9.9

ME5701: Homework #6

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Problem 1

Given two fair (unbiased) dice, let the state of die 1 be called X and the state of die 2 be called Y . The discrete probability distribution for each one is of the form

$$p_X(n) = \frac{1}{6}\sum_{i=1}^{6}\delta_{i,n} \text{ and } p_Y(n) = \frac{1}{6}\sum_{i=1}^{6}\delta_{i,n}$$

(where δ_{ij} is the Kronecker delta function, equal to 1 if $i = j$ and zero otherwise). Since the state of each die is independent of the other, the joint probability of X and Y is

$$p_{X,Y}(m,n) = p_X(m)p_Y(n)$$

Using the above facts, do the following:
a) Find the conditional distribution for X given Y ;
b) Work out the details of the probability distribution for $X + Y$ by considering all possible combinations of values of X and Y ;
c) Compute the result in (b) by directly using the convolution formula

$$(p_X * p_Y)(n) = \sum_m p_X(m)p_Y(n-m)$$

d) Re-compute the result in (c) using the DFT and inverse DFT built in Matlab

Solution
a)

$$p_{X|Y}(m) = \frac{p_{X,Y}(m,n)}{p_Y(n)} = P_X(m) = \frac{1}{6}\sum_{i=1}^{6}\delta_{i,m}$$

b)

Problem 1 continued on next page...

	$X+Y$	$PossibleX$	$PossibleY$	$\#Possibilities$
1	2	1	1	1
2	3	1	2	
3	3	2	1	2
4	4	1	3	
5	4	2	2	
6	4	3	1	3
7	5	1	4	
8	5	2	3	
9	5	3	2	
10	5	4	1	4
11	6	1	5	
12	6	2	4	
13	6	3	3	
14	6	4	2	
15	6	5	1	5
16	7	1	6	
17	7	2	5	
18	7	3	4	
19	7	4	3	
20	7	5	2	
21	7	6	1	6
22	8	2	6	
23	8	3	5	
24	8	4	4	
25	8	5	3	
26	8	6	2	5
27	9	3	6	
28	9	4	5	
29	9	5	4	
30	9	6	3	4
31	10	4	6	
32	10	5	5	
33	10	6	4	3
34	11	5	6	
35	11	6	5	2
36	12	6	6	1

So we know $X+Y$ with probability density distribution. The total number of possibilities is 36.

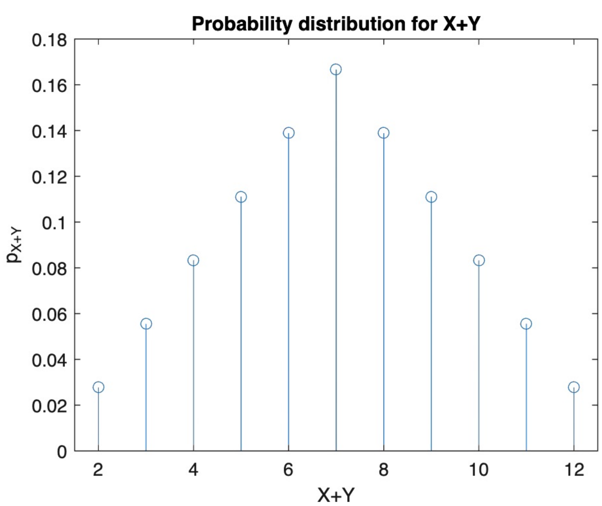
Problem 1 continued on next page. . .

$$P_{X+Y}(n) = \begin{cases} \frac{1}{36} & \text{for} \quad n = 2 \\ \frac{1}{18} & \text{for} \quad n = 3 \\ \frac{1}{12} & \text{for} \quad n = 4 \\ \frac{1}{9} & \text{for} \quad n = 5 \\ \frac{5}{36} & \text{for} \quad n = 6 \\ \frac{1}{6} & \text{for} \quad n = 7 \\ \frac{5}{36} & \text{for} \quad n = 8 \\ \frac{1}{9} & \text{for} \quad n = 9 \\ \frac{1}{12} & \text{for} \quad n = 10 \\ \frac{1}{18} & \text{for} \quad n = 11 \\ \frac{1}{36} & \text{for} \quad n = 12 \end{cases}$$

