Beyond Black Scholes - Local Volatility and Stochastic Volatility

A Comprehensive Analysis of Theory, Calibration, and Practical Application in Exotic options Derivatives

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Abstract

This research report presents an exhaustive examination of advanced volatility modeling frameworks within modern quantitative finance, specifically addressing the transition from constant volatility assumptions to local volatility (LV) and stochastic volatility (SV). Motivated by the persistent empirical failure of the Black–Scholes–Merton (BSM) model to capture the implied volatility skew and smile observed in equity markets, this document provides the theoretical understanding, numerical implementation, and empirical performance of industry-standard models. We rigorously analyze Dupire's local volatility model and Heston's stochastic volatility model. Special emphasis is placed on accurate pricing of exotic derivatives such as barrier and vanilla options. The report details robust calibration protocols using Monte Carlo and Fast Fourier Transform (FFT) methods.

1 Introduction

1.1 Option Pricing

Option contracts give the holder the right but not the obligation to buy (or sell) an underlying asset at some point in the future at a predetermined price. These contracts themselves have a value at expiration that is easy to determine, but what should we pay today for this contract?

We model stock prices as a random variable; option prices are functions of this underlying randomness so they are also random.

Under the risk-neutral measure $\mathbb{Q}$ the price of a European call with strike K and maturity T is

$$C(K,T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}\big[\max(S_T - K, 0)\big], \tag{1}$$

where r is the risk-free rate. Equivalently, using the risk-neutral density $f_{\mathbb{Q}}(S_T)$,

$$C(K,T) = e^{-rT}\int_K^{\infty} (S_T - K)\, f_{\mathbb{Q}}(S_T)\, dS_T. \tag{2}$$

Some exotic options, such as barrier options and vanilla options, will be discussed in later sections.

1.2 Mathematical Framework

1.2.1 Fourier Transforms

The Fourier transform maps a function from the time domain to the frequency domain:

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt. \tag{3}$$

- $f(t)$ is the input signal in the *time domain.*
- $\mathcal{F}(\omega)$ is the output signal in the *frequency domain.*
- $e^{-i\omega t}$ is the complex exponential, the basis function of the transform.

1.2.2 Fast Fourier Transforms

The Fast Fourier Transform (FFT) is an efficient algorithm for computing the Discrete Fourier Transform (DFT)

$$X_k = \sum_{n=0}^{N-1} x_n \, e^{-2\pi i \frac{kn}{N}}, \tag{4}$$

which reduces computational complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(N \log N)$ by:

- recursively dividing the transform into smaller sub-transforms,
- exploiting symmetries in the complex exponentials,
- using a divide-and-conquer approach.

Key properties:

- The input size is typically required to be a power of two, $N = 2^m$.
- Widely used in signal processing, numerical computing, and financial applications.
- The most common variant is the Cooley–Tukey FFT algorithm.

1.2.3 Inverse Fast Fourier Transform (IFFT)

The Inverse FFT (IFFT) reverses the FFT operation, transforming data from the frequency domain back to the time domain:

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \, e^{2\pi i \frac{kn}{N}}, n \qquad n = 0, 1, \ldots, N-1. \tag{5}$$

1.2.4 Characteristic Functions

The characteristic function of a random variable X is

$$\phi_X(t) = \mathbb{E}\left[e^{itX}\right]. \tag{6}$$

If X admits a density $f_X(x)$, then

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} \, f_X(x) \, dx. \tag{7}$$

Characteristic functions are central in FFT-based option pricing methods because they provide closed-form transforms for many stochastic models.

1.2.5 Stochastic Differential Equations

Most stochastic processes are described by stochastic differential equations (SDEs), typically of the form

$$dX_t = a(t, X_t) \, dt + b(t, X_t) \, dW_t, \tag{8}$$

where a and b are $\mathcal{F}_t$-measurable coefficients and W_t is a standard Brownian motion.

2 Background: Black–Scholes and the Implied Volatility Surface

The foundational Black–Scholes–Merton model posits that the asset price S_t follows a geometric Brownian motion under the risk-neutral measure $\mathbb{Q}$:

$$dS_t = (r - q)S_t\,dt + \sigma S_t\,dW_t, \tag{9}$$

where r is the risk-free rate, q the dividend yield, σ the constant volatility, and W_t a standard Brownian motion.

Realistically, the market (and contract selection) determine (S, K, r, T), and the Black–Scholes pricing map is

$$\mathrm{BS} : (\sigma \mid S, K, r, T) \mapsto C, \tag{10}$$

i.e. given market and contract parameters we choose σ to price the option.

First, let's see what a volatility surface looks like if we use the Black–Scholes model and compare it with the market volatility surface.

