HW 3 of ME5701. Sun Zhanhong. A0225282J

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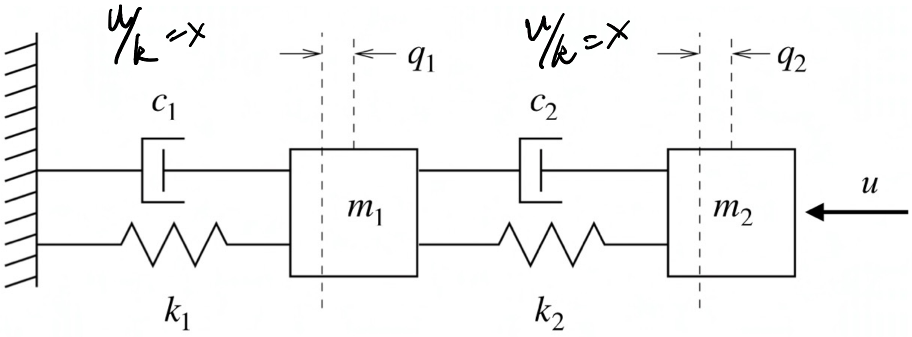
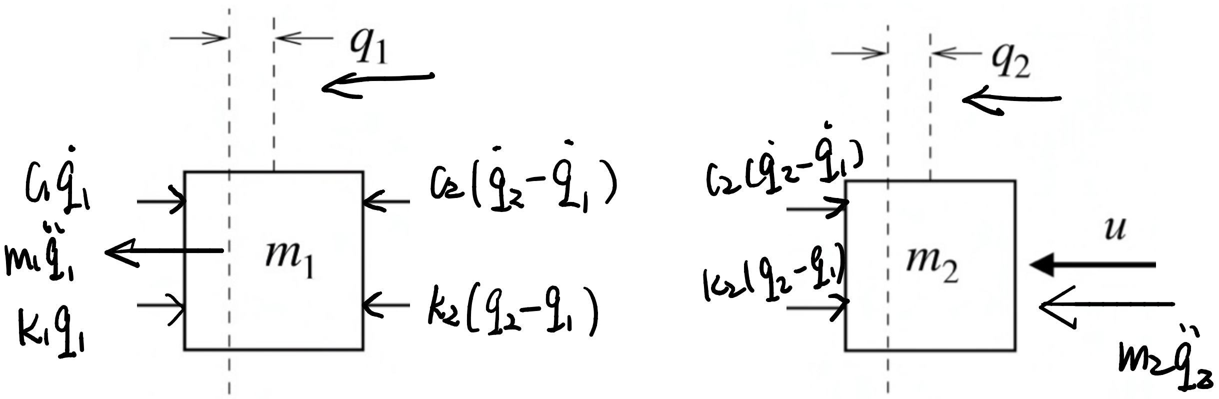


Given the above model of e.g. a suspension system:

1. Write the linear equations of motions
2. Write the corresponding linear dynamical system
3. Solve the linear dynamical system via ODE45 in Matlab
4. Discuss the stability of the system for different damping parameters c_1 and c_2
5. Discuss the observability of the system assuming that you can measure $q1$ and $q2$, $q1$ only and $q2$ only, respectively
6. Discuss the controllability of the system for different damping parameters c_1 and c_2
7. What happens to the controllability of the system if we apply the input u on the first mass only?
8. Write the transfer function of the system and its impulse response in Matlab

1. First draw the free body diagram of two masses



eqn of motion of m_1:

$$m_1\ddot{q}_1 - k_1 q_1 + k_2(q_2-q_1) + c_2(\dot{q}_2-\dot{q}_1) - c_1\dot{q}_1 = 0$$

eqn of motion of m_2:

$$m_2\ddot{q}_2 + u = k_2(q_2-q_1) + c_2(\dot{q}_2-\dot{q}_1)$$

simplify =>

(direction setted differently)

$$\begin{cases} m_1\ddot{q}_1 - (c_1+c_2)\dot{q}_1 + c_2\dot{q}_2 - (k_1+k_2)q_1 + k_2 q_2 = 0 \\ m_2\ddot{q}_2 + c_2\dot{q}_1 - c_2\dot{q}_2 + k_2 q_1 - k_2 q_2 = -u \end{cases}$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} -c_1-c_2 & c_2 \\ c_2 & -c_2 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} -k_1-k_2 & k_2 \\ k_2 & -k_2 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} u$$

