

Child Precieux Kounieka

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Exploring the Brachistochrone Problem and Solving it Using Calculus II Knowledge

INTRODUCTION

For many people, mathematics is a cryptic language of numbers, variables, and formulas that make no sense. For me, however, mathematics not only makes sense, but it is the most beautiful language of all. Mathematics is not just about logic and numbers; it is about curiosity, creativity, and challenge. It invites us to see the world in different ways, to ask why and how, and to enjoy the excitement of solving problems just for the fun of it. By following these principles, mathematicians throughout history have made discoveries that changed how we understand the universe.

The brachistochrone problem is a perfect example of how playful and curiosity-driven problems can lead to meaningful discoveries. The brachistochrone problem asks one simple question: If an object slides between two distant points at different heights, what path will get it there in the shortest time? One may intuitively think that the fastest path is the path of shortest distance - the straight line. But what mathematics teaches us is that intuition alone can often be misleading, and it is only by doing the math that we can know the real answer. To find the solution to this problem, mathematicians of the time had to be very creative and see things in a different way. This led to a new way of thinking and solving problems of the same kind - calculus of variations. Unlike normal calculus, which finds the point where a function is

optimized, calculus of variations looks for the function that optimizes a given quantity. Using calculus of variation to find the function that minimizes the time of descent, mathematicians found that the best path is not a straight line, but a cycloid - the path formed by a point on a rolling circle.

In this project, I will explore how the Brachistochrone problem grew from a playful challenge among mathematicians into one of the most important discoveries in mathematics. I will discuss its history, find its real-life applications, and work on the solution to understand the mathematical ideas behind it. Finally, I will make a 3D model of the brachistochrone curve and test the theory in real life.

HISTORY

In June 1696, Johann Bernoulli, a Swiss mathematician, published an article in the scholarly journal *Acta Eruditorum Lipsiæ* (Transactions of scholars) where he showed that calculus would be helpful in the field of geometry to help solve problems that were too hard using just normal geometry techniques (Rickey). At the end of the article, however, he presents a problem aimed at challenging the best mathematicians at the time (Rickey). This problem was the Brachistochrone problem, derived from the Greek words "brachistos chronos," which stand for "shortest time" (Rickey). The problem he published was as follows: "Given two points A and B in a vertical plane, assign a path AMB to the moving body M, along which the body will arrive at point B, falling by its own gravity and beginning from A, in the least time" (De Icaza-Herrera).

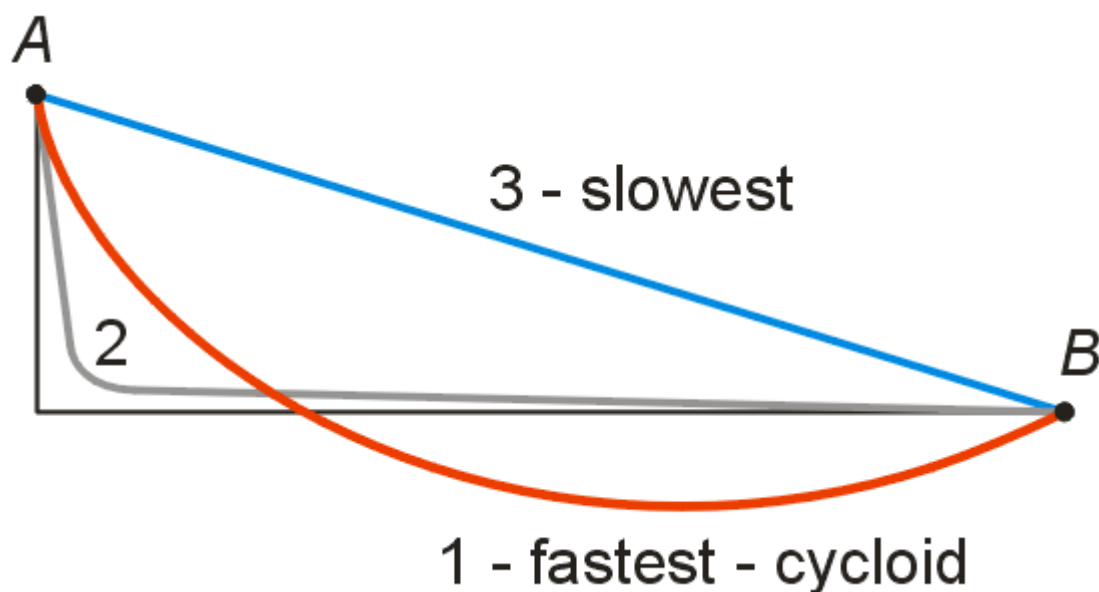


Fig. 1. Brachistochrone curve picture (Math on Web).

