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THE ABSORPTION AND DISSOCIATIVE OR IONIZING EFFECT OF MONOCHROMATIC RADIATION IN AN ATMOSPHERE ON A ROTATING EARTH

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ABSTRACT. The absorption of monochromatic radiation from the sun in an atmosphere of which the density varies exponentially with height is considered; the energy of the radiation, or a definite fraction of it, is supposed to dissociate or ionize the air, and the dissociation products are supposed to recombine with one another only, and not to diffuse away from the element in which they were formed. The resulting distribution of density of the dissociation products is determined, a constant recombination coefficient being assumed, while account is taken of the variation in rate of dissociation due to the earth's rotation. The results are illustrated by numerous diagrams, showing the density of the dissociation-products as a function of height, time, latitude and season.

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§ 1. INTRODUCTION

The main object of this paper is to consider the following idealized problem: (*a*) A uniform beam of monochromatic radiation from a sun falls upon a rotating earth, the rate of rotation of the earth and the changing declination of the sun being the same as for the actual earth and sun; the radiation is absorbed (before reaching the ground) in an atmosphere of uniform composition, in which the density varies exponentially with height. (*b*) The absorption of radiation at each point is proportional to the air-density and the intensity of the radiation reaching that point. (*c*) The energy of the absorbed radiation, or a constant fraction of it, is expended in dissociating some constituent A of the air into two components A_1, A_2 which may be electrically neutral or not. (*d*) The two components recombine with
α one another (alone) to re-form A, at a rate αn^2, where α is a constant independent of
n height and time, and n is the number of A_1 or A_2 particles per unit volume. (*e*) The particles A_1, A_2 do not move from the element of volume in which they were formed.

The problem is to determine (*f*) the rate of absorption and dissociation or ionization at each point, as a function of height, time of day, season of year, and latitude, and (*g*) the value of n at each point, as a function of the same variables.

This problem is easy to formulate in mathematical terms, and the differential equations can be partly, but not fully, integrated; at a certain point it is necessary to resort to numerical calculation. A feature of the present treatment is that, by suitable choice of units, the analysis and discussion are expressed in general terms, without the assumption of any special values for the numerical quantities involved, until the latest possible stage; when this can no longer be avoided, particular numerical values are considered, but, even so, the differential equation is written so that only a single constant (σ_0, § 9) comes in question. This procedure makes it possible to apply the present analysis and results to different types of radiation (which may be absorbed at different levels in the atmosphere, and have different dissociative or ionizing effects) when the particular numerical data for each such case become available.

On the other hand, the problem is an ideal one which will scarcely represent adequately all the factors of importance in any actual case. It is thought likely, however, to be of value as an approximation, and as a starting point for further investigation into the influence of factors here neglected.

The number of independent variables involved is four, namely height, time of day, season of year, and latitude; this renders it difficult to gain a comprehensive view of the results. It has seemed most convenient to represent them graphically, rather than by numerical tables, though a considerable number of diagrams are required, each of which includes several curves. Figures 1–5 relate to the absorption and rate of dissociation or ionization; Figures 6–11 represent n as a function of the time of day, for various heights, seasons, and latitudes (for the three adopted values of σ_0). The seasons considered are midsummer, midwinter, and the equinoxes; the latitudes are 0° (the equator) and 60°.

In § 14 the results are briefly considered in relation to the ionization of the upper atmosphere, though as regards the conditions (*d*) and (*e*) the present problem only approximately illustrates the ionization changes. But as this is the most interesting application of the results at the present time, the whole paper has been written as though the dissociated components A_1 and A_2 were ions and electrons; by merely verbal changes the discussion could be expressed relative to the corresponding dissociation phenomena in which A_1 and A_2 are neutral atoms or molecules.

§ 2. THE RATE OF ION-PRODUCTION

Consider a point P at height h above the ground at an angular distance θ (the colatitude) from the north pole. Let t, or ϕ, denote the local time reckoned from noon, that is, the longitude of P measured eastward from the noon meridian at the instant, ϕ being in angular measure (radians), and t in time-measure (seconds). Clearly

h, θ t, ϕ

$$t = 86400\,\phi/2\pi = 1{\cdot}37.10^4\,\phi, \qquad \ldots\ldots(1),$$

there being 86400 seconds in a day.

Over the range of level considered, the density of the atmosphere will be supposed to vary exponentially, i.e. $\rho = \rho_0 \exp(-h/H)$, where ρ_0 is the density that would exist at ground level if the formula were valid at all heights. In the actual atmosphere this formula is valid only if H itself is a slowly varying function of height, but the probable variation over the range of level that is important in this investigation is not great, and may in a first approximation be neglected. Constancy of H would result if the atmosphere were of uniform composition and temperature, but these conditions are only sufficient, not necessary. When they are fulfilled, $H \doteq kT/mg$,

ρ_0, H

k, T, m, g where k, T, m and g denote respectively Boltzmann's constant $1{\cdot}37.10^{-16}$, the absolute temperature, the mean molecular mass, and the acceleration of gravity; for example, if $T = 300°$ and the composition is the same as in the lower atmosphere (mainly nitrogen N_2 and oxygen O_2), $H = 8{\cdot}4$ km. This figure may be borne in mind in considering the following results, which, however, are independent of any particular numerical value of H, because heights will in general be expressed in terms of H as unit. It is quite possible that, above 100 km., the value of H differs from 8·4 km., but the value, when found, can at once be applied to the results of the general theory here given. The theory may also be relevant to different strata in the atmosphere, for which H has different values.

h' Thus, when H is taken as the unit of height, $\rho = \rho_0 \exp(-h')$, where h' $(= h/H)$ denotes height measured in this unit.

§ 3. THE INCIDENCE OF THE RADIATION

χ Since the sun is on the noon meridian, its zenith distance χ at P is given by

$$\cos\chi = \sin\delta\cos\theta + \cos\delta\sin\theta\cos\phi \qquad \text{......(2)},$$

δ where δ denotes the north declination of the sun.

At the equinoxes $\delta = 0$ and

$$\cos\chi = \sin\theta\cos\phi \text{ (equinox)} \qquad \text{......(3)}.$$

At noon ($\phi = 0$), denoting χ by χ_0, we have

$$\chi_0 = \tfrac{1}{2}\pi - (\theta + \delta) \text{ (noon)} \qquad \text{......(4)}.$$

When it is desired to indicate the variables on which χ depends, it will be denoted by $\chi(\delta, \theta, \phi)$. Then, by (2),

$$\chi(\delta, \tfrac{1}{2}\pi, \phi) = \chi(0, \tfrac{1}{2}\pi - \delta, \phi) \qquad \text{......(5)},$$

i.e. the sun's zenith distance, at any local time ϕ, is the same for the equator at the season when the sun's declination is δ, as at a point in latitude δ (or colatitude $\frac{1}{2}\pi - \delta$) at the equinoxes ($\delta = 0$).

§ 4. THE ABSORPTION OF RADIATION

Consider a beam of monochromatic ionizing solar radiation, of unit cross-section and of intensity S_∞ outside the atmosphere, passing through the layer between h and $h - dh$, above the point θ, ϕ, and therefore at the inclination χ to the vertical. Subject to the usual assumption that the absorption is proportional to the intensity S at that height, and to the mass $\rho_0 \sec\chi \exp(-h/H).dh$ of air traversed, the change of intensity dS after crossing the layer will be given by

$$dS = AS\rho_0 \sec\chi \exp(-h/H)\, dh \qquad \text{......(6)},$$

where A is the coefficient of absorption. The solution of this differential equation is*

$$S = S_\infty \exp\{-A\rho_0 H \sec\chi . \exp(-h/H)\} \qquad \text{......(7)}.$$

* P. Lenard. *Sitz. d. Heidelberger Ak. d. W.*, 12 Abh. (1911).

