HW4 of ME5701. Sun Zhanhong A0225282J

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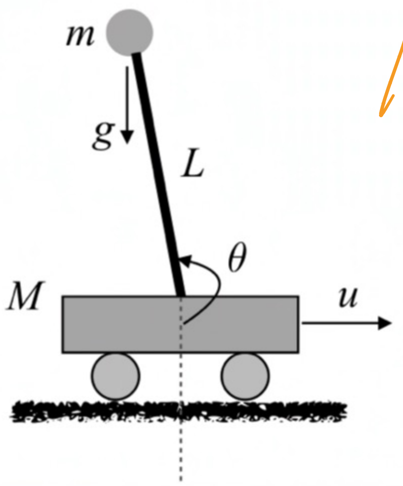


Given the above simplified model of e.g. a Segway, along with the equations of motions and parameters provided in the next slide:

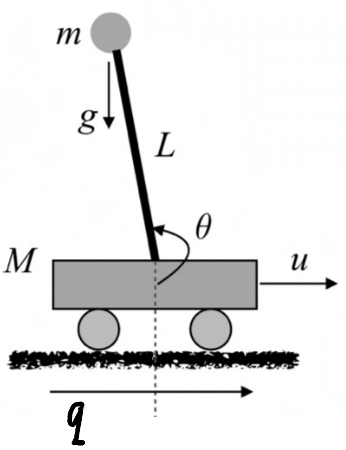
1. Linearize the equations around the pendulum in the upward position ($\theta = \pi$)
2. Write the corresponding linear dynamical system
3. Write the transfer function
4. Solve the linear dynamical system via
    1. Impulse response in Matlab
    2. ODE45 in Matlab
5. Discuss the stability of the system
6. Design an optimal control (LQR) for the system in Matlab

Cart

$$mL^2(M + m(1 - \cos(\theta)^2)\ddot{q} + m^2L^2 g\cos(\theta)\sin(\theta) + \dots$$
$$- mL^2(mL\dot{\theta}^2\sin(\theta) - c\dot{q}) = mL^2 u$$

Pendulum

$$mL^2(M + m(1 - cos(\theta)^2)\ddot{\theta} + mL\cos(\theta)(mL\dot{\theta}^2\sin(\theta) - c\dot{q}) + \dots$$
$$- (m + M)mgL\sin(\theta) = mL\cos(\theta)u$$



q = position of the cart
θ = angle of the pendulum
M = mass of the cart = **5**
m = mass at the pendulum tip = **2**
g = gravity = **-9.81**
L = length of the pendulum = **2**
c = damping on the cart = **0.1**
u = input

1. Linearize the eqn around $\theta = \pi$

① set $\underline{x}$, calculate $\frac{d\underline{x}}{dt}$.

$$\cancel{mL^2}(M + m(1-\cos(\theta)^2)\ddot{q} + m^{\cancel{2}}\cancel{L^2}g\cos(\theta)\sin(\theta) + \ldots$$
$$- \cancel{mL^2}(mL\dot{\theta}^2\sin(\theta) - c\dot{q}) = \cancel{mL^2}u$$

$$\cancel{m}L^{\cancel{2}}(M + m(1-cos(\theta)^2)\ddot{\theta} + \cancel{mL}\cos(\theta)(mL\dot{\theta}^2\sin(\theta) - c\dot{q}) + \ldots$$
$$-(m+M)\cancel{m}g\cancel{L}\sin(\theta) = \cancel{mL}\cos(\theta)u$$

set $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q \\ \dot{q} \\ \theta \\ \dot{\theta} \end{bmatrix}$

then $\frac{d\underline{x}}{dt} = \begin{bmatrix} \dot{q} \\ \ddot{q} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \underline{\Phi}(\underline{X})$

$$= \begin{bmatrix} X_2 \\ (mg\cos X_3\sin X_3 + mLX_4^2\sin X_3 - cX_2 + u) / [M + m(1-\cos^2 X_3)] \\ X_4 \\ (m+M)g\sin X_3 - \cos X_3(mLX_4^2\sin X_3 - cX_2) + u\cos X_3 / L(M + m(1-\cos^2 X_3)) \end{bmatrix}$$

② The fixed point to be calculated is:

q fixed, $\theta = \pi$, $\dot{q} = 0$, $\dot{\theta} = 0$

③ Find Jacobian: (set $u=0$)

Find analytical expression → plug in fixed points

As calculated by matlab, (code attached)

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.02 & 3.924 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.01 & 6.867 & 0 \end{bmatrix}$$

Therefore,

$$\begin{cases} \ddot{q} = -0.02\,\dot{q} + 3.924\,\theta + 0.2u \\ \ddot{\theta} = -0.01\,\dot{q} + 6.867\,\theta - 0.1u \end{cases}$$