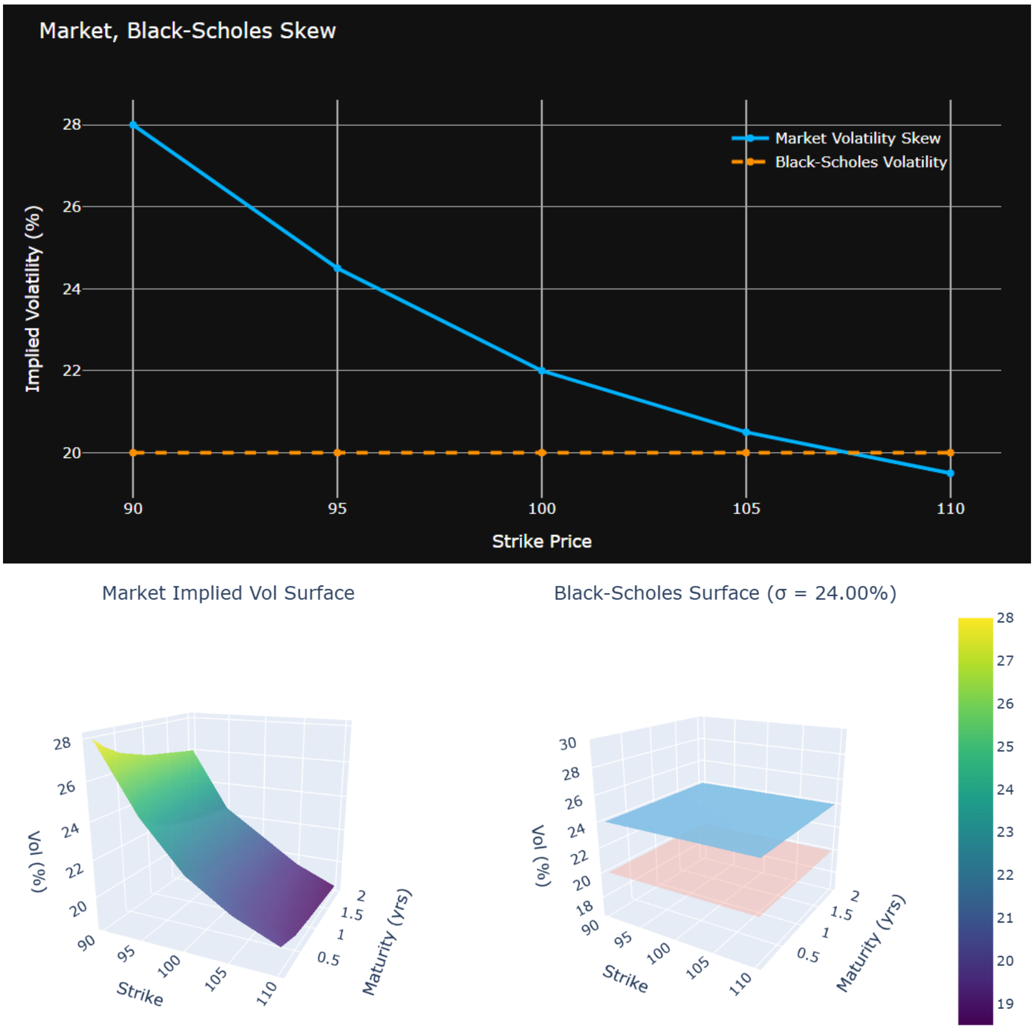

Figure 1: Black–Scholes constant-volatility surface vs. observed market implied-volatility surface.

How Black–Scholes compares to the market. The Black–Scholes model assumes a single constant volatility parameter σ for pricing, so the model-implied volatility surface is *flat* in strike and maturity when expressed as implied vol:

$$\mathrm{BS} : \;(\sigma \mid S, K, r, T) \;\mapsto\; C, \qquad \text{with } \sigma = \text{constant (independent of } K, T). \tag{11}$$

In contrast, market-implied volatilities typically exhibit:

- **Smile / skew:** implied volatility varies with strike (often higher for deep out-of-the-money puts — a skew).
- **Term structure:** implied volatility varies with maturity.
- **Local curvature:** small-scale curvature and non-flat surfaces that cannot be captured by a single constant σ.

Consequences of the mismatch.

- *Static mispricing:* A constant-volatility model will underprice some strikes and overprice others.
- *Calibration failure:* You cannot calibrate a single σ to match an entire market surface — at best you can match one point (a particular K, T).
- *Hedging risk:* Delta/vega hedging strategies derived from Black–Scholes will perform poorly when the true underlying volatility is strike- and time-dependent.

What to do about it. To obtain consistent prices across strikes and tenors we must move beyond constant-volatility Black–Scholes models. Two common directions are:

- **Local volatility models (Dupire):** construct a local volatility surface $\sigma_{\text{loc}}(S, t)$ that reproduces the market implied-volatility surface (Dupire's formula links $\partial_T C, \partial_{KK} C$ with σ_{loc}).
- **Stochastic volatility models (e.g. Heston):** model volatility as a separate stochastic process, which captures smile dynamics and provides more realistic hedging behavior (at the expense of extra parameters and calibration complexity).

Both approaches require robust numerical tools (FFT/COS, PDE solvers, Monte Carlo) and careful calibration (regularisation, arbitrage-free interpolation) to produce surfaces that are both *consistent* with observed market prices and *stable* for quoting and hedging.

3 Local Volatility Model

The Local Volatility (LV) model, introduced by Dupire (1994) and Derman and Kani (1994), represents the most exclusive extension of Black-Scholes capable of perfectly fitting the observed implied volatility surface.

The local volatility framework is arguably the simplest extension of Black–Scholes. It assumes that the stock's risk-neutral dynamics satisfy

$$dS_t = (r - q)S_t \, dt + \sigma_\ell(t, S_t) \, S_t \, dW_t, \tag{12}$$

so that the instantaneous volatility, $\sigma_\ell(t, S_t)$, is now a function of both time and the stock price. The key result of the local volatility framework is the *Dupire formula*, which links the local volatilities $\sigma_\ell(t, S_t)$ to the market-implied volatility surface.

The Dupire Formula

Let $C = C(K, T)$ be the price of a European call option as a function of strike K and time-to-maturity T. Then the local volatility function satisfies

$$\sigma_\ell^2(T, K) \; = \; \frac{\dfrac{\partial C}{\partial T} + (r - q) \, K \, \dfrac{\partial C}{\partial K} + q \, C}{\dfrac{K^2}{2} \, \dfrac{\partial^2 C}{\partial K^2}}. \tag{13}$$

Assumptions required for the above inversion to be valid: (a) continuum of arbitrage-free call prices with enough smoothness in K and T (b) price surface generated by a continuous diffusion (no jumps).

Given the implied volatility surface we can easily compute the corresponding call option price surface which is the graph of C(K, T) as a function of K and T. It is then clear from The Dupire Formula that we need to take first and second derivatives of this latter surface with respect to strike and first derivatives with respect to time-to-maturity in order to compute the local volatilities. Calculating the local volatilities from The Dupire Formula is therefore difficult and can be unstable as computing derivatives numerically can itself be very unstable. As a result, it is necessary to use a sufficiently smooth Black-Scholes implied volatility surface when calculating local volatilities using The Dupire Formula.

It is also possible to write the Dupire formula in terms of the implied volatilities rather than the call option prices. One can then work directly with the implied volatility surface to compute the local volatilities.

