10

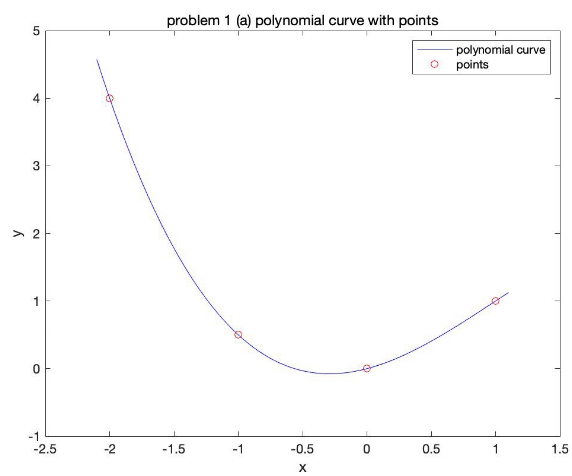
ME5701: Homework #9

Sun Zhanhong A0225282J

Problem 1

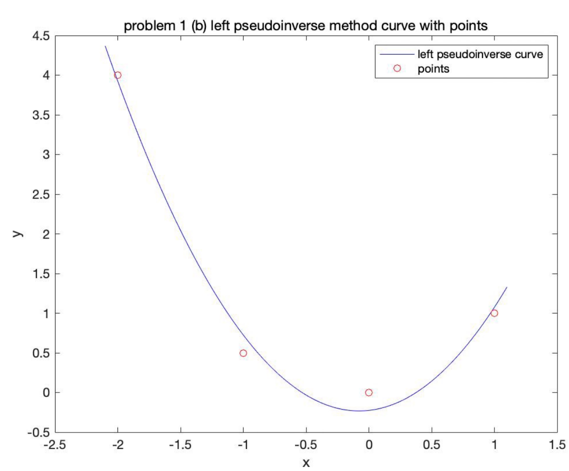
a) Given $n+1$ points $\{x_1, ..., x_{n+1}\}$ and associated data at those points $\{y_1, ..., y_{n+1}\}$, write a progam to fit the polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ to this data such that $y_i = f(x_i)$ for all $i = 1, ..., n+1$ and illustrate your program when $n = 3$ on the pairs $\{(x_i, y_i)\} = \{(0,0), (1,1), (-1, 0.5), (-2, 4)\}$ by plotting the points and the polynomial curve.

```
%% a
points = [0,0;1,1;-1,0.5;-2,4];
num_of_points = size(points,1);
highest_order = 3;
x_matrix = zeros(num_of_points,highest_order+1);
y_vector = zeros(num_of_points,1);
for i = 1:num_of_points
    for j = 0:highest_order
        x_matrix(i,j+1) = points(i,1)^j;
    end
    y_vector(i) = points(i,2);
end
a_vector = inv(x_matrix)*y_vector
syms x y
y = 0;
for k = 1:highest_order+1
    y = y+a_vector(k)*x^(k-1);
end
x_value = -2.1:0.01:1.1;
y_value = subs(y,x,x_value);
figure;
plot(x_value,y_value,'b');
hold on;
scatter(points(:,1),points(:,2),'r')
hold off;
xlabel('x');
ylabel('y');
legend('polynomial curve','points')
title('problem 1 (a) polynomial curve with points')
```



b) For the same four data points as in part a, use the left pseudoinverse with weight $W = \mathbb{I}$ to obtain the best fit parabola $y = ax^2 + bx + c$ (which we know does not have enough freedom to hit all points exactly).

```
%% b
points = [0,0;1,1;-1,0.5;-2,4];
num_of_points = size(points,1);
highest_order = 2;
x_matrix = zeros(num_of_points,highest_order+1);
y_vector = zeros(num_of_points,1);
for i = 1:num_of_points
    for j = 0:highest_order
        x_matrix(i,j+1) = points(i,1)^j;
    end
    y_vector(i) = points(i,2);
end
J = x_matrix;
J_T = transpose(J);
J_M = inv(J_T*J)*J_T;
a_vector = J_M*y_vector;
syms x y
y = 0;
for k = 1:highest_order+1
    y = y+a_vector(k)*x^(k-1);
end
y
x_value = -2.1:0.01:1.1;
y_value = subs(y,x,x_value);
figure;
plot(x_value,y_value,'b');
hold on;
scatter(points(:,1),points(:,2),'r')
hold off;
xlabel('x');
ylabel('y');
legend('left pseudoinverse curve','points')
title('problem 1 (b) left pseudoinverse method curve with points')
```



c) Again using the same four data points as in part a, now fit with a fourth order polynomial