Set $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ q_1 \\ q_2 \end{pmatrix}$

then. $\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$

$$\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} \frac{-c_1-c_2}{m_1} & \frac{c_2}{m_1} \\ \frac{c_2}{m_2} & \frac{-c_2}{m_2} \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} + \begin{pmatrix} \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{k_2}{m_2} & \frac{-k_2}{m_2} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{-1}{m_2} \end{pmatrix} u$$

So the system is:

$$4\times1 = 4\times4 * 4\times1 + 4\times1 * 1\times1$$

$$2\times1 = 2\times4 * 4\times1$$

$$\dot{x} = Ax + Bu$$

$$y = Cx,$$

where,

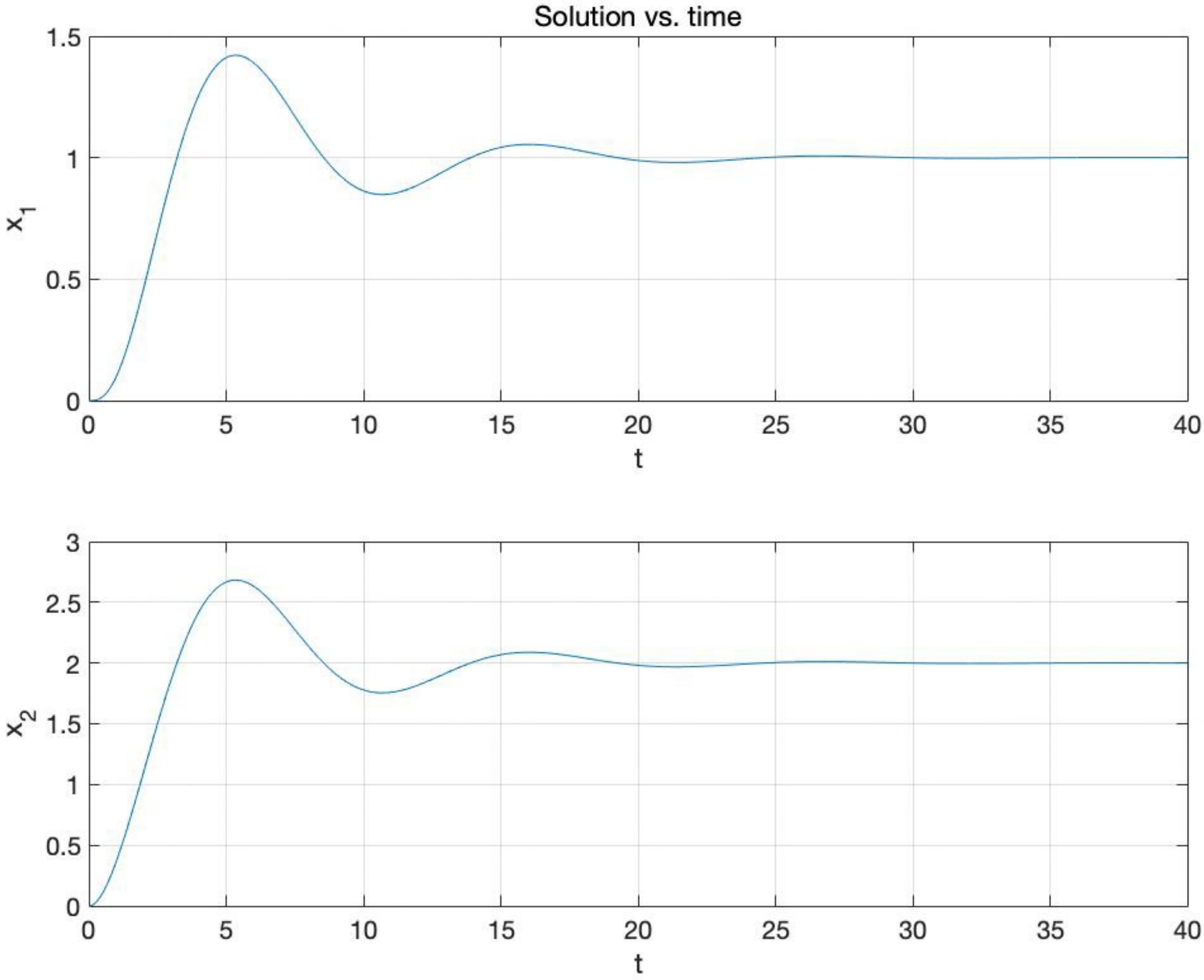
$$A = \begin{bmatrix} \frac{-c_1-c_2}{m_1} & \frac{c_2}{m_1} & \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{c_2}{m_2} & \frac{-c_2}{m_2} & \frac{k_2}{m_2} & \frac{-k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \frac{-1}{m_2} \\ 0 \\ 0 \end{bmatrix}$$

