Homework 12: ME5701, 9 November 2020, Due 16 November 2020

Problem 1. For an invertible 2-tensor, prove that

$$I_1(A^{-1}) = \frac{I_2(A)}{I_3(A)}$$

$$I_2(A^{-1}) = \frac{I_1(A)}{I_3(A)}$$

$$I_3(A^{-1}) = \frac{1}{I_3(A)}$$

$$I_3(A) = \frac{1}{6}\left([\mathrm{tr}(A)]^3 - 3\mathrm{tr}(A)\mathrm{tr}(A^2) + 2\mathrm{tr}(A^3)\right)$$

$$I_3(A)A^{-1} = A^2 - I_1(A)A + I_2(A)\mathbb{I}$$

Problem 2. Simplify the following expressions written in summation notation

$$\epsilon_{ijk}\delta_{jk}$$

$$\epsilon_{ijk}\epsilon_{mjk}\delta_{im}$$

$$\epsilon_{ijk}\delta_{km}\delta_{jn}$$

$$\epsilon_{ijk}\epsilon_{imn}\delta_{jm}$$

Problem 3. Given vector fields $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ and a scalar function $\phi(\mathbf{x})$, determine which of the following are true, and prove the ones that are

$$\nabla \times (\nabla\phi) = \mathbf{0}$$

$$\nabla(\mathbf{f}\cdot\mathbf{g}) = (\nabla\mathbf{f})\mathbf{g} + (\nabla\mathbf{g})\mathbf{f}$$

$$\nabla(\mathbf{f}\times\mathbf{g}) = (\nabla\mathbf{f})\times\mathbf{g} - (\nabla\mathbf{g})\times\mathbf{f}$$

where

$$\nabla\mathbf{f} \doteq \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}$$

and

$$\nabla\cdot\mathbf{f} \doteq \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3}$$

and

$$\nabla\phi \doteq \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3$$

and

$$(\nabla \times \mathbf{f})_k \doteq \epsilon_{ijk} \frac{\partial f_i}{\partial x_j}$$

and for a matrix $A = A(\mathbf{x})$,

$$\nabla \times A \doteq [\nabla \times (A\mathbf{e}_1), \nabla \times (A\mathbf{e}_2), \nabla \times (A\mathbf{e}_3)]$$

Note: in cases when the variables with which gradients are being computed are not clear, it is sometimes convenient to denote them by the variable. For example, the above gradients are with respect to x and can be denoted as ∇_x.

Problem 4. Consider the deformation

$$\mathbf{y} = \mathbf{f}(\mathbf{x}, t) = A(t)\mathbf{x}$$

where A is a 3×3 special linear matrix.

a) Show that this deformation is locally volume preserving

b) Assume $\|A - \mathbb{I}\|$ is small and compute the infinitesimal strain tensor ε

c) Given a solid unit disk in referential coordinates $\mathbf{x}$, defined by $\|\mathbf{x}\| \leq 1$, what will its shape be after the deformation ?

d) If the strain energy (potential energy due to deformation) at each point in the deformed disk is $dE = (1/2)\varepsilon_{ij} C_{ijkl} \varepsilon_{kl}$, then compute the total strain energy in the whole deformed disk assuming that the rules of infinitesimal elasticity apply.

Problem 5. Given the flow field $\mathbf{v}(\mathbf{x}, t)$ where $\mathbf{x}$ now describes spatial (Eulerian) Cartesian coordinates, the condition that a fluid is incompressible is $\nabla_x \cdot \mathbf{v} = 0$, and the condition that it is irrotational is $\nabla_x \times \mathbf{v} = \mathbf{0}$. Which (if any) of the following flow fields are incompressible, and which are irrotational ?

$$a) \quad \mathbf{v}(\mathbf{x}, t) = A(t)\mathbf{x}$$

where A is upper triangular with 0's on the diagonal

$$b) \quad \mathbf{v}(\mathbf{x}, t) = [a(t)x_2, b(t)x_3, c(t)x_1]^T$$

Problem 6. Which of the following structured matrices form groups under the operation of matrix multiplication

$$\begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & a_1 & a_2 \\ 0 & e^b & a_3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$