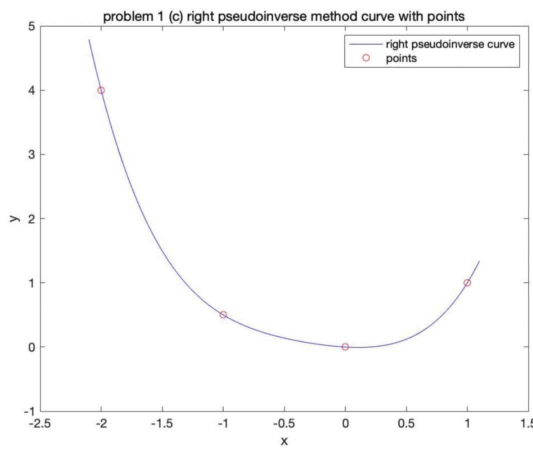
$$y = g(x) \doteq ax^4 + bx^3 + cx^2 + dx + e$$

such that all of the data points are hit, $y_i = g(x_i)$, while minimizing the cost

$$C_1(a,b,c,d,e) \doteq \int_{-2}^{1} \left(\frac{dg}{dx}\right)^2 dx$$

What W matrix does this produce for use with the right pseudoinverse ?

```
%% c
points = [0,0;1,1;-1,0.5;-2,4];
num_of_points = size(points,1);
highest_order = 4;
x_matrix = zeros(num_of_points,highest_order+1);
y_vector = zeros(num_of_points,1);
for i = 1:num_of_points
    for j = 0:highest_order
        x_matrix(i,j+1) = points(i,1)^j;
    end
    y_vector(i) = points(i,2);
end
syms a b c d e g x C integrated_function
g = a*x^4+b*x^3+c*x^2+d*x+e;
integrated_function = diff(g)^2;
C_func = int(integrated_function,x,-2,1);
coef = coeffs(C_func)*2;
coo=flipud(coef);
a = 1;
W = zeros(5);
for i = 1:4
    for j = 1:4
        if i<=j
            W(i,j) = coo(a)/2;
            a = a+1;
        else
            W(i,j) = W(j,i);
        end
    end
end
W = W + 0.01*eye(highest_order + 1);
W_inv = inv(W);
J = x_matrix;
J_T = transpose(J);
J_W = W_inv*J_T*inv(J*W_inv*J_T);
a_vector = J_W*y_vector;
syms x y
y = 0;
for k = 1:highest_order+1
    y = y+a_vector(k)*x^(k-1);
end
x_value = -2.1:0.01:1.1;
y_value = subs(y,x,x_value);
figure;
plot(x_value,y_value,'b');
hold on;
scatter(points(:,1),points(:,2),'r')
hold off;
xlabel('x');
ylabel('y');
legend('right pseudoinverse curve','points')
title('problem 1 (c) right pseudoinverse method curve with points')
```




```
W =

    3.0100   -6.0000   12.0000   18.0000         0
   -6.0000  -44.9900   59.4000  -30.0000         0
   12.0000   59.4000  105.6100 -252.0000         0
   18.0000  -30.0000 -252.0000  294.8671         0
         0         0         0         0    0.0100
```

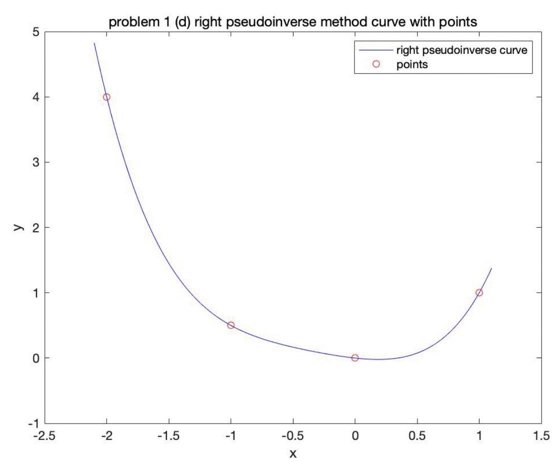
d) Do the same as in part c, but with the cost

$$C_2(a,b,c,d,e) \doteq \int_{-2}^{1} \left(\frac{d^2g}{dx^2}\right)^2 dx$$

What W matrix does this produce for use with the right pseudoinverse ?

Plot your results for all parts. Which one do you like the best, and why ? Why did I choose the range of integration $x \in [-2, 1]$?

