Section 4.1 we defined a unary operation on a set S as associating a unique member of S, $x^{\#}$, with each member x of S. This means that a unary operation on S is a function with domain and codomain S. We also defined a binary operation $\circ$ on a set S as associating a unique member of S, $x \circ y$, with every (x, y) pair of elements of S. Therefore a binary operation on S is a function with domain $S \times S$ and codomain S.

Again, domain values and codomain values are not always numbers.

EXAMPLE 30 Let S be the set of all character strings of finite length. Then the association that pairs each string with the number of characters in the string is a function with domain S and codomain $\mathbb{N}$ (we allow the "empty string," which has zero characters).

EXAMPLE 31 Any propositional wff with n statement letters defines a function with domain $\{T, F\}^n$ and codomain $\{T, F\}$. The domain consists of all n-tuples of T-F values; with each n-tuple is associated a single value of T or F. The truth table for the wff gives the association. For example, if the wff is $A \vee B'$, then the truth table

A	B	B'	$A \vee B'$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

says that the image of the 2-tuple (F, T) under this function is F. If we call this function f, then $f(\text{F}, \text{T}) = \text{F}$.

PRACTICE 25 Let the function defined by the wff $A \wedge (B \vee C')$ be denoted by f. What is $f(\mathrm{T}, \mathrm{T}, \mathrm{F})$? What is $f(\mathrm{F}, \mathrm{T}, \mathrm{F})$? ■

The next example defines two functions that are sometimes useful in analyzing algorithms.

EXAMPLE 32 The **floor function** $\lfloor x \rfloor$ associates with each real number x the greatest integer less than or equal to x. The **ceiling function** $\lceil x \rceil$ associates with each real number x the smallest integer greater than or equal to x. Thus $\lfloor 2.8 \rfloor = 2$, $\lceil 2.8 \rceil = 3$, $\lfloor -4.1 \rfloor = -5$, and $\lceil -4.1 \rceil = -4$. Both the floor function and the ceiling function are functions from $\mathbb{R}$ to $\mathbb{Z}$. ●

PRACTICE 26

a. Sketch a graph of the function $\lfloor x \rfloor$.
b. Sketch a graph of the function $\lceil x \rceil$. ■

EXAMPLE 33 For any integer x and any positive integer n, the **modulo function**, denoted by $f(x) = x \bmod n$, associates with x the nonnegative remainder when x is divided by n. We can write x as $x = qn + r$, $0 \le r < n$, where q is the quotient and r is the remainder, so the value of $x \bmod n$ is r.

$25 = 12 \cdot 2 + 1$ so $25 \bmod 2 = 1$

$21 = 3 \cdot 7 + 0$ so $21 \bmod 7 = 0$

$15 = 3 \cdot 4 + 3$ so $15 \bmod 4 = 3$

$-17 = (-4) \cdot 5 + 3$ so $-17 \bmod 5 = 3$ (it is true that $-17 = (-3)5 + (-2)$ but remember that the remainder must be nonnegative)

Section 5.6 discusses some of the many applications of the modulo function. ●

The definition of a function $f: S \to T$ includes three parts—the domain set S, the codomain set T, and the association itself. Is all this necessary? Why can't we simply write an equation, like $g(x) = x^3$, to define a function?

The quickest answer is that not all functional associations can be described by an equation (see Example 30, for instance). But there is more to it—let's limit our attention to situations where an equation can be used to describe the association, such as $g: \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3$. Even in algebra and calculus, it is common to say "consider the function $g(x) = x^3$," implying that the equation *is* the function. Technically, the equation only describes a way to compute associated values. The function $h: \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3 - 3x + 3(x + 5) - 15$ is the same function as g because it contains the same ordered pairs. However, the equation is different in that it says to process any given x value differently.

On the other hand, the function $f: \mathbb{Z} \to \mathbb{R}$ given by $f(x) = x^3$ is not the same function as g. The domain has been changed, which changes the set of ordered pairs. The graph of $f(x)$ would consist of discrete (separate) points (Figure 5.17). Most of the functions in which we are interested have this feature. Even in situations where one quantity varies continuously with another, in a digital computer we approximate by taking data at discrete, small intervals, much as the graph of $g(x)$ (see Figure 5.12) is approximated by the graph of $f(x)$ (see Figure 5.17).

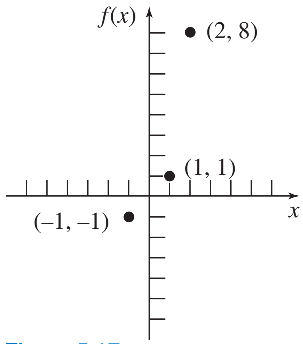


Figure 5.17

Finally, let's look at the function $k: \mathbb{R} \to \mathbb{C}$ given by $k(x) = x^3$. The equation and domain is the same as for $g(x)$; the codomain has been enlarged, but the change does not affect the ordered pairs. Is this function considered the same function as $g(x)$? It is not, but to see why, we'll have to wait until we discuss the *onto* property of functions. Then we will see that g has the onto property while k does not, so we do not want to consider them the same function.

In summary, a complete definition of a function requires giving its domain, its codomain, and the association, where the association may be given by a verbal description, a graph, an equation, or a collection of ordered pairs.

DEFINITION **EQUAL FUNCTIONS**

Two functions are **equal** if they have the same domain, the same codomain, and the same association of values of the codomain with values of the domain.

Suppose we are trying to show that two functions with the same domain and the same codomain are equal. Then we must show that the associations are the same. This can be done by showing that, given an arbitrary element of the domain, both functions produce the same associated value for that element; that is, they map it to the same place.

PRACTICE 27 Let $S = \{1, 2, 3\}$ and $T = \{1, 4, 9\}$. The function $f: S \to T$ is defined by $f = \{(1, 1), (2, 4), (3, 9)\}$. The function $g: S \to T$ is defined by the equation

$$g(n) = \frac{\sum_{k=1}^{n} (4k - 2)}{2}$$

Prove that $f = g$. ■

Properties of Functions

Onto Functions

Let $f: S \to T$ be an arbitrary function with domain S and codomain T (Figure 5.18). Part of the definition of a function is that every member of S has an image under f and that all the images are members of T; the set R of all such images is called the **range** of the function f. Thus, $R = \{f(s) | s \in S\}$, or $R = f(S)$. Clearly, $R \subseteq T$; the range R is shaded in Figure 5.19. If it should happen that $R = T$, that is, that the range coincides with the codomain, then the function is called an *onto* function.

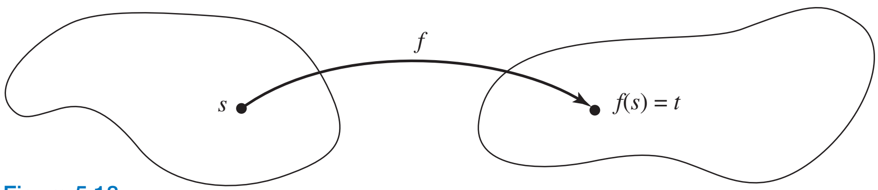


Figure 5.18

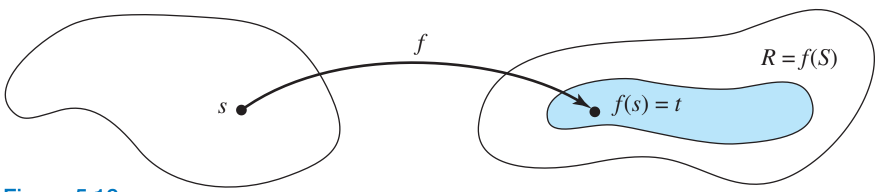


Figure 5.19

DEFINITION ONTO (SURJECTIVE) FUNCTION

A function $f: S \to T$ is an **onto** or **surjective** function if the range of f equals the codomain of f.

REMINDER

To show that a function is onto, pick an arbitrary element in the codomain and show that it has a preimage in the domain.

In every function with range R and codomain T, $R \subseteq T$. To prove that a given function is onto, we must show that $T \subseteq R$; then it will be true that $R = T$. We must therefore show that an arbitrary member of the codomain is a member of the range, that is, that it is the image of some member of the domain. On the other hand, if we can produce one member of the codomain that is not the image of any member of the domain, then we have proved that the function is not onto.

EXAMPLE 34

The function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$ is an onto function. To prove that $g(x)$ is onto, let r be an arbitrary real number, and let $x = \sqrt[3]{r}$. Then x is a real number, so x belongs to the domain of g and $g(x) = (\sqrt[3]{r})^3 = r$. Hence, any member of the codomain is the image under g of a member of the domain. The function $k: \mathbb{R} \to \mathbb{C}$ given by $k(x) = x^3$ is not onto. There are many complex numbers (i, for example) that cannot be obtained by cubing a real number. Thus, g and k are not equal functions.

