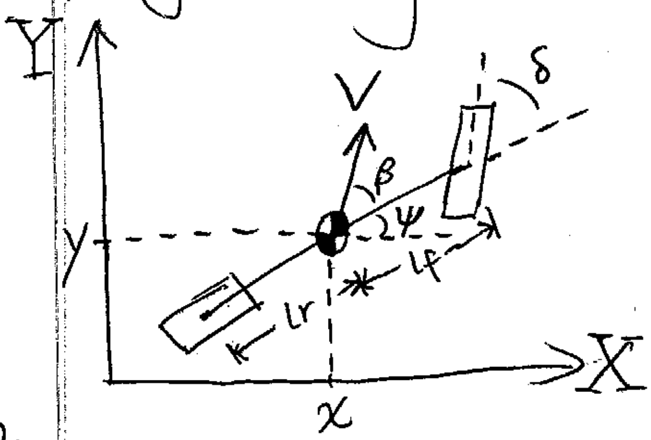
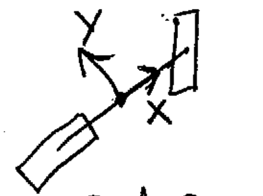
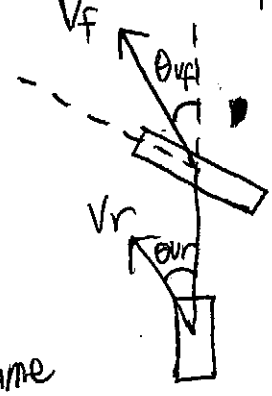
XingYu Wang | 2D Bicycle Model 7/28/2019





Body frame



FBD:

X, Y = Global coordinate frame

x, y = body coordinate frame

δ = steering angle

l_r, l_f = distance from rear/front wheel to center of mass

ψ = body yaw angle

β = velocity angle w.r.t body

θ_{vf}, θ_{vr} = front/rear ~~tire slip angle~~ velocity angle

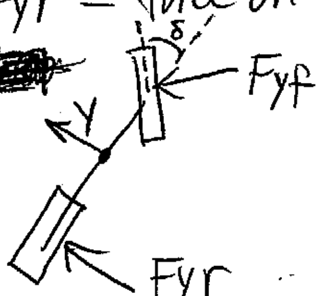
- **Lateral Dynamics:**

$$\Sigma F_x = 0 : \quad m a_y = F_{yf} \cos\delta + F_{yr} \qquad (1)$$

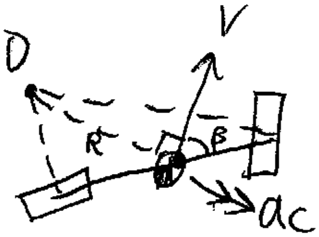
m = mass of car

a_y = acceleration in body y direction

F_{yf}, F_{yr} = force on front/rear tire in y-direction



$$\beta = \tan^{-1}\left(\frac{\dot{y}}{\dot{x}}\right) \qquad (2)$$



O = center of rotation

a_c = centripetal acceleration $= \frac{V^2}{R} = V \cdot \dot{\psi}$ (3)

$a_y = \ddot{y} + a_c \cos\beta = \ddot{y} + V\psi\cos\beta = \ddot{y} + \dot{x}\dot{\psi}$ (4)

From (1) & (4) $\Rightarrow \ddot{y} = -\dot{\psi}\dot{x} + \frac{1}{m}(F_{yf}\cos\delta + F_{yr})$ (5)

$\ddot{\psi} I_z = l_f F_{yf} - l_r F_{yr}$ (6), I_z = total moment of inertia

α_f, α_r = front/rear tire slip angle

$\alpha_f = \delta - \theta_{vf}$, $\alpha_r = 0 - \theta_{vr}$

Using small angle approximation:

$$\theta_{vf} = \frac{\dot{y}_f}{\dot{x}_f} = \frac{\dot{y} + l_f\dot{\psi}}{\dot{x}}, \quad \theta_{vr} = \frac{\dot{y} - l_r\dot{\psi}}{\dot{x}} \quad (7)$$

Linear approximation of tire force is:

$$F_{yf} = 2C_\alpha\left(\delta - \frac{\dot{y} + l_f\dot{\psi}}{\dot{x}}\right) \quad (8)$$

$$F_{yr} = 2C_\alpha\left(-\frac{\dot{y} - l_r\dot{\psi}}{\dot{x}}\right) \quad (9)$$