To encourage other mathematicians to work on the problem, Bernoulli stressed how fascinating this problem was to work on and how its solution was surprising (Knobloch). He also brought to light the applications that working on the problem would have in other scientific fields (Knobloch). Bernoulli promised to make public his solution to the problem if he had received no other solutions by the end of the year (Knobloch).

Despite being the one to publish it and make it famous, Johann Bernoulli wasn't the first one to work on the brachistochrone problem, contrary to what he thought. In 1638, the Italian polymath Galileo Galilei worked on a similar version of the problem in his last work titled "Discorsi e dimostrazioni matematiche, intorno à due nuoue scienze" (Discourses and Mathematical Demonstrations Concerning Two New Sciences) (Rickey). The limited knowledge and techniques of his time didn't enable him to find the solution to the problem; however, Galileo made a lot of progress and was on the right path to finding the solution (Knobloch). Galileo was

able to prove that an arc of a circle would provide a faster descent time than a straight line for a particle between two points, DC (Knobloch). Therefore, Galileo showed that the fastest possible path wasn't the shortest path, the straight line.

After publishing his article and the problem on June 19, 1696, Johann Bernoulli then sent a letter to one of his closest friends, Gottfried Wilhelm Leibniz, a German polymath, encouraging him to work on the problem (Knobloch). Leibniz was captivated by the problem and described that he was attracted by "its beauty like Eve before the apple" (Knobloch). Leibniz thus worked on the problem and was able to find its solution in just one evening (Rickey). One week later, on June 26, Bernoulli received Leibniz's work on the problem (Knobloch). Leibniz had found the correct answer; however, he just stopped at the differential equations and didn't notice that these were the equations of an inverted cycloid, which Bernoulli later pointed out to him (Rickey). Amazed by the problem and its solution, Leibniz sent the problem to his friend, the Italian mathematician Rudolf Christian von Bodenhausen, motivating him and other mathematicians to work on the problem (Knobloch). Other mathematicians, such as Johann Bernoulli's brother Jacob Bernoulli in Switzerland and Pierre Varignon in France, were also informed of the problem and started working on it (Knobloch). Close to the end of the year 1696, only Johann Bernoulli, Leibniz, and Jacob Bernoulli had found an answer to the problem (Knobloch). Leibniz then pleaded to Johann Bernoulli to push the deadline of the release of the solution until Easter of 1697 to give time to more people to solve the problem and to spread the problem to more regions (Knobloch). Bernoulli accepted Leibniz's offer (De Icaza-Herrera). He published another article in the journal *Acta Eruditorum*, where he announced the extension of the deadline (Knobloch). In this article, he also rewrote the problem as follows, "Find the path connecting two fixed points, chosen at different heights, not in the same vertical, along which a

moving body, falling by its own gravity and starting from the higher point, will descend most quickly to the lower one", and he included a second problem which is "Given a fixed point P, a curve is sought, such that for each straight [line] PKL cutting it in two points K and L, the sum of the distances [of the line] PK and PL, risen to a given power n , be constant" (De Icaza-Herrera).