Now, let's try to simulate Local Volatility surface for a given market volatility surface

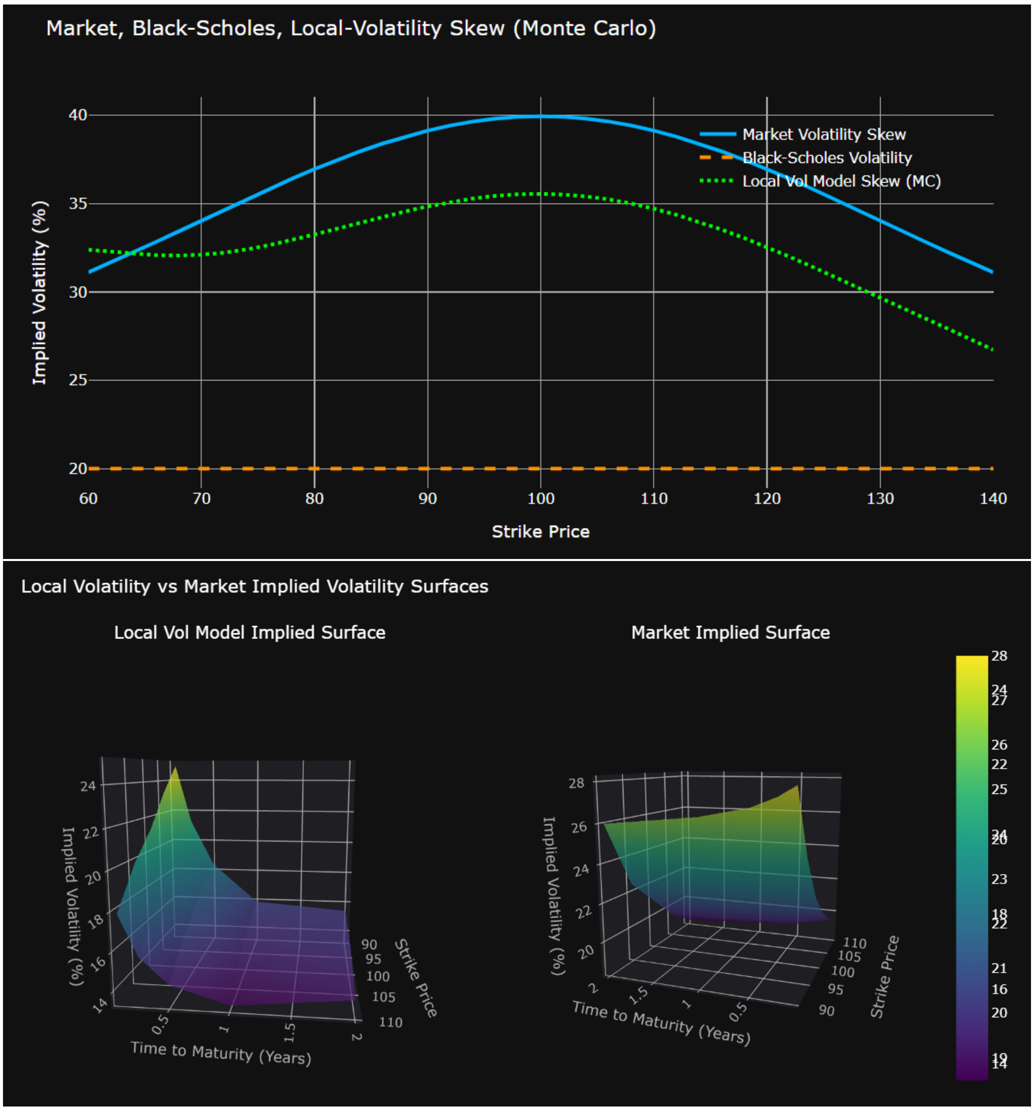

Figure 2: Local Volatility Dupire surface vs. observed market implied-volatility surface.

We observe that the constructed local volatility surface exhibits variation in implied volatility across both strike and time to maturity. However, the parameters of the current model do not yet fit the market-implied volatility surface.

This behaviour is entirely expected: the model has not been calibrated to market data. Until a calibration step is performed, the local volatility surface merely reflects the chosen initial parameterisation rather than the structure implied by observed option prices.

The market dictates prices for a set of contracts ζ (a vector of m instruments) $\zeta \in \mathbb{R}^m$.

Given a parameter set $\theta \in \Theta \subset \mathbb{R}^n$ and a model framework $\mathcal{M}$ we can produce model prices for the given contracts via the pricing map

$$P : \mathcal{M}(\theta, \zeta) \longmapsto \mathbb{R}^m, \qquad P\big(\mathcal{M}(\theta, \zeta)\big) \mapsto \mathbb{R}^m. \tag{14}$$

For a *Local Volatility* model we may write the risk–neutral dynamics as

$$dS_t = (r - q)S_t\,dt + \sigma_{\text{loc}}(t, S_t)\,S_t\,dW_t, \tag{15}$$

so the model class is $\mathcal{M} = \mathcal{L}$ (Local Volatility) and the natural "parameter" is the local volatility function itself. Hence one can characterise the parameter set by

$$\theta = \sigma_{\text{loc}}(\cdot, \cdot) \quad \Longrightarrow \quad \mathcal{M} = \mathcal{L}, \qquad P\big(\mathcal{L}(\sigma_{\text{loc}}, \zeta)\big) \mapsto \mathbb{R}^m. \tag{16}$$

In practice we do not work with an infinite-dimensional object directly; common parametrisations include

- a discretised surface $\{\sigma_{\text{loc}}(S_i, t_j)\}_{i,j}$ on a grid (interpolated between points),
- a finite-dimensional parametric family $\sigma_{\text{loc}}(S, t; \phi)$ with parameters $\phi \in \mathbb{R}^n$,
- a basis expansion (splines, PCA basis, neural network) with coefficients forming θ.

We therefore often write the parameter vector as $\theta = \phi$ (the coefficients / grid values).