EXAMPLE 35

Let $f: \mathbb{Q} \rightarrow \mathbb{Q}$ be defined by $f(x) = 3x + 2$. To test whether f is onto, let $q \in \mathbb{Q}$. We want an $x \in \mathbb{Q}$ such that $f(x) = 3x + 2 = q$. When we solve this equation for x, we find that $x = (q - 2)/3$ is the only possible value and is indeed a member of $\mathbb{Q}$. Thus, q is the image of a member of $\mathbb{Q}$ under f, and f is onto. However, the function $h: \mathbb{Z} \rightarrow \mathbb{Q}$ defined by $h(x) = 3x + 2$ is not onto because there are many values $q \in \mathbb{Q}$, for example 0, for which the equation $3x + 2 = q$ has no integer solution.

PRACTICE 28 Which of the functions found in Practice 23 are onto functions?

PRACTICE 29 Suppose a function $f: \{T, F\}^n \rightarrow \{T, F\}$ is defined by a propositional wff P (see Example 31). Give the two conditions on P under each of which f will fail to be an onto function.

One-to-One Functions

The definition of a function guarantees a unique image for every member of the domain. A given member of the range may have more than one preimage, however. In our very first example of a function (salary increases), both physical sciences and liberal arts were preimages of 1.5%. This function was not one-to-one.

DEFINITION **ONE-TO-ONE (INJECTIVE) FUNCTION**

A function $f: S \rightarrow T$ is **one-to-one**, or **injective**, if no member of T is the image under f of two distinct elements of S.

REMINDER

To show that a function f is one-to-one, assume $f(s_1) = f(s_2)$ and show that $s_1 = s_2$.

The one-to-one idea here is the same as for binary relations in general, as discussed in Section 5.1, except that every element of S must appear as a first component in an ordered pair. To prove that a function is one-to-one, we assume that there are elements s_1 and s_2 of S with $f(s_1) = f(s_2)$ and then show that $s_1 = s_2$. To prove that a function is not one-to-one, we produce a counterexample, an element in the range with two preimages in the domain.

EXAMPLE 36

The function $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = x^3$ is one-to-one because if x and y are real numbers with $g(x) = g(y)$, then $x^3 = y^3$ and $x = y$. The function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$ is not one-to-one because, for example, $f(2) = f(-2) = 4$. However, the function $h: \mathbb{N} \rightarrow \mathbb{N}$ given by $h(x) = x^2$ is one-to-one because if x and y are nonnegative integers with $h(x) = h(y)$, then $x^2 = y^2$; because x and y are both nonnegative, $x = y$.

PRACTICE 30 Which of the functions found in Practice 23 are one-to-one functions?

EXAMPLE 37 The floor function and the ceiling function of Example 32 are clearly not one-to-one. This is evident also in the graphs of these functions (Practice 26), which have a number of horizontal sections, indicating that many different domain values in R are mapped by the function to the same codomain value in $\mathbb{Z}$.

Figure 5.20 gives simple illustrations about functions and their properties. In each case, the domain is on the left and the codomain is on the right.

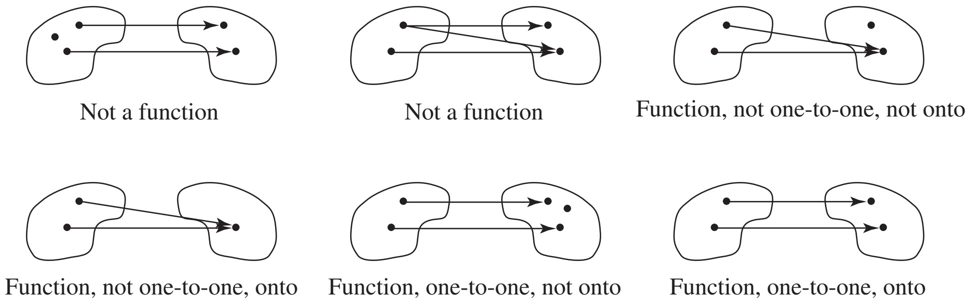


Figure 5.20

Bijections

DEFINITION **BIJECTIVE FUNCTION**

A function $f: S \to T$ is **bijective** (a **bijection**) if it is both one-to-one and onto.

EXAMPLE 38 The function $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3$ is a bijection. The function in part (g) of Practice 23 is a bijection. The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is not a bijection (not one-to-one), and neither is the function $k: \mathbb{R} \to \mathbb{C}$ given by $k(x) = x^3$ (not onto).

Composition of Functions

REMINDER

To prove that a function is a bijection requires proving two things—onto and one-to-one.

Suppose that f and g are functions with $f: S \to T$ and $g: T \to U$. Then for any $s \in S$, $f(s)$ is a member of T, which is also the domain of g. Thus, the function g can be applied to $f(s)$. The result is $g(f(s))$, a member of U (Figure 5.21). Taking an arbitrary member s of S, applying the function f, and then applying the function g to $f(s)$ is the same as associating a unique member of U with s. In short, we have created a function $S \to U$, called the composition function of f and g and denoted by $g \circ f$ (Figure 5.22).

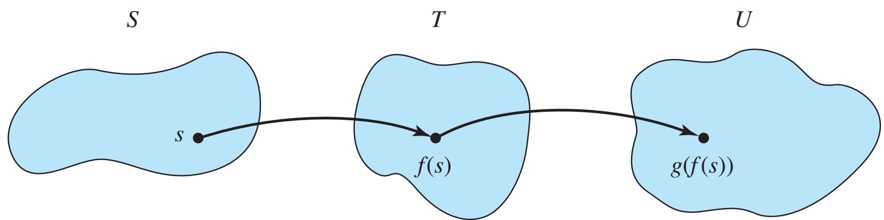


Figure 5.21

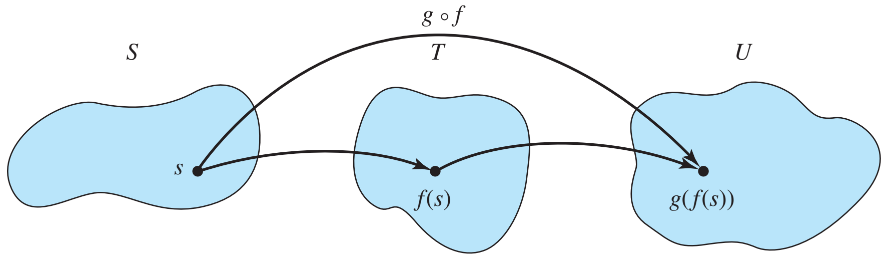


Figure 5.22

DEFINITION **COMPOSITION FUNCTION**

Let $f: S \to T$ and $g: T \to U$. Then the **composition function**, $g \circ f$, is a function from S to U defined by $(g \circ f)(s) = g(f(s))$.

Note that the function $g \circ f$ is applied right to left; function f is applied first and then function g.

The diagram in Figure 5.23 also illustrates the definition of the composition function. The corners indicate the domains and codomains of the three functions. The diagram says that, starting with an element of S, if we follow either path $g \circ f$ or path f followed by path g, we get to the same element in U. Diagrams illustrating that alternate paths produce the same effect are called **commutative diagrams**.

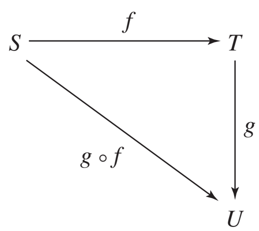


Figure 5.23

It is not always possible to take any two arbitrary functions and compose them; the domains and ranges have to be "compatible." For example, if $f: S \to T$ and $g: W \to Z$, where T and W are disjoint, then $(g \circ f)(s) = g(f(s))$ is undefined because $f(s)$ is not in the domain of g.

PRACTICE 31 Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \lfloor x \rfloor$.

a. What is the value of $(g \circ f)(2.3)$?
b. What is the value of $(f \circ g)(2.3)$? ■

From Practice 31 we see that order is important in function composition, which should not be surprising. If you make a deposit in your checking account and then write a large check, the effect is not the same as if you write a large check and later make a deposit! Your bank is very sensitive to these differences.

Function composition preserves the properties of being onto and being one-to-one. Again, let $f: S \to T$ and $g: T \to U$, but also suppose that both f and g are onto functions. Then the composition function $g \circ f$ is also onto. Recall that $g \circ f: S \to U$, so we must pick an arbitrary $u \in U$ and show that it has a preimage under $g \circ f$ in S. Because g is onto, there exists $t \in T$ such that $g(t) = u$. And because f is onto, there exists $s \in S$ such that $f(s) = t$. Then $(g \circ f)(s) = g(f(s)) = g(t) = u$, and $g \circ f$ is an onto function.

PRACTICE 32 Let $f: S \to T$ and $g: T \to U$, and assume that both f and g are one-to-one functions. Prove that $g \circ f$ is a one-to-one function. (*Hint*: Assume that $(g \circ f)(s_1) = (g \circ f)(s_2)$.) ■

We have now proved the following theorem.

THEOREM ON COMPOSING TWO BIJECTIONS
The composition of two bijections is a bijection.