The absorption of radiation per cc. of atmosphere is $dS/(\sec\chi . dh)$, and if the number of ions produced by the absorption of unit quantity of the radiation is β, β
the rate of production of ions per cc. will be $\beta \cos\chi . dS/dh$; this, being a function of h and χ, will be denoted by $I(\chi, h)$. By (7), I

$$I(\chi, h) = \beta \cos\chi \frac{dS}{dh} = \beta A S_\infty \rho_0 \exp\left\{-\frac{h}{H} - A\rho_0 H \sec\chi \exp\left(-\frac{h}{H}\right)\right\} \quad \ldots\ldots(8).$$

The total number of ions produced in a square cm. column of air by the complete absorption of the radiation (S_∞) concerned is $\beta S_\infty \cos\chi$.

This rate of production of ions per c.c. has a maximum at the height $h(\chi)$ given by

$$\exp\frac{h(\chi)}{H} = A\rho_0 H \sec\chi \quad \ldots\ldots(9),$$

and the corresponding maximum value $I(\chi)$ of $I(\chi, h)$ is given by

$$I(\chi) = \beta S_\infty \cos\chi / H \exp 1 \quad (10),$$

where exp 1 is written instead of e (or 2·718 ...) to avoid confusion with the usual symbol for the electronic charge. Let the values of $h(\chi)$ and $I(\chi)$ for $\chi = 0$ be denoted by h_0 and I_0. Then h_0, I_0

$$\exp(h_0/H) = A\rho_0 H, \quad I_0 = \beta S_\infty / H \exp 1 \quad (11);$$

consequently

$$h(\chi) = h_0 + H \ln \sec\chi \quad \ldots\ldots(12),$$

where ln denotes the Napierian logarithm, and ln

$$I(\chi) = I_0 \cos\chi \quad \ldots\ldots(13).$$

In terms of I_0 and h_0, (8) may be written

$$I(\chi, h) = I_0 \exp\left\{\frac{h_0 + H - h}{H} - \sec\chi \exp\frac{h_0 - h}{H}\right\} \quad \ldots\ldots(14).$$

It should be noted that these equations are not valid for very large values of $\sec\chi$ (corresponding to grazing incidence of the beam of radiation), because then the level surfaces traversed by the beam can no longer be treated as parallel planes, as they were when the distance along the beam between h and $h - dh$ was taken as $\sec\chi . dh$. The approximation is probably sufficiently accurate up to $\sec\chi = 12$, or $\chi = 85°$, which along the equator corresponds to about 20 minutes after sunrise or before sunset. In a succeeding paper the special conditions existing near sunrise will be examined more accurately; it will be shown that at and near the level of maximum ion-content the results of the present paper are very nearly true except near dawn, at the equator; but that in higher latitudes, *in winter*, the necessary corrections are appreciable up to noon.

§ 5. THE DATUM UNIT FOR DIFFERENCES OF LEVEL

It is convenient (cf. § 2) to measure heights in terms of H as unit, and reckoned from h_0 as datum; H will have the value appropriate to the height h_0, and its change with height over the ionized layer, at and near the level h_0, will be assumed small and neglected. Thus, let

$$z = (h - h_0)/H \qquad \text{......(15).}$$

Then if $z(\chi)$ corresponds to the level $h(\chi)$, we have, by (12),

$$z(\chi) = \ln \sec \chi \qquad \text{......(16),}$$

and by (14),

$$I(\chi, h) = I_0 \exp\{1 - z - \sec \chi . \exp(-z)\} \equiv I(\delta, \theta, z, \phi) \qquad \text{......(17),}$$

where the notation $I(\delta, \theta, z, \phi)$ is used to indicate the independent variables of which $I(\chi, h)$ is a function.

§ 6. THE NOON DISTRIBUTION OF ABSORPTION

At any point P the minimum value of χ and $\sec \chi$, and consequently the maximum value of $I(\chi, h)$ or $I(\delta, \theta, z, \phi)$, occur at noon ($\phi = 0$), when, by (4), $\chi = \frac{1}{2}\pi - (\theta + \delta)$. Hence, by (17),

$$I(\delta, \theta, z, 0) = I_0 \exp\{1 - z - \operatorname{cosec}(\theta + \delta) \exp(-z)\} \text{ (noon)} \qquad \text{......(18).}$$

This has its maximum value, with respect to height, of amount

$$I(\delta, \theta) = I_0 \sin(\theta + \delta) \qquad \text{......(19),}$$

by (13), at the level

$$z(\delta, \theta) = \ln \operatorname{cosec}(\theta + \delta) \qquad \text{......(20),}$$

by (16).

These formulae are exemplified by the graphs in figure 1, which represent $I(\delta, \theta, z, 0)/I_0$ as a function of z; that is, they indicate, for a number of values of $\theta + \delta$, the ratio of the noon rate of ion-production at various heights, to the noon-rate at the equator at height h_0. The curves relate to the values (1) 6·5°, (2) 15°, (3) 30°, (4) 45°, (5) 53·5°, (6) 66·5° and (7) 90°.

At the equinoxes ($\delta = 0$) they refer to points having these colatitudes, or to latitudes varying from 83·5° (curve 1) to zero (the equator, curve 7). They show how the ion-production at noon decreases as we recede from the equator, while at the same time the level of maximum production rises, at first slowly, then more rapidly as the pole is approached.

The curve (6) shows the noon-rate of ion-production at the equator ($\theta = \frac{1}{2}\pi$) at the solstices ($\delta = \pm 23{\cdot}5°$), and by comparison with the equinoctial curve (7) for the equator indicates how slight is the seasonal change in the height distribution of ion-production above the equator.

At latitude 60° ($\theta = 30°$), however, the seasonal change is very considerable, as is shown by curves (1), (3) and (5), which refer to this latitude at midwinter ($\delta = -23{\cdot}5°$, $\theta + \delta = 6{\cdot}5°$), the equinoxes ($\delta = 0, \theta + \delta = 30°$), and midsummer ($\delta = 23{\cdot}5, \theta + \delta = 53{\cdot}5°$) respectively. The level of maximum production varies

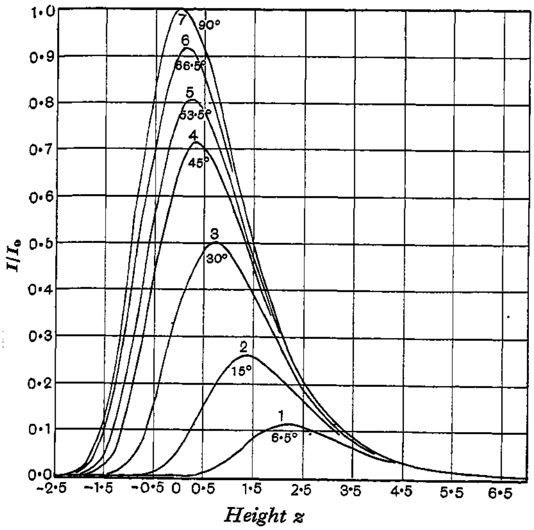


Fig. 1. Ion-production at noon for various values of $(\theta + \delta)$

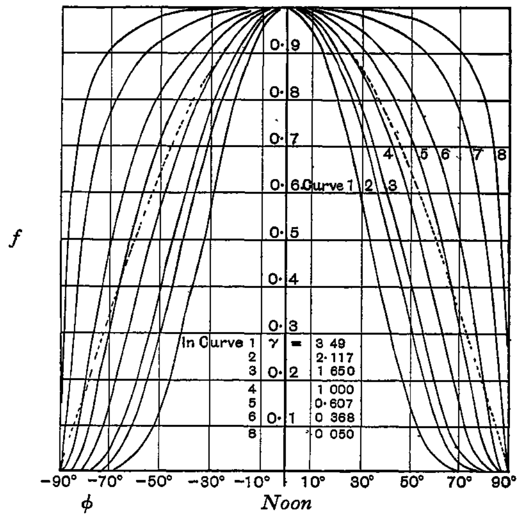


Fig. 2. Ion-production at the equinoxes

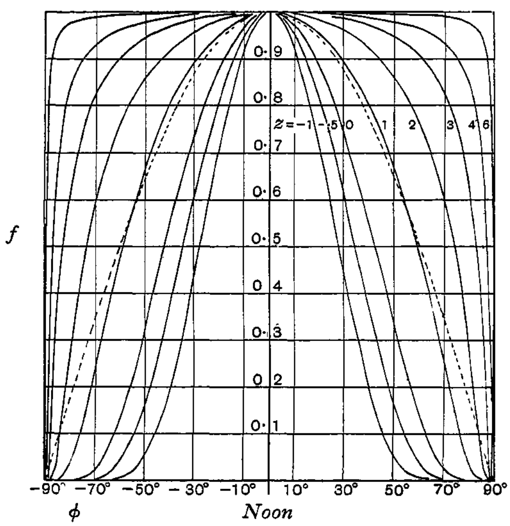


Fig. 3. Ion-production at the equinoxes in latitude 60°

from $z = 2{\cdot}2$ in midwinter to $z = 0{\cdot}2$ in midsummer (a change of 17 km. if $H = 8{\cdot}4$ km.), and the maximum rate of production is increased in the ratio $\sin 53{\cdot}5^\circ/\sin 6{\cdot}5^\circ$ or $7{\cdot}1$; the total rate of ion-production, integrated over all heights (i.e. $\beta S \cos \chi_0$, cf. § 4), is altered in the same ratio.