Exact (theoretical) inversion – Dupire. In the ideal frictionless, continuously-observed limit the Dupire formula provides an exact relationship between the market call price surface $C(K, T)$ and the local variance:

$$\sigma^2_{\text{loc}}(T, K) = \frac{\partial_T C(K, T) + (r - q)K\,\partial_K C(K, T) + qC(K, T)}{\frac{1}{2}K^2\,\partial_{KK} C(K, T)}. \tag{17}$$

Thus, if one knows the arbitrage-free call-price surface (smooth enough to differentiate), one can recover a local volatility surface $\sigma_{\text{loc}}(T, K)$ that *exactly* reproduces market European option prices:

$$\mathcal{C}_{\text{Dupire}} : \ C(\cdot, \cdot) \longmapsto \sigma_{\text{loc}}(\cdot, \cdot), \qquad P\big(\mathcal{L}(\sigma_{\text{loc}}, \zeta)\big) = \text{market prices (theoretical equality)}. \tag{18}$$

Calibration and numerical reality. In real markets the call surface is noisy and only observed at discrete (K, T) points. Thus calibration is an inverse problem and is usually performed by optimisation or numerical inversion:

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} \ \mathcal{L}\Big(P\big(\mathcal{L}(\theta, \zeta)\big),\ \text{market}(\zeta)\Big) + \lambda \mathcal{R}(\theta), \tag{19}$$

where $\mathcal{L}(\cdot, \cdot)$ is a loss (e.g. weighted least squares), $\mathcal{R}$ a regulariser (enforcing smoothness / no-arbitrage), and λ a penalty weight.

Once a parametrisation is chosen, pricing under the local-vol model requires numerical methods:

- PDE solvers (finite-difference / Crank–Nicolson) for the forward/backward Kolmogorov PDE,
- Monte Carlo simulation (Euler discretisation with local $\sigma_{\text{loc}}(t, S)$),

- transform / eigenfunction methods in special cases.

Consequently, the computed model prices $\widetilde{P}$ are approximations:

$$\widetilde{P}\big(\mathcal{L}(\theta,\zeta)\big) \approx P\big(\mathcal{L}(\theta,\zeta)\big). \tag{20}$$

Practical remarks.

- **Fit:** Local volatility can be calibrated to exactly match European option prices.
- **Stability:** Dupire inversion is sensitive to noise; smoothing and regularisation are essential.
- **Dynamics:** While vanillas are matched, path-dependent products may be mispriced.
- **Implementation:** Results depend strongly on grid design, interpolation, and numerical schemes.

Summary diagram

$$\text{market prices} \xrightarrow{\text{interpolate / smooth}} C(K,T) \xrightarrow{\text{Dupire / optimization}} \widehat{\sigma}_{\text{loc}}(T,K) \xrightarrow{\text{PDE/MC}} \widetilde{P}\big(\mathcal{L}(\widehat{\sigma}_{\text{loc}},\zeta)\big) \approx \text{market p} \tag{21}$$

4 Heston Stochastic Volatility Model

4.1 Model Dynamics

The Heston (1993) model is a widely used stochastic volatility model which introduces a mean-reverting square-root variance process coupled to the asset price. It is popular because it admits a semi-closed form expression for the characteristic function of log-price under risk-neutral measure, which in turn gives efficient numerical evaluation of European option prices (via numerical integration or Fourier methods).

Under the usual risk-neutral measure the Heston model for the asset price S_t and instantaneous variance v_t is

$$dS_t = (r-q)S_t\,dt + \sqrt{v_t}\,S_t\,dW_t^{(s)}, \tag{22}$$

$$dv_t = \kappa(\theta - v_t)\,dt + \xi\sqrt{v_t}\,dW_t^{(vol)}, \tag{23}$$

with correlation

$$d\langle W^{(s)}, W^{(vol)}\rangle_t = \rho\,dt, \tag{24}$$

where $W_t^{(s)}$ and $W_t^{(\mathrm{vol})}$ are standard $\mathbb{Q}$-Brownian motions with constant correlation coefficient ρ.

where the parameters are:

- r: risk-free rate,
- q: continuous dividend yield,
- $\kappa > 0$: mean-reversion speed of variance,
- $\theta > 0$: long-run variance (level),
- $\xi > 0$: volatility of variance ("vol-of-vol"),
- $\rho \in [-1,1]$: correlation between returns and variance,
- v_0: initial variance.

Whereas the local volatility model is a complete model, Heston's stochastic volatility model is an incomplete model. This should not be too surprising as there are two sources of uncertainty in the Heston model, $W_t^{(s)}$ and $W_t^{(\mathrm{vol})}$, but only one risky security and so net security is replicable. Put another way, while the drift in (22) must be $r-q$ under any EMM with the cash account as numeraire, we could use Girsanov's Theorem to change the drift in (23) in infinitely many different ways without changing the drift in (22).

To see this let us first suppose that the $\mathbb{P}$-dynamics of S_t and v_t satisfy

$$dS_t = \mu_t S_t\, dt + \sqrt{v_t}\, S_t\, dW_t^{(1)}, \tag{25}$$
$$dv_t = \nu_t\, dt + \gamma\sqrt{v_t}\big(\rho\, dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(2)}\big), \tag{26}$$

where μ_t and ν_t are some $\mathcal{F}_t$-adapted processes, and $W_t = (W_t^{(1)}, W_t^{(2)})$ is a standard 2-dimensional $\mathbb{P}$-Brownian motion. Let us now define

$$L_t := \exp\left(-\int_0^t \eta_s^{(1)}\, dW_s^{(1)} - \frac{1}{2}\int_0^t (\eta_s^{(1)})^2\, ds\right), \tag{27}$$

for $t \in [0,T]$ and where $\eta_t = (\eta_t^{(1)}, \eta_t^{(2)})$ is a 2-dimensional adapted process. Then Girsanov's Theorem implies

$$\widetilde{W}_t := W_t + \int_0^t \eta_s\, ds \tag{28}$$

is a standard 2-dimensional $\mathbb{Q}$-Brownian motion where $d\mathbb{Q}/d\mathbb{P} = L_T$. In particular the $\mathbb{Q}$-dynamics of S_t and v_t satisfy

$$dS_t = \big(\mu_t - \sqrt{v_t}\,\eta_t^{(1)}\big) S_t\, dt + \sqrt{v_t}\, S_t\, d\widetilde{W}_t^{(1)}, \tag{29}$$
$$\begin{aligned} dv_t = {}& \Big(\nu_t - \gamma\sqrt{v_t}\rho\,\eta_t^{(1)} - \gamma\sqrt{v_t}\sqrt{1-\rho^2}\,\eta_t^{(2)}\Big)dt \\ & + \gamma\sqrt{v_t}\big(\rho\, d\widetilde{W}_t^{(1)} + \sqrt{1-\rho^2}\, d\widetilde{W}_t^{(2)}\big). \end{aligned} \tag{30}$$