Inverse Functions

Bijective functions have another important property. Let $f: S \to T$ be a bijection. Because f is onto, every $t \in T$ has a preimage in S. Because f is one-to-one, that preimage is unique. We can associate with each element t of T a unique member of S, namely, that $s \in S$ such that $f(s) = t$. This association describes a function g, $g: T \to S$. The picture for f and g is given in Figure 5.24. The domains and codomains of g and f are such that we can form both $g \circ f: S \to S$ and $f \circ g: T \to T$. If $s \in S$, then $(g \circ f)(s) = g(f(s)) = g(t) = s$. Thus, $g \circ f$ maps each element of S to itself. The function that maps each element of a set S to itself, that is, that leaves each element of S unchanged, is called the **identity function** on S and denoted by i_S. Hence, $g \circ f = i_S$.

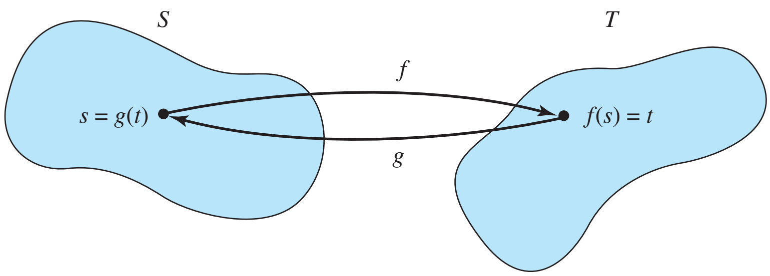


Figure 5.24

PRACTICE 33 Show that $f \circ g = i_T$. ■

We have now seen that if f is a bijection, $f\colon S \to T$, then there is a function $g\colon T \to S$ with $g \circ f = i_S$ and $f \circ g = i_T$. The converse is also true. To prove the converse, suppose $f\colon S \to T$ and there exists $g\colon T \to S$ with $g \circ f = i_S$ and $f \circ g = i_T$. We can prove that f is a bijection. To show that f is onto, let $t \in T$. Then $t = i_T(t) = (f \circ g)(t) = f(g(t))$. Because $g\colon T \to S$, $g(t) \in S$, and $g(t)$ is the preimage under f of t. To show that f is one-to-one, suppose $f(s_1) = f(s_2)$. Then $g(f(s_1)) = g(f(s_2))$ and $(g \circ f)(s_1) = (g \circ f)(s_2)$ implying $i_S(s_1) = i_S(s_2)$, or $s_1 = s_2$. Thus, f is a bijection.

DEFINITION **INVERSE FUNCTION**

Let f be a function, $f\colon S \to T$. If there exists a function $g\colon T \to S$ such that $g \circ f = i_S$ and $f \circ g = i_T$, then g is called the **inverse function** of f, denoted by f^{-1}.

We have proved the following theorem.

THEOREM **ON BIJECTIONS AND INVERSE FUNCTIONS**

Let $f\colon S \to T$. Then f is a bijection if and only if f^{-1} exists.

Actually, we have been a bit sneaky in talking about *the* inverse function of f. What we have shown is that if f is a bijection, this is equivalent to the existence of *an* inverse function. But it is easy to see that there is only one such inverse function. *When you want to prove that something is unique, the standard technique is to assume that there are two different such things and then obtain a contradiction*. Thus, suppose f has two inverse functions, f_1^{-1} and f_2^{-1} (existence of either means that f is a bijection). Both f_1^{-1} and f_2^{-1} are functions from T to S; if they are not the same function, then they must act differently somewhere. Assume that there is a $t \in T$ such that $f_1^{-1}(t) \neq f_2^{-1}(t)$. Because f is one-to-one, it follows that $f(f_1^{-1}(t)) \neq f(f_2^{-1}(t))$, or $(f \circ f_1^{-1})(t) \neq (f \circ f_2^{-1})(t)$. But both $f \circ f_1^{-1}$ and $f \circ f_2^{-1}$ are i_T, so $t \neq t$, which is a contradiction. We are therefore justified in speaking of f^{-1} as *the* inverse function of f. If f is a bijection, so that f^{-1} exists, then f is the inverse function for f^{-1}; therefore, f^{-1} is also a bijection.

PRACTICE 34 $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 4$ is a bijection.

Describe f^{-1}. ■

We've introduced a lot of terminology about functions. Table 5.2 gives an informal summary of these terms.

TABLE 5.2

Term	Meaning
function	Mapping from one set to another that associates with each member of the starting set exactly one member of the ending set
domain	Starting set for a function
codomain	Ending set for a function
image	Point that results from a mapping
preimage	Starting point for a mapping
range	Collection of all images of the domain
onto (surjective)	Range is the whole codomain; every codomain element has a preimage
one-to-one (injective)	No two elements in the domain map to the same place
bijection	One-to-one and onto
identity function	Maps each element of a set to itself
inverse function	For a bijection, a new function that maps each codomain element back where it came from

Permutation Functions

Bijections that map a set to itself are given a special name.

DEFINITION **PERMUTATIONS OF A SET**

For a given set A, $S_A = \{f \mid f: A \to A \text{ and } f \text{ is a bijection}\}$. S_A is thus the set of all bijections of set A into (and therefore onto) itself; such functions are called **permutations** of A.

If f and g both belong to S_A, then they each have domain = range = A. Therefore the composition function $g \circ f$ is defined and maps $A \to A$. Furthermore, because f and g are both bijections, our theorem on composing bijections says that $g \circ f$ is a bijection, a (unique) member of S_A. Thus, function composition is a binary operation on the set S_A.

In Section 4.4 we described a permutation of objects in a set as being an ordered arrangement of those objects. Is this now a new use of the word "permutation"? Not exactly; permutation functions represent ordered arrangements of the objects in the domain. If $A = \{1, 2, 3, 4\}$, one permutation function of A, call it f, is given by $f = \{(1, 2), (2, 3), (3, 1), (4, 4)\}$. We can also describe function f in array form by listing the elements of the domain in a row and, directly beneath, the images of these elements under f. Thus,

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

The bottom row is an ordered arrangement of the objects in the top row.

A shorter way to describe the permutation f shown in array form is to use *cycle notation* and write $f = (1, 2, 3)$—understood to mean that f maps each element listed to the one on its right, the last element listed to the first, and an element of the domain not listed to itself. Here 1 maps to 2, 2 maps to 3, and 3 maps to 1. The element 4 maps to itself because it does not appear in the cycle. The cycle $(2, 3, 1)$ also represents f. It says that 2 maps to 3, 3 maps to 1, 1 maps to 2, and 4 maps to itself, the same information as before. Similarly, $(3, 1, 2)$ also represents f.

PRACTICE 35

a. Let $A = \{1, 2, 3, 4, 5\}$, and let $f \in S_A$ be given in array form by

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix}$$

Write f in cycle form.

b. Let $A = \{1, 2, 3, 4, 5\}$, and let $g \in S_A$ be given in cycle form by $g = (2, 4, 5, 3)$. Write g in array form. ■

If f and g are members of S_A for some set A, then $g \circ f \in S_A$, and the action of $g \circ f$ on any member of A is determined by applying function f and then function g. If f and g are cycles, $g \circ f$ is still computed the same way.

EXAMPLE 39

If $A = \{1, 2, 3, 4\}$ and $f, g \in S_A$ are given by $f = (1, 2, 3)$ and $g = (2, 3)$, then $g \circ f = (2, 3) \circ (1, 2, 3)$. But what does this composition function look like? Let's see what happens to element 1 of A. Working from right to left (first f, then g), $1 \to 2$ under f and then $2 \to 3$ under g, so $1 \to 3$ under $g \circ f$. If we want to write $g \circ f$ as a cycle, we see that it can start with

$$(1, 3$$

and we next need to see what happens to 3. Under f, $3 \to 1$ and then under g, $1 \to 1$ (because 1 does not appear in the cycle notation for g), so $3 \to 1$ under $g \circ f$. Thus we can close the above cycle, writing it as $(1, 3)$. But what happens to 2 and 4? If we consider 2, $2 \to 3$ under f and then $3 \to 2$ under g, so $2 \to 2$ under $g \circ f$. Similarly, $4 \to 4$ under f and $4 \to 4$ under g, so $4 \to 4$ under $g \circ f$. We conclude that $g \circ f = (1, 3)$. ●

In Example 39, if we were to compute $f \circ g = (1, 2, 3) \circ (2, 3)$, we would get $(1, 2)$. (We already know that order is important in function composition.) If, however, f and g are members of S_A and f and g are **disjoint cycles**—the cycles have no elements in common—then $f \circ g = g \circ f$.

PRACTICE 36 Let $A = \{1, 2, 3, 4, 5\}$. Compute $g \circ f$ and $f \circ g$ for the following cycles in S_A.

a. $f = (5, 2, 3)$; $g = (3, 4, 1)$. Write the answers in cycle form.
b. $f = (1, 2, 3, 4)$; $g = (3, 2, 4, 5)$. Write the answers in array form.
c. $f = (1, 3)$; $g = (2, 5)$. Write the answers in array form. ■

Let $A = \{1, 2, 3, 4\}$ and consider the cycle $f \in S_A$ given by $f = (1, 2)$. If we compute $f \circ f = (1, 2) \circ (1, 2)$, we see that each element of A is mapped to itself. The permutation that maps each element of A to itself is the identity function on A, i_A, also called the **identity permutation**.

