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THE ABSORPTION AND DISSOCIATIVE OR IONIZING EFFECT OF MONOCHROMATIC RADIATION IN AN ATMOSPHERE ON A ROTATING EARTH

PART II. GRAZING INCIDENCE

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ABSTRACT. The absorption of radiation from the sun in an atmosphere varying exponentially with the height is considered, as in a former paper; but here the earth's curvature, and that of the level layers in the atmosphere, is taken into account. The values of this absorption as previously calculated are valid so long as the sun's zenith distance does not exceed 75°, but for greater zenith distances the necessary corrections are of importance. It is shown that the absorption, and resulting ionization or dissociation of the air, should begin to increase before ground sunrise, the interval varying from about 10 minutes at the equator to about an hour at 60° latitude.

§ 1. INTRODUCTION

THIS paper is a sequel to one* with the same (main) title, which will here be referred to as part I. In § 4 of that paper, dealing with the absorption of radiation, it was pointed out that the equation of absorption there used was not mathematically exact (except for direct incidence) as a representation of the physical assumptions adopted at the outset; it was suggested that the approximation was probably sufficiently accurate for beams inclined to the vertical at angles up to 85°, but that for greater angles, corresponding to nearly grazing incidence, the error became appreciable. It was further stated that the values of the ion-content deduced in the paper were very nearly true, in low latitudes, except near dawn, but that in higher latitudes, *in winter*, the necessary corrections are appreciable up to noon. In the present paper the absorption of radiation at nearly grazing incidence is considered in detail, and the consequent corrections to the former results, relating to the variation of ion-content with respect to height and time, are examined.

The notation of part I will in general be used in this paper without being re-defined. The tables and diagrams in this paper are numbered in continuation of those of part I.

§ 2. THE ABSORPTION OF RADIATION

In considering the absorption of radiation we shall ignore the refraction of the beam; that is, the radiation will be regarded as travelling along a straight line.

In figure 18 let O denote the earth's centre, OS the line from O to the sun, P' any point in the earth's atmosphere, and NTM part of the boundary of the

* S. Chapman, *Proc. Phys. Soc.* 43, 26 (1931).

section of the earth by the plane SOP'; N is thus the point on the earth immediately "below" the sun, and M the antipodal point. T is a point on the "twilight circle" of the earth, dividing the day hemisphere, from which the sun is visible, from the night hemisphere.

λ The flow of radiation is parallel to SO, and if $\angle\, SOP' = \lambda$, the angle of incidence
s of the beam at P' is λ. Let the beam through P' cut OT in T', and let distance s along the beam be measured from T' as origin, in the direction of travel of the
a, h', p beam. If a denotes the radius of the earth, and h' the height of P', then p, representing the distance from O of the beam through P', is given by

$$p = OT' = (a + h') \sin \lambda \qquad \text{......(1).}$$

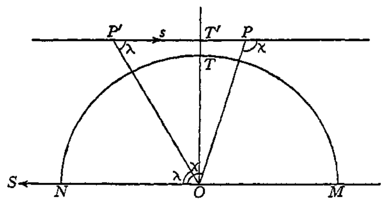

Fig. 18.

Along any particular beam p is constant, while for different points P' along it, specified by the corresponding angle λ, the height h' is given by

$$h' = p \operatorname{cosec} \lambda - a \qquad \text{......(2).}$$

For the value of s at P' we have

$$s = -p \cot \lambda \qquad \text{......(3);}$$

this is negative if P' is to the left of OT (as in figure 18) and positive if to the right.

Consider the absorption $-dS'$ between the point P' and a neighbouring point
ρ', S' on the beam, given by $s + ds$ or $\lambda + d\lambda$. If ρ' is the air density at P', and S' the intensity of radiation,

$$\begin{aligned} dS' &= -AS'\rho' ds \\ &= -AS'\rho_0 \exp(-h/H)\, p \operatorname{cosec}^2 \lambda\, d\lambda \\ &= -AS'\rho_0 \exp\{(a - p \operatorname{cosec} \lambda)/h\}\, p \operatorname{cosec}^2 \lambda\, d\lambda \qquad \text{......(4).} \end{aligned}$$

χ Let P be any point on the beam, such that $\angle\, SOP = \chi$; in figure 18 this point is
S shown to the right of OT, but it may be anywhere along the beam. The intensity S of the beam, at P, is obtainable by integration along it from the extreme left, outside the atmosphere (where $s = -\infty$, $\lambda = 0$) to P. We get

$$S/S_\infty = \exp\left[-A\rho_0 \int_0^\chi \exp\{(a - p \operatorname{cosec} \lambda)/H\}\, p \operatorname{cosec}^2 \lambda\, d\lambda\right] \qquad \text{......(5).}$$

This equation replaces equation (7) of part I, which is not strictly true for an atmosphere arranged in concentric spherical layers of equal density.

The absorption of radiation per cm.[3] of atmosphere is $-dS/ds$; we shall consider, instead of this, the quantity $I \equiv -\beta dS/ds$, where β denotes the number of ions produced by the absorption of unit quantity of the radiation (or the number of molecules dissociated, when the effect of the radiation is dissociative rather than ionizing). By putting $\beta = 1$ in I we can obtain the actual absorption if required. I, β

From (4) the value of I at P is found, on substituting for S from (5), to be given by

$$I = \beta A S_\infty \rho_0 \exp(-h/H) \exp\left[-A\rho_0 \int_0^\chi \exp\{(a - p \operatorname{cosec}\lambda)/H\} p \operatorname{cosec}^2\lambda \, d\lambda\right] \qquad \text{......(6).}$$

In terms of I_0 ($\equiv \beta S_\infty/H \exp 1$) and h_0 (given by $\exp(h_0/H) = A\rho_0 H$) this may be written I_0, h_0

$$\frac{I}{I_0} = \exp\left(1 + \frac{h_0 - h}{H}\right) \exp\left[-\frac{p}{H}\int_0^\chi \exp\left(\frac{a + h_0 - p\operatorname{cosec}\lambda}{H}\right)\operatorname{cosec}^2\lambda\, d\lambda\right], \qquad \text{......(7)}$$

or, taking $z \equiv (h - h_0)/H$, as in part 1, and substituting $(a + h)\sin\chi$ or $(a + h_0 + zH)\sin\chi$ for p,

$$\frac{I}{I_0} = \exp\left[1 - z - \left(\frac{a + h_0}{H} + z\right)\sin\chi \int_0^\chi \exp\left\{\frac{a + h_0}{H} - \left(\frac{a + h_0}{H} + z\right)\frac{\sin\chi}{\sin\lambda}\right\}\operatorname{cosec}^2\lambda\, d\lambda\right] \qquad \text{......(8).}$$

This equation replaces (14) or (17) in part I, and differs from those equations in one important respect; it contains a quantity $(a + h_0)/H$ which did not occur in the former discussion. Thus the proper consideration of the earth's curvature involves the introduction of a new parameter into the formulae, and if for no other reason than this it seemed justifiable, in a first general discussion of the subject, to defer the consideration of the curvature, particularly since this made it possible to put the analysis in a very general form involving only one parameter (σ_0).