In order for $\mathbb{Q}$ to be an EMM it is only necessary that

$$\mu_t - \sqrt{v_t}\,\eta_t^{(1)} = r - q, \tag{31}$$

and we are free to choose $\eta_t^{(2)}$ subject only to the usual integrability constraints. This means that there are infinitely many EMMs and so the model is incomplete. In the Heston model we choose $\eta_t^{(2)}$ so that

$$\nu_t - \gamma\sqrt{v_t}\rho\,\eta_t^{(1)} - \gamma\sqrt{v_t}\sqrt{1-\rho^2}\,\eta_t^{(2)} = \kappa(\theta - v_t) \tag{32}$$

is satisfied. We therefore recover (22) and (23) once we identify $\widetilde{W}_t^{(1)}$ with $W_t^{(s)}$ and $\rho\,\widetilde{W}_t^{(1)} + \sqrt{1-\rho^2}\,\widetilde{W}_t^{(2)}$ with $W_t^{(\mathrm{vol})}$ (using Lévy's Theorem).

4.2 The Pricing PDE

Because Heston's model is incomplete it is not possible to price options using the replication arguments that apply to complete market models. Instead, one assumes $\mathbb{Q}$-dynamics as in (22) and (23) and then prices all securities using this EMM. (The model parameters are chosen by calibrating model prices to observable market prices.) The pricing PDE that the price, $C(t, S_t, \sigma_t)$, of any derivative security must satisfy in Heston's model is given by

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma\gamma S\frac{\partial^2 C}{\partial S \partial \sigma} + \tfrac{1}{2}\gamma^2\frac{\partial^2 C}{\partial \sigma^2} + (r-q)S\frac{\partial C}{\partial S} + \kappa(\theta - \sigma)\frac{\partial C}{\partial \sigma} = rC. \tag{33}$$

4.3 Simulating the Heston Model

Now, let us try to simulate the stochastic Heston Volatility surface for a given market volatility surface

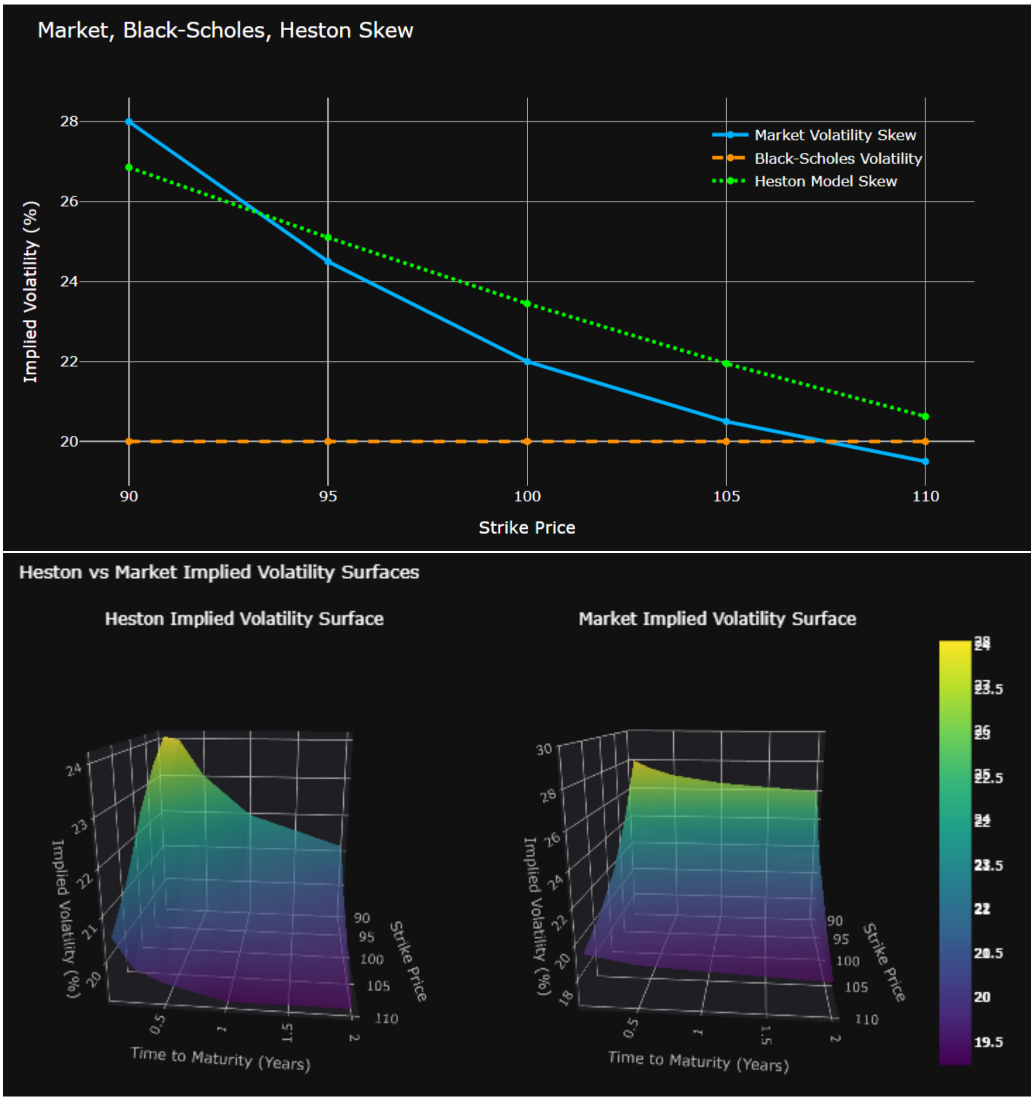

Figure 3: Stochastic Heston Volatility surface vs. observed market implied-volatility surface.