If A is an infinite set, not every permutation of A can be written as a cycle. But even when A is a finite set, not every permutation of A can be written as a cycle; for example, the permutation $g \circ f$ of Practice 36(b) cannot be written as a cycle. However, every permutation on a finite set that is not the identity permutation can be written as a composition of one or more disjoint cycles. The permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 5 & 1 & 3 \end{pmatrix}$$

of Practice 36(b) is $(1, 4) \circ (3, 5)$ or $(3, 5) \circ (1, 4)$.

PRACTICE 37 Write

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{pmatrix}$$

as a composition of disjoint cycles. ■

Among the permutations of A, some will map certain elements of A to themselves, while others will so thoroughly mix elements around that no element in A is mapped to itself. A permutation on a set that maps no element to itself is called a **derangement**.

EXAMPLE 40 The permutation f on $A = \{1, 2, 3, 4, 5\}$ given in array form by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 4 & 1 & 3 \end{pmatrix}$$

is a derangement. Members of S_A that are *not* derangements, if written as a cycle or a product of cycles, will have at least one element of A that is not listed. Thus $g \in S_A$ defined as $g = (1, 4) \circ (3, 5)$ maps 2 to itself, so g is not a derangement. ●

How Many Functions

Suppose S and T are finite sets, say $|S| = m$ and $|T| = n$. What can we say about the number of functions with various properties that map S to T? First, let's just count the number of functions $f: S \to T$, assuming no special properties about the functions. The multiplication principle can be used here because we can think of defining a function by assigning an image to each of the m elements of S. This gives us a sequence of m tasks. Each task has n outcomes because each element of S can map to any element in T. Therefore the number of functions is

$$\underbrace{n \times n \times n \times \cdots \times n}_{m \text{ factors}} = n^m$$

How many one-to-one functions are there from S to T? We must have $m \leq n$ or we can't have any one-to-one functions at all. (All the elements of S must be mapped to T, and if $m > n$ there are too many elements in S to allow for a one-to-one mapping. Actually, this is the pigeonhole principle at work.) We can again solve this problem by carrying out the sequence of tasks of assigning an image to each element in S, but this time we cannot use any image we have used before. By the multiplication principle, we have a product that begins with the factors

$$n(n-1)(n-2)\cdots$$

and must contain a total of m factors, so the result is

$$\begin{aligned} n(n-1)(n-2)\cdots[n-(m-1)] &= n(n-1)(n-2)\cdots(n-m+1) \\ &= \frac{n!}{(n-m)!} = P(n,m) \end{aligned}$$

How many onto functions are there from S to T? This time we must have $m \geq n$ so that there are enough values in the domain to provide preimages for every value in the codomain. (By the definition of a function, an element in S cannot be a preimage of more than one element in T.) Our overall plan is to subtract the number of non-onto functions from the total number of functions, which we know. To count the number of non-onto functions, we'll use the principle of inclusion and exclusion.

Enumerate the elements of set T as $t_1, \ldots, t_n$. For each i, $1 \leq i \leq n$, let A_i denote the set of functions from S to T that do not map anything to element t_i. (These sets are not disjoint, but every non-onto function belongs to at least one such set.) By the principle of inclusion and exclusion, we can write

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| \\ - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| \qquad (1) \end{aligned}$$

For any i, $|A_i|$ is the number of functions that do not map anything to t_i but have no other restrictions. By the multiplication principle, we can count the number of such functions by counting for each of the m domain elements its $n - 1$ possible images. The result is that $|A_i| = (n - 1)^m$. Therefore the first summation in Equation (1) adds together terms that are all of the same size. There is one such term for each distinct individual set A_i out of the n sets, so there are $C(n, 1)$ such terms.

For any i and j, $|A_i \cap A_j|$ is the number of functions that do not map anything to t_i or t_j, leaving $n - 2$ possible images for each of the m elements of S. Thus $|A_i \cap A_j| = (n - 2)^m$. The second summation adds one such term for each distinct group of two sets out of n, so there are $C(n, 2)$ such terms.

A similar result holds for all the intersection terms. If there are k sets in the intersection, then there are $(n - k)^m$ functions in the intersection set and there are $C(n, k)$ distinct groups of k sets to form the intersection. Equation (1) can thus be written as

$$|A_1 \cup \cdots \cup A_n| = C(n, 1)(n-1)^m - C(n, 2)(n-2)^m + C(n, 3)(n-3)^m \\ - \cdots + (-1)^{n+1}C(n, n)(n-n)^m \tag{2}$$

Now the expression on the left of Equation (2) represents the number of all functions that fail to map to at least one of the elements of T, that is, all the non-onto functions. If we subtract the value of this expression from the total number of functions, which we know is n^m, we will have the number of onto functions. Thus the number of onto functions is

$$n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m - C(n, 3)(n-3)^m \\ + \cdots + (-1)^{n-1}C(n, n-1)[n-(n-1)]^m + (-1)^n C(n, n)(n-n)^m$$

where we've added the next-to-last term. The last term is zero, so the final answer is

$$n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m - C(n, 3)(n-3)^m \\ + \ldots + (-1)^{n-1}\, C(n, n-1)(1)^m$$

We'll summarize these results.

THEOREM **ON THE NUMBER OF FUNCTIONS WITH FINITE DOMAINS AND CODOMAINS**

If $|S| = m$ and $|T| = n$, then

1. The number of functions $f: S \to T$ is n^m.
2. The number of one-to-one functions $f: S \to T$, assuming that $m \le n$, is
$$\frac{n!}{(n-m)!}$$
3. The number of onto functions $f: S \to T$, assuming that $m \ge n$, is
$$n^m - C(n, 1)(n-1)^m + C(n, 2)(n-2)^m - C(n, 3)(n-3)^m \\ + \ldots + (-1)^{n-1}C(n, n-1)(1)^m$$

EXAMPLE 41 Let $S = \{A, B, C\}$ and $T = \{a, b\}$. Find the number of functions from S onto T.

Here $m = 3$ and $n = 2$. By our theorem on the number of functions, there are

$$2^3 - C(2, 1)(1)^3 = 8 - 2 \cdot 1 = 6$$

such functions.

PRACTICE 38 One of the six onto functions in Example 41 can be illustrated by the following diagram:

Draw diagrams for the remaining five onto functions.

If A is a set with $|A| = n$, then the number of permutations of A is $n!$. This number can be obtained by any of three methods:

1. A combinatorial argument (each of the n elements in the domain must map to one of the n elements in the range with no repetitions)
2. Thinking of such functions as permutations on a set with n elements and noting that $P(n, n) = n!$
3. Using result (2) in the previous theorem with $m = n$

We propose to count the number of derangements on A. Our plan is similar to the one we used in counting onto functions. We'll use the principle of inclusion and exclusion to compute the number of permutations that are not derangements and then subtract this value from the total number of permutation functions.

Enumerate the elements of set A as $a_1, \ldots, a_n$. For each i, $1 \le i \le n$, let A_i be the set of all permutations that leave a_i fixed. (These sets are not disjoint, but every permutation that is not a derangement belongs to at least one such set.) By the principle of inclusion and exclusion, we can write

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n| \tag{3}$$

For any i, $|A_i|$ is the number of permutations that leave a_i fixed. By the multiplication principle we can count the number of such functions by counting for each of the n domain elements, beginning with a_i, its possible images. There is only one choice of where to map a_i because it must map to itself; the next element can map anywhere except to a_i, so there are $n - 1$ outcomes; the next element can map anywhere except the two images already used, so there are $n - 2$ outcomes, and so on. Continuing, there are

$$(1)(n - 1)(n - 2) \cdots (1) = (n - 1)!$$

elements in A_i for each i. Therefore the first summation in equation (3) adds together terms that are all of the same size. The number of such terms equals the number of ways to pick one set A_i out of the n such sets, or $C(n, 1)$.

In the second summation, the terms count the number of permutations on n elements that leave two of those elements fixed. There are

$$(1)(1)(n-2)\cdots(1) = (n-2)!$$

such functions in a given $A_i \cap A_{j,}$ and $C(n, 2)$ ways to choose the two sets out of n. In general, if there are k sets in the intersection, then k elements must be held fixed, so there are $(n-k)!$ functions in the intersection set, and there are $C(n, k)$ ways to choose the k sets to form the intersection. Therefore equation (3) becomes

$$\begin{aligned}|A_1 \cup \cdots \cup A_n| = {} & C(n,1)(n-1)! - C(n,2)(n-2)! + C(n,3)(n-3)! \\ & - \cdots + (-1)^{n+1}C(n,n)(n-n)!\end{aligned}$$

This expression represents the number of all possible nonderangement permutations. We subtract this value from the total number of permutation functions, which is $n!$:

$$\begin{aligned}n! - C(n,1)(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! \\ + \cdots + (-1)^n C(n,n)(n-n)!\end{aligned}$$

Rewriting this expression,

$$\begin{aligned}& n! - \frac{n!}{1!(n-1)!}(n-1)! + \frac{n!}{2!(n-2)!}(n-2)! - \frac{n!}{3!(n-3)!}(n-3)! \\ & + \cdots + (-1)^n \frac{n!}{n!0!}0! \\ & = n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \cdots + (-1)^n \frac{n!}{n!} \\ & = n!\left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right] \qquad (4)\end{aligned}$$

EXAMPLE 42 For $n = 3$, Equation (4) says that the number of derangements is

$$3!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = \frac{3!}{2!} - \frac{3!}{3!} = 3 - 1 = 2$$

Written in array form, the two derangements are

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Equivalent Sets

DEFINITIONS **EQUIVALENT SETS AND CARDINALITY**

A set S is **equivalent** to a set T if there exists a bijection $f: S \to T$. Two sets that are equivalent have the same **cardinality**.