The further parameter now introduced, $(a + h_0)/H$, is equal to the distance, *expressed in terms of H as unit of length*, from the earth's centre to the level at which, at noon at the equator, the absorption of radiation is a maximum. This parameter will be denoted by R. R

It is of interest to consider the numerical value of R in the case of the ionized layers of which the heights have been measured by E. V. Appleton and his collaborators*. The value of a is 6370 km. For the lower of the observed ionized layers h_0 is about 100 km., which is small, though not negligibly so, compared with a; thus $a + h_0$ is about 6470 km. for this layer. The value of H at this height in the atmosphere is unknown; if the temperature there is 300° K. and the composition is the same as in the lower atmosphere, $H = 8{\cdot}4$ km. It is possible that H may vary with the season, and even throughout the day and night; temperatures as high as 1000° K. have been suggested as existing at great heights, and for the same com-

* E. V. Appleton, *Proc. R.S.* A, **126**, 542 (1930); *ibid.* **128**, 133 and 159 (1930), with J. A. Ratcliffe and A. L. Green respectively; and earlier papers there cited.

position as before this would correspond to $H = 28$ km. At great heights (probably much above 100 km.) the composition is likely to differ considerably from that near the ground, and I have suggested elsewhere* that in the outermost layer atomic oxygen and nitrogen, possibly ionized, may be the main constituents; if so, at these heights H would be increased, on this account, in the ratio 2 (if the atoms are not ionized) or 4 (if wholly ionized, the negative charges being electrons). Thus for very high levels values of H as large as 112 km. might require consideration. The range of R or $(a + h_0)/H$, as H ranges from 8·4 to 112 km., is from about 770 to 58, if $h_0 = 100$, or slightly more for higher values of h_0. It is convenient, in the subsequent calculations, to bear in mind this possible extreme range of the new parameter R; but for the ionized layer at about 100 km. height, H is probably about 10 km., corresponding to R about 647 or, in round figures, 650.

In terms of R, (8) may be written

$$I/I_0 = \exp\left[1 - z - \exp(-z) f(R + z, \chi)\right] \qquad \text{......(9)},$$

where

$$f(x, \chi) = x \sin\chi \int_0^\chi \exp\{x(1 - \sin\chi/\sin\lambda)\} \operatorname{cosec}^2\lambda \, d\lambda \qquad \text{......(10)}.$$

Equation (9) is identical with equation (17) of part I, except that in (17) $\sec\chi$ occurs in the place of $f(R + z, \chi)$. It is of interest to show that $f(R + z, \chi)$ actually has the value $\sec\chi$ when R tends to infinity, because of course the neglect of the earth's curvature, in part 1, corresponds to taking R (or $R + z$) as infinite.

Since

$$d \operatorname{cosec}\lambda = -\cos\lambda \operatorname{cosec}^2\lambda \, d\lambda,$$

(10) may be rewritten as

$$f(x, \chi) = \int_{\lambda=0}^{\lambda=\chi} \sec\lambda \, d\left[\exp\{x(1 - \sin\chi \operatorname{cosec}\lambda)\}\right] \qquad \text{......(11)},$$

and, by a partial integration, this gives

$$\begin{aligned} f(x, \chi) &= \Big[\sec\lambda \exp\{x(1 - \sin\chi \operatorname{cosec}\lambda)\}\Big]_{\lambda=0}^{\lambda=\chi} \\ &\qquad - \int_0^\chi \exp\{x(1 - \sin\chi \operatorname{cosec}\lambda)\} \sec\lambda \tan\lambda \, d\lambda \\ &= \sec\chi - \int_0^\chi \exp\{x(1 - \sin\chi \operatorname{cosec}\lambda)\} \sec\lambda \tan\lambda \, d\lambda \qquad \text{......(12)}. \end{aligned}$$

It is not difficult to prove that as $x \to \infty$ the second term in the last line of (12) tends to zero provided that χ is less than 90°; such values were the only ones that came in question in the former paper. Thus when R is taken as infinite, as was done there, $f(x, \chi)$ is equal to $\sec\chi$. The revised discussion in the present paper depends essentially on the replacement of $\sec\chi$ in equation (17) of part 1 by $f(R + z, \chi)$ as here defined, and it is necessary to consider the nature of this function of the two variables x (or $R + z$) and χ.

It should be added that for any beam for which values of χ exceeding 90° have to be considered, p cannot be less than a, the radius of the solid body of the earth.

* *Phil. Mag.* **10**, 369 (1930).

This imposes an upper limit on the possible values of χ for any height h, since $p \geqslant a$ is equivalent to $(a+h)\sin\chi \geqslant a$. For example, for $h = 100$ km., the limiting value of χ is about $100°$. This restriction has little actual importance, however, for the problem of upper atmospheric ionization, because the absorption of the beams for which p only slightly exceeds a is practically completed well before the limiting value of χ is attained.

§ 3. THE FUNCTION $f(x, \chi)$ WHEN $\chi = \frac{1}{2}\pi$

When $\chi = \frac{1}{2}\pi$, $\sec\chi$ is infinite, whereas $f(x, \chi)$ is finite; this infinity of $\sec\chi$ in (17) renders I zero, corresponding to complete absorption of radiation by the time the beam reaches the twilight plane (i.e. the plane through O perpendicular to SO). Actually the value of I is still positive on reaching this plane, and also for points "behind" this plane, corresponding to values of χ greater than $\frac{1}{2}\pi$.

For the special value $\chi = \frac{1}{2}\pi$, $f(x, \chi)$ may be expressed in terms of Bessel functions. For

$$f(x, \tfrac{1}{2}\pi) = x\int_0^{\frac{1}{2}\pi} \exp\{x(1-\operatorname{cosec}\lambda)\}\operatorname{cosec}^2\lambda\, d\lambda \qquad \text{......(13)}.$$

On making the substitution $\qquad \sin\lambda = \operatorname{sech} u \qquad$ ......(14),

so that

$$\left.\begin{aligned} \cos\lambda &= \tanh u, & \tan\lambda &= \operatorname{cosech} u \\ \sec\lambda &= \cosh u, & \sec\lambda &= \coth u \\ \cot\lambda &= \sinh u, & d\lambda &= -\operatorname{sech} u\, du \end{aligned}\right\} \qquad \text{......(15)},$$

we find that (13) becomes

$$\begin{aligned} f(x, \tfrac{1}{2}\pi) &= x\int_0^\infty e^{x(1-\cosh u)}\cosh u\, du \\ &= -xe^x \frac{d}{dx}\int_0^\infty e^{-x\cosh u}\, du \\ &= -xe^x \frac{d}{dx} K_0(x) = -xe^x K_1(x) \end{aligned} \qquad \text{......(16)},$$

by well-known formulae in the theory of Bessel functions*. Tables of the function $e^x K_1(x)$ up to $x = 16$ are given at the end of Watson's treatise on these functions.

Since in the present application x is large (50 or more), the well-known asymptotic formula for $K_1(x)$ may be used, which gives

$$f(x, \tfrac{1}{2}\pi) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}}\left\{1 + \sum_{n=1} a_n x^{-n}\right\} \qquad \text{......(17)},$$

where

$$a_n = [\{1 \cdot 3 \dots (2n-3)\}^2 (4n^2-1)]/(n!\, 2^{3n}) \qquad \text{......(18)},$$

so that

$$a_1 = \tfrac{3}{8}, \quad a_2 = -\tfrac{15}{128}, \quad a_3 = \tfrac{105}{1024}, \dots \qquad \text{......(19)}.$$

Thus for $x > 50$, $f(x, \frac{1}{2}\pi) = (\frac{1}{2}\pi x)^{\frac{1}{2}}$ correct to within 1 per cent. For $x = 50$ it is 8·93, while for $x = 800$ it is 35·47; these values replace infinity, the value of $\sec\chi$, when (9) replaces (17) of part I.

* Cf. Whittaker and Watson, *Modern Analysis* (2nd ed.), p. 377, ex. 40.