We observe that the constructed local volatility surface exhibits variation in implied volatility across both strike and time to maturity and also it captures the market skew. However, the parameters of the current model do not yet fit the market-implied volatility surface.

An interesting question that arises is whether or not Heston's model accurately represents the dynamics of stock prices. This question is often reduced in practice to the less demanding question of how well the Heston model captures the volatility skew. By "capturing" the skew we have in mind the following: once the Heston model has been calibrated, then Exotic option prices can be computed using numerical techniques such as Monte-Carlo, PDE or FFT transform methods. The resulting option prices can then be used to determine the corresponding Black-Scholes implied volatilities. These volatilities can then be graphed to create the model's implied volatility surface which can then be compared to the market's implied volatility surface.

4.4 Solutions to Pricing Functionals

Pricing Function

For a European call option with strike K and maturity T the no–arbitrage price under the Heston model is

$$C(S_t, v_t, t) = \mathbb{E}_t\big[e^{-r(T-t)}\,\max(S_T - K, 0)\big], \tag{34}$$

where (S_t, v_t) follow the Heston dynamics under the risk–neutral measure.

If we introduce the pricing function $C = C(S, v, t)$ and apply standard arguments (Feynman–Kac), C satisfies the Heston pricing PDE:

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}vS^2\frac{\partial^2 C}{\partial S^2} + \rho\,\xi\,vS\frac{\partial^2 C}{\partial S\partial v} + \tfrac{1}{2}\xi^2 v\frac{\partial^2 C}{\partial v^2} + rS\frac{\partial C}{\partial S} + \kappa(\theta - v)\frac{\partial C}{\partial v} - rC = 0. \tag{35}$$

Ways to compute the expectation / price:

- Monte Carlo simulation of paths.
- Semi–analytical Fourier methods using characteristic functions (Heston's closed–form formula).
- Finite difference (PDE) solvers for (35).

Remark: Desks typically do not blindly use a single model price or hedge solely with model greeks. Pragmatic adjustments and heuristics are used to account for modelling assumptions and uncertainty.

Pricing Functional: notation (Horvath et al., 2019)

Let ζ be a vector of m traded instruments (contracts) and the market assigns prices in $\mathbb{R}^m$.

Given a model class $\mathcal{M}$ and a parameter set $\theta \in \Theta \subset \mathbb{R}^d$ we obtain a pricing map

$$P:\ \mathcal{M}(\theta, \zeta) \longmapsto \mathbb{R}^m. \tag{36}$$

For the Heston model we may write

$$\theta = (\kappa,\ \theta,\ \xi,\ \rho,\ v_0), \qquad \mathcal{M} = \mathcal{H}\ \text{(Heston)}. \tag{37}$$

When using a numerical solver or simulation we denote the approximate pricing map by $\widehat{P}(\mathcal{H}(\theta, \zeta)) \approx P(\mathcal{H}(\theta, \zeta))$.

Model Calibration to an Implied Volatility Surface

Once a pricing routine (closed–form, quasi–closed, simulation, or PDE) is chosen, calibration finds parameter values θ such that model prices match market prices as closely as possible.

A common formulation is the minimization problem

$$\theta^* = \underset{\theta \in \Theta}{\operatorname{argmin}}\ \delta\big(\widehat{P}(\mathcal{H}(\theta, \zeta)),\ P^{\mathrm{MKT}}(\zeta)\big), \tag{38}$$

where $\delta(\cdot, \cdot)$ is a chosen distance (for example weighted squared errors in price or implied volatility space) and $P^{\mathrm{MKT}}(\zeta)$ are the observed market prices.

Interpretation: θ^* are the Heston parameters $(\kappa, \theta, \xi, \rho, v_0, K, T)$ to use based on market prices.

5 Pricing Exotic Options using Calibrated Model

Informally, exotics are contracts with interesting payoff structures:

- Barrier Options (Knock-Out, Knock-In)
- Cliquet Options (Ratchets, Series of Forward-Start Options)
- Asian Options (Mean Price)

5.1 Barrier Options

A knock-out barrier option requires the price path to not hit a certain price threshold otherwise the payoff is zero.

Let B be a barrier level and τ the first hitting time of S_t to B,

$$\tau = \inf\{t \geq 0 : S_t = B\}. \tag{39}$$

For an *up–and–out* barrier call option with maturity T the payoff is knocked out once the barrier is hit. Its price under the Heston model can be written as

$$V(S_0, v_0; \theta) = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT}(S_T - K)^+ \mathbf{1}_{\{\tau > T\}} \mid S_0, v_0\right], \tag{40}$$

where (S_t, v_t) follow the Heston dynamics with parameters θ.

Closed–form solutions for such path–dependent claims are generally unavailable under stochastic volatility. A natural approach is therefore Monte Carlo simulation:

- Simulate joint paths (S_t, v_t) under the risk–neutral measure.
- For each path, monitor whether the barrier B is breached.
- Compute the discounted payoff $(S_T - K)^+$ if $\tau > T$, and zero otherwise.
- Average over many paths to approximate $V(S_0, v_0; \theta)$.