Considering curve (7) for the equator at the equinoxes, it may be seen that I/I_0 falls from 1 at height h_0 ($z = 0$) to $\frac{1}{8}$ at approximately $z = 1{\cdot}5$ and $z = 3{\cdot}0$; thus the greater part of the ion-production occurs in a layer of thickness $4{\cdot}5\, H$* (or about 39 km. if $H = 8{\cdot}4$ km.). Here, and at other latitudes and seasons, the rate of ion-production at noon decreases much more rapidly in passing downwards than in *ascending* from the level of maximum production.

The higher the level above h_0, the more uniform is the noon rate of ion-production at nearly all latitudes and seasons; this is already marked at $z = 3$, where I/I_0 is approximately $\frac{1}{8}$ for all values of $\theta + \delta$ between 15° and 165°. Over this range the term $\operatorname{cosec}(\theta + \delta) \exp(-z)$ in the exponential formula (18) is very small when $z > 3$, and I then becomes approximately $I_0 \exp(1 - z)$.

The variation of $z(\delta, \theta)$ is also shown in table 1 for various values of χ_0 or $\frac{1}{2}\pi - (\theta + \delta)$; the corresponding values in km. are also given, H being taken as 8·4 km.

Table 1.

$\chi_0 = \frac{1}{2}\pi - (\theta + \delta)$	0°	30°	45°	60°	75°	80°	82·5°	85°
$z(\delta, \theta)$	0	0·062	0·151	0·301	0·587	0·760	0·884	1·060
z (km.)	0	1·2	2·9	5·8	11·3	14·7	17·1	20·5

§ 7. THE PROPORTIONATE DAILY VARIATION OF ABSORPTION

It is of interest to indicate also how the rate of ion-production varies with respect to ϕ, that is, throughout the day at different heights, seasons and latitudes. This can conveniently be done by representing graphically, as a function of ϕ, the ratio of the rate at time ϕ to that at noon ($\phi = 0$), i.e.

$$\begin{aligned} f(\delta, \theta, z, \phi) &= I(\delta, \theta, z, \phi)/I(\delta, \theta, z, 0) \\ &= \exp\{(\sec \chi_0 - \sec \chi) . \exp(-z)\} \\ &= \exp[\{\operatorname{cosec}(\theta + \delta) - \sec \chi\} . \exp(-z)] \qquad \text{......(21).} \end{aligned}$$

The range of ϕ is that for which $\cos \chi > 0$, so that the limiting values of ϕ are given by

$$\cos \phi = -\tan \delta \cot \theta \qquad \text{......(22).}$$

At the equinoxes $\delta = 0$ and ϕ ranges between $\pm \frac{1}{2}\pi$, while (21) takes the form

$$f(0, \theta, z, \phi) = \exp\{-\operatorname{cosec} \theta \exp(-z)(\sec \phi - 1)\} \qquad \text{......(23),}$$

in which the dependence on each variable is specially simple; z and θ are involved together in the factor γ where

$$\gamma = \operatorname{cosec} \theta \exp(-z) \qquad \text{......(24),}$$

in terms of which

$$f(0, \theta, z, \phi) = \exp\{-\gamma(\sec \phi - 1)\} \qquad \text{......(25).}$$

* *H* itself is assumed to have a negligible variation throughout a layer of thickness about 5*H*.

The values of γ for various values of z and θ are given in table 2.

Table 2

$\gamma = \text{cosec}\,\theta . \exp(-z)$

z	θ							
	6·5°	15°	30°	45°	53·5°	60°	66·5°	90° (eqr.)
4·0	0·162	0·071	0·037	0·026	0·023	0·021	0·020	0·018
3·0	0·440	0·192	0·100	0·070	0·062	0·057	0·054	0·050
2·0	1·196	0·523	0·271	0·191	0·168	0·156	0·148	0·135
1·5	1·971	0·862	0·446	0·316	0·278	0·258	0·243	0·223
1·0	3·250	1·421	0·736	0·520	0·458	0·425	0·401	0·368
0·5	5·358	2·343	1·213	0·858	0·755	0·700	0·661	0·607
0	8·834	3·864	2·000	1·414	1·244	1·155	1·090	1·000
−0·5	14·56	6·37	3·297	2·332	2·051	1·904	1·798	1·649
−0·75	18·70	8·18	4·23	2·99	2·63	2·445	2·308	2·117
−1·0	24·01	10·50	5·44	3·84	3·38	3·14	2·96	2·72
−1·25	30·83	13·49	6·98	4·94	4·34	4·03	3·81	3·49
−1·5	39·59	17·32	8·96	6·34	5·58	5·18	4·89	4·48

In figure 2, curves representing $f(0, \theta, z, \phi)$ as a function of ϕ, equation (25), are drawn for various values of γ. They indicate the proportionate diurnal variation in the rate of ion-production, *at the equinoxes* ($\delta = 0$), for any point at height z (above or below h_0) in any colatitude θ; the value of γ corresponding to θ and z may be found by interpolation from table 2, and the corresponding curve can then be obtained from figure 2 by interpolation.

At the level h_0 ($z = 0$), at the equator ($\theta = \frac{1}{2}\pi$), $\gamma = 1$; the curve $\gamma = 1$ in figure 2 resembles, though except at $\phi = 0$ it is below, the curve of cos ϕ (shown by a dotted line for comparison). For $\gamma > 1$, corresponding, *inter alia*, to levels below h_0 at the equator, or to levels extending somewhat above h_0 at higher latitudes (cf. table 2), the excess of cos ϕ over $f(0, \theta, z, \phi)$ is greater, indicating that the part of the day during which these levels absorb the radiation is more and more concentrated round the hour of noon. Conversely, for $\gamma < 1$, $f(0, \theta, z, \phi)$ approaches unity the more rapidly and over a wider range of ϕ, the smaller the value of γ, indicating that in the higher levels of the atmosphere the rate of ion-production is near its maximum and varies little, from near sunrise to near sunset, the rise and fall near these epochs being rapid.

§ 8. THE SEASONAL CHANGES IN THE PROPORTIONATE DAILY VARIATION

At times other than the equinoxes the curves showing $f(\delta, \theta, z, \phi)$ as a function of ϕ differ from those of figure 2, except for the equator ($\theta = \frac{1}{2}\pi$). There, by virtue of (5),

$$f(\delta, \tfrac{1}{2}\pi, z, \phi) = f(0, \tfrac{1}{2}\pi - \delta, z, \phi),$$

indicating that the equatorial curve for the season δ and for any height z is the same as the *equinoctial* curve for the same height in latitude δ. The slightness of the seasonal change in the proportionate diurnal variation of ion-production at any height above the equator is indicated by the small difference between the values of γ in the two last columns of table 2, which refer respectively, for any height z, to the solstices and equinoxes.

At latitudes far from the equator the proportionate daily variation in the rate of ion-production at any height changes considerably with the season, as also does the duration of ion-production. This is illustrated by the curves in figures 3, 4, 5, which refer to latitude 60° ($\theta = 30°$) at the equinoxes, the summer solstice, and the winter solstice ($\delta = 23{\cdot}5 . 0, -23{\cdot}5$), for a series of heights between $z = -2$ and $z = 7$; the curve representing $\cos\chi$ is also shown on each diagram by a dotted line.