§4. THE FUNCTION $f(x, \chi)$ FOR SMALL VALUES OF χ

The smaller the value of χ, the more nearly will $f(x, \chi)$ approximate to $\sec\chi$, so that for small values of χ a formula expressing this fact should be obtainable. It may be found as follows. Let

$$y = x \sin\chi \qquad \text{......(20)},$$

so that

$$\begin{aligned} f(x, \chi) &= y e^{x} \int_0^{\chi} e^{-y \operatorname{cosec}\lambda} \operatorname{cosec}^2\lambda \, d\lambda \\ &= - e^{x} \int_{\lambda=0}^{\lambda=\chi} \sec\lambda \, d\left(e^{-y \operatorname{cosec}\lambda}\right) \\ &= \sec\chi - e^{x} \int_0^{\chi} e^{-y \operatorname{cosec}\lambda} \sec^2\lambda \sin\lambda \, d\lambda \\ &= \sec\chi - (e^{x}/y) \int_{\lambda=0}^{\lambda=\chi} \tan^3\lambda \, d\left(e^{-y \operatorname{cosec}\lambda}\right) \qquad \text{......(21)}, \end{aligned}$$

by a partial integration similar to that which we used in obtaining (12). By further similar partial integrations we find that

$$f(x, \chi) = \sec\chi + \sum_{n=1} b_n/x^n \qquad \text{......(22)},$$

where

$$\left.\begin{aligned} &b_1 = -\sec\chi\tan^2\chi, \quad b_2 = 3\tan^2\chi\sec^3\chi, \quad b_3 = -(15\tan^4\chi + 12\tan^2\chi)\sec^3\chi \\ &b_4 = (105\tan^4\chi + 60\tan^2\chi)\sec^5\chi, \\ &b_5 = -(945\tan^6\chi + 1260\tan^4\chi + 360\tan^2\chi)\sec^5\chi \end{aligned}\right\} \qquad \text{......(23)},$$

and, in general,

$$b_n \sin^n\chi = -\sin^2\chi \sec\chi \, (d/d\chi)\,(b_{n-1}\sin^{n-1}\chi) \qquad \text{......(24)}.$$

This series is useful only so long as $(\tan^2\chi)/x$ is small; for $x = 800$ it is of service for values of χ up to about 80°, but for $x = 50$ its numerical convergence becomes slow at $\chi = 60°$. The following table gives values of the *ratio* $100\{1 - f(x, \chi)/\sec\chi\}$, calculated (except for $\chi = 75°$ and $R \leqslant 200$) in the above way for various values of R and χ; this indicates the *percentage* by which f falls short of $\sec\chi$ in the various cases.

Table 3. $100\{1 - f(x, \chi)/\sec\chi\}$

$\chi =$	30°	45°	60°	75°
$\sec\chi =$	1·155	1·414	2·000	3·864
$R =$ 50	0·62	1·2	4·6	16·4
100	0·32	0·95	2·7	9·1
200	0·16	0·49	1·4	5·6
300	0·11	0·32	1·0	4·1
400	0·08	0·23	0 7	3·2
500	0·06	0·20	0 6	2·6
600	0·05	0·16	0·5	2·2
650	0·05	0·15	0·5	2·0
700	0·05	0·14	0·4	1·9
800	0·04	0·13	0·4	1·7

It is clear from this table that if $R > 300$ (corresponding to $H < 20$ km. approximately), the curvature of the earth reduces the factor $\sec\chi$ of part 1, equations (8), (9), (12), (14), (17), by less than 5 per cent., even when χ is as large as 75°.

§ 5. GENERAL ASYMPTOTIC FORMULAE FOR $f(x, \chi)$

When χ is neither small nor equal to $\frac{1}{2}\pi$, $f(x, \chi)$ may be evaluated as follows. Using the substitutions (14) and (20), we find

$$f(x, \chi) = ye^{x-y} \int_U^\infty e^{y(1-\cosh u)} \cosh u \, du \qquad \text{......(25)},$$

where U is the value of u corresponding to $\lambda = \chi$, so that U

$$\operatorname{sech} U = \sin \chi \qquad \text{......(26)}.$$

If $\chi < \frac{1}{2}\pi$, the positive value of U must be taken, but if $\chi > \frac{1}{2}\pi$, the negative value is the appropriate one. If we write

$$\chi = \tfrac{1}{2}\pi - \chi' \qquad \text{......(27)},$$ χ'

U will have the same sign as χ', or, if we take χ' and U as always positive and given by

$$\operatorname{sech} U = \cos \chi' \qquad \text{......(28)},$$

(25) may be written in the form

$$\begin{aligned} f(x, \tfrac{1}{2}\pi \mp \chi') &= ye^{x-y} \int_{\mp U}^\infty e^{y(1-\cosh u)} \cosh u \, du \\ &= ye^{x-y} \left\{ \int_0^\infty \mp \int_0^U \right\} e^{y(1-\cosh u)} \cosh u \, du \\ &= e^{x-y} f(y, \tfrac{1}{2}\pi) \mp ye^{x-y} \int_0^U e^{y(1-\cosh u)} \cosh u \, du \qquad \text{...(29)}. \end{aligned}$$

Writing $\quad v = \sinh \frac{1}{2}u, \quad \cosh u - 1 = 2v^2, \quad du = (2\,dv)/(1 + v^2)^{\frac{1}{2}} \qquad$ ......(30) $\quad v$

we have

$$\begin{aligned} \int e^{y(1-\cosh u)} \cosh u \, du &= 2 \int \exp(-2yv^2) \frac{1 + 2v^2}{(1 + v^2)^{\frac{1}{2}}} dv \\ &= 2 \int \exp(-2yv^2)(1 + \Sigma c_n v^{2n}) \, dv \qquad \text{......(31)}, \end{aligned}$$

where

$$\left. \begin{aligned} c_n &= (-1)^{n-1}(2n+1)\{1 \cdot 3 \ldots (2n-3)\}/n!\,2^n \\ c_1 &= \tfrac{3}{2}, \quad c_2 = -\tfrac{5}{8}, \quad c_3 = \tfrac{7}{16}, \quad c_4 = -\tfrac{45}{128}, \ldots \end{aligned} \right\} \qquad \text{......(32)}.$$

Now it is readily shown that, for all positive values of y and V,

$$\begin{aligned} \int_0^V \exp(-2yv^2) v^{2n} \, dv &= \frac{1 \cdot 3 \ldots (2n-1)}{2^{2n+\frac{1}{2}} y^{n+\frac{1}{2}}} \operatorname{erf}(2yV^2)^{\frac{1}{2}} \\ &\quad - \exp(-2yV^2) \left\{ \frac{V^{2n-1}}{4y} + \frac{(2n-1) V^{2n-3}}{(4y)^2} + \frac{(2n-1)(2n-3) V^{2n-5}}{(4y)^3} + \ldots \right\} \\ &\qquad \text{......(33)}, \end{aligned}$$

where the series on the right is a terminating one, and where

$$\operatorname{erf} \eta = \int_0^\eta e^{-w^2} dw \qquad \text{......(34)}.$$ η

By well-known theorems, $\quad \dfrac{2}{\sqrt{\pi}} \operatorname{erf} \eta \to 1 \text{ as } \eta \to \infty \qquad$ ......(35), and, in the form of an asymptotic series, useful when η is large,

$$1 - \frac{2}{\sqrt{\pi}} \operatorname{erf} \eta = \frac{2}{\sqrt{\pi}} \int_\eta^\infty e^{-w^2} dw = \frac{e^{-\frac{1}{2}\eta^2}}{\sqrt{(\pi\eta)}} W_{-\frac{1}{4}, \frac{1}{4}}(\eta^2)$$

$$= \frac{e^{-\eta^2}}{\eta\sqrt{\pi}} \left[1 + \Sigma (-1)^n \frac{1 \cdot 3 \ldots (2n-1)}{2^n \eta^{2n}}\right] \qquad \ldots\ldots(36),$$