The reason for the extension of the deadline was to target one person particularly. This person was the English polymath Sir Isaac Newton. At the time of the publication of the brachistochrone problem in 1696, Newton was around 53 years old. Earlier that year, Johann Bernoulli accused Newton of plagiarizing Leibniz's method in one of his papers (De Icaza-Herrera). Thus, Bernoulli and Leibniz believed that Newton was no longer able to do math at a high level like he used to. When the first 6 months after the publication of the problem had passed, since they didn't receive any response from Newton, Bernoulli and Leibniz were convinced that the problem was too big for Newton and he couldn't solve it (Rickey). Thus, the main motivation of Leibniz in requesting the extension of the deadline was to show that they were now better than Newton (Rickey). However, Newton did not fall into their trap. The reason for Newton's silence for the six months wasn't because he was not capable of solving the problem, but rather he had not yet seen the problem (Rickey). Newton received a letter which was dated January 1, 1697, from Bernoulli, in which Bernoulli told that he did not receive any answer for his first announcement of the problem, and he repeated what the problem was (De Icaza-Herrera). Once he had finally received the problem, Newton was ready to solve it. The same day, as he returned to his house at 4 in the evening, Newton worked on the problem the whole night until he finally found the solution at 4 in the morning (Rickey). The following day, Newton sent a letter to his friend Charles Montague, president of the Royal Society of London, in which he included Bernoulli's letter as well as his solution, which he wanted to be published

anonymously due to the absence of a signature on the paper (De Icaza-Herrera). Charles Montague then released Newton's work anonymously in the February edition of the Transactions of the Royal Society of London (Rickey).

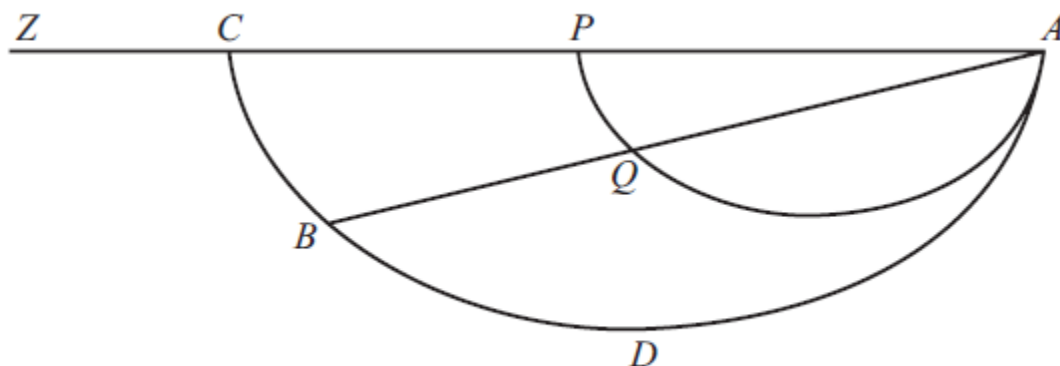


Figure 3. Newton's Construction

Fig. 2. Newton's construction of the Brachistochrone problem solution (Rickey).

Newton's paper was only seven lines long and contained no proof, just a construction of the solution, an inverted cycloid between the two given points (De Icaza-Herrera). When he received the anonymous solution from England, Bernoulli understood that it was the work of Newton from the "authority the paper displayed" as the classic Latin phrase says, "the lion is recognized from his print" (De Icaza-Herrera). Since he failed to show his superiority to Newton, the brachistochrone problem was no longer interesting to Bernoulli (Rickey). He gave the work of making public all the found solutions to Leibniz (Rickey). In total, 6 solutions were found (Rickey). Those solutions were those of Johann Bernoulli, Jakob Bernoulli, Leibniz, L'Hospital, Tschirnhaus, and the anonymous solution from Newton (Rickey). In the May 1697 issue of *Acta Eruditorum*, Leibniz published 5 of the solutions without his own (Rickey). Out of these

solutions, the most significant one was that of Jakob Bernoulli, Johann Bernoulli's brother, as he used methods that led to the development of a new branch of mathematics, calculus of variations (Rickey).

SOLUTION

This section is inspired by the work found on the papers by University of Tennessee, Knoxville and Markus Grasmair.