The different base-lengths in the three diagrams refer, of course, to the different periods of daylight at the three seasons. In summer, when the sun's rays are most direct, the rate of ion-production at the level $z = 0$ exceeds half its maximum (noon) value for about half the period of daylight; this is nearly true at the same level at the equinoxes also, but in winter the absorption of radiation at this level increases more slowly, and it exceeds half the noon value only for about a quarter of the period of daylight. At $z = 6$ at all seasons the absorption is nearly at its full value throughout almost the whole day.

To obtain a comprehensive idea of the variation of ion-production with respect to season, height, latitude and local time the curves in figure 1 should be considered in conjunction with those of figures 2–5.

§ 9. THE DISTRIBUTION AND VARIATIONS OF ION-DENSITY

n, n_e, n_- Let n, n_e, n_- denote the number per c.c. of positive ions, electrons, and negative ions, at a point ϕ, θ, z. The ions of either kind are, for simplicity, supposed to be all alike and simply charged; hence, the air being electrically neutral,

$$n = n_e + n_- \qquad \text{......(26).}$$

The number of positive ions produced per c.c. per second is denoted, as in the preceding sections, by $I(\delta, \theta, z, \phi)$. The equation of variation for n, due to new formation of positive ions, and their disappearance by recombination, is

$$dn/dt = I(\delta, \theta, z, \phi) - \alpha_e n n_e - \alpha_- n n_- \qquad \text{......(27),}$$

α_e, α_- where α_e, α_- denote coefficients of recombination. Their values are not known with any certainty, at the low pressures which obtain in the atmospheric region under consideration; for simplicity it will be assumed that they are equal, though this is a pure assumption. Thus we take

α
$$\alpha_e = \alpha_- = \alpha \qquad \text{......(28),}$$

so that, by virtue also of (26), (27) reduces to

$$dn/dt = I - \alpha n^2 \qquad \text{......(29).}$$

This is the equation that expresses the condition (*d*) of § 1.

Here, as has been shown in §§ 2–8, I is a function of δ, θ, z (or h) and ϕ, so that n also will depend on these variables; α may be a function of z, but will be supposed independent of δ, θ, and ϕ; the solutions of (29) which are obtained here will, however, only be discussed for values of α independent also of z.

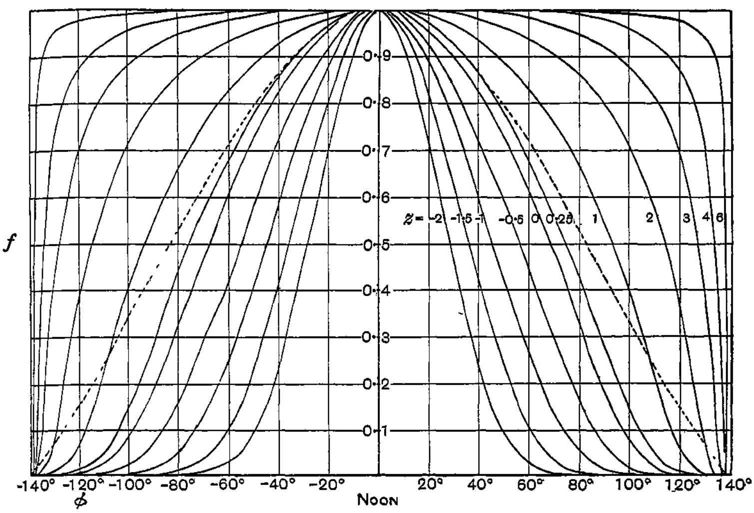


Fig. 4. Ion-production at the summer solstice in latitude 60°.

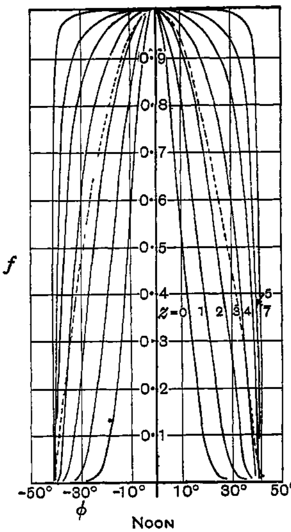


Fig. 5. Ion-production at the winter solstice in latitude 60°.

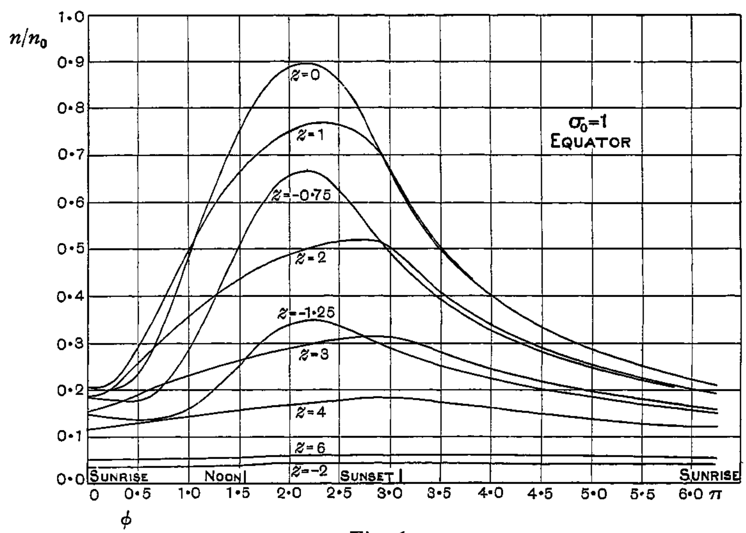


Fig. 6.

t It is convenient to express the local time t in (29) in terms of ϕ, by (1). Then (29) becomes, using (17):

$$(1/1{\cdot}37.10^4).dn/d\phi = I_0 \exp\{1 - z - \sec\chi.\exp(-z)\} - \alpha n^2 \quad \text{......(30)},$$

where χ depends on ϕ, θ and δ according to (2).

n_0 Let
$$n_0 = (I_0/\alpha)^{\frac{1}{2}} \quad \text{......(31)},$$
σ_0
$$1/\sigma_0 = 1{\cdot}37.10^4\,(I_0\alpha)^{\frac{1}{2}} \quad \text{......(32)},$$

so that n_0 is the steady value which n would attain at the level h_0 ($z = 0$) at the equator at midday (when $z = 0$, $\chi = 0$) if the earth did not revolve. Then n_0, σ_0 are alternative modes of specifying α and I_0, i.e.

$$1/\alpha = 1{\cdot}37.10^4\,\sigma_0 n_0 \quad \text{......(33)},$$
$$I_0 = n_0/(1{\cdot}37.10^4\,\sigma_0) \quad \text{......(34)}.$$

In terms of n_0, σ_0, and ν, defined by

$$\nu \equiv n/n_0 \quad \text{......(35)},$$

(30) may be written in the form

F
$$\sigma_0.d\nu/d\phi + \nu^2 = \exp\{1 - z - \sec\chi\exp(-z)\} \quad \text{(day)}$$
$$= F(z, \chi) \quad \text{......(36)}.$$

This represents the variation of ν, or n/n_0, during the hours of daylight; during the hours of darkness the right-hand side must be replaced by zero, i.e.

$$\sigma_0.d\nu/d\phi + \nu^2 = 0 \quad \text{(night)} \quad \text{......(37)}.$$

The solution of this equation is

$$\nu = \sigma_0/(\phi + C) \quad \text{(night)} \quad \text{......(38)},$$

C where C is an arbitrary constant.

r, s Distinguishing the values of ν and ϕ at sunrise and sunset by the suffixes r and s, we have

$$\frac{1}{\nu_r} - \frac{1}{\nu_s} = \frac{\phi_r - \phi_s}{\sigma_0} \quad \text{......(39)}.$$

The solution of (36) has to be found subject to this condition, which determines the arbitrary parameter involved in the general solution of (36).

It does not seem possible to solve the non-linear equation (36) in terms of elementary functions, and numerical methods must be adopted. If one solution of (36) has been found numerically, not satisfying (39), the true solution can be obtained from it by a process of quadratures, but the latter is not appreciably easier than finding the correct solution of (36), subject to (39), by successive direct trials; it is therefore unnecessary to explain the method referred to.

