HW 8 of ME5701. Sun Zhanhong. A0225282J

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[**Optional**.] Work out the differential operators, Gradient, Divergence and Laplacian in Cylindrical and Spherical coordinates

Assume $x = (x_1, x_2, x_3)^T$ in cartesian coordinate,

and $x = (u_1, u_2, u_3)^T$ in other coordinate.

Therefore, by chain rule, (full differential)

$$dx = \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 + \frac{\partial x}{\partial u_3} du_3$$

Let $h_i = \left\| \frac{\partial x}{\partial u_i} \right\|$ for $i = 1, 2, 3$.

then $e_i = \frac{\frac{\partial x}{\partial u_i}}{h_i}$,

and $dx = h_1 e_1 du_1 + h_2 e_2 du_2 + h_3 e_3 du_3$

Therefore, $\nabla X = \frac{1}{h_1} \frac{\partial X}{\partial u_1} e_1 + \frac{1}{h_2} \frac{\partial X}{\partial u_2} e_2 + \frac{1}{h_3} \frac{\partial X}{\partial u_3} e_3$.

$$\nabla \cdot X = \frac{1}{h_1 h_2 h_3} \left( \frac{\partial X_1 h_2 h_3}{\partial u_1} + \frac{\partial (X_2 h_3 h_1)}{\partial u_2} + \frac{\partial (X_3 h_1 h_2)}{\partial u_3} \right)$$

$$\nabla^2 X = \nabla \cdot \nabla X$$

For cylindrical coordinate,

$h_1 = 1$, $h_2 = r$. $h_3 = 1$

For spherical coordinate

$h_1 = 1$ $h_2 = r\sin\theta$ $h_3 = r$.

Therefore, for cylindrical coordinate,

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial r} ; \frac{\partial}{\partial x_2} = \frac{\partial}{r \partial \varphi} ; \frac{\partial}{\partial x_3} = \frac{\partial}{\partial z}.$$

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \phi} e_\phi + \frac{\partial f}{\partial z} e_z$$

$$\nabla \cdot f = \frac{1}{r} \frac{\partial (r F_r)}{\partial r} + \frac{1}{r} \frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

For spherical coordinate,

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial r} ; \frac{\partial}{\partial x_2} = \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} ; \frac{\partial}{\partial x_3} = \frac{\partial}{r d\theta}$$

$$\nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} e_\phi$$

$$\nabla \cdot f = \frac{1}{r^2} \frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi} + \frac{1}{r \sin \theta} \frac{\partial (F_\theta \sin \theta)}{\partial \theta}$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right)$$

Use Taylor expansion to obtain the coefficients $\alpha$, $\beta$ and $\gamma$ for the forward and backward difference formulae

Partial difference here is only with respect to $x$. So we can write it as a function of $x$

By Taylor expansion,

$$f(a+\Delta x) = f(a) + \Delta x f'(a) + \frac{\Delta x^2}{2} f''(a) + O^3(\Delta x)$$

$$\Rightarrow \frac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a) + \Delta x f''(a) + \underbrace{\frac{O^3(\Delta x)}{\Delta x}}_{=O^2}$$

$$\Rightarrow f'(a) = \frac{f(a+\Delta x) - f(a)}{\Delta x} \underline{-\Delta x f''(a)} + O^2(\Delta x)$$

the first error term is proportional to $\Delta x$

That means the full difference is first-order accurated in $\Delta x$.

$$\Rightarrow f'(a) = \frac{f(a+\Delta x) - f(a)}{\Delta x} + O(\Delta x)$$

$$\Rightarrow \begin{cases} \alpha = 1 \\ \beta = -1 \\ \gamma = 0 \end{cases}$$

By Taylor expansion,

$$f(a-\Delta x) = f(a) - \Delta x f'(a) + \frac{\Delta x^2}{2} f''(a) + O^3(\Delta x)$$

$$\Rightarrow \frac{f(a-\Delta x) - f(a)}{-\Delta x} = f'(a) - \frac{\Delta x}{2} f''(a) + O^2(\Delta x)$$

$$\Rightarrow f'(a) = \frac{f(a) - f(a-\Delta x)}{\Delta x} + \frac{\Delta x}{2} f''(a) + O^2(\Delta x)$$

$$= \frac{f(a) - f(a-\Delta x)}{\Delta x} + O(\Delta x)$$

$$\Rightarrow \begin{cases} \alpha = 0 \\ \beta = 1 \\ \gamma = -1 \end{cases}$$