V where W denotes the confluent hypergeometric function*. Hence if V is the value of v corresponding to $u = U$, it follows that

$$y e^{x-y} \int_0^U e^{y(1-\cosh u)} \cosh u \, du$$

$$= e^{x-y} (\tfrac{1}{2}\pi y)^{\frac{1}{2}} \left\{\frac{2}{\sqrt{\pi}} \operatorname{erf} (2yV^2)^{\frac{1}{2}}\right\} \left\{1 + \frac{c_1}{4y} + \frac{1 \cdot 3 c_2}{(4y)^2} + \ldots\right\}$$

$$- \frac{1}{2}\left[\sum_1^\infty c_n V^{2n-1} + \frac{1}{4y} \sum_2^\infty (2n-1) c_n V^{2n-3} + \frac{1}{(4y)^2} \sum_3^\infty (2n-1)(2n-3) c_n V^{2n-5} + \ldots\right]$$

$$= - y e^x K_1(y) \frac{2}{\sqrt{\pi}} \operatorname{erf} (2yV^2)^{\frac{1}{2}} - \tfrac{1}{2} \sum_{m=0}^\infty (4y)^{-m} \sum_{n=m+1}^\infty (2n-1)(2n-3) \ldots$$

$$\ldots (2n - 2m + 1) c_n V^{2n-2m-1} \qquad \ldots\ldots(37).$$

Hence by (29) and (16),

$$f(x, \tfrac{1}{2}\pi \mp \chi') = - y e^x K_1(y) \left\{1 \mp \frac{2}{\sqrt{\pi}} \operatorname{erf} (2yV^2)^{\frac{1}{2}}\right\}$$

$$\pm \tfrac{1}{2} \sum_{m=0}^\infty (4y)^{-m} \sum_{n=m+1}^\infty (2n-1)(2n-3) \ldots (2n-2m+1) c_n V^{2n-2m-1} \quad \ldots(38).$$

In this formula

$$\left.\begin{aligned} V &= \sinh \tfrac{1}{2} U = \sqrt{\{\tfrac{1}{2}(\operatorname{cosec} \chi - 1)\}} = \sqrt{\{\tfrac{1}{2}(\sec \chi' - 1)\}} \\ y &= x \sin \chi = x \cos \chi' \end{aligned}\right\} \quad \ldots\ldots(39),$$

$$(2yV^2)^{\frac{1}{2}} = \{x(1 - \sin \chi)\}^{\frac{1}{2}} = \{x(1 - \cos \chi')\}^{\frac{1}{2}} \qquad \ldots\ldots(40);$$

in all cases the positive square roots are to be taken.

The formula (38) is useful when χ is too great for (22) to be rapidly convergent; when χ' is positive ($\chi < 90°$) and such that $(2yV^2)^{\frac{1}{2}}$ is 3 or more, the expansion (36) is helpful in evaluating (38). The binomial expansion in (31) is valid so long as $V < 1$, or $\sin \chi > \frac{1}{3}$, $\chi > 20°$; in calculating $f(x, \chi)$ for smaller values of χ the formulae of § 3 are available.

When χ' is positive, and x tends to infinity, the first term of $f(x, \frac{1}{2}\pi - \chi')$, in (38), may be shown, with the aid of (36), to tend to $-1/2V$. The second part of (38) tends to $\frac{1}{2} \sum_{n=1}^\infty c_n V^{2n-1}$, which is equal to $1/2V + \sec \chi$. Thus this expression for $f(x, \frac{1}{2}\pi - \chi')$ tends to $\sec \chi$, as it should do, when $x \to \infty$, $0 < \chi' < \frac{1}{2}\pi$.

* Cf. Whittaker and Watson, *Modern Analysis* (2nd ed.), p. 335.

In calculating the first term of (38), use may be made either of tables of $e^y K_1(y)$, or of the asymptotic formula

$$-ye^x K_1(y) = (\tfrac{1}{2}\pi y)^{\frac{1}{2}} e^{x-y} \left\{1 + \sum_1 a_n y^{-n}\right\}$$

$$= (\tfrac{1}{2}\pi x \sin\chi)^{\frac{1}{2}} e^{x(1-\sin\chi)} \left\{1 + \sum_1 a_n x^{-n} \operatorname{cosec}^n \chi\right\} \quad \text{......(40).}$$

Table 4 gives values of $f(R, \chi)$ and of $\ln f(R, \chi)$ for various values of R and χ; it also gives values of $\sec\chi$ and $\ln \sec\chi$ for comparison.

When $\chi = 0$, $f(R, \chi) = \sec\chi = 1$, and $\ln\sec\chi = \ln f(R, \chi) = 0$ for all values of R.

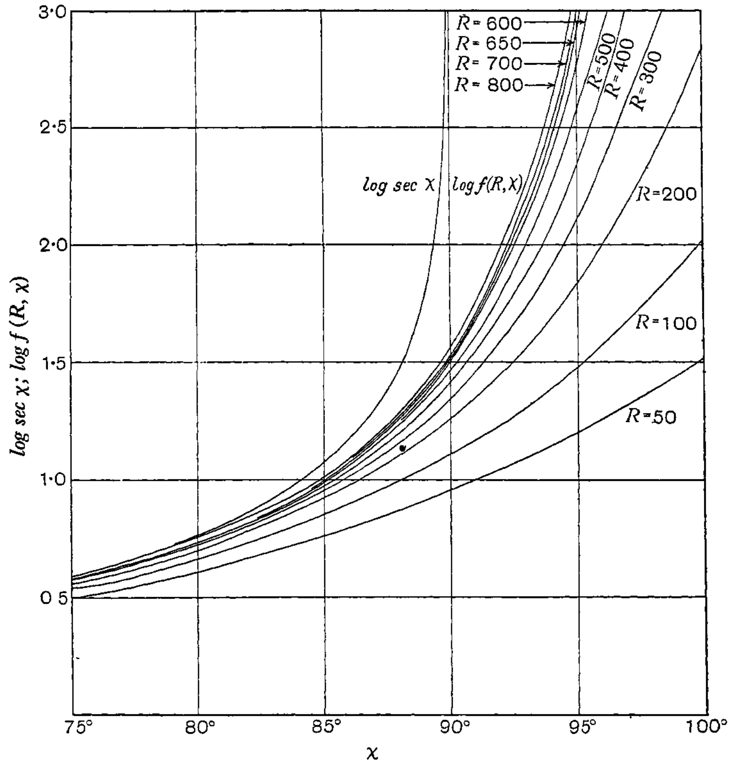

Fig. 19.

Figure 19 illustrates the functions $\log\sec\chi$ and $\log f(R, \chi)$, for various values of R, over the range of χ from 75° to 100°. If the ordinates are multiplied by 2·303 this figure also gives the values of $\ln\sec\chi$ and $\ln f(R, \chi)$.

§ 6. PROPERTIES OF THE FUNCTION $f(x, \chi)$

When $\chi < 90°$, or $\chi' > 0$, the principal factors in the first and most important term of $f(x, \chi)$, according to (38), are $e^{x(1-\sin\chi)}$ and $1 - (2/\sqrt{\pi}) \operatorname{erf}\{2yV^2\}^{\frac{1}{2}}$; the former increases as χ decreases, while the latter decreases; these tendencies approximately neutralize one another in their product, and as χ decreases from $\frac{1}{2}\pi$, $f(x, \chi)$ steadily decreases from $\sqrt{(\frac{1}{2}\pi x)}$, approximately, and tends, from below, to $\sec\chi$ for sufficiently small values of χ.

