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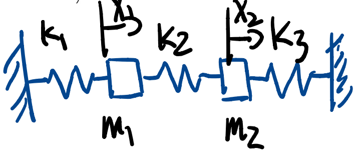
Homework 2: ME5701, 21 August 2020

1. Derive the equations of motion for the two-mass three-spring system presented on slide 23 using: a) Lagrange's equations of motion; b) Newton's laws.



a) The kinetic energy $T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$

potential energy $V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_3 x_2^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$

Substitute T and V into Lagrange's equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} = 0 \qquad \text{for } i = 1, 2$$

$$\frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1 \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1 \ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0 \qquad \frac{\partial V}{\partial x_1} = k_1 x_1 - k_2 (x_2 - x_1) = (k_1 + k_2) x_1 - k_2 x_2$$

$$\frac{\partial T}{\partial \dot{x}_2} = m_2 \dot{x}_2 \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2 \ddot{x}_2$$

$$\frac{\partial T}{\partial x_2} = 0 \qquad \frac{\partial V}{\partial x_2} = k_3 x_2 + k_2 (x_2 - x_1) = (k_2 + k_3) x_2 - k_2 x_1$$

Therefore, for x_1,

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$

for x_2,

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 = 0$$

Thus,

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

(b) By F.B.D.

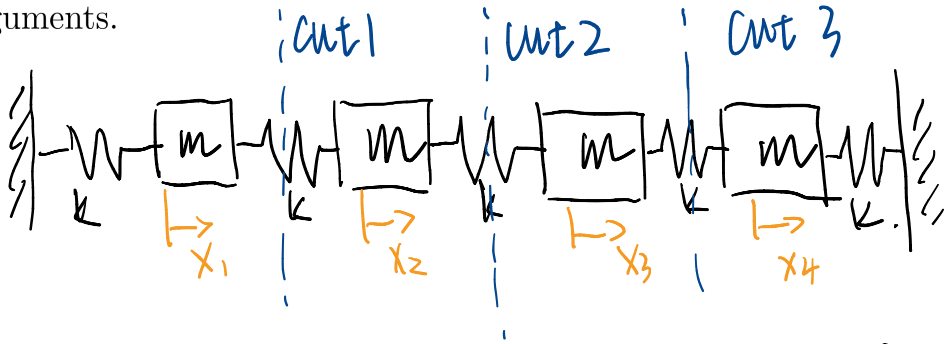
here use scalars rather than vectors.

$$\begin{cases} k_2(x_2-x_1)-k_1x_1=m_1\ddot{x}_1 \\ k_2(x_2-x_1)+k_3x_2=-m_2\ddot{x}_2 \end{cases} \Rightarrow \begin{cases} m_1\ddot{x}_1+(k_1+k_2)x_1-k_2x_2=0 \\ m_2\ddot{x}_2-k_2x_1+(k_2+k_3)x_2=0 \end{cases}$$

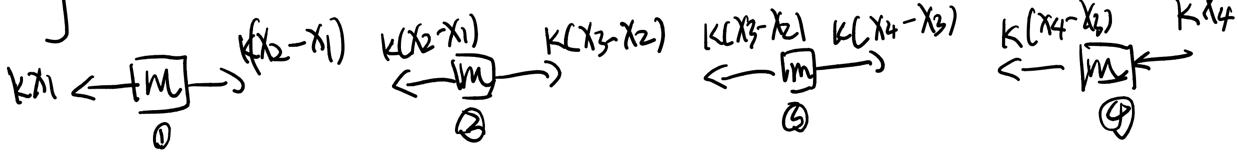
Thus.

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix}+\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=0$$

2. For the Toda lattice with four equal masses and five equal springs, attempt to derive as many eigenvectors (a.k.a. normal modes) as possible using symmetry arguments.



Firstly, try to derive the equ of motion by F.B.D.



$$\begin{cases} k(x_2-x_1)-kx_1 = m\ddot{x}_1 \\ k(x_3-x_2)-k(x_2-x_1) = m\ddot{x}_2 \\ k(x_4-x_3)-k(x_3-x_2) = m\ddot{x}_3 \\ -kx_4-k(x_4-x_3) = m\ddot{x}_4 \end{cases} \Rightarrow \begin{cases} m\ddot{x}_1+2kx_1-kx_2=0 \\ m\ddot{x}_2-kx_1+2kx_2-kx_3=0 \\ m\ddot{x}_3-kx_2+2kx_3-kx_4=0 \\ m\ddot{x}_4-kx_3+2kx_4=0 \end{cases}$$

Therefore,

$$m\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{pmatrix} + k\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