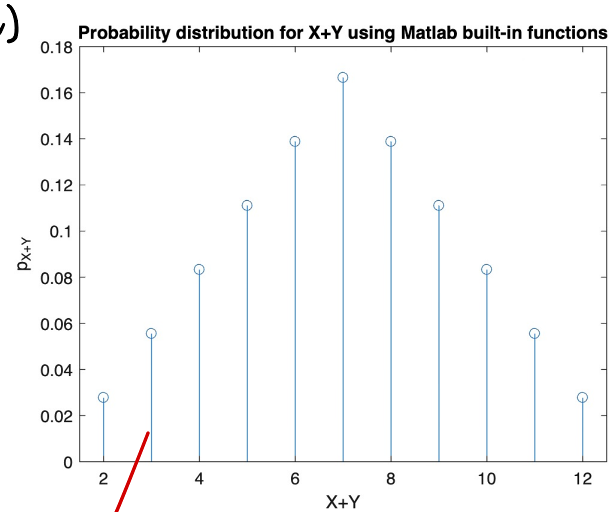
c)

$$p_X(n) = \frac{1}{6}\sum_{i=1}^{6} \delta_{i,n} \text{ and } p_Y(n) = \frac{1}{6}\sum_{i=1}^{6} \delta_{i,n}$$

$$(p_X * p_Y)(n) = \sum_m p_X(m) p_Y(n-m)$$



d)



code attached

Problem 2

Given a Gaussian distribution on $\mathbb{R}^n$ where $n = n_1 + n_2$, prove

$$\int_{\mathbb{R}^{n_2}} \rho\left(\left[\mathbf{x}_1^T, \mathbf{x}_2^T\right]^T; \boldsymbol{\mu}, \Sigma\right) d\mathbf{x}_2 = \rho\left(\mathbf{x}_1; \boldsymbol{\mu}_1, \Sigma_{11}\right)$$

Proof

$$\rho\left(\left[\mathbf{x}_1^T, \mathbf{x}_2^T\right]^T; \boldsymbol{\mu}, \Sigma\right) = \frac{1}{(2\pi)^{n/2}|det(\Sigma)|^{1/2}} exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\}$$

Assume

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

,
and

$$\Sigma^{-1} = \begin{bmatrix} \Sigma_a & \Sigma_b \\ \Sigma_c & \Sigma_d \end{bmatrix}$$

. We can get the relation between the entries of the two matrices:

$$\begin{aligned}
\Sigma_a &= \left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^{\top}\right)^{-1} = \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}\left(\Sigma_{22} - \Sigma_{12}^{\top}\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1}\Sigma_{12}^{\top}\Sigma_{11}^{-}{-1} \\
\Sigma_d &= \left(\Sigma_{22} - \Sigma_{12}^{T}\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1} = \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{12}^{T}\left(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^{T}\right)^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\
\Sigma_b &= -\Sigma_{11}^{-1}\Sigma_{12}\left(\Sigma_{22} - \Sigma_{12}^{T}\Sigma_{11}^{-1}\Sigma_{12}\right)^{-1} = (\Sigma_c)^T
\end{aligned}$$

Then we consider $(x-\mu)^T\Sigma^{-1}(x-\mu)$

$$\begin{aligned}
(x-\mu)^{\top}\Sigma^{-1}(x-\mu) &= \left[(x_1-\mu_1)^{\top}, (x_2-\mu_2)^{\top}\right]\left[\begin{array}{cc} \Sigma_a & \Sigma_b \\ \Sigma_c & \Sigma_d \end{array}\right]\left[\begin{array}{c} x_1-\mu_1 \\ x_2-\mu_2 \end{array}\right] \\
&= (x_1-\mu_1)^T\Sigma_a(x_1-\mu_1) + (x_1-\mu_1)^T\Sigma_b(x_2-\mu_2) \\
&\quad + (x_2-\mu_2)^T\Sigma_c(x_1-\mu_1) + (x_2-\mu_2)^T\Sigma_d(x_2-\mu_2)
\end{aligned}$$

We can assume $x_1 - \mu_1 = u$ and $x_2 - \mu_2 = v$. Then,

$$\begin{aligned}
(x-\mu)^T\Sigma^{-1}(x-\mu) = u^T\Sigma_{11}^{-1}u &+ u^T\Sigma_{11}^{-1}\Sigma_{12}(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}\Sigma_{12}^T\Sigma_{11}^{-1}u \\
&- 2\, u^T\Sigma_{11}^{-1}\Sigma_{12}(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}v \\
&+ v^T(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}v
\end{aligned}$$