§ 10. SOLUTIONS FOR THREE VALUES OF σ_0

The equation (36) has been solved, subject to the condition (39), for three values of σ_0, namely 1, $\frac{1}{5}$ and $\frac{1}{25}$, for various heights (at distances z above and below h_0) at the equator ($\theta = \frac{1}{2}\pi$) at the equinoxes ($\delta = 0$); and also, for $\sigma_0 = \frac{1}{25}$, at various

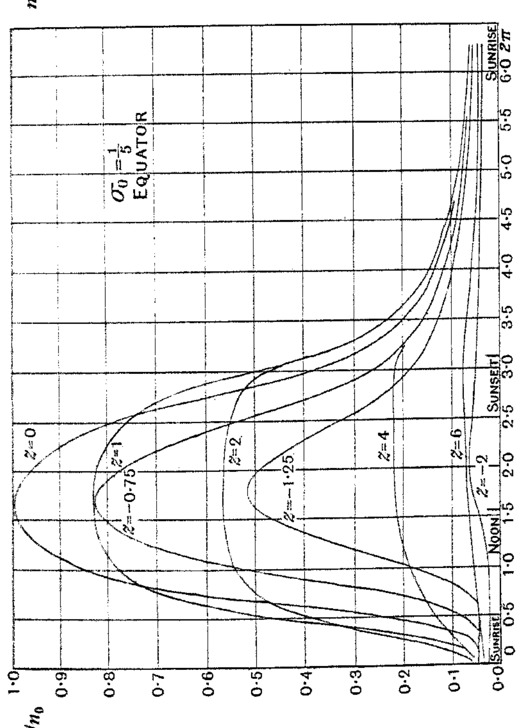


Fig. 7.

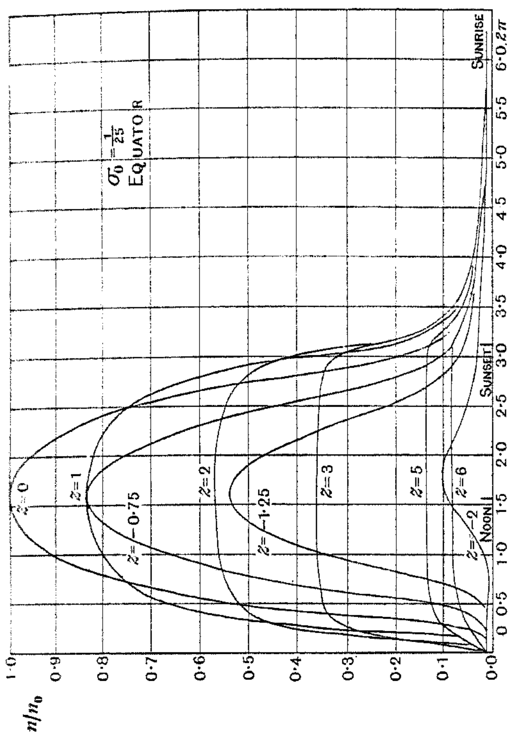


Fig. 8.

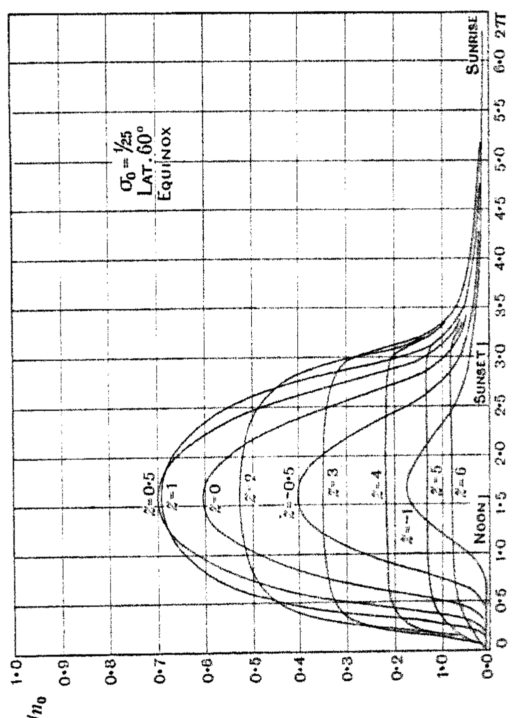


Fig. 9.

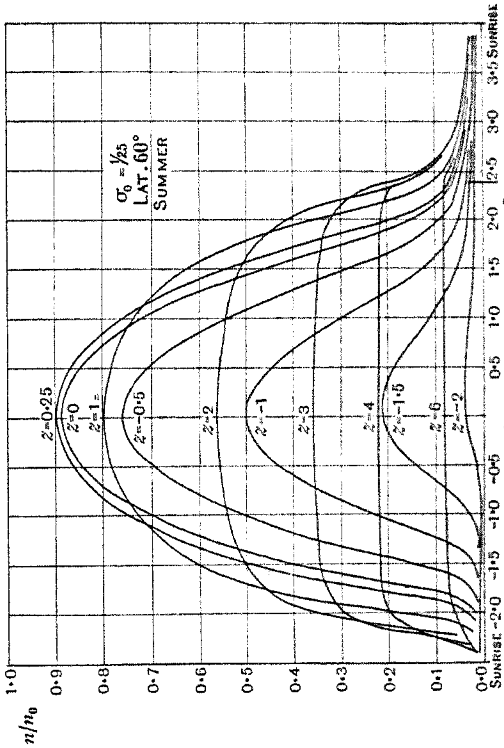


Fig. 10.

heights in latitude 60° ($\theta = 30°$), at the equinoxes ($\delta = 0$) and the solstices ($\delta = \pm 23{\cdot}5$). The corresponding values of ν or n/n_0 as a function of ϕ (in circular measure) are shown by series of graphs in figures 6–11. Further graphs have been derived from these, showing, in figures 12–17, the values of n/n_0 as a function of height at various local times, for the equator ($\sigma_0 = 1, \frac{1}{5}, \frac{1}{25}$) and latitude 60° ($\sigma_0 = \frac{1}{25}$). In the "equinoctial" figures 6–9 and 12–15, ϕ is reckoned from *sunrise*, not from *noon* as in figures 2–5, 10, 11, 16, 17, and elsewhere in this paper. These curves, and further general conclusions which can be inferred from them, or from the differential equations (36), (37), will now be discussed.

§ 11. THE ION-DENSITY AS A FUNCTION OF TIME

At and between sunset and sunrise ν varies according to (37), i.e.

$$d\nu/d\phi = -\nu^2/\sigma_0;$$

thus ν is decreasing at and between these epochs. Between sunrise and sunset ν varies according to (36), the right-hand side of which is essentially positive, and has its maximum at noon (when $\chi = \frac{1}{2}\pi - \theta - \delta$); during part of the hours of sunlight must increase, and the reversal from the decreasing rate $\nu_r{}^2/\sigma_0$ at sunrise occurs after an interval of time which is shorter, the smaller the value of σ_0. In many of the graphs in figures 6–10 this interval is too small to be shown, though for heights *below* h_0 (z negative) a considerable time elapses after sunrise before the increase in ν becomes noteworthy.