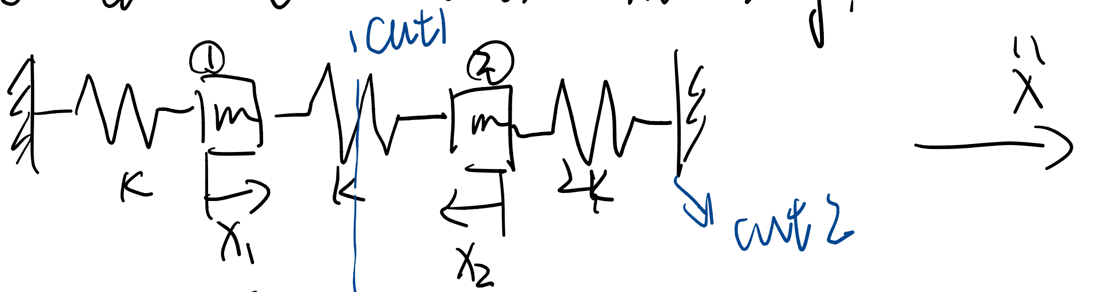
Set $m=1$, $k=1$

Thus, matrix $\underline{K} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$

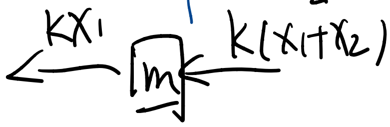
$$\text{Set } C = (\lambda I - \underline{K}) = \begin{pmatrix} \lambda-2 & -1 & 0 & 0 \\ -1 & \lambda-2 & -1 & 0 \\ 0 & -1 & \lambda-2 & -1 \\ 0 & 0 & -1 & \lambda-2 \end{pmatrix}$$

Guess 1: The total system is symmetric to cut 2

① and ② moves inversely.



for ①



$$kx_1 + k(x_1+x_2) = -m\ddot{x}_1$$

②

$$k(x_1+x_2) + 2kx_2 = -m\ddot{x}_2$$

$$\Rightarrow \begin{cases} m\ddot{x}_1 + 2kx_1 + kx_2 = 0 \\ m\ddot{x}_2 + kx_1 + 3kx_2 = 0 \end{cases}$$

$$\Rightarrow \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \begin{pmatrix} 2k & k \\ k & 3k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$K = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

$$(w^2-2)(w^2-3) - 1 = 0$$

$$(w^2)^2 - 5w^2 + 5 = 0$$

$$w^2 = \frac{5 \pm \sqrt{5}}{2}$$

when $w^2 = \frac{5-\sqrt{5}}{2}$

$$C = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & -1 & 0 & 0 \\ -1 & \frac{1-\sqrt{5}}{2} & -1 & 0 \\ 0 & -1 & \frac{1-\sqrt{5}}{2} & -1 \\ 0 & 0 & -1 & \frac{1-\sqrt{5}}{2} \end{pmatrix}, \quad C\vec{V_1} = 0 \Rightarrow \vec{V_1} = a \begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

$-V_3 + \frac{1-\sqrt{5}}{2} V_4 = 0$

$\frac{1-\sqrt{5}}{2} V_1 = V_2$

$-V_1 + \frac{(1-\sqrt{5})^2}{4} V_1 = V_3$

$V_3 =$

$V_3 = \frac{1-\sqrt{5}}{2} V_1$

$V_4 = \frac{2}{1-\sqrt{5}} \cdot \frac{1-\sqrt{5}}{2} V_1$

— guess 1.1

when $w^2 = \frac{5+\sqrt{5}}{2}$

$$C = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & -1 & 0 & 0 \\ -1 & \frac{1+\sqrt{5}}{2} & -1 & 0 \\ 0 & -1 & \frac{1+\sqrt{5}}{2} & -1 \\ 0 & 0 & -1 & \frac{1+\sqrt{5}}{2} \end{pmatrix} \quad C\vec{V_2} = 0 \Rightarrow \vec{V_2} = a \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$$

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$V_2 = \frac{1+\sqrt{5}}{2} V_1$

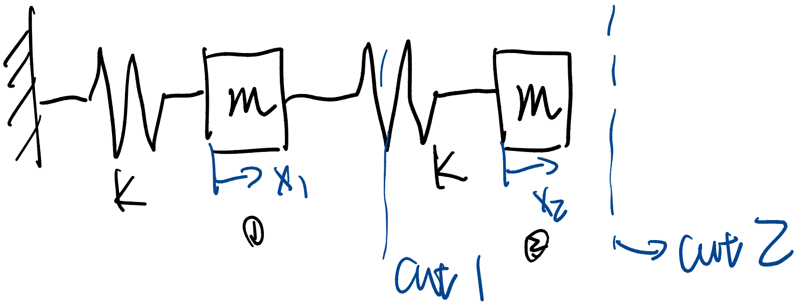
$V_3 = \frac{(1+\sqrt{5})^2}{4} V_1 - V_1$