Assume $\Sigma_{12}^T\Sigma_{11}^{-1}u = w$, thus $u^T\Sigma_{11}^{-1}\Sigma_{12} = w^T$. Then assume $\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12} = A$. Therefore,

$$\begin{aligned}
(x-\mu)^T\Sigma^{-1}(x-\mu) &= u^T\Sigma_{11}^{-1}u + w^TA^{-1}w - 2w^TA^{-1}v + v^TA^{-1}v \\
&= u^T\Sigma_{11}^{-1}u + (v-w)^TA^{-1}(v-w)
\end{aligned}$$

Then we consider $det(\Sigma)$. If

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$$

Then,

$$\begin{aligned}
det(\Sigma) &= det(\Sigma_{11})det(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12}) \\
&= det(\Sigma_{11})det(A^{-1})
\end{aligned}$$

Problem 2 continued on next page...

Therefore,

$$
\begin{aligned}
\rho(x;\mu,\Sigma) &= \frac{1}{(2\pi)^{n/2}|\det(\Sigma)|^{1/2}} exp\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\} \\
&= \frac{1}{(2\pi)^{n/2}|\det(\Sigma_{11})|^{1/2}|det(A^{-1})|^{1/2}} exp\{-\frac{1}{2}[u^T\Sigma_{11}^{-1}u + (v-w)^T A^{-1}(v-w)]\} \\
&= \frac{1}{(2\pi)^{n_1/2}|\det(\Sigma_{11})|^{1/2}} exp\{-\frac{1}{2}u^T\Sigma_{11}^{-1}u\}\cdot \\
&\quad \frac{1}{(2\pi)^{n_2/2}|\det(A^{-1})|^{1/2}} exp\{-\frac{1}{2}(v-w)^T A^{-1}(v-w)\} \\
&= \rho(x_1;\mu_1,\Sigma_{11})\rho(x_2;\mu_2+\Sigma_{12}^T\Sigma_{11}^{-1}u, A^{-1})
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_{\mathbb{R}^{\ltimes_{\not\asymp}}} \rho([x_1^T, x_2^T]^T;\mu,\Sigma)dx_2 &= \int_{\mathbb{R}^{\ltimes_{\not\asymp}}} \rho(x_1;\mu_1,\Sigma_{11})\rho(x_2;\mu_2+\Sigma_{12}^T\Sigma_{11}^{-1}u, A^{-1})dx_2 \\
&= \rho(x_1;\mu_1,\Sigma_{11})\int_{\mathbb{R}^{\ltimes_{\not\asymp}}} \rho(x_2;\mu_2+\Sigma_{12}^T\Sigma_{11}^{-1}u, A^{-1})dx_2 \\
&= \rho(x_1;\mu_1,\Sigma_{11})
\end{aligned}
$$

Get proved

3. Given a Gaussian distribution on $\mathbb{R}^n$ where $n = n_1 + n_2$, prove (2.29) in Vol 1.

$$p\left([x_1^T, x_2^T]^T; \mu, \Sigma\right) = \frac{1}{(2\pi)^{n/2} |\det(\Sigma)|^{1/2}} \exp\left\{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)\right\}$$

Assume $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ $\quad \Sigma^{-1} = \begin{bmatrix} \Sigma_a & \Sigma_b \\ \Sigma_c & \Sigma_d \end{bmatrix}$

So
$$\begin{cases} \Sigma_a = \underline{(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}} = \Sigma_{11}^{-1} + \Sigma_{11}^{-1}\Sigma_{12}(\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1}\Sigma_{12}^T\Sigma_{11}^{-1} \\ \Sigma_d = (\Sigma_{22} - \Sigma_{12}^T\Sigma_{11}^{-1}\Sigma_{12})^{-1} = \underline{\Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{12}^T(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}\Sigma_{12}\Sigma_{22}^{-1}} \\ \Sigma_c = -\Sigma_{22}^{-1}\Sigma_{12}^T(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1} = (\Sigma_b)^T \end{cases}$$

Then we consider $(x-\mu)^T\Sigma^{-1}(x-\mu)$

$$(x-\mu)^T\Sigma^{-1}(x-\mu)$$

$$= [(x_1-\mu_1)^T, (x_2-\mu_2)^T] \begin{bmatrix} \Sigma_a & \Sigma_b \\ \Sigma_c & \Sigma_d \end{bmatrix} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

$$= (x_1-\mu_1)^T\Sigma_a(x_1-\mu_1) + (x_1-\mu_1)^T\Sigma_b(x_2-\mu_2) + (x_2-\mu_2)^T\Sigma_c(x_1-\mu_1) + (x_2-\mu_2)^T\Sigma_d(x_2-\mu_2)$$