2. Linear dynamic system

so the system becomes:

$$\begin{bmatrix} \dot{q} \\ \ddot{q} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.02 & 3.924 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.01 & 6.867 & 0 \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{ML} \end{bmatrix} u$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{ML} \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ -0.1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

3. Transfer function

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

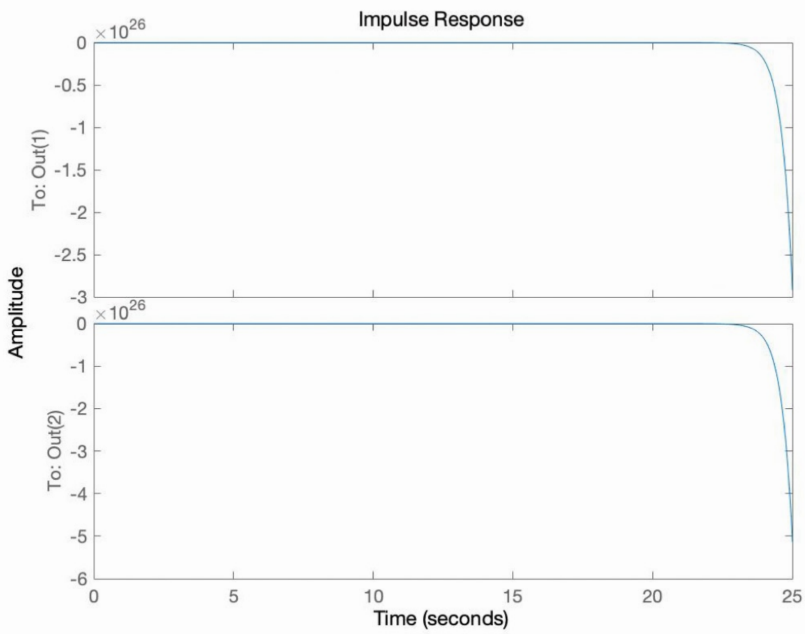
$$G(s) = \left[C(sI-A)^{-1} B + D \right] \quad \rightarrow D = 0$$

As calculated in Matlab,

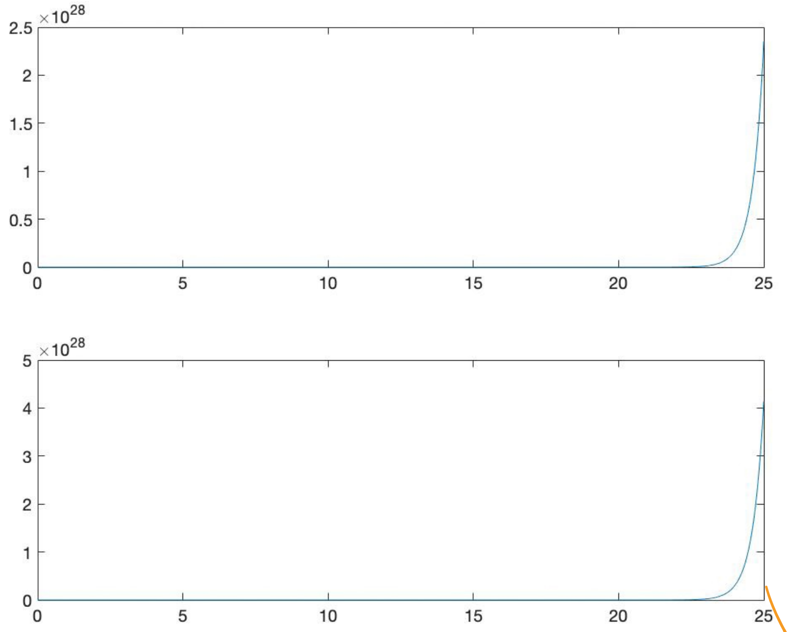
$$G(s) = \begin{bmatrix} \dfrac{0.2 s^6 + 0.004 s^5 - 3.139 s^4 - 0.05494 s^3 + 12.13 s^2 + 0.1732 s - 1.234e{-18}}{s^8 + 0.04 s^7 - 13.73 s^6 - 0.4709 s^5 + 47.15 s^4 + 1.347 s^3 + 0.009624 s^2 - 1.28e{-19} s + 4.189e{-37}} \\ \\ \dfrac{-0.1 s^4 - 0.006 s^3 + 0.6866 s^2 + 0.03728 s + 0.0003924}{s^6 + 0.04 s^5 - 13.73 s^4 - 0.4709 s^3 + 47.15 s^2 + 1.347 s + 0.009624} \end{bmatrix}$$

4.