As χ *increases* from $\frac{1}{2}\pi$, the increase of $e^{x(1-\sin\chi)}$ is no longer compensated by a rapidly decreasing factor, because the factor involving $\operatorname{erf}(2yV^2)^{\frac{1}{2}}$ is now $1 + (2/\sqrt{\pi}) \operatorname{erf}(2yV^2)^{\frac{1}{2}}$, which increases from 1, when $\chi = \frac{1}{2}\pi$, and rapidly approaches the limiting value 2 when $(2yV^2)^{\frac{1}{2}}$ exceeds the value 3.

Throughout the whole range of χ, $f(x, \chi) > f(x', \chi)$, if $x > x'$.

The rate of variation of $f(x, \chi)$ with respect to χ is readily calculable:

$$\frac{\partial f(x, \chi)}{\partial \chi} = x \cos\chi \int_0^\chi e^{x(1-\sin\chi \operatorname{cosec}\lambda)} \operatorname{cosec}^2\lambda \, d\lambda - x^2 \sin\chi \cos\chi \int_0^\chi e^{x(1-\sin\chi \operatorname{cosec}\lambda)} \operatorname{cosec}^3\lambda \, d\lambda \quad \text{......(41).}$$

This expression could, if desired, be expressed in forms similar to those derived, for suitable ranges of x and χ, in §§ 3, 4. When $\chi = \frac{1}{2}\pi$ the result is specially simple, i.e.

$$\left\{\frac{\partial f(x, \chi)}{\partial \chi}\right\}_{\chi=\frac{1}{2}\pi} = x \quad \text{......(42).}$$

The variation of $f(x, \chi)$ with respect to x is also of interest:

$$\frac{\partial f(x, \chi)}{\partial x} = \frac{x+1}{x} f(x, \chi) - x \sin^2\chi \int_0^\chi e^{x(1-\sin\chi \operatorname{cosec}\lambda)} \operatorname{cosec}^3\lambda \, d\lambda \quad \text{...(43).}$$

This can be evaluated by the methods adopted for $f(x, \chi)$ itself. But a general idea of the value of $\partial f/\partial x$ can be simply obtained as follows. When $\chi = \frac{1}{2}\pi$, differentiation of $(\frac{1}{2}\pi x)^{\frac{1}{2}}$, the approximate value of f (cf. § 2), gives

$$\left(\frac{\partial f}{\partial x}\right)_{\chi=\frac{1}{2}\pi} = \tfrac{1}{2}(\pi/2x)^{\frac{1}{2}} \text{ approximately} \quad \text{......(44).}$$

This is less, the greater the value of x. When $\chi = 0$, $f(x, \chi) = 1$ and $\partial f/\partial x = 0$; § 3 shows that for small values of χ the difference between $f(x, \chi)$ and $\sec\chi$ is less, the greater the value of x. This suggests that $\partial f/\partial x$ increases from 0 to approximately $\frac{1}{2}(\pi/2x)^{\frac{1}{2}}$ as χ varies from 0 to $\frac{1}{2}\pi$, being throughout smaller, the greater the value of x; this could be confirmed by a detailed discussion of (43), and is borne out by inspection of table 4. It is less easy to see how $\partial f/\partial x$ varies with x for values of χ greater than 90°, but when χ has increased sufficiently beyond $\frac{1}{2}\pi$ for the factor $1 + (2/\sqrt{\pi}) \operatorname{erf}(2yV^2)^{\frac{1}{2}}$ to have become practically 2, the principal

Table 4. Values of $f(R, \chi)$ and of $\ln f(R, \chi)$

$\chi =$	30°	45°	60°	75°	80°	83°	85°	87°	90°	93°	95°	97°	100°
$\sec \chi =$	1·155	1·414	2·000	3·864	5·758	8·206	11·474	19·107	∞				
$\ln \sec \chi =$	0·1438	0·3466	0·6932	1·352	1·751	2·105	2·440	2·950	∞				
$R = 50$, $f =$	1·148	1·389	1·908	3·232	4·098	5·050	5·825	6·813	8·928	12·30	15·73	20·78	32·94
$\ln f =$	0·1377	0·3284	0·6463	1·173	1·410	1·619	1·762	1·919	2·189	2·509	2·756	3·034	3·495
$R = 100$, $f =$	1·151	1·401	1·946	3·512	4·608	5·906	7·068	8·677	12·58	20·16	29·67	49·91	106·9
$\ln f =$	0·1406	0·3370	0·6658	1·256	1·528	1·776	1·956	2·161	2·532	3·004	3·390	3·848	4·672
$R = 200$, $f =$	1·153	1·407	1·972	3·646	5·010	6·656	8·276	10·73	17·76	35·95	67·59	150·5	714·1
$\ln f =$	0·1422	0·3417	0·6788	1·294	1·611	1·895	2·113	2·373	2·877	3·582	4·214	5·014	6·571
$R = 300$, $f =$	1·153	1·410	1·981	3·706	5·281	7·020	8·918	11·96	21·74	53·57	126·9	398·2	4108
$\ln f =$	0·1427	0·3433	0·6834	1·310	1·664	1·949	2·188	2·482	3·079	3·981	4·844	5·987	8·321
$R = 400$, $f =$	1·154	1·411	1·985	3·742	5·380	—	9·327	12·82	25·09	73·94	220·1	978·4	2169×10
$\ln f =$	0·1430	0·3441	0·6858	1·320	1·683	—	2·233	2·551	3·222	4·303	5·394	6·886	9·985
$R = 500$, $f =$	1·154	1·411	1·988	3·764	5·439	—	9·620	13·48	28·05	97·92	365·6	2315	1108×10^2
$\ln f =$	0·1431	0·3446	0·6873	1·326	1·694	—	2·264	2·601	3·334	4·584	5·902	7·747	11·62
$R = 600$, $f =$	1·154	1·412	1·990	3·780	5·492	—	9·837	13·97	30·72	125·7	591·5	—	5543×10^2
$\ln f =$	0·1433	0·3449	0·6882	1·330	1·703	—	2·286	2·637	3·425	4·834	6·383	—	13·23
$R = 650$, $f =$	1·154	1·412	1·991	3·786	5·510	—	9·930	14·18	31·97	141·5	747·0	—	1233×10^3
$\ln f =$	0·1433	0·3450	0·6886	1·331	1·707	—	2·296	2·652	3·465	4·952	6·616	—	14·03
$R = 700$, $f =$	1·154	1·412	1·992	3·791	5·526	—	10·01	14·38	33·18	158·7	940·0	—	2734×10^3
$\ln f =$	0·1433	0·3452	0·6889	1·333	1·709	—	2·303	2·666	3·502	5·067	6·846	—	14·82
$R = 800$, $f =$	1·154	1·412	1·993	3·800	5·552	—	10·15	14·73	35·46	197·5	1476	—	1336×10^4
$\ln f =$	0·1434	0·3453	0·6894	1·335	1·714	—	2·317	2·690	3·569	5·286	7·297	—	16·41

factor in f, which governs the value of $\partial f/\partial x$, is $e^{x(1-\sin\chi)}$; thus $\partial f/\partial x$ is approximately equal to $(1-\sin\chi)f(x,\chi)$ for such values of χ. Since f increases with x, $\partial f/\partial x$ changes, between $\chi=\frac{1}{2}\pi$ and somewhat greater values, from a decreasing function of x to an increasing one. The change is completed, over the range of R here considered, before $\chi=93^\circ$, as may be seen by inspection of table 4.

