ME5701

Homework #5

1. Continuous Fourier Analysis + probability question. Consider a pulse function f(x) which is equal to 1 on the interval [-1/2,1/2] and zero everywhere else.
   a. Is this a probability density function ? If so, compute its mean and covariance.
   b. Compute by hand (f*f)(x) and plot.
   c. Compute the Fourier transform f_hat(omega) and compute the convolution using the continuous Fourier transform. If you cannot do the integral by hand for the inverse Fourier transform, you can do it numerically. Plot your answer and compare to part b.
   d. Now imagine that the function f(x) is defined on the interval [-pi,pi] and is perodic outside that intevral. Calculate the Fourier coefficients and plot the bandlimited Fourier series approximation to f(x) with B=10
   e. Now use the Fourier series convolution theorem to compute (f*f)(x) and compare with plots from part b and c.

2. For the same function as in Problem 1 above,
   a. sample the function at 64 equally spaced points on the range [-pi,pi] and use the built-in discrete convolution in Matlab to evaluate a sampled version of (f*f)(x)
   b. input the sampled version of f(x) into the built-in FFT program in Matlab, and use the discrete convolution theorem and inverse FFT to recover the discretely sampled version of (f*f)(x)
   c. compare the results here with those of problem 1.

Do the following problems from the appendix of ``Stochastic Models, Information Theory, and Lie Groups, Vol 1''

In A.2-A.4, ||A|| denotes the Frobenious norm.

A.2. Show that for a 2×2 matrix, A, the 2-norm is related to the Frobenius norm and determinant of A as [14]

$$\|A\|_2^2 = \frac{1}{2}\left(\|A\|^2 + \sqrt{\|A\|^4 - 4|\det(A)|^2}\right).$$

A.3. Solve the above expression for $\|A\|^2$ as a function of $\|A\|_2^2$ and $\det A$.

A.4. Show that the 2-norm is the same as max eigenvalue

$$\|A\|_2 = \sqrt{\lambda_{max}(A^* A)}.$$

A.27. Determine the stability of the following system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 6 \\ 0 & 0 & -3 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} e^{-2t} \\ \cos 12t \\ 0 \end{bmatrix}.$$

A.28. Let $A \in \mathbb{R}^{n\times n}$ be a constant matrix and $\mathbf{x}(t), \mathbf{g}(t) \in \mathbb{R}^n$ be vector-valued functions of time. The solution to

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{g}(t) \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0$$

is

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0 + \int_0^t \exp(A(t-\tau))\mathbf{g}(\tau)d\tau. \tag{A.140}$$

Similarly, given

$$\frac{d\mathbf{x}}{dt} = (A + B(t))\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0$$

(where $B(t) \in \mathbb{R}^{n\times n}$ is a matrix-valued function of time) it is possible to write

$$\mathbf{x}(t) = \exp(At)\mathbf{x}_0 + \int_0^t \exp(A(t-\tau))B(\tau)\mathbf{x}(\tau)d\tau. \tag{A.141}$$

(a) Prove Equation (A.91).
(b) Prove Equation (A.94).

A.29. Use Equation (A.91) and/or Equation (A.94) and/or the Bellman–Gronwall inequality to determine the behavior of $x(t)$ governed by the following equations as $t \to \infty$:
(a) $\ddot{x} + \dot{x} + (1 + e^{-t})x = 0$
(b) $\ddot{x} + \dot{x} + (1 + 0.2\cos t)x = 0$
(c) $\ddot{x} + \dot{x} + x = \cos t$
(d) $\ddot{x} + \dot{x} + x = e^{-t}$
(e) $\ddot{x} + \dot{x} + x = e^{2t}$.
Hint: Rewrite the above second-order differential equations as a system of first-order differential equations in terms of the vector $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$ where $x_1 = x$ and $x_2 = \dot{x}$.

Regarding the proof for A.91 = A.140 that I did in class, the derivative of the integral of a function from 0 to t is just the function inside the integral evaluated at t. So there is no g(0) term.