Homework 6: ME5701, 19 September 2020

1. Given two fair (unbiased) dice, let the state of die 1 be called X and the state of die 2 be called Y. The discrete probability distribution for each one is of the form

$$p_X(n) = \frac{1}{6}\sum_{i=1}^{6}\delta_{i,n} \text{ and } p_Y(n) = \frac{1}{6}\sum_{i=1}^{6}\delta_{i,n}$$

(where δ_{ij} is the Kronecker delta function, equal to 1 if $i = j$ and zero otherwise). Since the state of each die is independent of the other, the joint probability of X and Y is

$$p_{X,Y}(m,n) = p_X(m)p_Y(n)\,.$$

Using the above facts, do the following:
a) Find the conditional distribution for X given Y;
b) Work out the details of the probability distribution for $X+Y$ by considering all possible combinations of values of X and Y;
c) Compute the result in (b) by directly using the convolution formula

$$(p_X * p_Y)(n) = \sum_m p_X(m)p_Y(n-m)$$

d) Re-compute the result in (c) using the DFT and inverse DFT built in Matlab

2. Given a Gaussian distribution on $\mathbb{R}^n$ where $n = n_1 + n_2$, prove (2.27) in Vol 1.

3. Given a Gaussian distribution on $\mathbb{R}^n$ where $n = n_1 + n_2$, prove (2.29) in Vol 1.

4. Using properties of Gaussian integrals, a) show by direct calculation that the convolution of two one-dimensional Gaussians has the property that

$$\mu_{1*2} = \mu_1 + \mu_2 \text{ and } \sigma^2_{1*2} = \sigma_1^2 + \sigma_2^2$$

(Even though this is a nonparametric result which does not depend on the probability densities being Gaussian, do it for the specific case of Gaussians);
b) Show the same thing by calculating the Fourier transforms of the two 1-D Gaussians and computing their convolution by the convolution theorem.

5. Consider the ramp-like function on the domain $[0,1]$ of the form

$$f(x) = ax$$

where a is a positive real number.
a) In order for this to be a probability density function on the domain $[0,1]$, what must the value of a be ?
b) Compute the cumulative distribution function for the resulting pdf;
c) Write a short program to implement the ITM method to randomly sample from this pdf;

d) Create a histogram of the samples generated (normalized by the total number of samples) and plot it together with your pdf.

6. Using the reasoning behind Liapunov's Direct Method, reason about the stability of the following systems:
a)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

b) $\ddot{x} - x^2\dot{x} + x^3 = 0\,.$