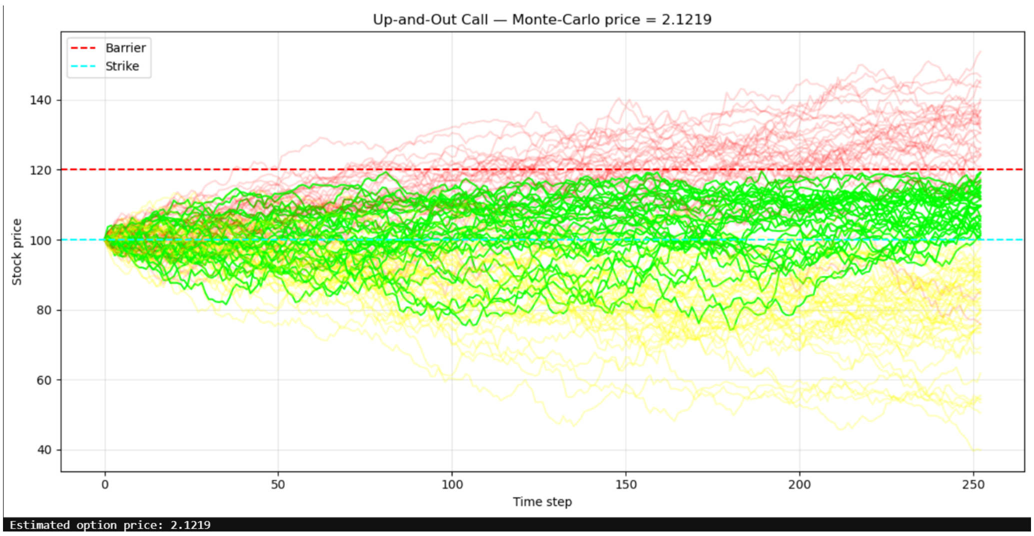

Figure 4: Monte Carlo simulation for Barrier Option

Monte Carlo thus provides a *functional approximation* of the exotic pricing map. Intuitively, the law of large numbers ensures convergence of the empirical average to the true expectation. A rigorous justification (and variance–reduction techniques) is typically covered in dedicated treatments of simulation methods.

Similarly, we can also price Vanilla Options

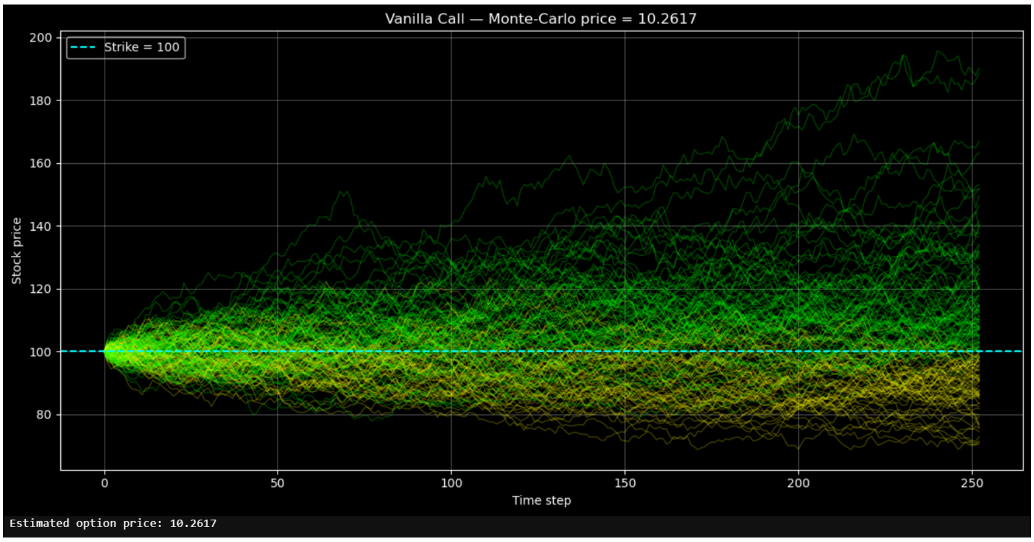

Figure 5: Monte Carlo simulation for Vanilla Call Option

6 Carr–Madan formula

Monte carlo is a slow calibration techinque, therefore we can use concepts of FFT

But, the problem is we do not know how to solve call price intergral in time domain so what we can do is apply foriuer transform go from time domain to frequency domain and then apply inverse Fourier transform to go back to time domain

Revisiting the option pricing problem now in the Heston framework, we have the following pricing problem

$$C(K,T) = e^{-rT}\,\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] \tag{41}$$

where the expectation is defined as

$$\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] = \int_{-\infty}^{\infty} \max(S_T - K, 0) f_{S_T}(x)\, \underset{\cdot}{x} = \int_{K}^{\infty} (S_T - K) f_{S_T}(x)\, \underset{\cdot}{x}. \tag{42}$$

What if we were to apply a Fourier transform to this call option value?

Well we run into an issue – the standard call option pricing function is NOT square-integrable which is necessary for the Fourier transform. This is due to the log-strike approaching a constant at $-\infty$ which is not zero.

Damping Factor $e^{\alpha k}$

By introducing a damping factor, Carr–Madan enable square-integrability of the pricing function and thus the Fourier transform applies.

Let $k = \ln(K)$ and $s = \ln(S_0)$, then the damped call price is:

$$c_T(k) = e^{\alpha k} C_T(k) \tag{43}$$

where $\alpha > 0$ is the damping parameter. The modified call price $c_T(k)$ is now square-integrable.

The Fourier transform of the damped call price is:

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{vk} c_T(k)\, \underset{\cdot}{k}. \tag{44}$$

And the inverse transform gives us the original call price:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-vk} \psi_T(v)\, \underset{\cdot}{v}. \tag{45}$$

Key Equation

The key equation for the Fourier transform of the damped call price in terms of the characteristic function is:

Starting with the Fourier transform of the damped call price:

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{vk} c_T(k)\,\text{\d{k}} = \int_{-\infty}^{\infty} e^{vk} e^{\alpha k} e^{-rT} \int_{\ln K}^{\infty} (e^y - e^k) f_{X_T}(y)\,\text{\d{y}}\,\text{\d{k}}. \tag{46}$$

Changing order of integration and evaluating the inner integral (standard Carr–Madan steps) yields the compact relation:

$$\boxed{\psi_T(v) = \frac{e^{-rT}}{(\alpha+v)(\alpha+1+v)}\,\phi_{X_T}\big(v-(\alpha+1)\big)} \tag{47}$$

where $\phi_{X_T}(u)$ is the characteristic function of the log-price process $X_T = \ln S_T$.

Heston characteristic function

Fortunately, the characteristic equation is known for the Heston model.