The notion of equivalent sets allows us to extend our definition of cardinality from finite to infinite sets. The cardinality of a finite set is the number of elements in the set. If S is equivalent to T, then all the members of S and T are paired off by f in a one-to-one correspondence. If S and T are finite sets, this pairing off can happen only when S and T are the same size. With infinite sets, the idea of size gets a bit fuzzy, because we can sometimes prove that a given set is equivalent to what seems to be a smaller set. The cardinality of an infinite set is therefore given only in a comparative sense; for example, we may say that an infinite set A has (or does not have) the same cardinality as the set $\mathbb{N}$.

PRACTICE 39 Describe a bijection $f: \mathbb{Z} \to \mathbb{N}$, thus showing that $\mathbb{Z}$ is equivalent to $\mathbb{N}$ ($\mathbb{Z}$ and $\mathbb{N}$ have the same cardinality) even though $\mathbb{N} \subset \mathbb{Z}$. ■

If we have found a bijection between a set S and $\mathbb{N}$, we have established a one-to-one correspondence between the members of S and the nonnegative integers. We can then name the members of S according to this correspondence, writing s_0 for the value of S associated with 0, s_1 for the value of S associated with 1, and so on. Then the list

$$s_0, s_1, s_2, \ldots$$

includes all the members of S. Since this list constitutes an enumeration of S, S is a denumerable set. Conversely, if S is denumerable, then a listing of the members of S exists and can be used to define a bijection between S and $\mathbb{N}$. Therefore a set is denumerable if and only if it is equivalent to $\mathbb{N}$.

For finite sets, we know that if S has n elements, then $\wp(S)$ has 2^n elements. Of course, $2^n > n$, and we cannot find a bijection between a set with n elements and a set with 2^n elements. Therefore S and $\wp(S)$ are not equivalent. This result is also true for infinite sets.

THEOREM **CANTOR'S THEOREM**

For any set S, S and $\wp(S)$ are not equivalent.

Proof: We will do a proof by contradiction and assume that S and $\wp(S)$ are equivalent. Let f be the bijection between S and $\wp(S)$. For any member s of S, $f(s)$ is a member of $\wp(S)$, so $f(s)$ is a set containing some members of S, possibly containing s itself. Now we define a set $X = \{x \in S \mid x \notin f(x)\}$. Because X is a subset of S, it is an element of $\wp(S)$ and therefore must be equal to $f(y)$ for some $y \in S$.

Then y either is or is not a member of X. If $y \in X$, then by the definition of X, $y \notin f(y)$, but since $f(y) = X$, then $y \notin X$. On the other hand, if $y \notin X$, then since $X = f(y)$, $y \notin f(y)$, and by the definition of X, $y \in X$. In either case, there is a contradiction, and our original assumption is incorrect. Therefore S and $\wp(S)$ are not equivalent. *End of Proof*

The proof of Cantor's theorem depends on the nature of set X, which was carefully constructed to provide the crucial contradiction. In this sense, the proof is similar to the diagonalization method (see Example 23 in Chapter 4) used to prove the existence of an uncountable set. Indeed, the existence of an uncountable set can be shown directly from Cantor's theorem.

EXAMPLE 43 The set $\mathbb{N}$ is, of course, a denumerable set. By Cantor's theorem, the set $\wp(\mathbb{N})$ is not equivalent to $\mathbb{N}$ and is therefore not a denumerable set, although it is clearly infinite.

SECTION 5.4 REVIEW

TECHNIQUES

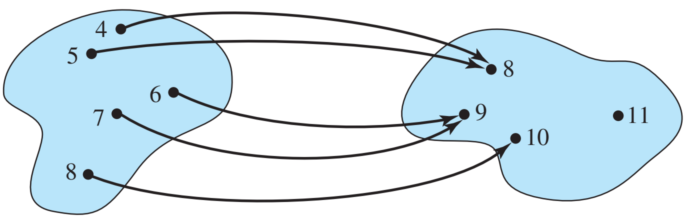
- Test whether a given relation is a function.
- Test a function for being one-to-one or onto.
- Find the image of an element under function composition.
- Write permutations of a set in array or cycle form.
- Count the number of functions, one-to-one functions, and onto functions from one finite set to another.

MAIN IDEAS

- The concept of function, especially bijective function, is extremely important.
- Composition of functions preserves bijectiveness.
- The inverse function of a bijection is itself a bijection.
- Permutations are bijections on a set.

EXERCISES 5.4

1. The accompanying figure represents a function.



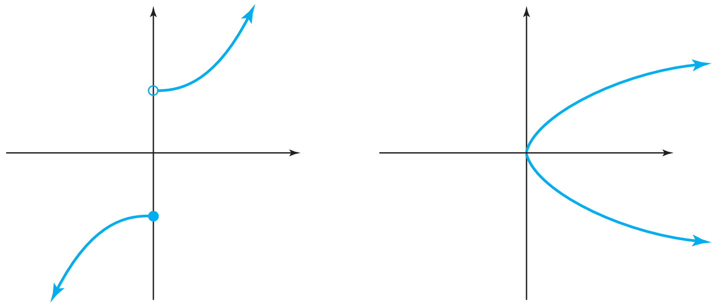
a. What is the domain? What is the codomain? What is the range?

b. What is the image of 5? of 8?

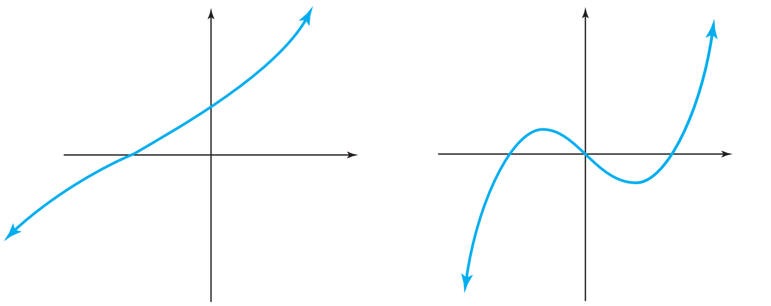
c. What are the preimages of 9?

d. Is this an onto function? Is it one-to-one?

2. The accompanying figure illustrates various binary relations from $\mathbb{R}$ to $\mathbb{R}$. Which are functions? For those that are functions, which are onto? Which are one-to-one?



(a) (b)



(c) (d)

3. Using the equation $f(x) = 2x - 1$ to describe the functional association, write the function as a set of ordered pairs if the codomain is R and the domain is
   a. $S = \{0, 1, 2\}$.
   b. $S = \{1, 2, 4, 5\}$.
   c. $S = \{\sqrt{7}, 1.5\}$.
4. Using the equation $f(x) = x^2 + 1$ to describe the functional association, write the function as a set of ordered pairs if the codomain is $\mathbb{Z}$ and the domain is
   a. $S = \{1, 5\}$.
   b. $S = \{-1, 2, -2\}$.
   c. $S = \{-\sqrt{12}, 3\}$.
5. If $f: \mathbb{Z} \to \mathbb{Z}$ is defined by $f(x) = 3x$, find $f(A)$ for
   a. $A = \{1, 3, 5\}$.
   b. $A = \{x \mid x \in \mathbb{Z} \text{ and } (\exists y)(y \in \mathbb{Z} \text{ and } x = 2y)\}$.
6. If $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, describe
   a. $f(\mathbb{N})$.
   b. $f(\mathbb{Z})$.
   c. $f(\mathbb{R})$.

7. The function f: {all English words} $\rightarrow \mathbb{Z}$. In each case, find $f(S)$.
   a. S = {dog, cat, buffalo, giraffe}, $f(x)$ = the number of characters in x
   b. S = {goose, geese, moose, Mississippi}, $f(x)$ = the number of double-letter pairs in x
   c. S = {cheetah, seal, porpoise, koala}, $f(x)$ = the number of e's in x

8. The function f: {binary strings} $\rightarrow$ {binary strings}. In each case, find $f(S)$.
   a. $S = \{000, 1011, 10001\}$, $f(x)$ = the second bit in x
   b. $S = \{111, 100, 0111\}$, $f(x)$ = the binary string that is the sum of the first and last bit
   c. $S = \{001, 11, 101\}$, $f(x)$ = the binary string that is equal to $x + 1$

9. True or false:
   a. An onto function means that every element in the codomain must have a unique preimage.
   b. A one-to-one function means that every element in the codomain must have a unique preimage.
   c. A one-to-one function means that no two elements in the domain map to the same element in the codomain.
   d. An onto function means that (the range) $\cap$ (the codomain) $= \emptyset$.