Starting from the law of conservation of energy,

Where $v(x)$ is a function of the velocity in terms of x , and $y(x)$ is a function of the height in terms of x ,

$$\frac{1}{2}mv(x)^2 = mgy(x)$$

$$\frac{1}{2}v(x)^2 = gy(x)$$

$$v(x)^2 = 2gy(x)$$

$$v(x) = \sqrt{2gy(x)}$$

Using arc length formula to find the length of the path from 0 to $(x, y(x))$,

$$s(x) = \int_0^x \sqrt{1 + y'(x)^2} dx$$

$$\therefore \frac{ds}{dx} = \sqrt{1 + y'(x)^2}$$

$$ds = \sqrt{1 + y'(x)^2} dx$$

Formula for velocity,

$$v(t) = \frac{ds}{dt}$$

$$\frac{dt}{ds} = \frac{1}{v(s)}$$

$$dt = \frac{1}{v(s)} ds$$

We can express the time (T) as,

$$T = \int_0^T dt$$

Substituting dt by its value shown above, and changing the bounds in terms of distance,

We have

$$T = \int_0^L \frac{1}{v(s)} ds$$

Substituting v(s) by its value of v(x) and ds by its value, and changing the bounds to be in terms of x as the path goes from x=0 to x=a,

We have

$$T = \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} dx$$

Therefore, minimizing the travel time (T) involves minimizing the function,

$$F(y) := \int_0^a \frac{\sqrt{1 + y'(x)^2}}{\sqrt{2gy(x)}} dx$$

Euler-Lagrange equation for variational problems.

For a function of 3 variables $f(x, y, y')$ in the form:

$$F(y) = \int_a^b f(x, y(x), y'(x)) dx$$

The function $y(x)$ which has the optimal value must satisfy this condition:

$$\frac{\partial}{\partial y} f(x, y(x), y'(x)) = \frac{d}{dx} \left[\frac{\partial}{\partial y'} f(x, y(x), y'(x)) \right]$$

This is the Euler-Lagrange equation.

In our case (from now on defining $y(x)$ as just y),

$$F(y) = \int_0^a \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}} dx$$

$$f(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2gy}}$$

Focusing on the left side of the condition,

$$f(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2g}} (y^{-1/2})$$

$$\frac{\partial}{\partial y} f(x, y, y') = \frac{\sqrt{1 + y'^2}}{\sqrt{2g}} \times \frac{-1}{2} \times y^{-3/2}$$

$$\frac{\partial}{\partial y} f(x, y, y') = \frac{-1}{2} \sqrt{\frac{1 + y'^2}{2g}} \times \frac{1}{y^{3/2}}$$

Focusing on the right side of the condition,

$$f(x, y, y') = \frac{1}{\sqrt{2gy}} (1 + y'^2)^{1/2}$$

$$\frac{\partial}{\partial y'} f(x, y, y') = \frac{1}{\sqrt{2gy}} \times 2y' \times \frac{1}{2} \times (1 + y'^2)^{-1/2}$$

$$\frac{\partial}{\partial y'} f(x, y, y') = \frac{1}{\sqrt{2gy}} \times \frac{y'}{\sqrt{1 + y'^2}}$$

$$\frac{\partial}{\partial y'} \left(\frac{1}{\sqrt{2gy}} (1 + y'^2)^{1/2} \right) = \frac{1}{\sqrt{2gy}} \times \frac{y'}{\sqrt{1 + y'^2}}$$

Combining both sides,

$$\frac{-1}{2} \sqrt{\frac{1 + y'^2}{2g}} \times \frac{1}{y^{3/2}} = \frac{d}{dx} \left[\frac{1}{\sqrt{2gy}} \times \frac{y'}{\sqrt{1 + y'^2}} \right]$$

$$\frac{-1}{2} \sqrt{\frac{1 + y'^2}{y^3}} \times \frac{1}{\sqrt{2g}} = \frac{1}{\sqrt{2g}} \times \frac{d}{dx} \left[\frac{y'}{\sqrt{y(1 + y'^2)}} \right]$$

$$\therefore \frac{-1}{2} \sqrt{\frac{1 + y'^2}{y^3}} = \frac{d}{dx} \left[\frac{y'}{\sqrt{y(1 + y'^2)}} \right]$$