At the equator, for $\sigma_0 = 1$ the maximum value of n/n_0 occurs distinctly after noon at heights adjacent to h_0, and at greater heights the maximum is deferred till near sunset. But for $\sigma_0 = \frac{1}{5}$ and $\sigma_0 = \frac{1}{25}$ the value of n/n_0 at noon is very near the maximum value; this is true likewise for $\sigma_0 = \frac{1}{25}$, at 60° latitude, and therefore also at intermediate latitudes. For such values of σ_0 it is possible to deduce a close approximation to $\nu_{\max}$; for if, at noon, when $\sec\chi = \operatorname{cosec}(\theta + \delta)$, $d\nu/d\phi$ is small or zero, then by (36)

$$\nu^2 = F(z, \chi) = \exp\{1 - z - \operatorname{cosec}(\theta+\delta)\exp(-z)\},$$

or

$$n_{\max}/n_0 = \exp\tfrac{1}{2}\{1 - z - \operatorname{cosec}(\theta+\delta).\exp(-z)\}$$
$$= \sqrt{(I/I_0)}\ \text{(noon)} \qquad \text{......(40)}$$

by (18). At height h_0 ($z = 0$) at the equator ($\theta = \frac{1}{2}\pi$) at the equinoxes ($\delta = 0$), the value of I/I_0 is unity (cf. figure 1), so that $n_{\max} = n_0$ almost exactly. At the solstices at the equator $(I/I_0)_{\text{noon, max}} = 0{\cdot}938$ (curve 6, figure 1), so that $n_{\max} = 0{\cdot}968\,n_0$; the maximum value occurs very shortly after noon, at the height $z = 0{\cdot}1$. At the equinox in 60° latitude $(I/I_0)_{\text{noon, max}} = 0{\cdot}5$ (curve 3, figure 1) so that $n_{\max}/n_0 = 0{\cdot}707$, occurring at $z = 0{\cdot}7$; at the same latitude the values of $n_{\max}/n_0$ at midsummer and midwinter (cf. curves 5 and 1, figure 1) are 0·90 and 0·33, occurring nearly at noon at $z = 0{\cdot}2$ and $z = 2{\cdot}2$ respectively. At this latitude the seasonal change in $n_{\max}$ is very great, though less than that of $(I/I_0)_{\text{noon, max}}$.

Before attaining its maximum, ν must be less than $\sqrt{F(x, \chi_0)}$, because $\nu^2 = F(z, \chi) - \sigma d\nu/d\phi$, and $d\nu/d\phi$ is positive; similarly $\nu > \sqrt{F(z, \chi)}$ after attaining

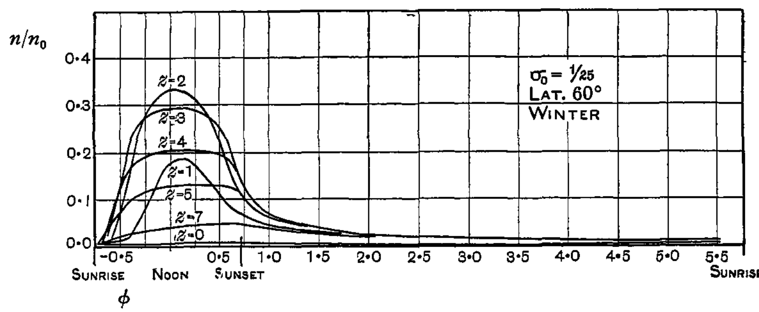


Fig. 11.

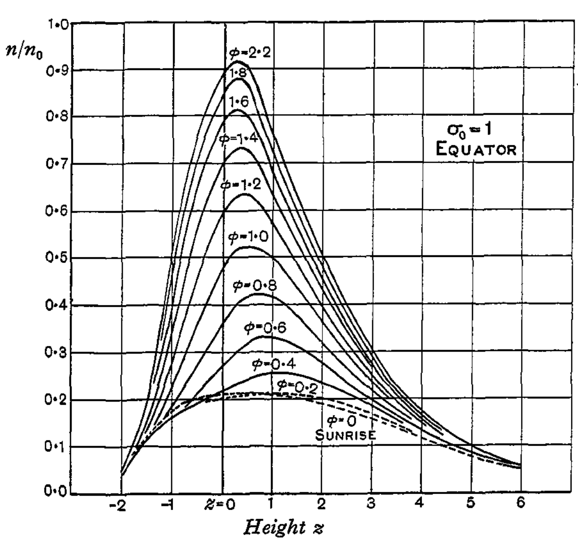


Fig 12*a*.

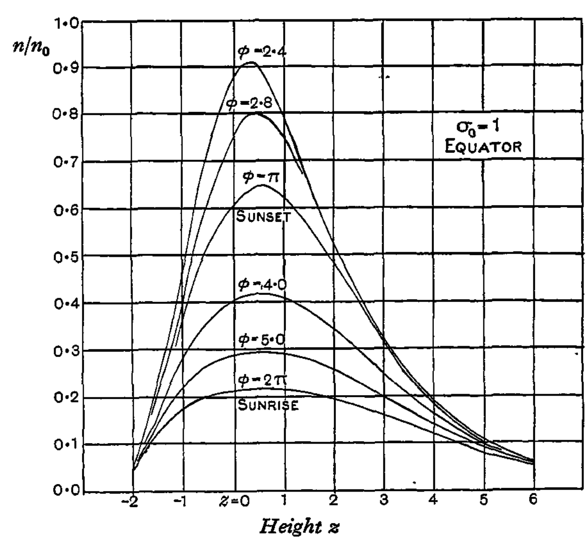


Fig. 12*b*.

its maximum. But there appears to be no easy general way of finding the amount of the difference.

At heights above the level of maximum noon ionization the graphs of n/n_0 as a function of ϕ (for $\sigma \leqslant \frac{1}{5}$) become increasingly flat; at the greater heights considered n/n_0 rises nearly to its maximum soon after sunrise, and varies little until near sunset; this is because $\sqrt{(I/I_0)}$ varies in a similar way (figures 2–5). At heights below the level of maximum noon (n/n_0), the graphs become increasingly narrow, the maximum ion density at such levels being approximated to over a short time only, near noon.

The seasonal variation of n/n_0, in latitude 60°, is much less for the upper levels than for the lower; for example for $\sigma_0 = \frac{1}{25}$ at $z = 6$, n_{max}/n_0 lies between 0·8 and 0·9 all through the year, while for $z = -1$ it is 0·6 in summer, 0·5 at the equinoxes, and less than 0·1 in winter. The seasonal variation of n/n_0 at the equator is so small that no graphs have been drawn to illustrate it.

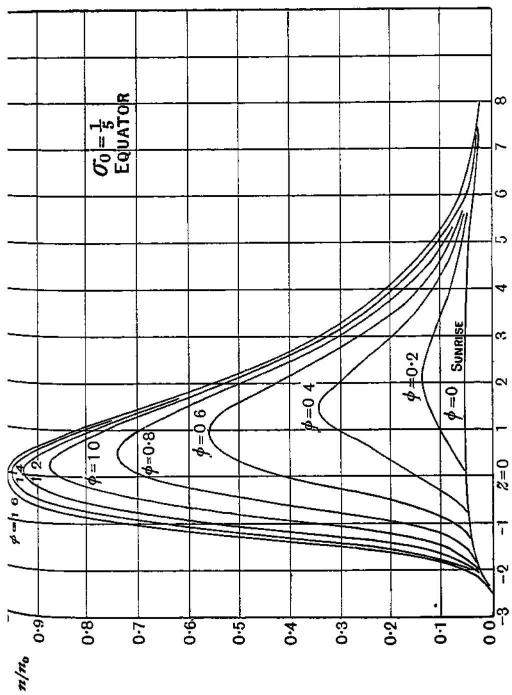
§ 12. THE ION-DENSITY AS A FUNCTION OF HEIGHT

The curves in figures 12–17, giving n/n_0 as a function of height at various local times, will next be considered.

Figures 12*a*, *b*, for $\sigma_0 = 1$ at the equator, show that in this case n has its maximum, at all heights, at about $2\frac{1}{2}$ hours after noon ($\phi = 2·2$); the maximum value of n occurs at about $z = \frac{1}{4}$, or about 2 km. (if $H = 8·4$ km.) above h_0, the level of maximum ion-production at midday; moreover the maximum value of n is distinctly less than n_0, being about 0·92 n_0. Before and after the epoch of maximum ionization, n varies fairly rapidly with respect both to height and time; at sunset ($\phi = \pi$) the ion-distribution has fallen from its maximum only by about one-third, and a considerable decrease of ionization proceeds during the night. But even at sunrise ($\phi = 0$ or 2π) there is a well-marked distribution of ionization, with maximum ion-density at about $z = \frac{1}{2}$. The level of maximum ion-density falls while n is rising, and *vice versa*.

