Isomorphism of $\mathfrak{so}(3)$ and $\mathbb{R}^3$

Oftentimes, we claim that $\mathbb{R}^3 \cong \mathfrak{so}(3)$, and we proceed to do all our math in $\mathbb{R}^3$. This document seeks to solidify that claim.

Lie Algebras

A lie algebra is a vector space $\mathfrak{g}$ endowed with a lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

1. Bilinearity: $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$
2. Alternativity: $[x, x] = 0$
3. Jacobi Identity: $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$

For $x, y, z \in \mathfrak{g}$, $\alpha, \beta \in \mathbb{R}$.
Two examples of lie algebras are $\mathfrak{so}(3)$ and $\mathbb{R}^3$.

The Lie Algebra $\mathbb{R}^3$

It is well known that $\mathbb{R}^3$ forms a vector space over the real numbers. We claim that when we endow this vector space with the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, it becomes a lie algebra.

Proposition 1. $\mathbb{R}^3$ *endowed with the cross product* $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ *is a lie algebra*

Proof. Let $x, y, z \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$. Using properties of the cross product, we can show that the properties of the lie bracket hold.

1. Bilinearity: $(\alpha x + \beta y) \times z = (\alpha x) \times z + (\beta y) \times z = \alpha(x \times z) + \beta(y \times z)$
2. Alternativity: $x \times x = 0$
3. Jacobi Identity: $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) =$
   $\langle x, z\rangle y - \langle x, y\rangle z + \langle y, x\rangle z - \langle y, z\rangle x + \langle z, y\rangle x - \langle z, x\rangle y = 0$

□

The Lie Algebra $\mathfrak{gl}(n)$

The set $\mathfrak{gl}(n)$ consisting of $n \times n$ matrices can be shown to be a vector space over the real numbers. We will endow this vector space with the operation $[X, Y] = XY - YX$, where XY denotes matrix matrix multiplication between $X, Y \in \mathfrak{gl}(n)$. This operation is commonly called the commutator.

Proposition 2. $\mathfrak{gl}(n)$ *endowed with the commutator* $[\cdot,\cdot] : \mathfrak{gl}(n) \times \mathfrak{gl}(n) \to \mathfrak{gl}(n)$ *is a lie algebra.*

Proof. Let $X, Y, Z \in \mathfrak{gl}(n)$ and $\alpha, \beta \in \mathbb{R}$. Using the properites of matrix multiplication, we can show that the properties of the lie bracket hold.

1. Bilinearity: $[\alpha X + \beta Y, Z] = (\alpha X + \beta Y)Z - Z(\alpha X + \beta Y) =$
   $\alpha XZ + \beta YZ - \alpha ZX - \beta ZY = \alpha(XZ - ZX) + \beta(YZ - ZY) = \alpha[X, Z] + \beta[Y, Z]$
2. Alternativity: $[X, X] = XX - XX = 0$
3. Jacobi Identity: $[X, [Y, Z]] + [Z, [X, Y] + [Y, [Z, X]] = [X, YZ - ZY] + [Z, XY - YX] + [Y, ZX - XZ] =$
   $XYZ - XZY - YZX + ZYX + ZXY - ZYX - XYZ + YXZ + YZX - YXZ - ZXY + XZY = 0$

□

At this point we note that the set of skew symmetric matrices are a subspace of $\mathfrak{gl}(n)$ and specifically we have that when $n = 3$ the set of skew-symmetric matrices forms a subspace of $\mathfrak{gl}(3)$. We call this space of 3×3 skew-symmetric matrices $\mathfrak{so}(3)$. Although, $\mathfrak{so}(3)$ is a subspace of $\mathfrak{gl}(3)$, it remains to be shown that it is a subalgebra. To do this we must show that $\mathfrak{so}(3)$ is closed under the commutator.