Focusing on the right side,

$$\frac{d}{dx} \left[\frac{y'}{\sqrt{y(1 + y'^2)}} \right]$$

Using quotient rule,

$$u = y'$$

$$u' = y''$$

$$v = (y + yy'^2)^{1/2}$$

$$v' = [y' + (y \cdot y'' \cdot 2 \cdot y' + y'^2 \cdot y')] \times \frac{1}{2} \times (y + yy'^2)^{-1/2}$$

$$v' = \frac{y' + 2yy'y'' + y'^3}{2(y + yy'^2)^{1/2}}$$

$$v' = \frac{y'(1 + y'^2) + 2yy'y''}{2(y + yy'^2)^{1/2}}$$

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) = \frac{(y + yy'^2)^{1/2} \cdot y'' - y' \left(\frac{y'(1 + y'^2) + 2yy'y''}{2(y + yy'^2)^{1/2}} \right)}{(y + yy'^2)}$$

$$= \frac{y'' \cdot (y + yy'^2)^{1/2}}{(y + yy'^2)} - y' \left\{ \frac{y'(1 + y'^2) + 2yy'y''}{2(y + yy'^2)^{1/2} \cdot (y + yy'^2)} \right\}$$

$$= \frac{y''}{(y + yy'^2)^{1/2}} - y' \left\{ \frac{y'(1 + y'^2) + 2yy'y''}{2[y^{1/2}(1 + y'^2)^{1/2} \times y(1 + y'^2)]} \right\}$$

$$= \frac{y''}{(y + yy'^2)^{1/2}} - y' \left\{ \frac{y'(1 + y'^2)}{2[y^{1/2}(1 + y'^2)^{1/2} \times y(1 + y'^2)]} + \frac{2yy'y''}{2[y^{1/2}(1 + y'^2)^{1/2} \times y(1 + y'^2)]} \right\}$$

$$= \frac{y''}{(y + yy'^2)^{1/2}} - y' \left\{ \frac{y'}{2y^{3/2}(1 + y'^2)^{1/2}} + \frac{y'y''}{y^{1/2}(1 + y'^2)^{3/2}} \right\}$$

$$\begin{aligned}
&= \frac{y''}{(y + yy'^2)^{1/2}} - \frac{y'^2}{2y^{3/2}(1 + y'^2)^{1/2}} - \frac{y'^2 y''}{y^{1/2}(1 + y'^2)^{3/2}} \\
&= \frac{y''}{\sqrt{y(1 + y'^2)}} - \frac{1}{2} \cdot \frac{y'^2}{\sqrt{y^3(1 + y'^2)}} - \frac{y'^2 y''}{\sqrt{y(1 + y'^2)^3}} \\
&= \frac{1}{\sqrt{y(1 + y'^2)}} \left(y'' - \frac{y'^2}{2\sqrt{y^2}} - \frac{y'^2 y''}{\sqrt{(1 + y'^2)^2}} \right) \\
&= \frac{1}{\sqrt{y(1 + y'^2)}} \left(y'' - \frac{y'^2}{2y} - \frac{y'^2 y''}{1 + y'^2} \right) \\
&= \frac{1}{\sqrt{y(1 + y'^2)}} \left(\frac{y'' + y'^2 y'' - y'^2 y''}{1 + y'^2} - \frac{y'^2}{2y} \right) \\
&= \frac{1}{\sqrt{y(1 + y'^2)}} \left(\frac{y''}{1 + y'^2} - \frac{y'^2}{2y} \right) \\
&\therefore \frac{d}{dx} \left[\frac{y'}{\sqrt{y(1 + y'^2)}} \right] = \frac{1}{\sqrt{y(1 + y'^2)}} \left(\frac{y''}{1 + y'^2} - \frac{y'^2}{2y} \right)
\end{aligned}$$

Substituting the simplified right side back into the full equation,

$$\frac{-1}{2} \sqrt{\frac{1 + y'^2}{y^3}} = \frac{1}{\sqrt{y(1 + y'^2)}} \left(\frac{y''}{1 + y'^2} - \frac{y'^2}{2y} \right)$$

$$\frac{-1}{2} \cdot \frac{\sqrt{1+y'^2}}{y\sqrt{y}} = \frac{1}{\sqrt{y(1+y'^2)}} \left(\frac{y''}{1+y'^2} - \frac{y'^2}{2y} \right)$$