(Assume $x_1 - \mu_1 = u \quad x_2 - \mu_2 = v$)

$$= v^T\Sigma_{22}^{-1}v + u^T(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}u - 2v^T\Sigma_{22}^{-1}\Sigma_{12}^T(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}u + v^T\Sigma_{22}^{-1}\Sigma_{12}^T(\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{12}^T)^{-1}\Sigma_{12}\Sigma_{22}^{-1}v$$

Assume $\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = B \quad (\Sigma_{12}^T = \Sigma_{21})$

$\Sigma_{12}\Sigma_{22}^{-1}v = t$

$$= v^T\Sigma_{22}^{-1}v + u^TB^{-1}u - 2t^TB^{-1}u + t^TB^{-1}t$$

$$= v^T\Sigma_{22}^{-1}v + (u-t)^TB^{-1}(u-t)$$

Similar to last question,

$$|\Sigma| = |\Sigma_{22}|\,|B^{-1}|$$

Therefore,

$$p([x_1^T, x_2^T]^T; \mu, \Sigma)$$

$$= \frac{1}{(2\pi)^{(n_1+n_2)/2}\left(\det(\Sigma_{22})\det(B^{-1})\right)^{1/2}} \exp\left\{-\frac{1}{2}\left(v^T \Sigma_{22}^{-1} v + (u-t)^T B^{-1} (u-t)\right)\right\}$$

$$= \frac{1}{(2\pi)^{n_1/2}(\det(B^{-1}))^{1/2}} \exp\left\{-\frac{1}{2}(u-t)^T B^{-1}(u-t)\right\} \frac{1}{(2\pi)^{n_2/2}(\det(\Sigma_{22}))^{1/2}} \exp\left\{-\frac{1}{2} v^T \Sigma_{22}^{-1} v\right\}$$

$$= \frac{1}{(2\pi)^{n_1/2}(\det(B^{-1}))^{1/2}} \exp\left\{-\frac{1}{2}(u-t)^T B^{-1}(u-t)\right\} p(x_2; \mu_2, \Sigma_{22})$$

Thus

$$p([x_1^T, x_2^T]^T; \mu, \Sigma) / p(x_2; \mu_2, \Sigma_{22})$$

$$= \frac{1}{(2\pi)^{n_1/2}(\det(B^{-1}))^{1/2}} \exp\left\{-\frac{1}{2}(u-t)^T B^{-1}(u-t)\right\}$$

$$= \frac{1}{(2\pi)^{n_1/2}\left(\det(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\right)^{1/2}} \exp\left\{-\frac{1}{2}\left[x_1-(\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2))\right]^T(\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})^{-1}\left[x_1-(\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2))\right]\right\}$$

$$= p(x_1; \mu_1+\Sigma_{12}\Sigma_{22}^{-1}(x_2-\mu_2), \Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

get proved ✓

4. Using properties of Gaussian integrals, a) show by direct calculation that the convolution of two one-dimensional Gaussians has the property that

$$\mu_{1*2} = \mu_1 + \mu_2 \text{ and } \sigma_{1*2}^2 = \sigma_1^2 + \sigma_2^2$$

(Even though this is a nonparametric result which does not depend on the probability densities being Gaussian, do it for the specific case of Gaussians); b) Show the same thing by calculating the Fourier transforms of the two 1-D Gaussians and computing their convolution by the convolution theorem.

a) Assume the two functions are:

$$f_1 = \rho(x;\mu_1,\sigma_1^2) = \frac{1}{\sqrt{2\pi}\,\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\} \qquad f_2 = \rho(x;\mu_2,\sigma_2^2) = \frac{1}{\sqrt{2\pi}\,\sigma_2}\exp\left\{-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right\}$$