(X set different (y))

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. graph: (code attached)



4. stability $\rightarrow c_1, c_2$.

$$A = \begin{bmatrix} \frac{-c_1-c_2}{m_1} & \frac{c_2}{m_1} & \frac{-k_1-k_2}{m_1} & \frac{k_2}{m_1} \\ \frac{c_2}{m_2} & \frac{-c_2}{m_2} & \frac{k_2}{m_2} & \frac{-k_2}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

As known, the stability depends on monotomic part, in other words, on real part of eigenvalues of A.

$$|\lambda I - A| = \begin{vmatrix} \lambda + \frac{c_1+c_2}{m_1} & \frac{-c_2}{m_1} & \frac{k_1+k_2}{m_1} & \frac{-k_2}{m_1} \\ \frac{-c_2}{m_2} & \lambda + \frac{c_2}{m_2} & \frac{-k_2}{m_2} & \frac{k_2}{m_2} \\ -1 & 0 & \lambda & 0 \\ 0 & -1 & 0 & \lambda \end{vmatrix} = 0$$

This eqn relys strongly of c_1 and c_2.

By solving this eqn and discuss c_1 and c_2, we can know the solns (stability) of system

should discuss with symbols.

5. ① when wanna measure q_1 and q_2,

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

assume $c_1, c_2, m_1, m_2, k_1, k_2$ are all 1.

$$A = \begin{bmatrix} -2 & 1 & -2 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C = CA = CA^2 = CA^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 2$$

So, this system is not observable

If only outputs q_1,

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 1$$

So the system is not observable

If only observes q_2.

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\operatorname{rank}\left(\begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}\right) = 1$$

So the system is not observable.

6. For the controllability,

we consider the matrix

$[B \quad AB \quad A^2B \quad A^3B]$.

if the rank of this matrix $= 4$,

it is controllable.

if not,

it is not controllable.

7. If u on first mass only,

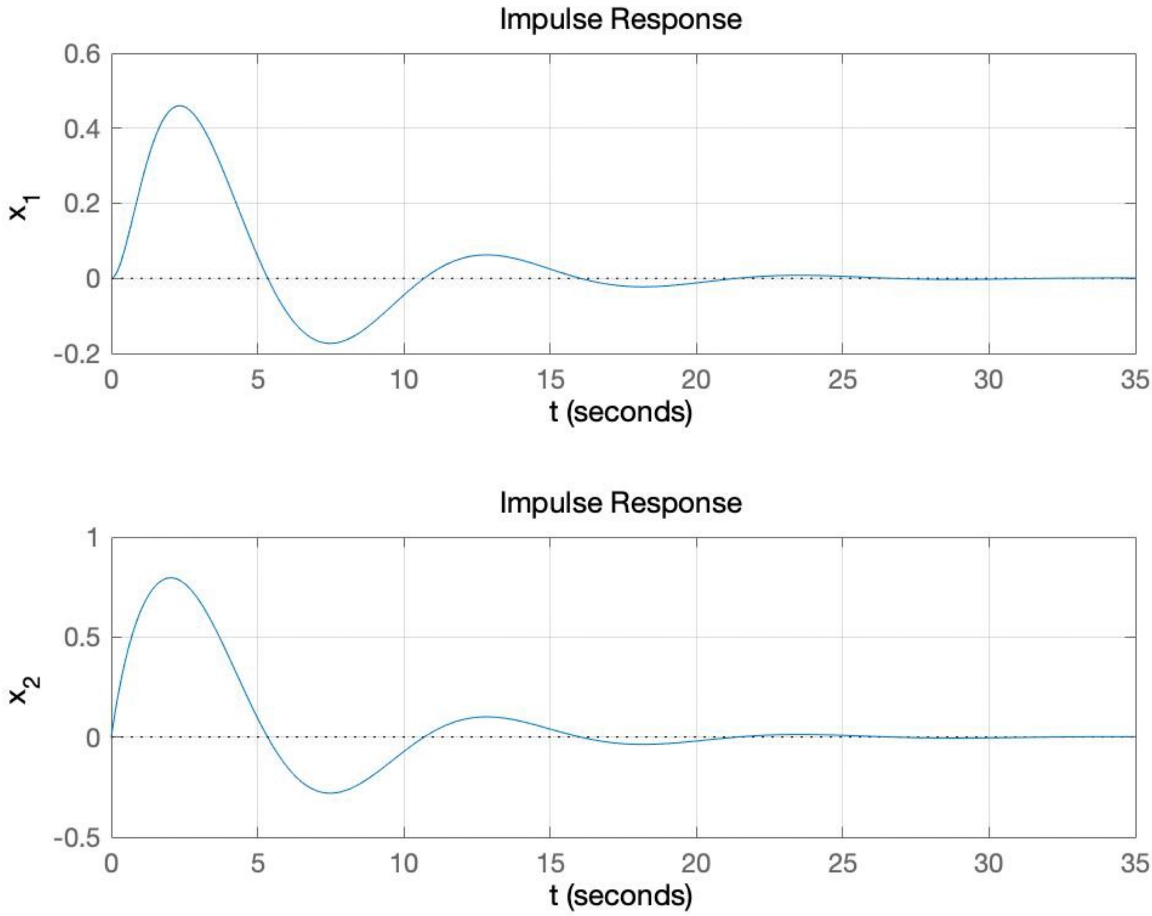
B changes into: $\begin{bmatrix} -1/m_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