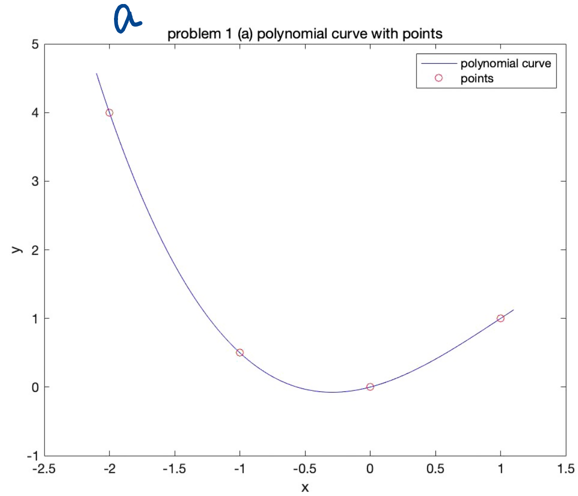
```
%% d
points = [0,0;1,1;-1,0.5;-2,4];
num_of_points = size(points,1);
highest_order = 4;
x_matrix = zeros(num_of_points,highest_order+1);
y_vector = zeros(num_of_points,1);
for i = 1:num_of_points
    for j = 0:highest_order
        x_matrix(i,j+1) = points(i,1)^j;
    end
    y_vector(i) = points(i,2);
end
syms a b c d e g x C integrated_function
g = a*x^4+b*x^3+c*x^2+d*x+e;
integrated_function = diff(diff(g))^2;
C_func = int(integrated_function,x,-2,1);
coef = coeffs(C_func)*2;
coo=flipud(coef);
a = 1;
W = zeros(5);
for i = 1:3
    for j = 1:3
        if i<=j
            W(i,j) = coo(a)/2;
            a = a+1;
        else
            W(i,j) = W(j,i);
        end
    end
end
W = W + 0.01*eye(highest_order + 1);
W_inv = inv(W);
J = x_matrix;
J_T = transpose(J);
J_W = W_inv*J_T*inv(J*W_inv*J_T);
a_vector = J_W*y_vector;
syms x y
y = 0;
for k = 1:highest_order+1
    y = y+a_vector(k)*x^(k-1);
end
x_value = -2.1:0.01:1.1;
y_value = subs(y,x,x_value);
figure;
plot(x_value,y_value,'b');
hold on;
scatter(points(:,1),points(:,2),'r')
hold off;
xlabel('x');
ylabel('y');
legend('right pseudoinverse curve','points')
title('problem 1 (d) right pseudoinverse method curve with points')
```



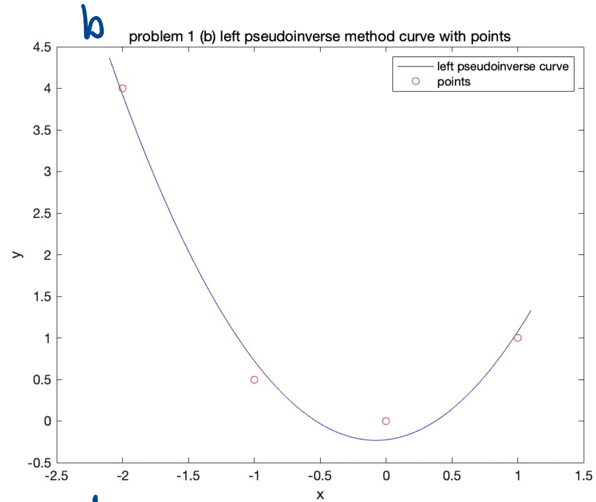

```
W =

   12.0100  -36.0000  108.0000         0         0
  -36.0000  144.0100 -540.0000         0         0
  108.0000 -540.0000  950.4100         0         0
         0         0         0    0.0100         0
         0         0         0         0    0.0100
```

Plot your results for all parts. Which one do you like the best, and why ? Why did I choose the range of integration $x \in [-2, 1]$?

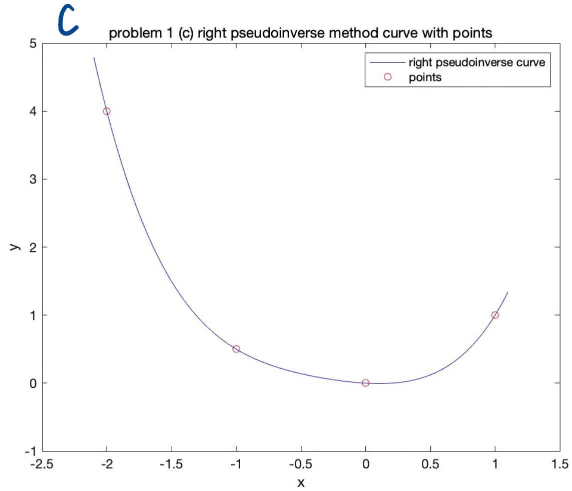
a



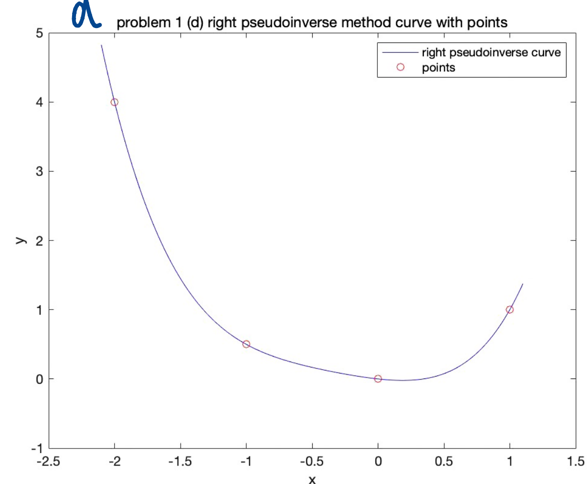
b



c



d



I like the first one since it seems not too redundant and also hits all points. However, we will not always meet this kind of problem as # points equal to # polynomial coefficients exactly...
I guess the reason of such range is that it should meet all the sample points and also do not care about the outer range. So it take exactly the smallest and the largest value of x.^ ^