§7. THE HEIGHT-DISTRIBUTION OF THE RATE OF ION-PRODUCTION

$z(\chi)$ In part I, equation (16), it was found that the value of z, written as $z(\chi)$, corresponding to the maximum value of I at a point from which the sun's zenith distance is χ, is $\ln\sec\chi$. The corresponding height $h(\chi)$ above the ground is $h_0+Hz(\chi)$;
Z if height measured from this level as datum, in terms of H as unit, be denoted by Z,

$$Z=z-z(\chi) \qquad \text{......(45).}$$

It is of interest to point out that in terms of Z the equation (17) of part I takes a specially simple form, namely,

$$I/I_0\cos\chi=\exp\{1-Z-\exp(-Z)\} \qquad \text{......(46).}$$

Thus the *proportionate* height-distribution of the rate of ion-production, to the accuracy afforded by the approximate formulae of part I, is the same at all points χ, relative to the *local* height of maximum ion-production; this height increases with χ to infinity at $\chi=\frac{1}{2}\pi$; the actual rate of production at points similarly situated with respect to the local level of maximum is, however, reduced, as compared with the value at the point immediately beneath the sun, in the ratio $\cos\chi$. Thus all the curves in figure 1 of part I are identical except in their scale of ordinates, and in being bodily shifted so that their maxima occur at different points along the scale of abscissae; this point was not noted in part I. The modifications in these results due to the curvature of the earth will now be considered.

The value $z(\chi)$ at which I, as given by (9), is a maximum, is given by

$$1-e^{-z}f(R+z,\chi)+e^{-z}\,\partial f(R+z,\chi)/\partial z=0 \qquad \text{......(47).}$$

For $\chi=\frac{1}{2}\pi$ it has been seen in § 5 that $\partial f(R+z,\chi)/\partial z$ is approximately equal to $f(R+z,\chi)/2(R+z)$; thus it is negligible compared with $f(R+z,\chi)$ when $R\geqslant 50$. Consequently (47) is approximately equivalent to

$$1-\exp(-z)f(R+z,\chi)=0,$$

or

$$z(\chi)=\ln f(R+z,\chi) \qquad \text{......(48).}$$

This is valid for $\chi=\frac{1}{2}\pi$, and also for smaller values, since the ratio of $\partial f/\partial z$ to f decreases with χ when $\chi<\frac{1}{2}\pi$. As $\chi\to 0$ this equation tends to the approximate equation $z(\chi)=\ln\sec\chi$ of part I.

Reckoning height from the local level of maximum I, by the substitution (45), an approximate equation analogous to (46) is obtained, namely

$$I=\{I_0/f(R+z,\chi)\}\exp\{1-Z-\exp(-Z)\} \qquad \text{......(49).}$$

Thus the height-distribution of I, relative to the local level of maximum I, is approximately the same as at the point immediately beneath the sun (as illustrated by curve 7 in figure 1 of part I) except for a reduction, the same at all relative heights, in the ratio $1/f(R+z,\chi)$, or, in terms of $z(\chi)$, in the ratio $e^{-z(\chi)}$.

In part I, by neglect of the curvature of the earth, $z(\chi)$ was found to be infinity at $\chi=\frac{1}{2}\pi$, where I was given as zero at all heights. The earth's curvature being taken into account and the values of $f(x,\chi)$ given in table 4 utilized, it appears that $z(\frac{1}{2}\pi)$ varies from 2·18 for $R=50$, to 3·45 for $R=650$; thus the change of height of the level of maximum I, from the equator to the twilight circle, is only a small multiple of H. It should be remembered, however, that R itself depends on H (§ 2); h_0 being neglected in comparison with a, the earth's radius, $R=50$ corresponds to a value of H 13 times as large as $R=650$, so that $z(\frac{1}{2}\pi)$ for $R=50$ represents $(2{\cdot}18\times 13)/(3{\cdot}45)$ or 8·2 times as great a distance, in kilometres, as $z(\frac{1}{2}\pi)$ for $R=650$. In the latter value, corresponding to $H=10$ km., $z(\frac{1}{2}\pi)$ represents a distance of 34·5 km.

When χ increases beyond $\frac{1}{2}\pi$, the approximate expression for $\partial f(R+z,\chi)/\partial z$ changes fairly rapidly from $f/2(R+z)$ to $(1-\sin\chi)f(R+z,\chi)$, while $f(R+z,\chi)$ rapidly increases, approximately in proportion to the factor $e^{(R+z)(1-\sin\chi)}$; as $f(R+z,\chi)$ increases, I decreases; for the values of R that require consideration (§ 2), I is appreciable only so long as $1-\sin\chi$ is small, so that $\partial f(R+z,\chi)/\partial z$ in (47) is still negligible in comparison* with $f(R+z,\chi)$ over this range. Consequently (48) and (49) remain valid.

Thus, over the range for which I is appreciable, the function $f(x,\chi)$, in which x is to be given the value $R+z$, suffices to determine, in a very simple way, the level of maximum I, and the reduction of I at this and other levels above and below it, as compared with the corresponding magnitude at the point immediately beneath the sun. Since $\ln f=2{\cdot}303\log f$, the graphs of $\log f(R,\chi)$ in figure 19 thus indicate the variation in the *level* of maximum absorption of radiation, or of maximum rate of ion-production, as a function of the sun's zenith distance χ; the unit of height is taken as H.

§ 8. THE DENSITY OF THE AIR AT THE LEVEL OF MAXIMUM ABSORPTION

It is of interest to consider the density of the air, say $\rho(\chi)$, at the level at which, for any value of χ, the rate of absorption, or of dissociation (I), is a maximum. This occurs at $z(\chi)=\ln f(R,\chi)$ approximately, or $h(\chi)=h_0+H\ln f(R,\chi)$: at this level $\rho(\chi)$

$$\begin{aligned}\rho(\chi)&=\rho_0\exp\{-h(\chi)/H\}=\rho_0\exp\{-(h_0/H)-z(\chi)\}\\&=\rho_0/\{f(R,\chi)\exp(-h_0/H)\}\\&=\rho_0/A\rho_0Hf(R,\chi)=1/AHf(R,\chi)\qquad\qquad\text{......(50).}\end{aligned}$$

Up to $\chi=75°$, this is approximately $(\cos\chi)/AH$.

Thus the density of the air at this level is inversely proportional to H, the height of the homogeneous atmosphere, and to A, the coefficient of absorption. The former depends on the temperature and composition of the air; also A depends on the composition.

* This may be verified also directly from table 4.

§ 9. THE SEASONAL VARIATION IN THE MAXIMUM RATE OF ABSORPTION AT NOON

The annual variation in the maximum value of I, which by (49) (putting $Z = 0$) is $I_0/f(R+z, \chi)$, over a station in latitude $l\,(=\frac{1}{2}\pi - \theta)$, may be illustrated by considering the ratio of the values at the summer and winter solstices. Reference to these two seasons will be indicated by the suffixes s and w. Since I has its maximum at noon

δ_0 $$\chi_s = l - \delta_0, \quad \chi_w = l + \delta_0,$$

where $\delta_0 = 23°\cdot 5$. Now $I_0 = \beta S_\infty/H \exp 1$, and while β and S_∞ are unlikely to undergo any annual variation, it is possible that H may do so. Taking account of this, the required ratio is

$$I_s/I_w = \{H_w f(R+z, \chi_w)\}/\{H_s f(R+z, \chi_s)\} \qquad \text{......(51)}.$$

It may be noted that, by (50),

$$I_s/I_w = \rho_s/\rho_w \qquad \text{......(52)},$$

where ρ_s and ρ_w are the densities $\rho(\chi)$ at the levels of maximum I at the two seasons.