Multiplying both sides by $\sqrt{y(1+y'^2)}$,

$$-\frac{\sqrt{y(1+y'^2)}}{2} \times \frac{\sqrt{1+y'^2}}{y\sqrt{y}} = \frac{\sqrt{y(1+y'^2)}}{\sqrt{y(1+y'^2)}} \left(\frac{y''}{1+y'^2} - \frac{y'^2}{2y} \right)$$

$$\frac{-\sqrt{y}}{2} \times \frac{1+y'^2}{y\sqrt{y}} = \frac{y''}{1+y'^2} - \frac{y'^2}{2y}$$

$$\frac{-(1+y'^2)}{2y} = \frac{y''}{1+y'^2} - \frac{y'^2}{2y}$$

$$\frac{y'^2}{2y} - \frac{1+y'^2}{2y} = \frac{y''}{1+y'^2}$$

$$\frac{y'^2 - 1 - y'^2}{2y} = \frac{y''}{1+y'^2}$$

$$\frac{-1}{2y} = \frac{y''}{1+y'^2}$$

$$y'' = -\left(\frac{1+y'^2}{2y}\right)$$

$$2yy'' = -1 - y'^2$$

$$1 + 2yy'' + y'^2 = 0$$

Multiply both sides by y' ,

$$y' + 2yy'y'' + y'^3 = 0$$

We have already seen something similar to the left side before.

$(y' + 2yy'y'' + y'^3)$ is this derivative $\frac{d}{dx}(y + yy'^2)$

We saw this when using the quotient rule, and it is similar to our v' .

So,

$$\frac{d}{dx}(y + yy'^2) = y' + 2yy'y'' + y'^3 = 0$$

$$\therefore \frac{d}{dx}(y + yy'^2) = 0$$

Therefore, $(y + yy'^2)$ must be a constant c as only the derivative of a constant gives 0 (i.e $\frac{d}{dx}(c) = 0$)

$$\therefore y + yy'^2 = c$$

$$y + yy'^2 = c$$

$$y(1 + y'^2) = c$$

$$1 + y'^2 = \frac{c}{y}$$

$$y'^2 = \frac{c}{y} - 1$$

$$y'^2 = \frac{c - y}{y}$$

$$y' = \sqrt{\frac{c - y}{y}}$$

since y' is just $\frac{dy}{dx}$

$$\frac{dy}{dx} = \sqrt{\frac{c - y}{y}}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{c - y}}$$

Note that the slope at a point (m or $\frac{dy}{dx}$) can also be represented in terms of the angle that the tangent line at that point makes with the horizontal x-axis.

$$\therefore \tan \theta = \frac{dy}{dx}$$

Now for $\frac{dx}{dy}$ (in terms of y), we would take the angle that the tangent line makes with the vertical y-axis.

$$\therefore \tan \varphi = \frac{dx}{dy}$$

$$\frac{dx}{dy} = \sqrt{\frac{y}{c - y}}$$

$$\tan \varphi = \sqrt{\frac{y}{c - y}}$$

$$\frac{\sin \varphi}{\cos \varphi} = \sqrt{\frac{y}{c - y}}$$

$$\frac{\sin^2 \varphi}{\cos^2 \varphi} = \frac{y}{c - y}$$

$$(c - y) \sin^2 \varphi = y(\cos^2 \varphi)$$

$$c \sin^2 \varphi - y \sin^2 \varphi = y \cos^2 \varphi$$

$$c \sin^2 \varphi = y \sin^2 \varphi + y \cos^2 \varphi$$

$$c \sin^2 \varphi = y(\sin^2 \varphi + \cos^2 \varphi)$$

$$c \sin^2 \varphi = y$$

$$\therefore y = c \sin^2 \varphi$$

Taking the derivative,

$$\frac{dy}{d\varphi} = 2c \cdot \sin \varphi \cdot \cos \varphi$$

We have found y in terms of φ , now time to find x in terms of φ .