Problem 2

a)
We know that

$$\tilde{\mathbf{x}}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ ht \end{pmatrix},$$

where r and h are constants. Then we can calculate:

$$\frac{dx_1}{dt} = -r\sin t, \quad \frac{dx_2}{dt} = r\cos t, \quad \frac{dx_3}{dt} = h$$

Then,

$$\begin{aligned} s(t) &= \int_0^t (\mathrm{x}'(t), \mathrm{x}'(t))^{\frac{1}{2}}\, d\tau \\ &= \int_0^t \sqrt{r^2+h^2} d\tau \\ &= \sqrt{r^2+h^2}\ t \end{aligned}$$

Therefore,

$$t = \frac{s}{\sqrt{r^2+h^2}}$$

Plug into the original representation:

$$\begin{aligned} \mathbf{x}(s) &\doteq \tilde{\mathbf{x}}\left(f^{-1}(s)\right) \\ &= \begin{pmatrix} r\cos\frac{s}{\sqrt{r^2+h^2}} \\ r\sin\frac{s}{\sqrt{r^2+h^2}} \\ \frac{hs}{\sqrt{r^2+h^2}} \end{pmatrix} \end{aligned}$$

b)
According to the definition,

$$\kappa(s) \doteq \left(\frac{d\mathbf{u}}{ds} \cdot \frac{d\mathbf{u}}{ds}\right)^{\frac{1}{2}} = \left(\frac{d^2\mathbf{x}}{ds^2} \cdot \frac{d^2\mathbf{x}}{ds^2}\right)^{\frac{1}{2}}$$

Then,

$$\begin{aligned} \frac{d^2x_1(s)}{ds^2} &= \frac{d^2\ r\cos\frac{s}{\sqrt{r^2+h^2}}}{ds^2} = \frac{-r}{r^2+h^2}\cos\frac{s}{\sqrt{r^2+h^2}} \\ \frac{d^2x_2(s)}{ds^2} &= \frac{-r}{r^2+h^2}\sin\frac{s}{\sqrt{r^2+h^2}} \\ \frac{d^2x_3(s)}{ds^2} &= 0 \end{aligned}$$

We consider $r > 0$,

$$\kappa(s) = \left(\frac{d^2\mathbf{x}}{ds^2} \cdot \frac{d^2\mathbf{x}}{ds^2}\right)^{\frac{1}{2}} = \frac{r}{r^2+h^2}$$

Then,

$$\mathbf{u}(s) = \frac{d\mathbf{x}}{ds} = \begin{pmatrix} \frac{-r}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}} \\ \frac{r}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}} \\ \frac{h}{\sqrt{r^2+h^2}} \end{pmatrix}$$

,and

$$\begin{aligned}
\mathbf{n}_1(s) &\doteq \frac{1}{\kappa(s)}\frac{d\mathbf{u}}{ds} \\
&= \frac{r^2+h^2}{r}\begin{pmatrix} \frac{-r}{r^2+h^2}\cos\frac{s}{\sqrt{r^2+h^2}} \\ \frac{-r}{r^2+h^2}\sin\frac{s}{\sqrt{r^2+h^2}} \\ \frac{h}{r^2+h^2} \end{pmatrix} \\
&= \begin{pmatrix} -\cos\frac{s}{\sqrt{r^2+h^2}} \\ -\sin\frac{s}{\sqrt{r^2+h^2}} \\ 0 \end{pmatrix}
\end{aligned}$$

Then,

$$n_2(s) = u(s)\times n_1(s) = \begin{pmatrix} \frac{h}{\sqrt{r^2+h^2}}\sin\frac{s}{\sqrt{r^2+h^2}} \\ \frac{-h}{\sqrt{r^2+h^2}}\cos\frac{s}{\sqrt{r^2+h^2}} \\ r \end{pmatrix}$$

So,

$$\begin{aligned}
\tau(s) &\doteq -\frac{d\mathbf{n}_2(s)}{ds}\cdot\mathbf{n}_1(s) \\
&= \frac{h}{r^2+h^2}\cos^2\frac{s}{\sqrt{r^2+h^2}} + \frac{h}{r^2+h^2}\sin^2\frac{s}{\sqrt{r^2+h^2}} \\
&= \frac{h}{r^2+h^2}
\end{aligned}$$

c)

$$\tilde{\mathbf{x}}'(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ ht \end{pmatrix} \quad \tilde{\mathbf{x}}''(t) = \begin{pmatrix} -r\sin t \\ r\cos t \\ h \end{pmatrix} \quad \tilde{\mathbf{x}}'''(t) = \begin{pmatrix} -r\cos t \\ -r\sin t \\ 0 \end{pmatrix}$$