10. True or false:
    a. If every element in the domain has an image, it must be an onto function.
    b. If every element in the codomain has an image, it must be an onto function.
    c. If every element in the codomain has a preimage, it must be an onto function.
    d. If the domain is the larger than the codomain, it can't be a one-to-one function.

11. Let $S = \{0, 2, 4, 6\}$ and $T = \{1, 3, 5, 7\}$. Determine whether each of the following sets of ordered pairs is a function with domain S and codomain T. If so, is it one-to-one? Is it onto?
    a. $\{(0, 2), (2, 4), (4, 6), (6, 0)\}$
    b. $\{(6, 3), (2, 1), (0, 3), (4, 5)\}$
    c. $\{(2, 3), (4, 7), (0, 1), (6, 5)\}$
    d. $\{(2, 1), (4, 5), (6, 3)\}$
    e. $\{(6, 1), (0, 3), (4, 1), (0, 7), (2, 5)\}$

12. For any bijections in Exercise 11, describe the inverse function.

13. Let S = the set of all U. S. citizens alive today. Which of the following are functions from domain S to the codomain given? Which functions are one-to-one? Which functions are onto?
    a. Codomain = the alphabet, f(person) = initial of person's middle name
    b. Codomain = the set of dates between January 1 and December 31, f(person) = person's date of birth
    c. Codomain = 9-digit numbers, f(person) = person's Social Security number

14. Let S = the set of people at a meeting, let T = the set of all shoes in the room. Let $f(x)$ = the left shoe x is wearing.
    a. Is this a function?
    b. Is it one-to-one?
    c. Is it onto?

15. Which of the following definitions describe functions from the domain to the codomain given? Which functions are one-to-one? Which functions are onto? Describe the inverse function for any bijective function.
   a. f: $\mathbb{Z} \to \mathbb{N}$ where f is defined by $f(x) = x^2 + 1$
   b. g: $\mathbb{N} \to \mathbb{Q}$ where g is defined by $g(x) = 1/x$
   c. h: $\mathbb{Z} \times \mathbb{N} \to \mathbb{Q}$ where h is defined by $h(z, n) = z/(n + 1)$
   d. f: $\{1, 2, 3\} \to \{p, q, r\}$ where $f = \{(1, q), (2, r), (3, p)\}$
   e. g: $\mathbb{N} \to \mathbb{N}$ where g is given by $g(x) = 2^x$
   f. h: $\mathbb{R}^2 \to \mathbb{R}^2$ where h is defined by $h(x, y) = (y + 1, x + 1)$
16. Which of the following definitions describe functions from the domain to the codomain given? Which functions are one-to-one? Which functions are onto? Describe the inverse function for any bijective function.
   a. f: $\mathbb{Z}^2 \to \mathbb{N}$ where f is defined by $f(x, y) = x^2 + 2y^2$
   b. f: $\mathbb{N} \to \mathbb{N}$ where f is defined by $f(x) = \begin{cases} x/2 & \text{if } x \text{ is even} \\ x + 1 & \text{if } x \text{ is odd} \end{cases}$
   c. g: $\mathbb{R} \to \mathbb{R}$ where g is defined by $g(x) = 1/\sqrt{(x + 1)}$
   d. f: $\mathbb{N} \to \mathbb{N}$ where f is defined by $f(x) = \begin{cases} x + 1 & \text{if } x \text{ is even} \\ x - 1 & \text{if } x \text{ is odd} \end{cases}$
   e h: $\mathbb{N}^3 \to \mathbb{N}$ where h is given by $h(x, y, z) = x + y - z$
   f. g: $\mathbb{N}^2 \to \mathbb{N}^3$ where g is defined by $g(x, y) = (y, x, 0)$
17. Let f: $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^n$, where n is a fixed, positive integer. For what values of n is f bijective?
18. Let f: $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = n2^x$, where n is a fixed, positive integer. For what values of n is f an onto function?
19. Let $A = \{x, y\}$ and let A^* be the set of all strings of finite length made up of symbols from A. A function f: $A^* \to \mathbb{Z}$ is defined as follows: For s in A^*, $f(s)$ = the length of s. Is f one-to-one? Prove or disprove. Is f onto? Prove or disprove.
20. Let $A = \{x, y\}$ and let A^* be the set of all strings of finite length made up of symbols from A. A function f: $A^* \to \mathbb{Z}$ is defined as follows: For s in A^*, $f(s)$ = the number of x's minus the number of y's. Is f one-to-one? Prove or disprove. Is f onto? Prove or disprove.
21. Let $A = \{x, y\}$ and let A^* be the set of all strings of finite length made up of symbols from A. A function f: $A^* \to A^*$ is defined as follows: For s in A^*, $f(s)$ is the string obtained by writing the characters of s in reverse order. Is f one-to-one? Prove or disprove. Is f onto? Prove or disprove.
22. Let $A = \{x, y\}$ and let A^* be the set of all strings of finite length made up of symbols from A. A function f: $A^* \to A^*$ is defined as follows: For s in A^*, $f(s) = xs$ (the single-character string x followed by s). Is f one-to-one? Prove or disprove. Is f onto? Prove or disprove.
23. Let P be the power set of $\{a, b, c\}$. A function f: $P \to \mathbb{Z}$ is defined as follows: For A in P, $f(A)$ = the number of elements in A. Is f one-to-one? Prove or disprove. Is f onto? Prove or disprove.
24. Let P be the power set of $\{a, b\}$ and let S be the set of all binary strings of length 2. A function f: $P \to S$ is defined as follows: For A in P, $f(A)$ has a 1 in the high-order bit position (left end of string) if and only if a is in A. $f(A)$ has a 1 in the low-order bit position (right end of string) if and only if b is in A. Is f one-to-one? Prove or disprove. Is f onto? Prove or disprove.
25. Let $S = \{x | x \in \mathbb{R} \text{ and } x \geq 1\}$, and $T = \{x | x \in \mathbb{R} \text{ and } 0 < x \leq 1\}$. Find a function f: $S \to T$ that is a bijection.

26. Let $S = \{a, b, c, d\}$ and $T = \{x, y, z\}$.
    a. Give an example of a function from S to T that is neither onto nor one-to-one.
    b. Give an example of a function from S to T that is onto but not one-to-one.
    c. Can you find a function from S to T that is one-to-one?
27. Compute the following values.
    a. $\lfloor 3.4 \rfloor$
    b. $\lceil -0.2 \rceil$
    c. $\lfloor 0.5 \rfloor$
28. Compute the following values.
    a. $\lceil -5 - 1.\ 2 \rceil$
    b. $\lceil -5 - \lceil 1.2 \rceil \rceil$
    c. $\lfloor 2 * 3.7 \rfloor$
    d. $\lceil 1 + 1/2 + 1/3 + 1/4 \rceil$
29. What can be said about x if $\lfloor x \rfloor = \lceil x \rceil$?
30. Prove that $\lceil x \rceil + 1 = \lceil x + 1 \rceil$.
31. Prove that $\lfloor x \rfloor = -\lceil -x \rceil$.
32. The ceiling function $f(x) = \lceil x \rceil: \mathbb{R} \to \mathbb{Z}$. Prove or disprove:
    a. f is one-to-one
    b. f is onto
33. Prove or disprove:
    a. $\lceil \lfloor x \rfloor \rceil = x$
    b. $\lfloor 2x \rfloor = 2\lfloor x \rfloor$
34. Prove or disprove:
    a. $\lfloor x \rfloor + \lfloor y \rfloor = \lfloor x + y \rfloor$
    b. $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + 1/2 \rfloor$
35. Prove that if $2^k < n < 2^{k+1}$ then $k = \lfloor \log n \rfloor$ and $k + 1 = \lceil \log n \rceil$. (Here $\log n$ means $\log_2 n$.)
36. Prove that if $2^k \le n < 2^{k+1}$ then $\lfloor \log n \rfloor + 1 = \lceil \log (n + 1) \rceil$. (Here $\log n$ means $\log_2 n$.)
37. Compute the value of the following expressions.
    a. 31 mod 11
    b. 16 mod 8
    c. 22 mod 6
    d. -7 mod 3
38. a. List five values x such that $x \bmod 7 = 0$.
    b. List five values x such that $x \bmod 5 = 2$.
39. Prove or disprove: For any integers x and y, $x \bmod 10 + y \bmod 10 = (x + y) \bmod 10$.
40. Prove that $x \equiv y \pmod{n}$ if and only if $x \bmod n = y \bmod n$. (Recall the definition of congruence modulo n from Section 5.1.)
41. Let S be a set and let A be a subset of S. The *characteristic function* of A is a function $c_A: S \to \{0, 1\}$ with $c_A(x) = 1$ exactly when $x \in A$.
    a. Let $S = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$. Give the ordered pairs that belong to c_A.
    b. Prove that for any set S and any subsets A and B of S, $c_{A \cap B}(x) = c_A(x) \cdot c_B(x)$.

c. Prove that $c_A(x) = 1 - c_{A'}(x)$.

d. Is it true that for any set S and any subsets A and B of S, $c_{A \cup B}(x) = c_A(x) + c_B(x)$? Prove or give a counterexample.