We have,

$$\frac{dx}{d\varphi} = \frac{dx}{dy} \times \frac{dy}{d\varphi}$$

$$\frac{dx}{d\varphi} = \sqrt{\frac{y}{c-y}} \times 2c \cdot \sin\varphi \cdot \cos\varphi$$

Substituting $y = c \sin^2 \varphi$,

$$\frac{dx}{d\varphi} = \sqrt{\frac{c \sin^2 \varphi}{c - c \sin^2 \varphi}} \times 2c \cdot \sin\varphi \cdot \cos\varphi$$

$$\frac{dx}{d\varphi} = \sqrt{\frac{c \sin^2 \varphi}{c(1 - \sin^2 \varphi)}} \times 2c \cdot \sin\varphi \cdot \cos\varphi$$

$$\frac{dx}{d\varphi} = \sqrt{\frac{c \sin^2 \varphi}{c \cos^2 \varphi}} \times 2c \cdot \sin\varphi \cdot \cos\varphi$$

$$\frac{dx}{d\varphi} = \sqrt{\frac{\sin^2 \varphi}{\cos^2 \varphi}} \times 2c \cdot \sin\varphi \cdot \cos\varphi$$

$$\frac{dx}{d\varphi} = \frac{\sin \varphi}{\cos \varphi} \times 2c \cdot \sin\varphi \cdot \cos\varphi$$

$$\frac{dx}{d\varphi} = 2c \sin^2 \varphi$$

$$dx = 2c \sin^2 \varphi \, d\varphi$$

Applying power reduction formula for sine (*i.e.* $\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$),

$$dx = 2c \left(\frac{1 - \cos(2\varphi)}{2} \right) d\varphi$$

$$dx = c (1 - \cos(2\varphi)) d\varphi$$

Integrating both sides,

$$\int dx = c \int 1 - \cos(2\varphi) \, d\varphi$$

$$x = c \left(\varphi - \frac{\sin 2\varphi}{2} \right)$$

$$\therefore x = \frac{c}{2} (2\varphi - \sin 2\varphi)$$

Further simplifying $y = c \sin^2 \varphi$,

$$y = c \sin^2 \varphi$$

Applying power reduction formula of sine,

$$y = c \left(\frac{1 - \cos(2\varphi)}{2} \right)$$

$$\therefore y = \frac{c}{2} (1 - \cos(2\varphi))$$

Therefore, we have:

$$x = \frac{c}{2}(2\varphi - \sin 2\varphi)$$

$$y = \frac{c}{2}(1 - \cos(2\varphi))$$

Conclusion:

Given the standard form of the parametric equations of a cycloid formed by a circle of radius (a) and having the angle (t) as the parameter is,

$$x = at - a\sin(t)$$

$$y = a - a\cos(t)$$

Simplifying our equations to match that form, we have,

$$x = c\varphi - \frac{c}{2}\sin 2\varphi$$

$$y = \frac{c}{2} - \frac{c}{2}\cos 2\varphi$$

We have $a = \frac{c}{2}$ and $t = 2\varphi$.

Therefore, we have shown that the path of shortest time will be a cycloid.

TESTING



Fig. 3. 3D print of the brachistochrone curve.

This is a 3d print of the brachistochrone curve and other curves. The 3D print contains 3 different curves. The first curve is a straight line, the second curve / middle curve is the brachistochrone/cycloid curve, and the last curve is a curve that has a sharp fall to make the moving body gain speed early in the hope of minimizing the time to get to the endpoint. This 3d print was made in the Durham Tech Newton MakerSpace using the 3d printers available there. It comes from the model by Taevinrude. This 3d print will be used during the Honors symposium presentation to test in real life if the brachistochrone curve is actually the fastest curve.

REAL LIFE APPLICATIONS

Finding the fastest possible path between two points is a concept that is useful in many areas. Thus, the brachistochrone problem, whether its solution or the methods used to find its solution, has a lot of different applications in real life.

One of these applications is in the design of roller coasters. Roller coaster designers want to find a shape that provides riders with feelings of speed and thrill, while still being safe. Thus, roller coaster designers want to find a shape that balances speed with steepness, which is what the brachistochrone/cycloid shape does. The brachistochrone shape enables designers to create tracks that maximize speed with no sharp drop or rise, which ensures an exciting and smooth ride while taking not much time for each ride (Your Physicist; Data Genetics; The Harsh World of Mechanics). The brachistochrone curve is also used in the design of other park attractions, such as water slides. Just as in a roller coaster, the brachistochrone curve is used to design water slides that are thrilling to the rider and do not last long for each ride (Your Physicist).