Then,

$$\|\tilde{\mathbf{x}}'(t)\times\tilde{\mathbf{x}}''(t)\| = \left\|\begin{pmatrix} rh\sin t \\ rh\cos t \\ r^2 \end{pmatrix}\right\| = r(r^2+h^2)^{1/2}$$

$$\|\tilde{\mathbf{x}}'(t)\|^3 = (r^2+h^2)^{3/2}$$

Thus,

$$\tilde{\kappa}(t) = \frac{\|\tilde{\mathbf{x}}'(t)\times\tilde{\mathbf{x}}''(t)\|}{\|\tilde{\mathbf{x}}'(t)\|^3} = \frac{r}{r^2+h^2}$$

Then,

$$\det\left[\tilde{\mathbf{x}}'(t), \tilde{\mathbf{x}}''(t), \tilde{\mathbf{x}}'''(t)\right] = \begin{vmatrix} -r\sin t & -r\cos t & r\sin t \\ r\cos t & -r\sin t & -r\cos t \\ h & 0 & 0 \end{vmatrix} = hr^2$$

Therefore,

$$\tilde{\tau}(t) = \frac{hr^2}{r^2(r^2+h^2)} = \frac{h}{r^2+h^2}$$

Obviously,

$$\begin{gathered}
\tilde{\kappa}(t) = \kappa(f(t)) \text{ and } \tilde{\tau}(t) = \tau(f(t)); \\
\tilde{\kappa}\left(f^{-1}(s)\right) = \kappa(s) \text{ and } \tilde{\tau}\left(f^{-1}(s)\right) = \tau(s)
\end{gathered}$$

Problem 3

Solution
a)

$$\tilde{\mathbf{x}}(\theta, \phi) = \begin{pmatrix} a\cos\phi\sin\theta \\ b\sin\phi\sin\theta \\ c\cos\theta \end{pmatrix},$$

$$\begin{aligned}\phi(\tilde{\mathbf{x}}) &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \\ &= cos^2(\phi)sin^2(\theta) + sin^2(\phi)sin^2(\theta) + cos^2(\theta) \\ &= sin^(\theta) + cos^2(\theta) \\ &= 1\end{aligned}$$

b)
First we calculate the partial derivatives:

$$\frac{\partial\tilde{\mathbf{x}}}{\partial\theta} = \begin{pmatrix} a\cos\theta\cos\phi \\ b\cos\theta\sin\phi \\ -c\sin\theta \end{pmatrix}, \quad \frac{\partial\tilde{\mathbf{x}}}{\partial\phi} = \begin{pmatrix} -a\sin\theta\sin\phi \\ b\sin\theta\cos\phi \\ 0 \end{pmatrix}$$

We know that

$$g_{ij} = \frac{\partial\mathbf{x}}{\partial q_i} \cdot \frac{\partial\mathbf{x}}{\partial q_j}$$

Therefore,

$$\begin{aligned}g_{11} &= a^2\cos^2\theta\cos^2\phi + b^2\cos^2\theta\sin^2\phi + c^2\sin^2\theta \\ g_{12} = g_{21} &= -a^2\sin\theta\cos\theta\sin\phi\cos\phi + b^2\sin\theta\cos\theta\sin\phi\cos\phi \\ g_{22} &= a^2\sin^2\theta\sin^2\phi + b^2\sin^2\theta\cos^2\phi\end{aligned}$$

$$G(\theta,\phi) = \begin{pmatrix} a^2\cos^2\theta\cos^2\phi + b^2\cos^2\theta\sin^2\phi + c^2\sin^2\theta & -a^2\sin\theta\cos\theta\sin\phi\cos\phi + b^2\sin\theta\cos\theta\sin\phi\cos\phi \\ -a^2\sin\theta\cos\theta\sin\phi\cos\phi + b^2\sin\theta\cos\theta\sin\phi\cos\phi & a^2\sin^2\theta\sin^2\phi + b^2\sin^2\theta\cos^2\phi \end{pmatrix}$$

Then we try to calculate $\mathbf{n}(\theta,\phi)$. We know that

$$\mathbf{n} = |G|^{-\frac{1}{2}}\left(\frac{\partial\mathbf{x}}{\partial q_1} \times \frac{\partial\mathbf{x}}{\partial q_2}\right)$$

Firstly, we calculatedet G

$$\begin{aligned}\det(G) &= (a^2\cos^2\theta\cos^2\phi + b^2\cos^2\theta\sin^2\phi + c^2\sin^2\theta)(a^2\sin^2\theta\sin^2\phi + b^2\sin^2\theta\cos^2\phi) \\ &\quad - ((b^2-a^2)\sin\theta\cos\theta\sin\phi\cos\phi)^2 \\ &= a^4\sin^2\phi\cos^2\phi\sin^2\phi\cos^2\theta + a^2b^2\sin^4\phi\cos^2\theta\sin^2\theta + a^2c^2\sin^2\phi\sin^4\theta \\ &\quad + a^2b^2\cos^4\phi\sin^2\theta\cos^2\theta + b^4\sin^2\phi\cos^2\phi\sin^2\theta\cos^2\theta + b^2c^2\cos^2\phi\sin^4\theta \\ &\quad - (a^4 - 2a^2b^2 + b^2)\sin^2\phi\cos^2\phi\sin^2\theta\cos^2\theta \\ &= \sin^2\theta\cos^2\theta(a^4\sin^2\phi\cos^2\phi + a^2b^2\sin^4\phi + a^2b^2\cos^4\phi + b^4\sin^2\phi\cos^2\phi - \\ &\quad ((a^4 - 2a^2b^2 + b^2)\sin^2\phi\cos^2\phi) + \sin^4\theta(a^2c^2\sin^2\phi + b^2c^2\cos\phi) \\ &= a^2b^2\sin^2\theta\cos^2\theta + c^2\sin^4\theta(a^2\sin^2\theta + a^2c^2\sin^2\phi\sin^2\theta + b^2c^2\cos^2\phi\sin^2\theta) \\ &= \sin^2\theta(a^2b^2\cos^2\theta + a^2c^2\sin^2\phi\sin^2\theta + b^2c^2\cos^2\phi\sin^2\theta)\end{aligned}$$