42. *Ackermann's function*[1], mapping $\mathbb{N}^2$ to $\mathbb{N}$, is a recursive function that grows very rapidly. It is given by

$$A(0, n) = n + 1 \text{ for all } n \in \mathbb{N}$$
$$A(m, 0) = A(m - 1, 1) \text{ for all } m \in \mathbb{N}, m > 0$$
$$A(m, n) = A(m - 1, A(m, n - 1)) \text{ for all } m \in \mathbb{N}, n \in \mathbb{N}, m > 0, n > 0$$

a. Compute (show all steps) the value of $A(1, 1)$.

b. Compute (show all steps) the value of $A(2, 1)$.

c. The value of $A(4, 0) = 13 = 2^{2^2} - 3$, still a small value. But $A(4,1) = 2^{2^{2^2}} - 3$. Compute this value.

d. Write a likely expression for the value of $A(4, 2)$.

43. Another rapidly growing function is the *Smorynski function*, which also maps $\mathbb{N}^2$ to $\mathbb{N}$. The definition is

$$S(0, n) = n^n \text{ for all } n \in \mathbb{N}$$
$$S(m, n) = S(m - 1, S(m - 1, n)) \text{ for all } m \in \mathbb{N}, n \in \mathbb{N}, m > 0$$

a. How does $S(0, n)$ compare to $A(0, n)$? (See Exercise 40.)

b. Find (show all steps) an expression for the value of $S(1, n)$.

c. A *googolplex* is a very large number, which if written in standard form (such as 1,000,000 …), even in 1-point font, would take more room to write than the diameter of the known universe. Look up the definition of the googolplex and write it as $S(m, n)$ for a specific value of m and n.

44. The *Dwyer function* also maps $\mathbb{N}^2$ to $\mathbb{N}$ and grows very rapidly, but it has a closed-form definition:

$$D(m, n) = n!\left[\frac{(2m + 1)!}{2^m m!}\right]^n$$

a. Compute the values of $D(1, 1)$, $D(2, 1)$, $D(3, 1)$ and $D(4, 1)$.

b. Verify that $D(2, 1) = (2*1 + 3)D(1, 1)$, that $D(3, 1) = (2*2 + 3)D(2, 1)$, and that $D(4, 1) = (2*3 + 3)D(3, 1)$.

c. Verify that $D(m, 1)$ satisfies the recurrence relation

$$D(m + 1, 1) = (2m + 3)D(m, 1) \text{ with } D(0, 1) = 1$$

(*Hint*: When you evaluate $D(m + 1, 1)$ and $D(m, 1)$ do not divide the denominator factorial into the numerator factorial. Instead, think of the numerator factorial as a product of even and odd factors.

d. Find (use a spreadsheet) the smallest value of m for which

$$D(m, 1) < m^m$$

[1]This is the most common of several versions of Ackermann's function, all of which are recursive with extremely rapid growth rates. To watch the tedious recursiveness of computations of Ackermann's function, go to http://www. gfredericks.com/sandbox/arith/ackermann

45. Let $S = \{1, 2, 3, 4\}$, $T = \{1, 2, 3, 4, 5, 6\}$, and $U = \{6, 7, 8, 9, 10\}$. Also, let $f = \{(1, 2), (2, 4), (3, 3), (4, 6)\}$ be a function from S to T, and let $g = \{(1, 7), (2, 6), (3, 9), (4, 7), (5, 8), (6, 9)\}$ be a function from T to U. Write the ordered pairs in the function $g \circ f$.

46. a. Let f: $\mathbb{R} \to \mathbb{Z}$ be defined by $f(x) = \lfloor x \rfloor$. Let g: $\mathbb{Z} \to \mathbb{N}$ be defined by $g(x) = x^2$. What is $(g \circ f)(-4.7)$?

    b. Let f map the set of books into the integers where f assigns to each book the number of words in its title. Let g: $\mathbb{Z} \to \mathbb{Z}$ be given by $g(x) = 2x$. What is $(g \circ f)$ (this book)?

    c. Let f map strings of alphabetical characters and blank spaces into strings of alphabetical consonants where f takes any string and removes all vowels and all blanks. Let g map strings of alphabetical consonants into integers where g maps a string into the number of characters it contains. What is $(g \circ f)$ (abraham lincoln)?

47. Let f: $\mathbb{N} \to \mathbb{N}$ be defined by $f(x) = x + 1$. Let g: $\mathbb{N} \to \mathbb{N}$ be defined by $g(x) = 3x$. Calculate the value of the following expressions.

    a. $(g \circ f)(5)$
    b. $(f \circ g)(5)$
    c. $(g \circ f)(x)$
    d. $(f \circ g)(x)$
    e. $(f \circ f)(x)$
    f. $(g \circ g)(x)$

48. The following functions map $\mathbb{R}$ to $\mathbb{R}$. Give an equation describing the composition functions $g \circ f$ and $f \circ g$ in each case.

    a. $f(x) = 6x^3$, $g(x) = 2x$
    b. $f(x) = (x - 1)/2$, $g(x) = 4x^2$
    c. $f(x) = \lceil x \rceil$, $g(x) = \lfloor x \rfloor$

49. Let f: $S \to T$ and g: $T \to U$ be functions.

    a. Prove that if $g \circ f$ is one-to-one, so is f.
    b. Prove that if $g \circ f$ is onto, so is g.
    c. Find an example where $g \circ f$ is one-to-one but g is not one-to-one.
    d. Find an example where $g \circ f$ is onto but f is not onto.

50. a. Let f be a function, f: $S \to T$. If there exists a function g: $T \to S$ such that $g \circ f = i_S$, then g is called a *left inverse* of f. Show that f has a left inverse if and only if f is one-to-one.

    b. Let f be a function, f: $S \to T$. If there exists a function g: $T \to S$ such that $f \circ g = i_T$, then g is called a *right inverse* of f. Show that f has a right inverse if and only if f is onto.

    c. Let f: $\mathbb{N} \to \mathbb{N}$ be given by $f(x) = 3x$. Then f is one-to-one. Find two different left inverse functions for f.

    d. Let f: $\mathbb{N}^+ \to \mathbb{N}^+$ be given by $f(x) = \left\lceil \frac{x}{2} \right\rceil$. Then f is onto. Find two different right inverse functions for f.

51. For each of the following bijections f: $\mathbb{R} \to \mathbb{R}$, find f^{-1}.

    a. $f(x) = 2x$
    b. $f(x) = x^3$
    c. $f(x) = (x + 4)/3$

52. Let f and g be bijections, f: $S \to T$ and g: $T \to U$. Then f^{-1} and g^{-1} exist. Also, $g \circ f$ is a bijection from S to U. Show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

53. Let $A = \{1, 2, 3, 4, 5\}$. Write each of the following permutations on A in cycle form.

a. $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 4 & 2 \end{pmatrix}$

b. $f = \{(1, 4), (2, 5), (3, 2), (4, 3), (5, 1)\}$

54. Let A $\{a, b, c, d\}$. Write each of the following permutations on A in array form.

a. $f = \{(a, c), (b, b), (c, d), (d, a)\}$

b. $f = (c, a, b, d)$

c. $f = (d, b, a)$

d. $f = (a, b) \circ (b, d) \circ (c, a)$

55. Let A be any set and let S_A be the set of all permutations of A. Let $f, g, h \in S_A$. Prove that the functions $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are equal, thereby showing that we can write $h \circ g \circ f$ without parentheses to indicate grouping.

56. Find the composition of the following cycles representing permutations on $A = \{1, 2, 3, 4, 5\}$. Write your answer as a composition of one or more disjoint cycles.

a. $(2, 4, 5, 3) \circ (1, 3)$

b. $(3, 5, 2) \circ (2, 1, 3) \circ (4, 1)$ (By Exercise 55, we can omit parentheses indicating grouping.)

c. $(2, 4) \circ (1, 2, 5) \circ (2, 3, 1) \circ (5, 2)$

57. Find the composition of the following cycles representing permutations on $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Write your answer as a composition of one or more disjoint cycles.

a. $(1, 3, 4) \circ (5, 1, 2)$

b. $(2, 7, 8) \circ (1, 2, 4, 6, 8)$

c. $(1, 3, 4) \circ (5, 6) \circ (2, 3, 5) \circ (6, 1)$

d. $(2, 7, 1, 3) \circ (2, 8, 7, 5) \circ (4, 2, 1, 8)$

58. Find the composition of the following cycles representing permutations on $\mathbb{N}$. Write your answer as a composition of one or more disjoint cycles.

a. $(3, 5, 2) \circ (6, 2, 4, 1) \circ (4, 8, 6, 2)$

b. $(1, 5, 13, 2, 6) \circ (3, 6, 4, 13) \circ (13, 2, 6, 1)$

c $(1, 2) \circ (1, 3) \circ (1, 4) \circ (1, 5)$

59. Find the composition of the following cycles representing permutations on $A = \{a, b, c, d, e\}$. Write your answer as a composition of one or more disjoint cycles.

a. $(a, d, c, e) \circ (d, c, b) \circ (e, c, a, d) \circ (a, c, b, d)$

b. $(e, b, a) \circ (b, e, d) \circ (d, a)$

c. $(b, e, d) \circ (d, a) \circ (e, a, c) \circ (a, c, b, e)$

60. Find a permutation on an infinite set that cannot be written as a cycle.

61. The function f written in cycle form as $f = (4, 2, 8, 3)$ is a bijection on the set $\mathbb{N}$. Write f^{-1} in cycle form.