For $\chi \leqslant 75°$, and therefore up to the latitude $75° - 23°\cdot 5$ or $51°\cdot 5$ (which happens to be the latitude of London), $f(R+z, \chi)$ is equal to $\sec\chi$ within 2 per cent. (taking R to be about 650). Thus, up to this latitude,

$$I_s/I_w = \{H_w \cos(l - \delta_0)\}/\{H_s \cos(l + \delta_0)\} \text{ approximately} \qquad \text{......(53)}.$$

The following are the values of the ratio $I_s H_s/I_w H_w$ (which is equal to I_s/I_w, if $H_s = H_w$) for various latitudes:

Latitude	0	15	30	45	50	55	60
$I_s H_s/I_w H_w$	1	1·26	1·67	2·54	3·15	4·27	7·09

§ 10. THE DAILY VARIATION IN THE RATE OF ION-PRODUCTION

The daily variation in the rate of ion-production at a given level z depends on the geographical latitude (l) of the point, and the season. If $R = 650$, then at times and places at which χ does not exceed $75°$, the figures 1–5 of part I remain valid, because $f(650, \chi)$ is practically equal to $\sec\chi$, as assumed in part I, up to $\chi = 75°$.

For greater values of χ, the values of I/I_0 given in part I are in error by more than 2 per cent., if R is 650 or less. In particular, I/I_0 does not vanish when $\chi = 90°$, but is still appreciable for even greater values of χ, corresponding to points at places where at ground level the sun is below the horizon; $\chi = 90°$ corresponds to ground sunrise or sunset.

The following method may be adopted to illustrate how I/I_0 depends on height and on χ when the sun is near (above or below) the horizon.

The equation (9) may readily be transformed to

G $$f(R+z, \chi) = e^z\{1 - z - \ln(I/I_0)\} \equiv G(z, I/I_0) \qquad \text{......(54)},$$

or, the small difference between $f(R+z, \chi)$ and $f(R, \chi)$ being neglected,

$$f(R, \chi) = G(z, I/I_0) \qquad \text{......(55)},$$

in which χ is involved only on the left, and z only on the right.

The graph of $G(z, I/I_0)$ as a function of z is readily plotted for any assigned value of I/I_0; G is positive from $z = -\infty$ to $z = z_0 \equiv 1 - \ln(I/I_0)$, which is positive, and beyond this value G is negative. As $z \to -\infty$, $G \to 0$ by positive values, while as $z \to \infty$, $G \to -\infty$; G has one maximum value, equal to I_0/I, for

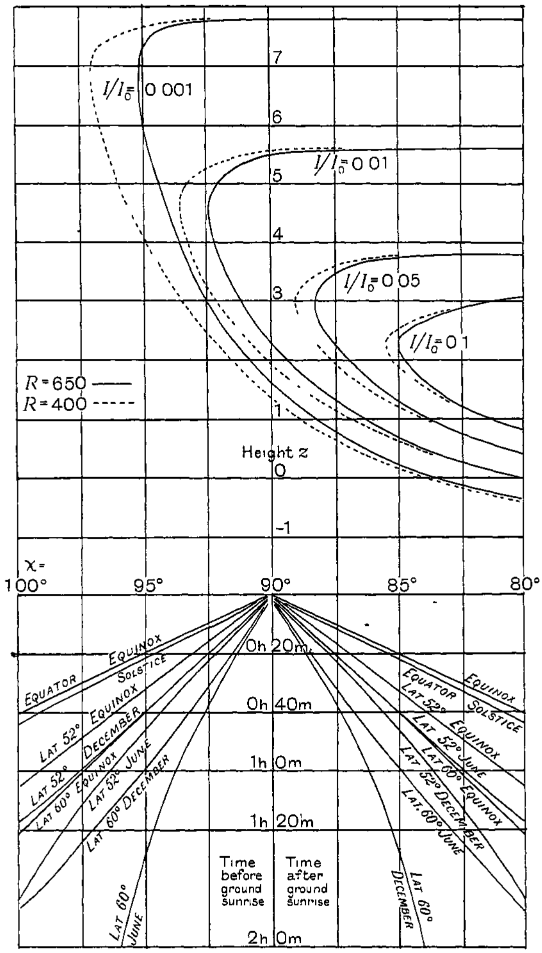

Fig. 20.

$z = z_m \equiv -\ln(I/I_0)$. Thus for any value of $G < I_0/I$, there are two corresponding values of z; moreover there is one value of χ such that $f(R, \chi)$ is equal to this value of G. The two values of z will be said to correspond to these values of χ and I/I_0; they indicate the heights at which I/I_0 has the assigned values, over any place from which the sun's zenith distance is χ. For any value of I/I_0, there is one value of χ, say χ_m, for which the two corresponding values of z coalesce; this is the χ corresponding to the maximum value, I_0/I, of G, and the value of z is z_m or $-\ln(I/I_0)$. All the other values of χ corresponding to I/I_0 are less than χ_m.

Figure 20 shows the heights and times for which I/I_0 has the values 0·001, 0·01, 0·05, and 0·1, for values of χ near 90°. The graphs are drawn for two values of R, viz. $R = 650$ and $R = 400$. The lower part of the diagram indicates the relation between χ and time—now measured from the epoch of ground sunrise ($\chi = 90°$)—for various latitudes and seasons; the lower left-hand curves refer to the period before sunrise, and the right-hand curves to the period after sunrise.

It appears that if $R = 650$, I/I_0 first attains the value 0·01 at a height $z = 4{\cdot}6$ approximately, at the time when $\chi = 92°{\cdot}4$; at the equator this time is about 10 minutes before ground sunrise, while in latitude 52° the corresponding time before sunrise varies from 16 minutes at the equinox to 20 in December and to 22 in June; the interval increases somewhat rapidly with increasing latitude. The value $I/I_0 = 0{\cdot}001$ is first attained at a height $z = 7$ approximately, at about 20 minutes before ground sunrise at the equator, or 40 minutes in latitude 52° in December. The figure well illustrates the rapid downward penetration of the radiation during the period of dawn.

§ 11. THE DISTRIBUTION AND VARIATIONS OF ION-DENSITY

The corrections here made in the value of I/I_0, allowing for the earth's curvature, modify the distribution and variations of ion-*density* found in part I, but so long as the same physical assumptions are adopted, the method of calculation is the same as indicated in §§ 9–11 of part I.

Since, for any assigned values of χ and z, the true value of I here calculated exceeds the value used in part I (except when $\chi = 0$, i.e. at noon at the point immediately beneath the sun, when the two are equal), the corresponding value of ν, or n/n_0, will be greater, at all times throughout the day and night, than was formerly deduced. The excess ion-production is due to the slightly greater absorbing area presented by the earth's atmosphere than was considered in part I; it is readily seen that the excess is proportionately least at the equator, and increases with latitude.

The changes in the values of n/n_0 formerly calculated, consequent on taking correct account of the earth's curvature, are best illustrated by graphs showing a few particular cases. This has been done for the equator and for latitude 60°, at the equinox (figures 21, 22), and for latitude 60° also in midsummer and midwinter (figures 23, 24); the parameter σ_0 has been taken as 1/25 throughout. The corrected curves are the full lines, and the original curves are shown by dotted lines, for comparison. The epoch of ground sunrise ($\chi = 0$) is indicated in each case; in the two equinoctial figures (21, 22) it is 6 a.m. It is found that the change in the pre-dawn value of n/n_0 is too small to be shown on the diagrams.

At the equator at the equinoxes the initial rise of n/n_0 occurs about 10 minutes earlier than was inferred in part I. From one hour after ground sunrise, the corrections to the old results are small, and the curves are not reproduced for the period after $7^h\,30^m$. The equatorial curves at the solstices would differ only very slightly from those shown for the equinox in figure 20.