$= \frac{1+\sqrt{5}}{2} V_1$

$V_4 = V_1$

— guess 1.2

Guess 2. the middle spring does not stretch or press.



for ①: $\xleftarrow{kx_1} \boxed{m} \xrightarrow{k(x_2-x_1)}$

$$k(x_2-x_1) - kx_1 = m\ddot{x}_1$$

for ②: $\xleftarrow{+k(x_2-x_1)} \boxed{m}$

$$+k(x_2-x_1) = -m\ddot{x}_2$$

$$\Rightarrow \begin{cases} m\ddot{x}_1 + 2kx_1 - kx_2 = 0 \\ m\ddot{x}_2 - kx_1 + kx_2 = 0 \end{cases}$$

$$\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + k\begin{pmatrix} 2 & -1 \\ -1 & +1 \end{pmatrix} = 0$$

$$K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$(w^2-2)(w^2-1) - 1 = 0$$

$$(w^2)^2 - 3w^2 + 1 = 0$$

$$w^2 = \frac{3 \pm \sqrt{5}}{2}$$

when $w^2 = \frac{3-\sqrt{5}}{2}$,

$$C = \begin{pmatrix} \frac{-1-\sqrt{5}}{2} & -1 & 0 & 0 \\ -1 & \frac{-1-\sqrt{5}}{2} & -1 & 0 \\ 0 & -1 & \frac{-1-\sqrt{5}}{2} & -1 \\ 0 & 0 & -1 & \frac{-1-\sqrt{5}}{2} \end{pmatrix}, \quad C\vec{V_3} = 0 \Rightarrow \vec{V_3} = a\begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} \\ -1 \end{pmatrix}$$

— guess 2.1

when $w^2 = \frac{3+\sqrt{5}}{2}$

$$C = \begin{pmatrix} \frac{\sqrt{5}-1}{2} & -1 & 0 & 0 \\ -1 & \frac{\sqrt{5}-1}{2} & -1 & 0 \\ 0 & -1 & \frac{\sqrt{5}-1}{2} & -1 \\ 0 & 0 & -1 & \frac{\sqrt{5}-1}{2} \end{pmatrix}, \quad C\vec{V_4} = 0 \Rightarrow \vec{V_4} = a\begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \\ \frac{1-\sqrt{5}}{2} \\ -1 \end{pmatrix}$$

— guess 2.2

In all, the 4 guesses eigenvectors are:

$$\begin{pmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \\ \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \frac{-1-\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} \\ -1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \\ \frac{1-\sqrt{5}}{2} \\ -1 \end{pmatrix}$$

3. Let A and B both be $n \times n$ matrices. Show that if $AB = BA$, then

$$\exp(A + B) = \exp A \exp B$$

Firstly, If $AB = BA$,

$$(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2,$$

$$(A+B)^3 = A^3 + A^2B + ABA + AB^2 + BA^2 + BAB + B^2A + B^3$$

$$= A^3 + 3A^2B + 3AB^2 + B^3$$

......

It means that the n^{th} power of such matrix sum $A+B$ is exactly the same in form of scalars.

Then, consider the R.H.S of eqn to be proved.

$$\exp(A)\exp(B)$$

$$= (I + A + \tfrac{1}{2}A^2 + \tfrac{1}{6}A^3 + \cdots)(I + B + \tfrac{1}{2}B^2 + \tfrac{1}{6}B^3 + \cdots)$$

$$= I + (A+B) + \tfrac{1}{2}(A^2 + 2AB + B^2) + \tfrac{1}{6}(A^3 + 3A^2B + 3AB^2 + B^3) + \cdots$$

as proved earlier, the eqn can be further simplified into

$$I + (A+B) + \tfrac{1}{2}(A+B)^2 + \tfrac{1}{6}(A+B)^3 + \cdots = \exp(A+B),$$

= LHS of eqn to be proved.

Therefore, if $AB = BA$, $\exp(A+B) = \exp(A)\exp(B)$

4. Consider the 100×100 version of my favorite symmetric matrix (i.e., 2's on the diagonal and -1's on the sub-diagonal and superdiagonal. (The stiffness matrix for the Toda lattice). Call this matrix A.