62. The "pushdown store," or "stack," is a storage structure that operates much like a set of plates stacked on a spring in a cafeteria. All storage locations are initially empty. An item of data is added to the top of the stack by a "push" instruction, which pushes any previously stored items farther down in the stack. Only the topmost item on the stack is accessible at any moment, and it is fetched and removed from the stack by a "pop" instruction.

Let's consider strings of integers that are an even number of characters in length; half the characters are positive integers, and the other half are zeros. We process these strings through a pushdown store as follows: As we read from left to right, the push instruction is applied to any nonzero integer, and a zero causes the pop instruction to be applied to the stack, thus printing the popped integer. Thus, processing the string 12030040 results in an output of 2314, and processing 12304000 results in an output of 3421. (A string such as 10020340 cannot be handled by this procedure because we cannot pop two integers from a stack containing only one integer.) Both 2314 and 3421 can be thought of as permutations,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

respectively, on the set $A = \{1, 2, 3, 4\}$.

a. What permutation of $A = \{1, 2, 3, 4\}$ is generated by applying this procedure to the string 12003400?

b. Name a permutation of $A = \{1, 2, 3, 4\}$ that cannot be generated from any string where the digits 1, 2, 3, and 4 appear in order, no matter where the zeros are placed.

63. Let $S = \{2, 4, 6, 8\}$ and $T = \{1, 5, 7\}$.

a. Find the number of functions from S to T.

b. Find the number of onto functions from S to T.

64. Let $S = \{P, Q, R\}$ and $T = \{k, l, m, n\}$.

a. Find the number of functions from S to T.

b. Find the number of one-to-one functions from S to T.

65. a. For $|S| = 2$, 3, and 4, respectively, use the theorem on the number of functions to show that the number of one-to-one functions from S to S equals the number of onto functions from S to S.

b. Argue that for $|S| = n$, f: $S \to S$ is one-to-one if and only if f is onto.

c. Find an infinite set S and a function f: $S \to S$ such that f is one-to-one but not onto.

d. Find an infinite set S and a function f: $S \to S$ such that f is onto but not one-to-one.

66. Let $A = \{a, b, c, d\}$. How many functions are in S_A? How many of these functions are derangements? Write all the derangements in array form.

67. Let $|S| = n$. In parts a-e, find the number of

a. functions from S to S.

b. one-to-one functions from S to S.

c. functions from S onto S (see Exercise 65).

d. permutations from S onto S.

e. derangements from S onto S.

f. Order the values obtained in parts (a) through (e) from smallest to largest and explain why this ordering is reasonable.

68. a. A system development project calls for five different tasks to be assigned to Maria, Jon, and Suzanne. In how many ways can the assignment be done if each of the three workers must get at least one task?

b. In how many ways can the projects be assigned if Maria must develop the test plan, which is one of the five tasks, but may do other tasks as well? (*Hint*: Consider the two cases where Maria does and does not do any of the other tasks.)

69. In a programming class of seven students, the instructor wants each student to modify the program from a previous assignment, but no student should work on his or her own program. In how many ways can the instructor assign programs to the students?

70. a. Find a calculus book and look up the Maclaurin series representation for the function e^x.
 b. Use the answer to part (a) to find a series representation for e^{-1}.
 c. Use a calculator to compute an approximate value for e^{-1} to about 5 decimal places.
 d. How can the answer to parts (b) and (c) help you approximate the number of derangements of n objects when n is large, say, $n \geq 10$? (*Hint*: Look at Equation (4) in this section.)
 e. Apply this approach to Exercise 69 and compare the results.
 f. Approximately how many derangements are there of 10 objects?
71. Let f be a function, $f: S \to T$.
 a. Define a binary relation ρ on S by $x \rho y \leftrightarrow f(x) = f(y)$. Prove that ρ is an equivalence relation.
 b. What can be said about the equivalence classes if f is a one-to-one function?
 c. For $S = T = \mathbb{Z}$ and $f(x) = 3x^2$, what is $[4]$ under the equivalence relation of part (a)?
72. Prove that $S(m, n)$, the number of ways to partition a set of m elements into n blocks, is equal to $1/n!$ times the number of onto functions from a set with m elements to a set with n elements. (*Hint*: Consider Exercise 71.)
73. By the definition of a function f from S to T, f is a subset of $S \times T$ where the image of every $s \in S$ under f is uniquely determined as the second component of the ordered pair (s, t) in f. Now consider any binary relation ρ from S to T. The relation ρ is a subset of $S \times T$ in which some elements of S may not appear at all as first components of an ordered pair and some may appear more than once. We can view ρ as a *nondeterministic function* from a subset of S to T. An $s \in S$ not appearing as the first component of an ordered pair represents an element outside the domain of ρ. For an $s \in S$ appearing once or more as a first component, ρ can select for the image of s any one of the corresponding second components.

 Let $S = \{1, 2, 3\}$, $T = \{a, b, c\}$, and $U = \{m, n, o, p\}$. Let ρ be a binary relation on $S \times T$ and σ be a binary relation on $T \times U$ defined by

$$\rho = \{(1, a), (1, b), (2, b), (2, c), (3, c)\}$$
$$\sigma = \{(a, m), (a, o), (a, p), (b, n), (b, p), (c, o)\}$$

 Thinking of ρ and σ as nondeterministic functions from S to T and T to U, respectively, we can form the composition $\sigma \circ \rho$, a nondeterministic function from S to U.
 a. What is the set of possible images of 1 under $\sigma \circ \rho$?
 b. What is the set of possible images of 2 under $\sigma \circ \rho$? of 3?
74. Let f be a function, $f: S \to T$.
 a. Show that for all subsets A and B of S, $f(A \cap B) \subseteq f(A) \cap f(B)$.
 b. Show that $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S if and only if f is one-to-one.
75. Let $\mathcal{C}$ be a collection of sets, and define a binary relation ρ on $\mathcal{C}$ as follows: For $S, T \in \mathcal{C}$, $S \rho T \leftrightarrow S$ is equivalent to T. Show that ρ is an equivalence relation on $\mathcal{C}$.
76. Group the following sets into equivalence classes according to the equivalence relation of Exercise 75.

$$A = \{2, 4\}$$
$$B = \mathbb{N}$$
$$\mathcal{C} = \{x \mid x \in \mathbb{N} \text{ and } (\exists y)(y \in \mathbb{N} \text{ and } x = 2*y)\}$$
$$D = \{a, b, c, d\}$$
$$E = \wp(\{1, 2\})$$
$$F = \mathbb{Q}^+$$

Exercises 77 and 78 involve programming with a functional language. Functional programming languages, as opposed to conventional (procedural) programming languages such as $C++$, Java, or Python, treat tasks in terms of mathematical functions. A mathematical function such as $f(x) = 2x$ transforms the argument 5 into the result 10. Think of a program as a big function to transform input into output. A functional programming language contains primitive functions as part of the language, and the programmer can define new functions as well. Functional programming languages support function composition, allowing for complex combinations of functions. Using the functional programming language Scheme, we can define the doubling function by

```
(define (double x)
   (* 2 x)
```

The user can then run the program and type

```
(double 5)
```

which produces an immediate output of 10.

77. a. Write a Scheme function to square a number.
    b. What is the output from the following user input?

```
(double (square 3))
```

78. Scheme also supports recursion, plus the usual control structures of procedural languages, such as conditional and iterative statements.
    a. Given the Scheme function

```
(define (mystery n)
   (cond ((= n 1) 1)
      (else (*n (mystery (- n 1))))))
```

what is the output of

```
(mystery 4)
```

    b. The "mystery" function is better known as _______.

SECTION 5.5 ORDER OF MAGNITUDE

Function Growth

Order of magnitude is a way of comparing the "rate of growth" of different functions. We know, for instance, that if we compute $f(x) = x$ and $g(x) = x^2$ for increasing values of x, the g values will be larger than the f values by an ever increasing amount. This difference in the rate of increase cannot be overcome by simply multiplying the f values by some large constant; no matter how large a constant we choose, the g values will eventually race ahead again. Our experience indicates that the f and g functions seem to behave in fundamentally different ways with respect to their rates of growth. In order to characterize this difference formally, we define a binary relation on functions.

Let S be the set of all functions with domain and codomain the nonnegative real numbers. We can define a binary relation on S by

$$f \rho g \leftrightarrow \text{there exist positive constants } n_0, c_1, \text{ and } c_2 \text{ such that, for all } x \geq n_0,$$
$$c_1 g(x) \leq f(x) \leq c_2 g(x)$$