So we should reconsider the controlability

matrix $[B \quad AB \quad A^2B \quad A^3B]$ of its

rank. $\begin{cases} \text{if } = 4 \Rightarrow \text{controlable} \\ \text{if } \neq 4 \Rightarrow \text{not controlable} \end{cases}$

8. result followed



Result transfer function of question 8

```
G =

  From input to output...
                -s - 1
   1:  -----------------------------
       s^4 + 3 s^3 + 4 s^2 + 2 s + 1

            -s^2 - 2 s - 2
   2:  -----------------------------
       s^4 + 3 s^3 + 4 s^2 + 2 s + 1

Continuous-time transfer function.
```

Appendix: Code

```
clc
clear
close all
%%
T0    = 0;         % initial time
TN    = 40;        % final time
dt    = 0.01;      % time-step
tspan = T0:dt:TN;  % time set
x1    = 0;         % initial condition for x1
x2    = 0;         % initial condition for x2
x3    = 0;
x4    = 0;
c1 = 1;
c2 = 1;
k1 = 1;
k2 = 1;
m1 = 1;
m2 = 1;
u = 1;
options = odeset('RelTol',1e-12,'AbsTol',1e-12);


%% Construct the right-hand side
rhs_mck = @(t,x)[
   ((-c2-c1)/m1) * x(1) +  (c2/m1) * x(2) +  ((-k1-k2)/m1) * x(3) +  (k2/m1) * x(4)           ; ...
        (c2/m2) * x(1) + (-c2/m2) * x(2) +       (k2/m2) * x(3) + (-k2/m2) * x(4) + -1/m2*u ; ...
                x(1)                                                        ; ...
                              x(2)                                          ; ...
];

%% Solve the system via the ODE package
[t,sol] = ode45(rhs_mck,tspan,[x1 x2 x3 x4]);
x1_s = -sol(:,3);
x2_s = -sol(:,4);

%% Plot results
figure()
subplot(2,1,1); plot(t,x1_s) ; grid on;  xlabel('t'); ylabel('x_1'); title('Solution vs. time')
subplot(2,1,2); plot(t,x2_s) ; grid on;  xlabel('t'); ylabel('x_2')

A = [
   ((-c2-c1)/m1) ,  (c2/m1) ,  ((-k1-k2)/m1) ,  (k2/m1) ; ...
       (c2/m2)  , (-c2/m2) ,       (k2/m2)  , (-k2/m2) ; ...
        1       ,      0   ,          0     ,    0     ; ...
        0       ,      1   ,          0     ,    0     ; ...
   ];
B = [0;-1/m2;0;0];
C = [0,0,1,0;0,0,0,1];
D = [0;0];

[b1,a1] = ss2tf(A,B,C,D);
G1 = tf(b1(1,:),a1);
G2 = tf(b1(2,:),a1);
G = [G1;G2]

figure()
subplot(2,1,1); impulse(-G1) ; grid on; xlabel('t'); ylabel('x_1'); title('Impulse Response')
subplot(2,1,2); impulse(-G2) ; grid on; xlabel('t'); ylabel('x_2')
```