Standard Left-Invariant Kinematics for Two Moving Cameras

The homography is a matrix that relates the two images of the same planar scene. The homography can be written as

$$H = \gamma \left(R + \frac{Tn^T}{d} \right)$$

Where R is a rotation matrix, T is the translation between camera frames, n is the vector normal to the scene, d is the orthogonal distance to the scene and γ is a scale factor. The goal of this document is to derive the standard left invariant kinematics on $SL(3)$ given by

$$\dot{H} = HX$$

Where $H \in SL(3)$ is the homography and $X \in \mathfrak{sl}(3)$ is the group velocity. Furthermore, we wish to derive this group velocity with respect to intertial body velocities of each camera. We begin by laying down our coordinate frames. Suppose that we have two camera frames $\{A\}$ and $\{B\}$. Let R_B^A be the rotation matrix expressing the orientation of $\{B\}$ with respect to $\{A\}$ expressed in $\{A\}$, $T_{B/A}^A$ be the translation of $\{B\}$ with respect to $\{A\}$ expressed in $\{A\}$, n^B the normal vector of the planar scene expressed in frame $\{B\}$, and d_B be the orthogonal distance from frame $\{B\}$ to the planar scene.

We can then write the homography as

$$H_B^A = \gamma \left(R_B^A + \frac{T_{B/A}^A (n^B)^T}{d_B} \right)$$

It is important to note that n is NOT expressed in the same frame as T and R. This fact becomes very apparent in the derivation of the homography (See An Invitation to 3-D Vision Chapter 5, Ma & Soatto). The problem with writing the homography in this fashion, is that we don't measure derivative of R_B^A or $T_{B/A}^A$ directly. Thus, we will introduce an intermediate frame $\{0\}$ and first derive the dynamics in that frame. Consider the homography

$$H_A^0 = \gamma_A \left(R_A^0 + \frac{T_{A/0}^0 (n^A)^T}{d_A} \right)$$

We wish to write the left invariant kinematics in the form $\dot{H}_A^0 = H_A^0 X$, where $X \in \mathfrak{sl}(3)$. Taking the derivative yields

$$\begin{aligned}
\dot{H}_A^0 &= \frac{\dot{\gamma}_A}{\gamma_A} H_A^0 + \gamma_A \left(\dot{R}_A^0 + \frac{\dot{T}_{A/0}^0 (n^A)^T + T_{A/0}^0 (\dot{n}^A)^T}{d_A} - \frac{\dot{d}_A T_{A/0}^0 (n^A)^T}{d_A^2} \right) \\
&= \frac{\dot{\gamma}_A}{\gamma_A} H_A^0 + \gamma_A \left(R_A^0 \left(\omega_{A/0}^A\right)^\wedge + \frac{R_A^0 V_{A/0}^A (n^A)^T + T_{A/0}^0 (n^A)^T \left(\omega_{A/0}^A\right)^\wedge}{d_A} + \frac{(n^A)^T V_{A/0}^A T_{A/0}^A (n^A)^T}{d_A^2} \right) \\
&= \frac{\dot{\gamma}_A}{\gamma_A} H_A^0 + \gamma_A \left(\left[R_A^0 + \frac{T_{A/0}^0 (n^A)^T}{d_A} \right] \left(\omega_{A/0}^A\right)^\wedge + \left[R_A^0 + \frac{T_{A/0}^A (n^A)^T}{d_A} \right] \frac{V_{A/0}^A (n^A)^T}{d_A} \right) \\
&= H_A^0 \left(\left(\omega_{A/0}^A\right)^\wedge + \frac{V_{A/0}^A (n^A)^T}{d_A} + \frac{\dot{\gamma}_A}{\gamma_A} I_3 \right)
\end{aligned}$$

Now, we recall the definition of $\mathfrak{sl}(3)$

$$\mathfrak{sl}(3) = \left\{ X \in \mathbb{R}^{3\times 3} \mid \mathrm{tr}(X) = 0 \right\},$$

which imposes the constraint

$$\mathrm{tr}\left(\left(\omega_{A/0}^A\right)^\wedge + \frac{V_{A/0}^A (n^A)^T}{d_A} + \frac{\dot{\gamma}_A}{\gamma_A} I_3 \right) = \frac{(n^A)^T V_{A/0}^A}{d_A} + 3\frac{\dot{\gamma}_A}{\gamma_A} = 0$$

Thus we have that

$$\frac{\dot{\gamma}_A}{\gamma_A} = -\frac{(n^A)^T V_{A/0}^A}{3 d_A}$$

Therefore, our standard left-invariant kinematics for H_A^0 are

$$\dot{H}_A^0 = H_A^0 \left(\left(\omega_{A/0}^A\right)^\wedge + \frac{V_{A/0}^A (n^A)^T}{d_A} - \frac{(n^A)^T V_{A/0}^A}{3 d_A} I_3 \right) = H_A^0 X_A$$

Similarly for H_B^0, we have

$$\dot{H}_B^0 = H_B^0 \left(\left(\omega_{B/0}^B \right)^\wedge + \frac{V_{B/0}^B (n^B)^T}{d_B} - \frac{(n^B)^T V_{B/0}^B}{3 d_B} I_3 \right) = H_B^0 X_B$$

Now, we have written two systems in terms of quantities that are easily measured. To combine these systems into one system, we define $H_B^A = (H_A^0)^{-1} H_B^0$, and take the derivative as follows

$$\begin{aligned}
\dot{H}_B^A &= \frac{d}{dt} \left((H_A^0)^{-1} \right) H_B^0 + (H_A^0)^{-1} \dot{H}_B^0 \\
&= -(H_A^0)^{-1} \dot{H}_A^0 (H_A^0)^{-1} H_B^0 + (H_A^0)^{-1} \dot{H}_B^0 \\
&= -X_A (H_A^0)^{-1} H_B^0 + (H_A^0)^{-1} H_B^0 X_B \\
&= H_B^A X_B - H_B^A (H_B^A)^{-1} X_A H_B^A \\
&= H_B^A \left(X_B - (H_B^A)^{-1} X_A H_B^A \right) \\
&= H_B^A \left(X_B - \mathrm{Ad}_{(H_B^A)^{-1}} (X_A) \right) \\
&= H_B^A \left(X_B - \mathrm{Ad}_{H_A^B} (X_A) \right)
\end{aligned}$$

Thus, we have standard left-invariant kinematics for a system of two moving cameras.