The brachistochrone is also found in sports where speed matters a lot. In many winter sports, the brachistochrone is used. In sports such as skiing, bobsled, skeleton, and luge, using the brachistochrone curve will lead to having the lowest time (Haake). It has been noticed that some skiers take a brachistochrone path when going down a hill at a sports competition (Haake). The brachistochrone curve is also used in surfing. Bruce Henry and Simon Watt have noticed that the brachistochrone path is used in a lot of surfing moves, such as those that involve "executing a turn down a wave to carve back up and rejoin the peel of a spilling wave or getting up to speed as quickly as possible to ride the barrel of a plunging wave" (Haake). Thus, without even consciously knowing it, surfers follow the brachistochrone principle to enhance their performance. Skateboarding is another sport where the brachistochrone is used. Here, skateparks

have the shape of the brachistochrone to enable skaters to go fast in a short amount of time (so). Therefore, whether consciously or unconsciously, the brachistochrone serves a lot of different sports to make athletes have the best possible performance.

The brachistochrone curve, more precisely the cycloid, is also useful for one of its other surprising features, its tautochrone property. This tautochrone property means that no matter where an object is placed on a cycloid shape, it will arrive at the bottom of the cycloid in the same amount of time (Data Genetics). This means that, for example, if three balls are placed on three different points on a cycloidal shape, one at the top, one at the middle, and another one close to the bottom of the shape, and released at the same time, these three balls will arrive at the bottom of the shape at the same time. This can be explained by the fact that objects close to the top will have a steeper path and thus will go faster, but have more distance to cover; while objects close to the bottom have less distance to cover but have a more straight path and thus will go slower, leading to both objects traveling in the same amount of time (Data Genetics). This tautochrone property makes objects that follow a cycloid path have the same frequency (Data Genetics). This property is then used to create pendulums that have the same frequency (Data Genetics). These pendulums are then used in different physics experiences, and also in clocks to record time.

The biggest application of the brachistochrone problem isn't in the direct use of the curve in the real world, but rather in the methods that were created to solve the problem. As mentioned before, Jakob Bernoulli and Newton used techniques while working on the brachistochrone problem that led to the development of a very important branch of mathematics, calculus of variations. Calculus of variations is a branch of mathematics that deals with problems involving finding the maximum or minimum function that solves that problem (The Harsh World of

Mechanics), unlike normal calculus, which involves finding the minimum or maximum point on a function. Euler and Lagrange in the 1700s developed the calculus of variations in the form that we use today (The Harsh World of Mechanics). Calculus of variations has many applications in many different fields. It is used in engineering problems that involve "differential geometry, computational geometry, analytic mechanics, and computational mechanics" (The Harsh World of Mechanics). The Euler-Lagrange differential equations found in calculus of variations are used to find "equations of motion, natural frequencies, and natural modes of multi-DOF systems" (The Harsh World of Mechanics). They are also used in "calculating geodesics of a surface, which are used to plan the shortest airplane trajectories around the Earth" (The Harsh World of Mechanics). Thus, calculus of variations is crucial in the modern world in which we live. All this started with a challenge among mathematicians, the brachistochrone problem.

CONCLUSION

To conclude, the Brachistochrone problem shows what mathematics is truly about - curiosity, creativity, and the joy of solving challenges just for the fun of it. What began as a simple question that may have seemed trivial to some people ended up leading to one of the most important discoveries in mathematics. By putting their minds together and accepting this challenge, the greatest mathematical minds of that time have shown that mathematics grows not from routine, but from wonder. The solution to the problem, the cycloid, is not just a beautiful mathematical curiosity; it is very useful in our real world. From roller coasters and clocks to physics experiments, engineering, and even sports, the brachistochrone curve applications are real. But perhaps the greatest legacy of the brachistochrone problem isn't the curve itself, but the new way of thinking about these types of problems that led to the creation of calculus of

variations, which is used in even more fields to solve real-world problems. Exploring the Brachistochrone problem made me even more realize how mathematics is not just formulas or equations, but rather a way of thinking and a form of creativity that connects logic with imagination.

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