Proposition 3. $\mathfrak{so}(3)$ *is a subalgebra of* $\mathfrak{gl}(3)$ *and therefore is a lie algebra.*

Proof. Let $X, Y \in \mathfrak{so}(3)$. Thus we have $X^T = -X$ and $Y^T = -Y$. Thus we have

$$[X, Y] = XY - YX = (-X^T)(-Y^T) - (-Y^T)(-X^T) = (YX)^T - (XY)^T = (YX - XY)^T = -(XY - YX)^T = -[X, Y]^T$$

Thus $[X, Y] \in \mathfrak{so}(3)$, and $\mathfrak{so}(3)$ is a subalgebra of $\mathfrak{gl}(3)$. □

Lie Algebra Isomorphisms

At this point we will explore transformations between lie algebras. As in most fields of abstract algebra, we can define *homomorphisms* and *isomorphisms* of lie algebras.

Homomorphisms of Lie Algebras

A lie algebra homomorphism is a mapping $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

1. φ is a linear map, $\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y)$.
2. $\varphi([x, y]_{\mathfrak{g}_1}) = [\varphi(x), \varphi(y)]_{\mathfrak{g}_2}$

Isomorphisms of Lie Algebras

A lie algebra isomorphism is a mapping $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ such that

1. φ is a lie algebra homomorhpism
2. φ is bijective

Isomorphism from $\mathbb{R}^3$ to $\mathfrak{so}(3)$

We define the following mapping from $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$, such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^\wedge = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

We claim that is mapping is a lie algebra isomorphism

Proposition 4. $\wedge : \mathbb{R}^3 \to \mathfrak{so}(3)$ *as defined above is a lie algebra isomorphism.*

Proof. It is straight-forward to show that $\wedge$ is a bijective mapping, so we will focus on showing that $\wedge$ is a lie algebra homomorphism. Let $x_1, x_2 \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$. We have that

1.
$$\begin{aligned}
(\alpha x_1 + \beta x_2)^\wedge &= \begin{bmatrix} 0 & -\alpha z_1 - \beta z_2 & \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 & 0 & -\alpha x_1 - \beta x_2 \\ -\alpha y_1 - \beta y_2 & \alpha x_1 + \beta x_2 & 0 \end{bmatrix} \\
&= \alpha \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & -z_2 & y_2 \\ z_2 & 0 & -x_2 \\ -y_2 & x_2 & 0 \end{bmatrix} \\
&= \alpha x_1^\wedge + \beta x_2^\wedge
\end{aligned}$$

2.
$$\begin{aligned}
(x_1 \times x_2)^\wedge &= \begin{bmatrix} -z_1 y_2 + y_1 z_2 \\ z_1 x_2 - x_1 z_2 \\ -y_1 x_2 + x_1 y_2 \end{bmatrix}^\wedge = \begin{bmatrix} 0 & y_1 x_2 - x_1 y_2 & z_1 x_2 - x_1 z_2 \\ -y_1 x_2 + x_1 y_2 & 0 & z_1 y_2 - y_1 z_2 \\ -z_1 x_2 + x_1 z_2 & -z_1 y_2 + y_1 z_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -z_1 z_2 - y_1 y_2 & y_1 x_2 & z_1 x_2 \\ x_1 y_2 & -z_1 z_2 - x_1 x_2 & z_1 y_2 \\ x_1 z_2 & y_1 z_2 & -y_1 y_2 - x_1 x_2 \end{bmatrix} - \begin{bmatrix} -z_1 z_2 - y_1 y_2 & x_1 y_2 & x_1 z_2 \\ y_1 x_2 & -z_1 z_2 - x_1 x_2 & y_1 z_2 \\ z_1 x_2 & z_1 y_2 & -y_1 y_2 - x_1 x_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -z_2 & y_2 \\ z_2 & 0 & -x_2 \\ -y_2 & x_2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -z_2 & y_2 \\ z_2 & 0 & -x_2 \\ -y_2 & x_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & -z_1 & y_1 \\ z_1 & 0 & -x_1 \\ -y_1 & x_1 & 0 \end{bmatrix} \\
&= [x_1^\wedge, x_2^\wedge]
\end{aligned}$$

□

At this moment we note that there is an inverse transformation, given by

$$\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}^\vee = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

This implies that in every sense $\mathfrak{so}(3) \cong \mathbb{R}^3$, because $\wedge$ and $\vee$ preserve all necessary properties of the lie algebra.