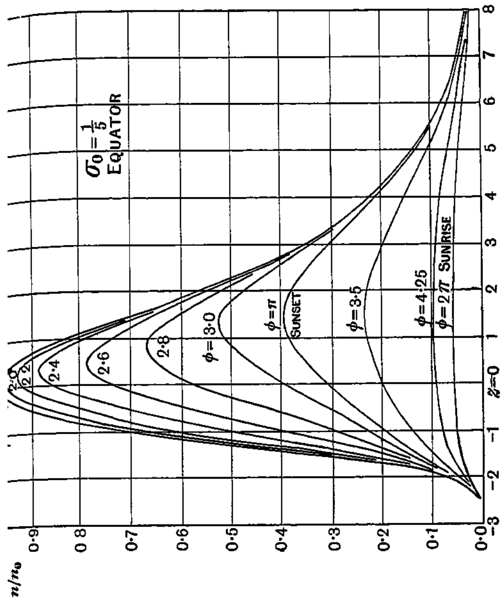
Figures 13*a*, *b*, for $\sigma_0 = \frac{1}{5}$ at the equator, show that in this case n/n_0 has a maximum value of almost exactly unity at or very near to the height h_0, and that the maximum ion-density, at all heights considered, is attained very shortly after noon. Before and after noon the level of maximum ion-density is above h_0, the value of z for maximum n, near sunrise or sunset, being about 2. The maximum ion-density at sunset occurs at about the level $z = 1·5$, and is approximately 0·4 n_0; the further reduction in n during the night is considerable, and at sunrise the ion-distribution consists of a thick layer of nearly uniform ion-density with a maximum of about $\frac{1}{20} n_0$ at about $z = 2$.

Figures 14*a*, *b*, for $\sigma_0 = \frac{1}{25}$ at the equator, show that in this case n attains the maximum value n_0 at the level h_0 at noon, almost exactly. At sunset the maximum ion-density is reduced to about 0·23 n_0 and occurs at the level $z = 2·5$ (about 21 km. above h_0, if $H = 8·4$ km.). During the night the ion-density decreases still further to about $\frac{1}{100} n_0$ at sunrise, when n is nearly uniform throughout a thick layer extending from about $z = -2·5$ to $z > 6$. These remarks apply to the equinoxes, but the reduction, and changes of distribution, of the ionization at the solstices are very slight.



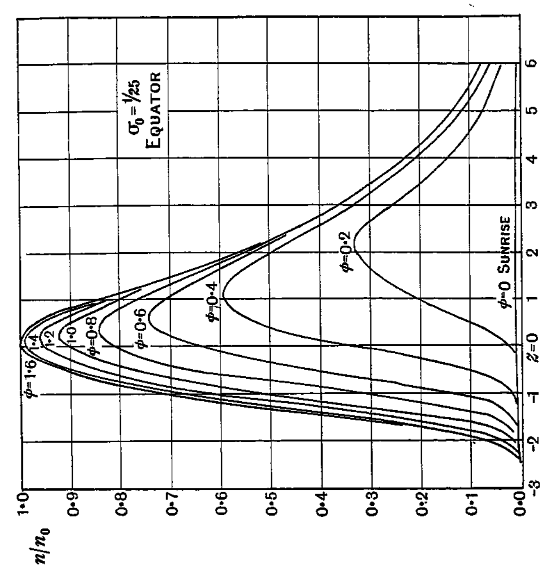
Height z

Fig. 13*a*.



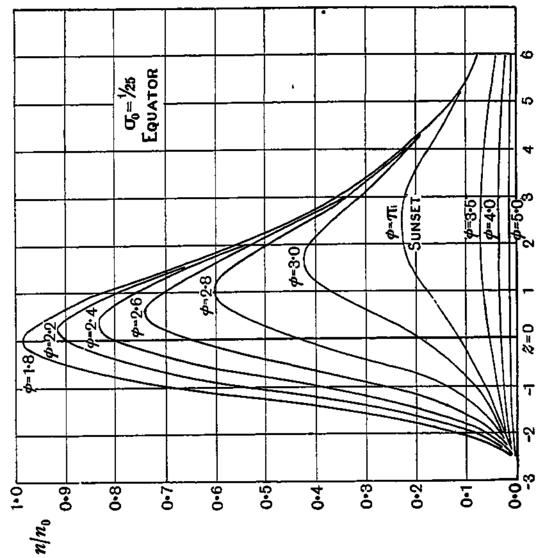
Height z

Fig. 13*b*.



Height z

Fig. 14*a*.



Height z

Fig. 14*b*.

Figures 17–18 for $\sigma_0 = \frac{1}{25}$ at latitude 60° show that at the equinox here, as at the equator, n has its maximum nearly at noon, but the maximum is less than at the equator, being about 0·71 n_0 instead of n_0; moreover it occurs at a somewhat higher level, where $z = 0·7$ or about 6 km. above h_0. Allowing for an all-round reduction in n at all times and heights, and for a rather higher level of $n_{\max}$ at any time (by about 0·5 to 0·7, or 4 to 6 km.) the features of this case resemble those for the equator ($\sigma_0 = \frac{1}{25}$).

Figures 16 and 17 show the height-distribution of n/n_0 at 60° latitude at midsummer and midwinter, for $\sigma_0 = \frac{1}{25}$; at midsummer the sunrise curve is a little above the sunrise curve for the equator (figure 14*a*), owing to the long day and short night in summer at 60° latitude; in midwinter the converse is true. In midsummer the noon maximum value of n is nearly as great at latitude 60° as at the equator; the sunset ionization at 60° latitude is greatest at about $z = 3·3$, and is about 0·16 n_0, as compared with 0·23 n_0 at $z = 2·5$ at the equator. At 60° latitude the sunset ionization in midwinter is very little less than in midsummer, though at noon the difference of ionization between the two seasons is large.

For still smaller values of σ_0 it is evident that the main changes from the curves of figures 14–17 would be as follows: the ion-density at sunrise and sunset, and throughout the night, would be reduced; the noon distribution of ion-density would scarcely be affected, but the rise to it would be initially less rapid, mainly taking place in a shorter interval before noon; similarly most of the fall from the noon maximum would be completed in a shorter interval, the period of high ion-density thus being concentrated more towards the middle of the day.

§ 13. UPPER ATMOSPHERIC IONIZATION

The preceding results will be briefly considered in relation to the ionization in the upper atmosphere; for further information, as regards both observational data and more detailed theory, reference may be made to the works of Pedersen*, Appleton†, Eckersley‡, Hulburt§ and other writers.

At present our knowledge of the actual values and variation of n as a function of height and time, at high levels in the atmosphere, is uncertain, though there is hope that it will later become possible to obtain detailed information of the kind by radio methods. There can scarcely be any doubt, however, that at least one strongly ionized layer exists in the atmosphere, at a height of the order 100 km., in which the ion-density undergoes a considerable daily variation; the evidence for this consists of various kinds of radio measurements, together with the daily, especially the lunar-diurnal, variations of the earth's magnetism. These suggest that the ion-density is greater by day than by night, rising from sunrise to about noon, and decreasing towards sunset, but still leaving at sunset a considerable distribution of ion-density,

* P. O. Pedersen, *The Propagation of Radio Waves* (Copenhagen, 1927).

† E. V. Appleton and collaborators, *Proc. R.S.* A, **128**, 133, 159 (1930) and earlier papers there cited.

‡ T. L. Eckersley, *Proc. Inst. Rad. Eng.* **18**, 106 (1930), and earlier references.

§ E. O. Hulburt, *Phys. Rev.* **31**, 1028 (1928); **34**, 1167 (1929); **35**, 240 (1930).

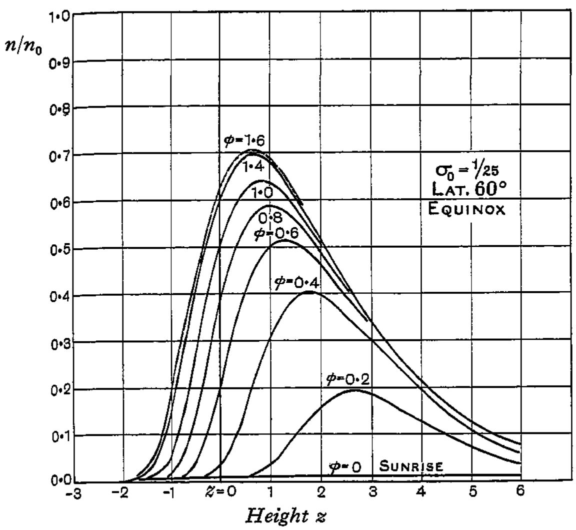


Fig. 15*a*.