Impulse response:



ODE result:



5. If we calculate the eigenvalues of A, we get

```
>> eig(A)

ans =

         0
   -0.0143
   -2.6234
    2.6177
```

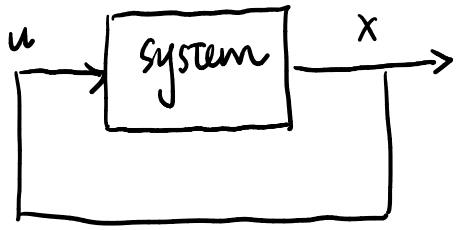
-0.0143, -2.6234 — stable

2.6177 — unstable

There exists unstable eigenvalues, therefore the system is unstable

6. Set $u = -Kx$

$$\Rightarrow \dot{x} = (A-BK)x$$




```
>> rank(ctrb(A,B))

ans =

     4
```
$= n$

First we should judge the controlability of the system. We found that the system is controlability

```
>> eig(A)

ans =

         0
   -0.0143
   -2.6234
    2.6177
```
$0 \to \times$, -0.0143 ✓, -2.6234 ✓, $2.6177 \to \times$

for now, if we calculate the eigenvalues of A, we'll find there exists unstable poles.

We use LQR

We can take $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 100 \end{bmatrix}$, $R = 0.001$

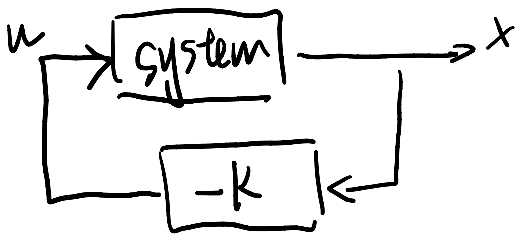
and use lqr. We can get the optimal K.

```
K =

   1.0e+03 *

   -0.0316   -0.0661   -1.0778   -0.4850
```

The optimal controlled:



Appendix : Matlab Code

```
%% The differential equation
function dx = cartpend(x,m,M,L,g,c,u)


dx(1,1) = x(2);
dx(2,1) = -1/50*x(2)+981/250*x(3);
dx(3,1) = x(4);
dx(4,1) = -0.01*x(2)+6867/1000*x(3);






clear all, close all, clc


%% Calculate A
syms x1 x2 x3 x4 m M g L u c D Sy Cy
Sy = sin(x3);
Cy = cos(x3);
D = m*L*L*(M+m*(1-Cy^2));
phi = [x2;...
(1/D)*(-m^2*L^2*g*Cy*Sy + m*L^2*(m*L*x4^2*Sy - c*x2) + m*L*L*u);...
x4;...
(1/D)*((m+M)*m*g*L*Sy - m*L*Cy*(m*L*x4^2*Sy - c*x2) - m*L*Cy*u)];


x = [x1,x2,x3,x4];
J = jacobian(phi,x);
J_pri = subs(J,[m M g L c u x2 x3 x4],[2,5,-9.81,2,0.1,0,0,pi,0])
A = J_pri

%% Calculate transfer function
% input A, B and C
A = [0 1 0 0;...
   0 -0.02 981/250 0;...
   0 0 0 1;...
   0 -1/100 6867/1000 0];
B = [0; 0.2; 0; -0.1];
C = [0,0,1,0;0,0,0,1];

% output eigenvalues of A to discuss the stability of system
eig(A)
% output rank of controllability matrix to discuss the controllability of
% system
rank(ctrb(A,B))

% build transfer function
s = tf('s');
G = simplify([1,0,0,0;0,0,1,0]*inv([s,0,0,0;0,s,0,0;0,0,s,0;...
   0,0,0,s]-[0,1,0,0;0,-0.02,3.924,0;0,0,0,1;...
   0,-0.01,6.867,0])*[0;0.2;0;-0.1]);

%% Impulse response
figure();
impulse(G);


%% solve by ODE
```

```
tspan = 0:.0005:25;
% if(s==-1)
%     y0 = [0; 0; 0; 1.5];
%     [yL,t,xL] = initial(sys,y0,tspan);
%     [t,yNL] = ode45(@(t,y)cartpend(y,m,M,L,g,d,0),tspan,y0);
% elseif(s==1)
    y0 = [0; 0; pi-.001; 0];
%     [yL,t,xL] = initial(sys,y0-[0; 0; pi; 0],tspan);
    [t,yNL] = ode45(@(t,y)cartpend(y,m,M,L,g,c,0),tspan,y0);
% else
% end
% figure;
figure(1);
title('ODE solver result')
xlabel('Times (seconds)')
subplot(2,1,1)
plot(t,yNL(:,1))
hold on
subplot(2,1,2)
plot(t,yNL(:,3))
hold off
% figure
% plot(t,yL);
% plot(t,yL+ones(10001,1)*[0; 0; pi; 0]');
% hold on
% plot(t,yNL);

%% (show movie)
figure(2);
for k=1:100:length(t)
    drawcartpend_bw(yNL(k,:),m,M,L);
end

%% optimal controller

Q = [1 0 0 0;...
     0 1 0 0;...
     0 0 10 0;...
     0 0 0 100];
 R = 0.001;
 K = lqr(A,B,Q,R)
```