Therefore,

$$(f_1 * f_2)(x) = \int_{-\infty}^{+\infty} f_1(\xi)\, f_2(x-\xi)\, d\xi$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{-\frac{(\xi-\mu_1)^2}{2\sigma_1^2} - \frac{(x-\xi-\mu_2)^2}{2\sigma_2^2}\right\} d\xi$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{-\frac{+\sigma_2^2(\xi-\mu_1)^2+\sigma_1^2(\xi+\mu_2-x)^2}{2\sigma_1^2\sigma_2^2}\right\} d\xi$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi\sigma_1\sigma_2}\exp\left\{\underbrace{-\frac{(\sigma_1^2+\sigma_2^2)\xi^2+(-2\sigma_2^2\mu_1+2\sigma_1^2(\mu_2-x))\xi+\sigma_2^2\mu_1^2+\sigma_1^2(\mu_2-x)^2}{2\sigma_1^2\sigma_2^2}}_{\sim}\right\} d\xi \quad \leftarrow \text{eqn 1}$$

We focus on the $\sim$ part.

Our goal is to seperate ξ out and form a pdf so that the integral of it is 1.

$$\sim = -\frac{(\sigma_1^2+\sigma_2^2)\xi^2+[2\sigma_1^2(\mu_2-x)-2\sigma_2^2\mu_1]\xi}{2\sigma_1^2\sigma_2^2} - \frac{\sigma_2^2\mu_1^2+\sigma_1^2(\mu_2-x)^2}{2\sigma_1^2\sigma_2^2}$$

$$= -\frac{\xi^2+2\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}(\mu_2-x)-\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\mu_1\right)\xi}{2\left(\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)} - \frac{\sigma_2^2\mu_1^2+\sigma_1^2(\mu_2-x)^2}{2\sigma_1^2\sigma_2^2}$$

$$= -\frac{\xi^2+2\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}(\mu_2-x)-\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\mu_1\right)\xi+\left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}(\mu_2-x)-\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\mu_1\right)^2}{2\left(\frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)}$$

$$-\frac{\sigma_2^2\mu_1^2+\sigma_1^2(\mu_2-x)^2-\frac{1}{\sigma_1^2+\sigma_2^2}\left(\sigma_1^2(\mu_2-x)-\sigma_2^2\mu_1\right)^2}{2\sigma_1^2\sigma_2^2}$$

Look at the $\sim$ part:

It is in form of $-\dfrac{(\xi-\mu_3)^2}{\sigma_3^2}$,

$$\mu_3 = \frac{-1}{\sigma_1^2+\sigma_2^2}\left(\sigma_1^2(\mu_2-x)-\sigma_2^2\mu_1\right), \quad \sigma_3 = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

So, $$\frac{1}{\sqrt{2\pi}\,\sigma_3} = \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{2\pi}\,\sigma_1\sigma_2}$$

Therefore eqn 1 can be further simplified into:

$$(f_1*f_2)(x) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}\exp\left\{-\underline{\frac{\sigma_2^2\mu_1^2+\sigma_1^2(\mu_2-x)^2-\frac{1}{\sigma_1^2+\sigma_2^2}\left(\sigma_1^2(\mu_2-x)-\sigma_2^2\mu_1\right)^2}{2\sigma_1^2\sigma_2^2}}\right\}\cdot\underbrace{\int \rho(\xi,\mu_3,\sigma_3^2)\,d\xi}_{=1}$$

Forcus on $\sim$ part

$$\sim = -\frac{1}{2}\left(\frac{\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2}\mu_1^2+\frac{\sigma_1^2+\sigma_2^2}{\sigma_2^2}(\mu_2-x)^2-\frac{\sigma_1^2}{\sigma_2^2}(\mu_2-x)^2-\frac{\sigma_2^2}{\sigma_1^2}\mu_1^2+2\mu_1(\mu_2-x)}{(\sigma_1^2+\sigma_2^2)}\right)$$

$$= -\frac{1}{2}\,\frac{\mu_1^2+(\mu_2-x)^2+2(\mu_2-x)\mu_1}{\sigma_1^2+\sigma_2^2}$$

$$= -\frac{1}{2}\,\frac{(x-(\mu_1+\mu_2))^2}{\sigma_1^2+\sigma_2^2}$$

Therefore,

$$(f_1*f_2)(x) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}}\exp\left\{-\frac{(x-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right\}$$

$$= \rho(x;\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)$$

get proved.

