Practice Sessions
Astrophysical Simulations

Part 6a: Two-body problem (Assignment)

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Two-body motion

Consider two point-like bodies with masses m_1 and m_2 gravitating about each other. Denoting the relative position vector as r(t), the motion can be described by the equations

$$\begin{aligned}\dot{\boldsymbol{r}} &= \boldsymbol{v}\\ \dot{\boldsymbol{v}} &= \boldsymbol{a}(\boldsymbol{r}) = -\mu\,\frac{\boldsymbol{r}}{|\boldsymbol{r}|^3}\end{aligned}$$

with v(t) the velocity, a(t) the acceleration, $\mu = G(m_1 + m_2)$ the reduced mass, and G the gravitational constant.

The constant energy of the system, per unit mass, is given by

$$\mathcal{E} = \frac{1}{2}\,|\boldsymbol{v}|^2 - \frac{\mu}{|\boldsymbol{r}|}$$

Since the motion is planar, we can – without loss of generality – choose the initial conditions such that the motion is restricted to the xy-plane, so that we can use vectors with only two components x and y.

Leapfrog integrator

With time step h, we denote $t_i \equiv t_0 + ih$, $r_i \equiv r(t_i)$, $v_i \equiv v(t_i)$ and $a_i \equiv a(t_i)$.

Given the initial conditions r_0 and v_0 , the leapfrog integrator scheme for the equations of motion on the previous slide can then be written as

$$\begin{aligned} \boldsymbol{r}_{\frac{1}{2}} &= \boldsymbol{r}_0 + \frac{1}{2}h\boldsymbol{v}_0 + \frac{1}{8}h^2\boldsymbol{a}_0 \\ \boldsymbol{v}_{n+1} &= \boldsymbol{v}_n + h\boldsymbol{a}_{n+\frac{1}{2}} \\ \boldsymbol{r}_{n+\frac{1}{2}} &= \boldsymbol{r}_{n-\frac{1}{2}} + h\boldsymbol{v}_n \end{aligned}$$

where we have neglected terms of order $O(h^3)$.

Note: to calculate the energy during the integration, you need a value for r and v at the same point in time. However the staggered leapfrog scheme only calculates values for r and v that are half a time step away from each other. Use a simple approximation to resolve this issue.

Assignment: solve the two-body problem

Part 1

- Use the leapfrog integrator to solve the two-body problem in the xy-plane for the following initial conditions and parameters

  $r_0 = (1, 0) \qquad v_0 = (0, 0.06) \qquad \mu = 0.01 \qquad h = 0.01$
- Plot the orbits in the xy-plane, assuming that one of the bodies stays at rest in the origin
- Plot the absolute value of the relative error on the energy in function of time for a couple consecutive orbits, and interpret the results

Part 2

- Change the initial conditions to obtain a much more eccentric orbit, and integrate over 15 or more orbits still using h=0.01; make the same plots as above
- Change the time step to h=0.001 and note the differences

Assignment: further guidance

Keep it simple

- Use a Vec class (see example) to represent r and v
- Use functions, e.g. to calculate acceleration and energy
- Hardcode all formulas and constants (i.e. no user input)
- Write data to output file as it is being computed (no memory arrays)

Work in steps, and verify your results step by step

- Implement the initial conditions and the first integration step; check the output manually (at least the approximate values)
- Add the integration loop and plot the orbit
- Calculate the relative energy error and plot it

Use log scale

Seek help

- Use the slides
- Look up things on the web
- Ask me to review your code or to help when you get stuck

Questions?