Also, we can calculate that

$$\frac{\partial \mathbf{x}}{\partial q_1} \times \frac{\partial \mathbf{x}}{\partial q_2} = \begin{pmatrix} bc\sin^2\theta\cos^2\phi \\ ac\sin\phi\sin^2\theta \\ ab\cos^2\phi\sin\theta\cos\theta + ab\sin^2\phi\sin\theta\cos\theta \end{pmatrix} = \sin\theta \begin{pmatrix} bc\sin\theta\cos\phi \\ ac\sin\phi\sin\theta \\ abcos\theta \end{pmatrix}$$

Therefore,

$$\begin{aligned}\mathbf{n} &= |G|^{-\frac{1}{2}} \left(\frac{\partial \mathbf{x}}{\partial q_1} \times \frac{\partial \mathbf{x}}{\partial q_2}\right) \\ &= \frac{1}{\sqrt{a^2b^2\cos^2\theta + a^2c^2\sin^2\phi\sin^2\theta + b^2c^2\cos^2\phi\sin^2\theta}} \begin{pmatrix} bc\sin\theta\cos\phi \\ ac\sin\phi\sin\theta \\ abcos\theta \end{pmatrix}\end{aligned}$$

Next, let's calculate $L(\theta,\phi)$. For L, we know that

$$L_{ij} = \left(\frac{\partial^2 \mathbf{x}}{\partial q_i \partial q_j}\right) \cdot \mathbf{n}$$

Then, if we set $1/\sqrt{a^2b^2\cos^2\theta + a^2c^2\sin^2\phi\sin^2\theta + b^2c^2\cos^2\phi\sin^2\theta} = \triangle$

$$L_{11} = \begin{pmatrix} -a\cos\phi\sin\theta \\ -b\sin\phi\sin\theta \\ -c\cos\theta \end{pmatrix} \mathbf{n} = -abc\triangle$$

$$L_{12} = \begin{pmatrix} -a\sin\phi\cos\theta \\ b\cos\phi\cos\theta \\ 0 \end{pmatrix} \mathbf{n} = 0$$

$$L_{21} = \begin{pmatrix} -a\cos\phi\cos\theta \\ -b\sin\phi\cos\theta \\ 0 \end{pmatrix} \mathbf{n} = 0$$

$$L_{22} = \begin{pmatrix} -a\cos\phi\sin\theta \\ -b\sin\phi\sin\theta \\ 0 \end{pmatrix} \mathbf{n} = -abc\sin^2\theta\triangle$$

Therefore,

$$L(\theta,\phi) = \frac{-abc}{\sqrt{b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta}} \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

c)

We know that

$$m = \frac{g_{11}L_{22} + g_{22}L_{11} - 2g_{12}L_{12}}{2\left(g_{11}g_{22} - g_{12}^2\right)} \quad \text{and} \quad k = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}$$

Luckily, all the coefficients in these two formulas have already been calculated. Firstly, we can calculate the mean curvature m: Taking a breef look, we can find that the denominator is exactly $2 \times \det(G)$, and also, $L_{12} = L_{21} = 0$. So,

$$\begin{aligned} m &= \frac{(a^2\cos^2\theta\cos^2\phi + b^2\cos^2\theta\sin^2\phi + c^2\sin^2\theta)(-abc\sin^2\theta\triangle) + (a^2\sin^2\theta\sin^2\phi + b^2\sin^2\theta\cos^2\phi)(-abc\triangle)}{2\sin^2\theta}\triangle^2 \\ &= \frac{(-abc)[a^2\cos^2\theta\cos^2\phi + b^2\cos^2\theta\sin^2\phi + c^2\sin^2\theta + a^2\sin^2\phi + b^2\cos^2\phi]}{2(b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta)^{3/2}} \\ &= \frac{(-abc)[a^2(\cos^2\theta\cos^2\phi + \sin^2\phi) + b^2(\cos^2\theta\sin^2\phi + \cos^2\phi) + c^2\sin^2\theta]}{2(b^2c^2\sin^2\theta\cos^2\phi + a^2c^2\sin^2\theta\sin^2\phi + a^2b^2\cos^2\theta)^{3/2}} \end{aligned}$$

Problem 3 continued on next page. . .