In latitude 60° at the equinoxes the initial rise of n/n_0 becomes appreciable at

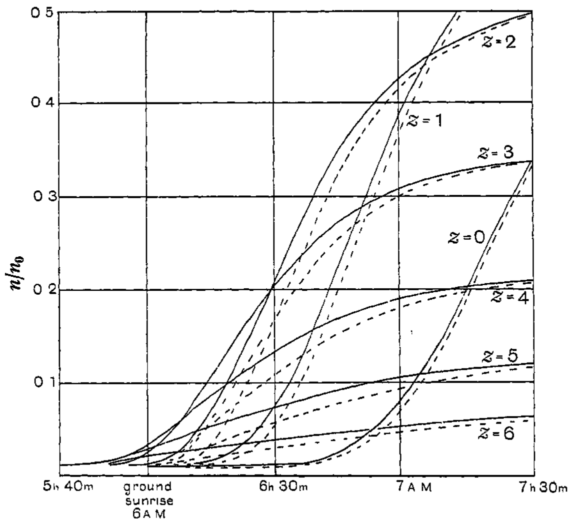

Fig. 21. Equator, equinox, $\sigma_0 = 1/25$.

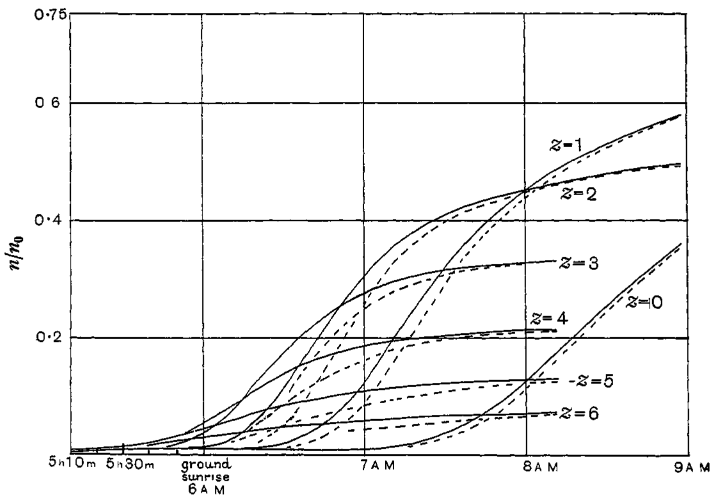

Fig. 22. Latitude 60°, equinox, $\sigma_0 = 1/25$.

20 or even 30 minutes before ground sunrise (figure 22); at ground sunrise n/n_0 is already 0·05 at $z = 4$. From about 8 a.m. the original and corrected curves are nearly identical.

In latitude 60° at midsummer (figure 23) the dawn rise of n/n_0 begins at about

1 hour before ground sunrise; at ground sunrise n/n_0 is almost 0·1 at the level $z = 4$. From 6 a.m. onwards the corrections to the original curves become small.

In the same latitude at midwinter (figure 24) the initial rise of n/n_0 begins at

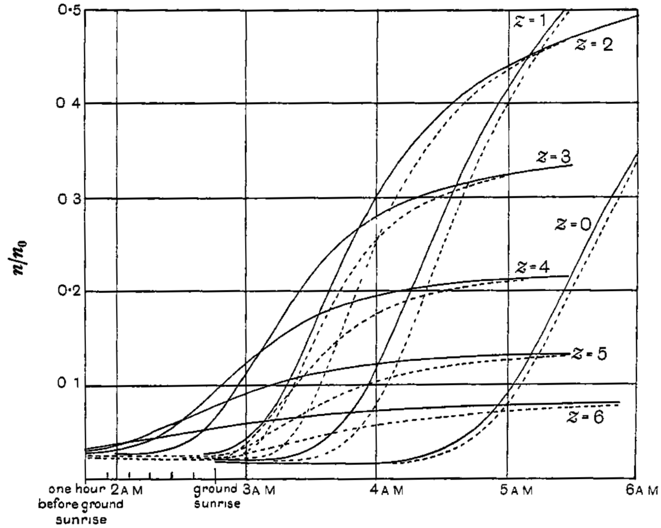

Fig. 23. Latitude 60°, summer, $\sigma_0 = 1/25$.

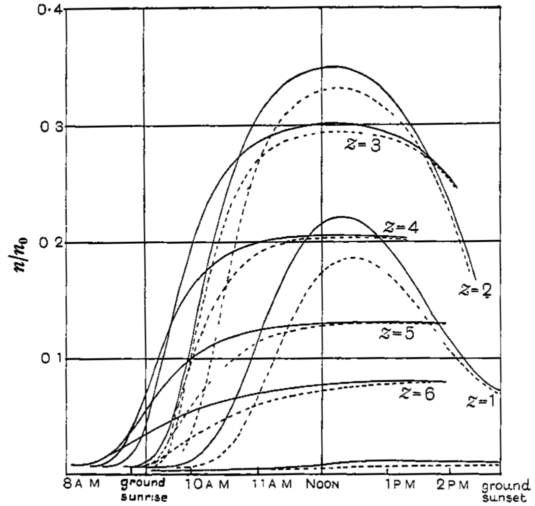

Fig. 24. Latitude 60°, winter, $\sigma_0 = 1/25$.

about 50 minutes before ground sunrise; at ground sunrise n/n_0 is about 0·07. At this season the corrections to the original curves are of importance throughout the period of daylight, though becoming small towards sunset; at noon they are quite considerable. This is because χ is large throughout the day, and so there is a material difference between $\sec\chi$ and $f(R, \chi)$ over this period, instead of only near dawn as in the other cases.

For the latitude of London ($51^\circ\cdot 5$) the interval before ground sunrise at which n/n_0 begins to increase will be somewhat less than that calculated here for latitude 60°. It would therefore appear to agree well with the interval (rather less than an hour) observed by Prof. E. V. Appleton* in his recent measurements of the electron density in the lower ionized layer of the upper atmosphere.

It is of interest to consider how n/n_0 or ν varies near dawn. The equation of change is

$$\sigma_0\,(d\nu/d\phi) = (I/I_0) - \nu^2;$$

during the night $I/I_0 = 0$, so that $d\nu/d\phi$ is negative. In the latter part of the night ν is nearly constant, and small; its value (ν', say) depends on the season, latitude, and on σ_0. In figures 21–24 ν' is about 0·01 at the equator (and, in winter, at latitude 60°), rising to 0·025 in latitude 60° in summer (these values depend on σ_0 and are purely illustrative: probably they are smaller than the true values in the ionized layers of the atmosphere). The value of ν becomes stationary when $I/I_0 = \nu'^2$, which is less than 1/1000 in the above cases; figure 20 shows that I/I_0 first attains the value 1/1000 when $\chi = 95^\circ$. From this time ν will increase, and at first, while ν^2 is still small, and I/I_0 several times as great as ν^2, the solution of the above equation will be approximately

$$\nu = \nu' + \sigma_0 \int_{\phi_0}^{\phi} (I/I_0)\, d\phi,$$

where ϕ_0 denotes the sun's hour angle at the instant when $I/I_0 = \nu'^2$.

§ 12. ACKNOWLEDGMENTS

In conclusion I wish to acknowledge assistance received from Miss M. C. Gray, M.A., and Miss V. F. White, M.Sc., who made most of the detailed numerical calculations involved in this paper, and to Mr W. Reeve, Miss V. Hatcher and Miss R. Rossiter, who assisted in the preparation of the diagrams. I wish also to thank the Department of Scientific and Industrial Research for a grant for payment for some of the assistance received in this work.

* E. V. Appleton, *Nature*, Feb. 7, 1931.