Homework 10: ME5701, 25 October 2020. Due 1 November 2020

Problem 1.

An *orthogonal matrix*, R, is a matrix with the property that $R^T R = RR^T = \mathbb{I}$. A *rotation matrix* is a "special" orthogonal matrix with the extra condition that its determinant is equal to $+1$. The set of all $n \times n$ matrices with these two conditions is called $SO(n)$.

(a) Show by direct calculation that the following matrices are rotation matrices:

$$R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

R_1,R_2, and R_3 given above have the physical interpretation of rotations by the angle ϕ about the x, y, and z axes respectively. Based on physical intuition or mathematical calculation, determine if the following are true or false for $\phi \in (0, \pi)$ (i.e., the open interval not containing 0 or π):

(b) $R_1(\phi)R_2(\phi) = R_2(\phi)R_1(\phi)$
(c) $R_1(\phi)R_3(\phi) = R_3(\phi)R_1(\phi)$
(d) $R_2(\phi)R_3(\phi) = R_3(\phi)R_2(\phi)$
(e) $R_1(\phi)R_1(\phi) = R_1(2\phi)$
(f) $R_2(\phi)R_2(-\phi) = \mathbb{I}$

NOTE: A rotation matrix which *rotates* a vector with respect to a fixed reference frame is the same as the matrix which *describes* the rotated vector with respect to the fixed reference frame. That is, if the vector $\mathbf{x}_0$ is rotated relative to a fixed frame by the matrix R, then the rotated vector is represented as $\mathbf{x} = R\mathbf{x}_0$ in the fixed reference frame.

Problem 2.

Let $x_1, x_2, x_3 \in \mathbb{R}$ and

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} = -X^T.$$

The set of all such matrices forms a vectorspace (called $so(3)$). Exponentiating matrices $X \in so(3)$ produces matrices $\exp(X) \in SO(3)$. If the $\vee$ operator is defined such that

$$X^\vee = \mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

prove for any $R \in SO(3)$, regardless of whether or not $R = \exp(X)$, that

$$(RXR^T)^\vee = R\mathbf{x}.$$

Hint: divide R and R^T into columns and rows.

Problem 3.

a) If $\mathbf{x}, \mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$, prove or disprove the following:

$$\|\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}\| = \|\mathbf{n} \times \mathbf{x}\|$$

b) If $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$, and $N = -N^T \in \mathbb{R}^{3\times 3}$ is defined by the condition $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^3$, let

$$R(\theta, \mathbf{n}) = I + \sin\theta N + (1 - \cos\theta)N^2$$

be the rotation matrix from the Rodrigues formula. Show that

$$R(\theta, \mathbf{n}) = Q\, R_3(\theta)\, Q^T$$

where $Q = [\mathbf{a}, \mathbf{b}, \mathbf{n}]$ and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary except for the fact that $Q \in SO(3)$.

Problem 4.

Recall the notation $N^\vee = \mathbf{n}$. Given a rotation matrix $R(\mathbf{q})$ parameterized with $\mathbf{q} = [q_1, q_2, q_3]^T$, the 'left' and 'right' Jacobians for $SO(3)$ are

$$J_l = \left[\left(\frac{\partial g}{\partial q_1} g^{-1}\right)^\vee, \left(\frac{\partial g}{\partial q_2} g^{-1}\right)^\vee, \left(\frac{\partial g}{\partial q_3} g^{-1}\right)^\vee \right]$$

and

$$J_r = \left[\left(g^{-1}\frac{\partial g}{\partial q_1}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial q_2}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial q_3}\right)^\vee \right].$$

Show that in general

$$J_l = RJ_r$$

and in the special case when $[q_1, q_2, q_3] = [\alpha, \beta, \gamma]$ are the ZXZ Euler angles, explicitly compute J_l and J_r.