Also,

$$
\begin{aligned}
k &= \left(\frac{abc\sin\theta\triangle^2}{\sin\theta}\right)^2 \\
&= \left(\frac{abc}{c^2\sin^2\theta(b^2\cos^2\phi + a^2\sin^2\phi) + a^2b^2\cos^2\theta}\right)^2
\end{aligned}
$$

d)
First we calculate the gradient:

$$
\begin{aligned}
\nabla\phi &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)^T \\
&= \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)^T \\
k(\mathbf{x}) &= \frac{1}{\|\nabla\phi\|^4}\det\begin{bmatrix} \nabla\nabla^T\phi & \nabla\phi \\ \nabla^T\phi & 0 \end{bmatrix} \\
&= \frac{\nabla^T\phi \cdot H^*(\phi)\cdot\nabla\phi}{\|\nabla\phi\|^4},
\end{aligned}
$$

where $H^*(\phi)$ is the adjactory matrix of $H(\phi) = \nabla\nabla^T\phi$

$$
H(\phi) = \begin{pmatrix} \frac{\partial^2\phi}{\partial x^2} & \frac{\partial^2\phi}{\partial x\partial y} & \frac{\partial^2\phi}{\partial x\partial z} \\ \frac{\partial^2\phi}{\partial y\partial x} & \frac{\partial^2\phi}{\partial y^2} & \frac{\partial^2\phi}{\partial y\partial z} \\ \frac{\partial^2\phi}{\partial z\partial x} & \frac{\partial^2\phi}{\partial z\partial y} & \frac{\partial^2\phi}{\partial z^2} \end{pmatrix} = \begin{pmatrix} \phi_{xx} & \phi_{xy} & \phi_{xz} \\ \phi_{yx} & \phi_{yy} & \phi_{yz} \\ \phi_{zx} & \phi_{zy} & \phi_{zz} \end{pmatrix} = \begin{pmatrix} 2/a^2 & 0 & 0 \\ 0 & 2/b^2 & 0 \\ 0 & 0 & 2/c^2 \end{pmatrix}
$$

$$
\begin{aligned}
H^*(\phi) &= \begin{pmatrix} \text{Cofactor }(\phi_{xx}) & \text{Cofactor }(\phi_{xy}) & \text{Cofactor }(\phi_{xz}) \\ \text{Cofactor }(\phi_{yx}) & \text{Cofactor }(\phi_{yy}) & \text{Cofactor }(\phi_{yz}) \\ \text{Cofactor }(\phi_{zx}) & \text{Cofactor }(\phi_{zy}) & \text{Cofactor }(\phi_{zz}) \end{pmatrix} \\
&= \begin{pmatrix} \phi_{yy}\phi_{zz} - \phi_{yz}\phi_{zy} & \phi_{yz}\phi_{zx} - \phi_{yx}\phi_{zz} & \phi_{yx}\phi_{zy} - \phi_{yy}\phi_{zx} \\ \phi_{xz}\phi_{zy} - \phi_{xy}\phi_{zz} & \phi_{xx}\phi_{zz} - \phi_{xz}\phi_{zx} & \phi_{xy}\phi_{zx} - \phi_{xx}\phi_{zy} \\ \phi_{xy}\phi_{yz} - \phi_{xz}\phi_{yy} & \phi_{yx}\phi_{xz} - \phi_{xx}\phi_{yz} & \phi_{xx}\phi_{yy} - \phi_{xy}\phi_{yx} \end{pmatrix} \\
&= \begin{pmatrix} 4/(b^2c^2) & 0 & 0 \\ 0 & 4/(a^2c^2) & 0 \\ 0 & 0 & 4/(a^2b^2) \end{pmatrix}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
k(\mathbf{x}) &= \frac{\nabla^T\phi\cdot H^*(\phi)\cdot\nabla\phi}{\|\nabla\phi\|^4} \\
&= \frac{\left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)\begin{pmatrix} 4/(b^2c^2) & 0 & 0 \\ 0 & 4/(a^2c^2) & 0 \\ 0 & 0 & 4/(a^2b^2) \end{pmatrix}\left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)^T}{(4x^2/a^4 + 4y^2/b^4 + 4z^2/c^4)^2} \\
&= \frac{\frac{16x^2}{a^4b^2c^2} + \frac{16y^2}{a^2b^4c^2} + \frac{16z^2}{a^2b^2c^4}}{(4x^2/a^4 + 4y^2/b^4 + 4z^2/c^4)^2} \\
&= \frac{1}{\left(\text{abc}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right)^2}
\end{aligned}
$$

(different eqn used. same result)