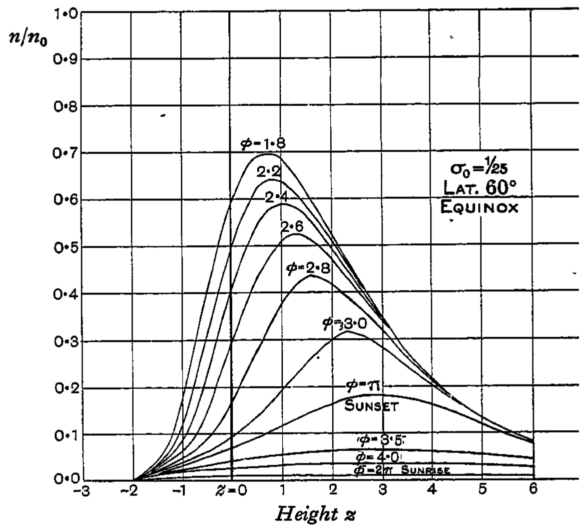


Fig. 15*b*.

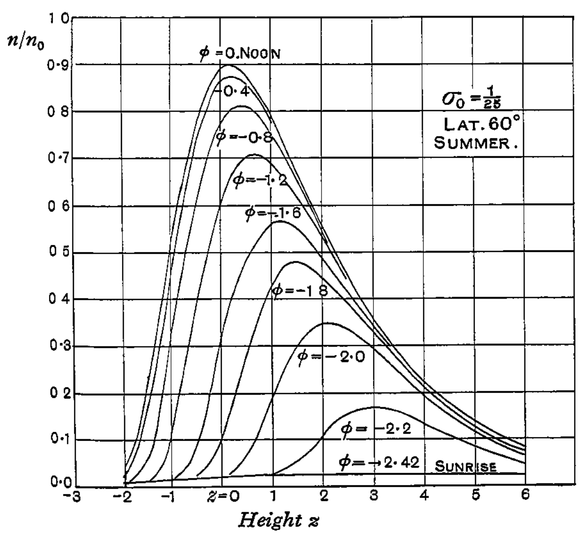


Fig. 16*a*.

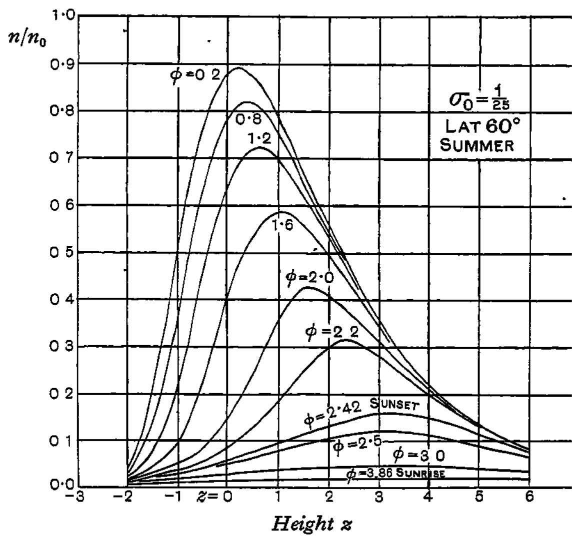


Fig. 16*b*.

which slowly decreases throughout the night. The theory of the lunar daily magnetic variations is not sufficiently well established to justify any safe estimate of the ratio of the values of n at noon, sunset and sunrise, but a guess may be hazarded, which is perhaps not likely to be wholly wrong as regards order of magnitude, that $n_{\text{noon}}/n_{\text{sunset}}$ is about 5. This, and the apparent absence of any appreciable lag in the maximum ion-density after noon, suggests, on the basis of figures 12 to 17, that the value of σ_0 lies between $\frac{1}{5}$ and $\frac{1}{25}$, being definitely less than 1, and probably not less, or at least not much less, than $\frac{1}{25}$.

n_0 The value of n_0, the maximum (noon) equatorial value of n for the positive ions, is uncertain, but such indications as exist (based on radio measures) suggest that the order of magnitude is 10^6 or 10^7.

By (33) or (34) it is possible from these very rough estimates of n_0 and σ_0 to derive corresponding estimates of α and I_0, as follows:

σ_0	n_0	α	I_0
$\frac{1}{25}$	10^6	$2\cdot10^{-9}$	$2\cdot10^3$
$\frac{1}{25}$	10^7	$2\cdot10^{-10}$	$2\cdot10^4$

The corresponding total rate of production of ions in a cm.^{2} column of air at the equator at noon is (cf. § 3) βS_∞ or, by (8), $HI_0 \exp 1$; if $H = 8\cdot4$ km., the values of βS_∞ corresponding to the above two values of I_0 are about $5\cdot10^9$ and $5\cdot10^{10}$.

These estimates of α, I_0 and βS_∞ are in general accordance with those of the same quantities (there denoted by α, q, and $\int q dh$) which I made in an earlier discussion of upper atmospheric ionization*; this discussion was based on data rather different in kind from those here considered. They seem also to be in general accordance with the observations of the effects of eclipses on radio transmission, which indicate a rapid change in at least the lower part of the ionized layer due to the temporary (total or partial) interception of solar radiation; if, at time $t = 0$, the ion-density at the point considered being n, the ion-producing agent is entirely removed, n will at later time be given by $n/(1 + \alpha n t)$, so that it will be halved in a time $1/\alpha n$. If $\alpha = 2\cdot10^{-9}$ and $n = 10^6$, this time is 500 seconds, or about 8 minutes, which is of the right order of magnitude; similarly if $\alpha = 2\cdot10^{-10}$ and $n = 10^7$.

The main purpose of the present paper, however, is not to discuss the actual state of ionization of the atmosphere on the basis of the scanty available data; it is intended to afford a means of discussing the value and variations of the ion-content of the upper atmosphere when reliable data become available. Its results are applicable not only to the ionized layer near 100 km., but also to the higher layer, at about 250 km., discovered by Appleton†; the values of the constants H, β, S_∞, α, ... for the two layers may, and probably will, be different. The present analysis is applicable also to the absorption of non-ionizing radiation, such as that which, by dissociating oxygen molecules, leads to the formation of ozone; but the work of §§ 9 *et seq.* is valid for dissociating-radiation only if the products of dissociation recombine according to the simple law (29), which may not be the case for ozone.

* *Q.J.R. Met. Soc.* **52**, 229 (1926). † *Loc. cit.*

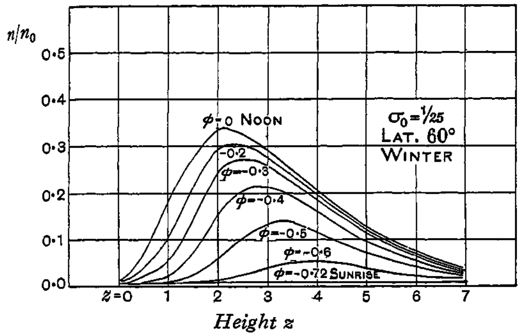


Fig. 17*a*.

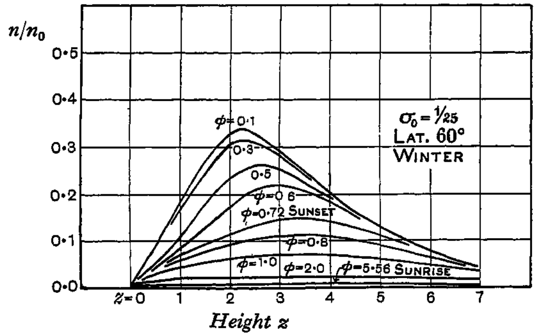


Fig. 17*b*.

§ 14. ACKNOWLEDGMENTS

In conclusion I have to acknowledge assistance received from Miss M. C. Gray, who made the detailed numerical calculations involved in this paper, and to Mr W. Reeve, Miss V. Hatcher and Miss R. Rossiter, who assisted in the preparation of the numerous diagrams.