(b) Fourier transform of a gaussian:

$$[\mathcal{F}(f)](w) = \int_{-\infty}^{+\infty} f(x) e^{-iwx} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2} - iwx\right\} dx$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x^2-2\mu x+\mu^2+i2\sigma^2 wx}{2\sigma^2}\right\} dx$$

$$= -\frac{(x+i\sigma^2 w-\mu)^2+\mu^2-(i\sigma^2 w-\mu)^2}{2\sigma^2}$$

$$= -\frac{(x+i\sigma^2 w-\mu)^2}{2\sigma^2} - \frac{\mu^2-(i\sigma^2 w-\mu)^2}{2\sigma^2}$$

$$= \exp\left\{-\frac{\mu^2-(i\sigma^2 w-\mu)^2}{2\sigma^2}\right\} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x+i\sigma^2 w-\mu)^2}{2\sigma^2}\right\} dx}_{=1}$$

$$= \exp\left\{\frac{-2i\mu w-\sigma^2 w^2}{2}\right\}$$

Therefore $[\mathcal{F}(f_1)](w)[\mathcal{F}(f_2)](w) = \hat{f}_1(w)\hat{f}_2(w)$

$$= \exp\left\{\frac{-2i\mu_1 w-\sigma_1^2 w^2}{2} + \frac{-2i\mu_2 w-\sigma_2^2 w^2}{2}\right\}$$

$$= \exp\left\{-\frac{1}{2}\left[(\sigma_1^2+\sigma_2^2)w^2 + 2i(\mu_1+\mu_2)w\right]\right\}$$

$$f_1(x)*f_2(x) = [\mathcal{F}^{-1}(\hat{f}_1\hat{f}_2)](x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}_1(w)\hat{f}_2(w) e^{iwx} dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{-\frac{1}{2}\left[(\sigma_1^2+\sigma_2^2)w^2 + 2i(\mu_1+\mu_2-x)w\right]\right\} dw.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\left\{\frac{w^2+2\overbrace{\boxed{\frac{i(\mu_1+\mu_2-x)}{\sigma_1^2+\sigma_2^2}}}^{=\Delta} w+\Delta^2-\Delta^2}{-2\frac{1}{\sigma_1^2+\sigma_2^2}}\right\}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sqrt{\frac{1}{\sigma_1^2+\sigma_2^2}}} \exp\left\{ \frac{(w+\Delta)^2}{-2\frac{1}{\sigma_1^2+\sigma_2^2}}\right.}_{\int \sim dw = 1} - \left.\left(-\frac{1}{2}(\sigma_1^2+\sigma_2^2)\Delta^2 \right) \right\} dw$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \exp\left\{ - \frac{(x-(\mu_1+\mu_2))^2}{2\,\sigma_1^2+\sigma_2^2} \right\}$$

$$= \rho\,(x;\ \mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2)$$

get proved √

5. Consider the ramp-like function on the domain $[0, 1]$ of the form

$$f(x) = ax$$

where a is a positive real number.
a) In order for this to be a probability density function on the domain $[0, 1]$, what must the value of a be ?
b) Compute the cumulative distribution function for the resulting pdf;
c) Write a short program to implement the ITM method to randomly sample from this pdf;
d) Create a histogram of the samples generated (normalized by the total number of samples) and plot it together with your pdf.

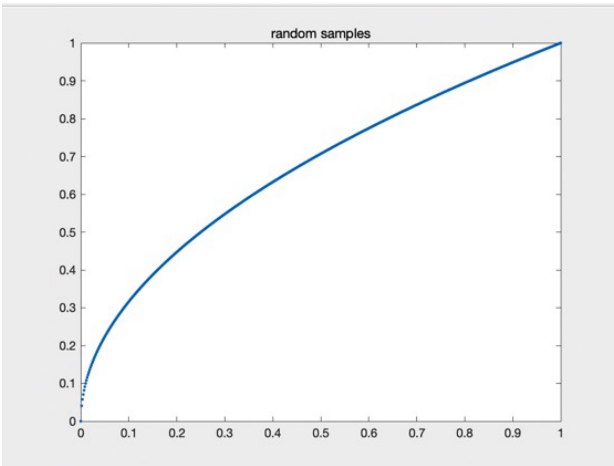
a) $\int_0^1 f(x)\,dx = 1 \Rightarrow \frac{1}{2}ax^2\Big|_{x=0}^{1} = 1 \Rightarrow \frac{1}{2}a = 1 \Rightarrow a = 2$