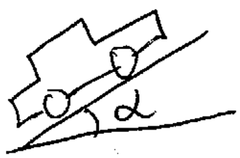
C_α is called conering stiffness

Longitutinal Dynamics

$$m a_x = F_x - F_{load} \quad (10)$$

a_x = acceleration due to ~~tog~~ longitutinal force

F_x = tireforce , F_{load} = total load force



$$F_{load} = F_{aero} + R_x + F_g$$

$$F_{aero} = \frac{1}{2} C_{aero} \rho A \dot{x}^2 = C_{aero} \dot{x}^2 \quad \text{(aerodynamic drag)}$$

$$R_x \approx C_r \dot{x} \quad \text{(rolling friction)}$$

F_g = gravitational force from an incline at angle α

$$F_g = mg \sin\alpha$$

$$\Rightarrow F_{load} = C_{aero} \dot{x}^2 + C_r \dot{x} + mg\sin\alpha \quad (11)$$

Let w_e = engine rotational speed

w_w = wheel rotational speed

GR = gear ratio , S = slip ration ,

x_θ = throttle position $\in [0, 1]$, F_{max} = max tire force

T_e = engine torque , r_e = effective radius of tire

we have: $w_w = (GR) w_e$

$$S = \frac{w_w \cdot r_e - \dot{x}}{\dot{x}}, \quad F_x = S \cdot F_{max} \quad (12)$$

Engine torque is computed from throttle input:

$$T_e = x_\theta (a_0 + a_1 w_e + a_2 w_e^2)$$

where a_0, a_1, and a_2 are coefficients

$$\dot{w}_e = \frac{1}{J_e} (T_e - GR \cdot r_e \cdot F_{load}) \quad (13)$$

where J_e is the combined power system inertia

From (10) ~~[illegible]~~ : we have $\ddot{x} = a_x + \dot{\psi} V \cdot \sin\theta$

$$\Rightarrow \ddot{x} = a_x + \dot{\psi}\dot{y} \quad (14)$$

From (14), (12), and (13), we have:

$$(15) \quad \ddot{x} = \dot{\psi}\dot{y} + \frac{1}{m}\left[\frac{(GR)w_e \cdot r_e - \dot{x}}{\dot{x}} \cdot F_{max} - C_{aero}\dot{x}^2 - C_r\dot{x} - mg\sin\alpha\right]$$

$$(16) \quad \dot{w}_e = \frac{1}{J_e}\left(X_\theta(a_0 + a_1 w_e + a_2 w_e^2) - (GR) \cdot r_e \cdot (C_{aero}\dot{x}^2 + C_r\dot{x} + mg\sin\alpha)\right)$$

- Global coordinates

$$\dot{X} = \dot{x}\cos\psi - \dot{y}\sin\psi$$

$$\dot{Y} = \dot{x}\sin\psi + \dot{y}\cos\psi$$

Complete system equation

$$
\begin{cases}
\ddot{y} = -\dot{\psi}\dot{x} + \dfrac{2C_a}{m}\left[\cos\delta\left(\delta - \dfrac{\dot{y}+l_f\dot{\psi}}{\dot{x}}\right) - \dfrac{\dot{y}-l_r\dot{\psi}}{\dot{x}}\right] \\
\ddot{\psi} = \dfrac{2l_fC_a}{I_z}\left(\delta - \dfrac{\dot{y}+l_f\dot{\psi}}{\dot{x}}\right) - \dfrac{2l_rC_a}{I_z}\left(-\dfrac{\dot{y}-l_r\dot{\psi}}{\dot{x}}\right) \\
\ddot{x} = \dot{\psi}\dot{y} + \dfrac{1}{m}\left[\dfrac{(GR)W_e\cdot r_e - \dot{x}}{\dot{x}}\cdot F_{max} - C_{aero}\dot{x}^2 - C_r\dot{x} - mg\sin\alpha\right] \\
\dot{W}_e = \dfrac{1}{J_e}\left[x_\theta(a_0+a_1W_e+a_2W_e^2) - (GR)\cdot r_e\cdot(C_{aero}\dot{x}^2 + C_r\dot{x} + mg\sin\alpha)\right] \\
\dot{X} = \dot{x}\cos\psi - \dot{y}\sin\psi \\
\dot{Y} = \dot{x}\sin\psi + \dot{y}\cos\psi
\end{cases}
$$

State variables:

- x = ~~body~~ location in body x frame
- y = location in body y frame
- X = ------ global X frame
- Y = ------ global y frame
- ψ = body yaw angle in global frame
- W_e = engine rotational speed

Controled variables:

- δ = steering angle
- x_θ = throttle position

(assuming no ~~breaking~~ braking)

All other variables are constants.