The characteristic function for the Heston model (log-stock prices) is:

$$\phi_{X_T}(u) = \exp\Big(u(\ln S_0 + rT) + \frac{\kappa\theta}{\sigma^2}\big((\kappa - \rho\sigma u - d)\,T - 2\ln\frac{1-ge^{-dT}}{1-g}\big) + \frac{v_0}{\sigma^2}\frac{1-e^{-dT}}{1-ge^{-dT}}(\kappa - \rho\sigma u - d)\Big) \tag{48}$$

where:

$$d = \sqrt{(\rho\sigma u - \kappa)^2 + \sigma^2(u + u^2)} \tag{49}$$

$$g = \frac{\kappa - \rho\sigma u - d}{\kappa - \rho\sigma u + d} \tag{50}$$

$$a = \kappa - \rho\sigma u \quad \text{(note: in some texts } a = \kappa\theta \text{ is used separately)} \tag{51}$$

Parameters:

- κ: mean reversion rate
- θ: long-run variance
- σ: volatility of variance (vol-of-vol)
- ρ: correlation between asset and variance processes
- v_0: initial variance
- s: log spot price (or $\ln S_0$)
- r: risk-free rate
- T: time to maturity

To obtain the option price, we need to invert the Fourier transform:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-vk} \psi_T(v)\,\text{\d{v}}. \tag{52}$$

Substituting the expression for $\psi_T(v)$ gives:

$$C_T(k) = \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-vk} \frac{e^{-rT}}{(\alpha+v)(\alpha+1+v)} \phi_{X_T}\big(v - (\alpha+1)\big)\,\text{\d{v}}. \tag{53}$$

This integral can be evaluated numerically using the Fast Fourier Transform (FFT).

The key steps are:

1. Choose a suitable damping parameter $\alpha > 0$ to ensure integrability.
2. Discretize the integral using a grid of points.
3. Apply FFT to compute the integral efficiently.
4. Undamp the result by multiplying by $e^{-\alpha k}$.

The resulting $C_T(k)$ gives us the call option price for log-strike k. For actual strikes K, we use the relationship $k = \ln(K)$.

General Recipe

The Carr–Madan FFT algorithm for Heston option pricing follows these steps:

1. **Formulate the Heston Characteristic Function.** The characteristic function $\phi_T(u)$ is the Fourier transform of the risk-neutral probability density:

$$\phi_T(u) = \exp\left(u(s+rT) + \frac{v_0}{\sigma^2}\frac{1-e^{-dT}}{1-ge^{-dT}}(a-\rho\sigma u - d)\right)\times\exp\left(\frac{\theta\kappa}{\sigma^2}\left[2\ln\left(\frac{1-ge^{-dT}}{1-g}\right)+(a-\rho\sigma u-d)T\right]\right) \tag{54}$$

where

$$d = \sqrt{(\rho\sigma u - a)^2 + \sigma^2(u+u^2)}, \qquad g = \frac{a-\rho\sigma u - d}{a-\rho\sigma u + d}, \qquad a = \kappa. \tag{55}$$

2. **Damping and the Carr–Madan Formula.** Introduce a damping factor $e^{\alpha k}$ and express the Fourier transform of damped prices:

$$\tilde{c}(v) = \frac{e^{-rT}\,\phi_T\big(v-(\alpha+1)\big)}{\alpha^2+\alpha-v^2+(2\alpha+1)v}. \tag{56}$$

3. **Discretize for the FFT.** Set up discrete grids:

   - Frequencies: $v_j = j\Delta_v$, for $j = 0,\dots,N-1$,
   - Log-strikes: $k_m = k_{\min} + m\Delta_k$, for $m = 0,\dots,N-1$,

   where $\Delta_k\,\Delta_v = \frac{2\pi}{N}$.

4. **Apply the Inverse FFT.** Transform from frequency domain to log-strike domain:

$$c(k_m) = \frac{1}{2\pi}\sum_{j=0}^{N-1} e^{-v_j k_m}\,\tilde{c}(v_j)\,\Delta_v\,w_j, \tag{57}$$

   where w_j are quadrature weights.

5. **Undamp and Correct.** Recover actual call prices:

$$C(k_m) = e^{-\alpha k_m}\,c(k_m), \tag{58}$$

   where $K_m = e^{k_m}$ gives option prices at strikes K_m.

Now, let's look at Call Option price generated by Carr-Madan FFT and compare it with monte carlo

Price Comparison

Strike	FFT Price	MC Price	MC 95% CI
80.0	22.3804	22.2519	[21.9297, 22.5741]
85.0	18.2617	18.1371	[17.8355, 18.4387]
90.0	14.4595	14.3671	[14.0908, 14.6434]
95.0	11.0425	10.9957	[10.7485, 11.2429]
100.0	8.0777	8.0875	[7.8724, 8.3027]
105.0	5.6161	5.6843	[5.5027, 5.8659]
110.0	3.6817	3.7925	[3.6441, 3.9408]
115.0	2.2629	2.3911	[2.2740, 2.5083]
120.0	1.2993	1.4366	[1.3473, 1.5259]

Table 1: Comparison of FFT and Monte Carlo (MC) option prices with 95% CI.

7 Price Adjustments

After calculating the base exotic option price using our calibrated model, several adjustments are typically made:

Counterparty Risk Adjustment

- Credit Value Adjustment (CVA) accounts for counterparty default risk.
- Debt Value Adjustment (DVA) accounts for own default risk.
- These adjustments reduce the price when default risk is higher.

Liquidity Adjustment

- Exotic options are less liquid than vanilla options.
- Wider bid–ask spreads reflect higher hedging costs.
- Size of adjustment depends on:
  - Option complexity,
  - Market conditions,
  - Position size.

Profit Margin

- Trading desks add profit margin to cover:
  - Operating costs,
  - Capital charges,
  - Return on equity requirements.
- Typical margins range from 2–10% depending on:

  - Client relationship,
  - Competition,
  - Market conditions.

The final quote sent to clients incorporates all these adjustments on top of the theoretical price. Let Q be the final quote sent to clients. Then:

$$Q = P + \mathrm{CVA} + \mathrm{DVA} + L + M \tag{59}$$

where:

- P is the theoretical price from the calibrated model,
- CVA is the credit value adjustment,
- DVA is the debt value adjustment,
- L is the liquidity adjustment,
- M is the profit margin.