Write a computer program which performs the following iterations

$$\mathbf{u}^{(k+1)} = \frac{A\mathbf{u}^{(k)}}{\|A\mathbf{u}^{(k)}\|}$$ → is a unit vector along $Au^{(k)}$

starting with a random 100-dimensional initial unit vector $\mathbf{u}^{(k)}$. Run the program for $k = 1$ to 100 and plot the quantity

$$c_k \doteq \mathbf{u}^{(k)} \cdot (A\mathbf{u}^{(k)})$$ → is the projection length of $Au^{(k)}$ on $u^{(k)}$

as a function of k. Does it converge ? Also plot

$$\epsilon_k \doteq \|\underline{c_k\mathbf{u}^{(k)}} - \underline{A\mathbf{u}^{(k)}}\|$$ → is the distance between

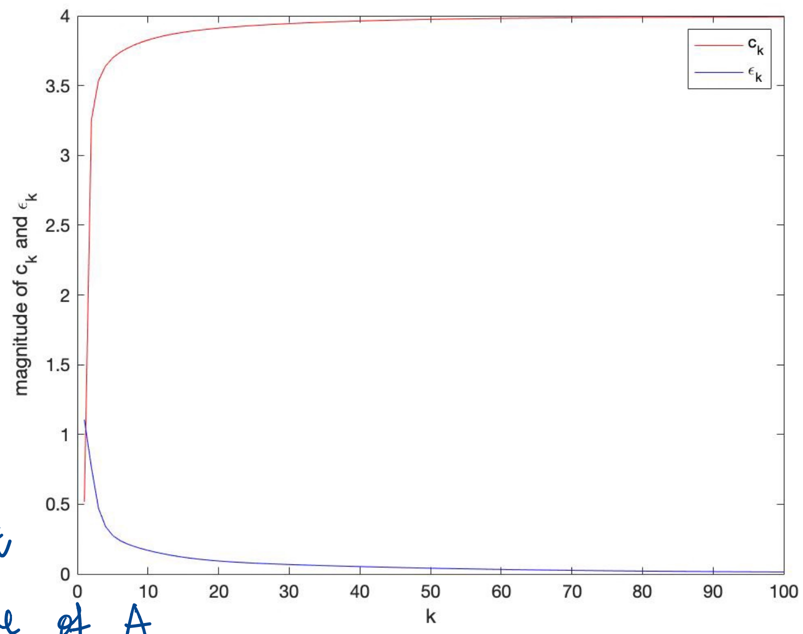
How do you interpret that result ?

As guessed,

$Au^{(k)}$ gives $u^{(k)}$ a kind of "stretch".

This kind of stretch is expressed by eigenvalues.

So, C_k actually describes the "stretching ratio" of the effect of A, which is the eigenvalue of A.



As calculated with MATLAB, the eigenvalue of A converges to 4, so C_k also converges to 4. And actually, u_k converges to the eigenvector of A, so $Au_k \approx C_k u_k$ after infinite times. Therefore, $\mathcal{E}_k$ converges to 0.

5. Compute the matrix exponentials $\exp(tJ_2(\lambda))$, $\exp(J_2(t\lambda))$, $\exp(tJ_3(\lambda))$, $\exp(J_3(t\lambda))$ where J_k is the $k \times k$ Jordan block.

$$J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad tJ_2(\lambda) = \begin{pmatrix} \lambda t & t \\ 0 & \lambda t \end{pmatrix}$$

$$J_2(t\lambda) = \begin{pmatrix} \lambda t & 1 \\ 0 & \lambda t \end{pmatrix}$$

$$J_3(\lambda) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \qquad tJ_3(\lambda) = \begin{pmatrix} \lambda t & 1 & 0 \\ 0 & \lambda t & 1 \\ 0 & 0 & \lambda t \end{pmatrix}$$

$$J_3(t\lambda) = \begin{pmatrix} \lambda t & t & 0 \\ 0 & \lambda t & t \\ 0 & 0 & \lambda t \end{pmatrix}$$

$$I + \frac{1}{2} A^2$$

$$\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\exp(tJ_2(\lambda)) = \exp\left(\begin{bmatrix} \lambda t & t \\ 0 & \lambda t \end{bmatrix}\right)$$

$$= \exp\left(\lambda t I + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}\right)$$

since $\exp(A+B) = \exp(A)\exp(B)$ if $AB = BA$,

$$= \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$ — Soln

$$\exp(J_2(\lambda t)) = \exp\left(\begin{bmatrix} \lambda t & 1 \\ 0 & \lambda t \end{bmatrix}\right)$$

$$= \exp\left(\lambda t I + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$ — Soln

$$\exp(tJ_3(\lambda)) = \exp\left(\begin{bmatrix} \lambda t & t & 0 \\ 0 & \lambda t & t \\ 0 & 0 & \lambda t \end{bmatrix}\right)$$

$$= \exp\left(\lambda t I + \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

— Soln

$$\exp(J_3(t\lambda)) = \exp\left(\begin{bmatrix} \lambda t & 1 & 0 \\ 0 & \lambda t & 1 \\ 0 & 0 & \lambda t \end{bmatrix}\right)$$

$$= \exp\left(\lambda t I + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{\lambda t} & 0 & 0 \\ 0 & e^{\lambda t} & 0 \\ 0 & 0 & e^{\lambda t} \end{bmatrix} \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & e^{\lambda t} & \frac{1}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & e^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$$

— Soln