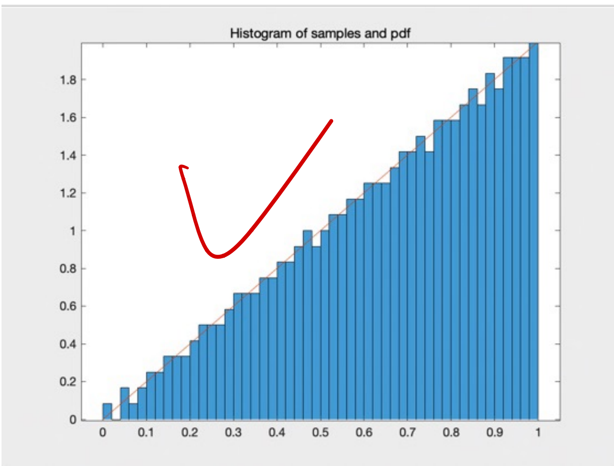
b) $F(x) = \int_0^x f(y)\,dy = \frac{1}{2}ay^2\Big|_{y=0}^{x} = \begin{cases} \frac{1}{2}ax^2 & 0 \le x \le 1 \\ \frac{1}{2}a & x > 1 \text{ or } x < 0 \end{cases}$

c) $F(x) = \frac{1}{2}ax^2$

$F^{-1}(x) = \sqrt{\frac{2x}{a}}$

```
close all;clc;clear all;
a = 2;
i = 1;
for x = sort(linspace(0,1,600))
    F(i) = sqrt(2*x/a);
    i = i+1;
end
F = sort(F)
figure
plot(linspace(0,1,length(F)),F,'.')
title('random samples')
figure
h = histogram(F,'BinWidth',0.02);
h.Normalization='pdf';
hold on;
plot(linspace(0,1,100),2*linspace(0,1,100));
hold off
title('Histogram of samples and pdf')
```





6. Using the reasoning behind Liapunov's Direct Method, reason about the stability of the following systems:

a)

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

b) $\ddot{x} - x^2\dot{x} + x^3 = 0$.

a) suppose $E = \frac{1}{2}\vec{x}^T P \vec{x}$

$$\Rightarrow \dot{E} = x^T\left[\tfrac{1}{2}(A^T P + PA)\right]\vec{x} = x^T(-Q)\vec{x}$$

set $(A^T P + PA) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} P + P \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -c & -d \\ a-c & b-d \end{pmatrix} + \begin{pmatrix} -b & a-b \\ -d & c-d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -b-c & a-b-d \\ a-c-d & b+c-2d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow P = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$ positive definite.

there exists $P = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

therefore, the system is stable.

(b) $\ddot{x} - x^2\dot{x} + x^3 = 0$

Set $E = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4$ positive definite

$\dot{E} = \dot{x}\ddot{x} + x^3\dot{x}$

$= \dot{x}(\ddot{x} + x^3)$

$= x^2\dot{x}^2$ positive definite

therefore the system is unstable

```
%% Question 1
clc;
close all;
clear all;
%% 1.c convolution

p1 = [1/6 1/6 1/6 1/6 1/6 1/6];
p2 = p1;
pxy = conv(p1,p2);
samp = 2:12;
sum(pxy)
figure
stem(samp,pxy)
% stem(pxy)
axis([1.5 12.5 0 0.18])
xlabel('X+Y')
ylabel('p_{X+Y}')
title('Probability distribution for X+Y using convolution')

%% 1.d built-in function
p1 = [0 0 0 1/6 1/6 1/6 1/6 1/6 1/6 0 0 0];
p2 = p1;
ffv1 = fft(p1);
ffv2 = fft(p2);
fre_dom = ffv1.*ffv2;
tim_dom = ifft(fre_dom);
fin = fftshift(tim_dom);
fin(length(fin)) = nan;
% fin = tim_dom;
samp = 2:(length(fin)+1);
figure
stem(samp,fin)
axis([1.5 12.5 0 0.18])
xlabel('X+Y')
ylabel('p_{X+Y}')
title('Probability distribution for X+Y using Matlab built-in functions')

%% question
a = 2;
i = 1;
for x = sort(linspace(0,1,600))
    F(i) = sqrt(2*x/a);
    i = i+1;
end
F = sort(F)
figure
plot(linspace(0,1,length(F)),F,'.')
title('random samples')
figure
h = histogram(F,'BinWidth',0.02);
h.Normalization='pdf';
hold on;
plot(linspace(0,1,100),2*linspace(0,1,100));
hold off
title('Histogram of samples and pdf')
```