Then we calculate m by calculating the trace first:

$$
tr(\nabla\nabla^T\phi) = \frac{2}{a^2} + \frac{2}{b^2} + \frac{2}{c^2}
$$

Then,

$$
\begin{aligned}
m(\mathbf{x}) &= \nabla \cdot \left(\frac{\nabla \phi}{\|\nabla \phi\|} \right) \\
&= \nabla \cdot \frac{\left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right)^T}{\sqrt{4x^2/a^4 + 4y^2/b^4 + 4z^2/c^4}} \\
&= \frac{\frac{2}{a^2}(\frac{4y^2}{b^4} + \frac{4z^2}{c^2}) + \frac{2}{b^2}(\frac{4x^2}{a^4} + \frac{4z^2}{c^4}) + \frac{2}{c^2}(\frac{4x^2}{a^4} + \frac{4y^2}{b^4})}{2(4x^2/a^4 + 4y^2/b^4 + 4z^2/c^4)^{3/2}} \\
&= \frac{b^2c^2(\frac{y^2}{b^4} + \frac{z^2}{c^2}) + a^2c^2(\frac{x^2}{a^4} + \frac{z^2}{c^4}) + a^2b^2(\frac{x^2}{a^4} + \frac{y^2}{b^4})}{2a^2b^2c^2(x^2/a^4 + y^2/b^4 + z^2/c^4)^{3/2}} \\
&= \frac{a^2(\frac{y^2}{b^2} + \frac{z^2}{c^2}) + b^2(\frac{x^2}{a^2} + \frac{z^2}{c^2}) + c^2(\frac{y^2}{b^2} + \frac{x^2}{a^2})}{2a^2b^2c^2(x^2/a^4 + y^2/b^4 + z^2/c^4)^{3/2}} \\
&= \frac{a^2(1 - \frac{x^2}{a^2}) + b^2(1 - \frac{y^2}{b^2}) + c^2(1 - \frac{z^2}{c^2})}{2a^2b^2c^2(x^2/a^4 + y^2/b^4 + z^2/c^4)^{3/2}} \\
&= \frac{a^2 + b^2 + c^2 - x^2 - y^2 - z^2}{2a^2b^2c^2(x^2/a^4 + y^2/b^4 + z^2/c^4)^{3/2}}
\end{aligned}
$$

(different eqn used, same result)

e)
According to:

$$
\tilde{\mathbf{x}}(\theta, \phi) = \begin{pmatrix} a \cos\phi \sin\theta \\ b \sin\phi \sin\theta \\ c \cos\theta \end{pmatrix},
$$

We can plug $x = a\cos\phi,\ y = b\sin\phi\sin\theta,$ and $z = c\cos\theta$ into the expression of m,k in Cartesian coordinate and check whether we get the same result as derivating directly.

$$
\begin{aligned}
m(\mathbf{x}) &= \frac{a^2 + b^2 + c^2 - x^2 - y^2 - z^2}{2a^2b^2c^2(x^2/a^4 + y^2/b^4 + z^2/c^4)^{3/2}} \\
&= \frac{(a^2 + b^2 + c^2 - a^2\cos^2\phi\sin^2\theta + b^2\sin^2\phi\sin^2\theta + c^2\cos^2\theta)}{2a^2b^2c^2(\cos^2\phi\sin^2\theta/a^2 + \sin^2\phi\sin^2\theta/b^2 + \cos^2\theta^2/c^2)^{3/2}} \\
&= \frac{(a^2(1 - \cos^2\phi\sin^2\theta) + b^2(1 - \sin^2\phi\sin^2\theta) + c^2(1 - \cos^2\theta))(abc)}{2(b^2c^2\cos^2\phi\sin^2\theta + a^2c^2\sin^2\phi\sin^2\theta + a^2b^2\cos^2\theta^2)^{3/2}} \\
&= \frac{(a^2(\sin^2\phi + \cos^2\phi - \cos^2\phi\sin^2\theta) + b^2(\cos^2\phi + \sin^2\phi - \sin^2\phi\sin^2\theta) + c^2\sin^2\theta)(abc)}{2(b^2c^2\cos^2\phi\sin^2\theta + a^2c^2\sin^2\phi\sin^2\theta + a^2b^2\cos^2\theta^2)^{3/2}} \\
&= \frac{(a^2(\sin^2\phi + \cos^2\phi\cos^2\theta) + b^2(\cos^2\phi + \sin^2\phi\cos^2\theta) + c^2\sin^2\theta)(abc)}{2(b^2c^2\cos^2\phi\sin^2\theta + a^2c^2\sin^2\phi\sin^2\theta + a^2b^2\cos^2\theta^2)^{3/2}} \\
&= m(\tilde{\mathbf{x}}(\theta, \phi))
\end{aligned}
$$

And,

$$
\begin{aligned}
k(\mathbf{x}) &= \frac{1}{\left(\text{abc}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right)^2} \\
&= \frac{1}{\left(abc(\frac{\cos^2\phi\sin^2\theta}{a^2} + \frac{\sin^2\phi\sin^2\theta}{b^2} + \frac{\cos^2\theta}{c^2})\right)^2} \\
& \left(\frac{abc}{c^2\sin^2\theta(b^2\cos^2\phi + a^2\sin^2\phi) + a^2b^2\cos^2\theta}\right)^2 \\
&= k(\tilde{\mathbf{x}}(\theta, \phi))
\end{aligned}
$$