Homework 9: ME5701, 16 October 2020

Problem 1. In this problem you will code up various forms of interpolation.

a) Given $n+1$ points $\{x_1, ..., x_{n+1}\}$ and associated data at those points $\{y_1, ..., y_{n+1}\}$, write a progam to fit the polynomial $f(x) = \sum_{k=0}^{n} a_k x^k$ to this data such that $y_i = f(x_i)$ for all $i = 1, ..., n+1$ and illustrate your program when $n = 3$ on the pairs $\{(x_i, y_i)\} = \{(0,0), (1,1), (-1, 0.5), (-2, 4)\}$ by plotting the points and the polynomial curve.

b) For the same four data points as in part a, use the left pseudoinverse with weight $W = \mathbb{I}$ to obtain the best fit parabola $y = ax^2 + bx + c$ (which we know does not have enough freedom to hit all points exactly).

c) Again using the same four data points as in part a, now fit with a fourth order polynomial

$$y = g(x) \doteq ax^4 + bx^3 + cx^2 + dx + e$$

such that all of the data points are hit, $y_i = g(x_i)$, while minimizing the cost

$$C_1(a,b,c,d,e) \doteq \int_{-2}^{1} \left(\frac{dg}{dx}\right)^2 dx$$

What W matrix does this produce for use with the right pseudoinverse ?
d) Do the same as in part c, but with the cost

$$C_2(a,b,c,d,e) \doteq \int_{-2}^{1} \left(\frac{d^2 g}{dx^2}\right)^2 dx$$

What W matrix does this produce for use with the right pseudoinverse ?

Plot your results for all parts. Which one do you like the best, and why ? Why did I choose the range of integration $x \in [-2, 1]$?

Problem 2. Given a right-circular helix parameterized as

$$\tilde{\mathbf{x}}(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ ht \end{pmatrix}$$

where r and h are constants,
a) compute the arclength s as a function of t, $s = f(t)$, and reparameterize the helix as $\mathbf{x}(s) \doteq \tilde{\mathbf{x}}(f^{-1}(s))$;
b) Use the arclength based definitions to compute $\kappa(s)$ and $\tau(s)$
c) compute curvature and torsion in the original parameterization as

$$\tilde{\kappa}(t) = \frac{\|\tilde{\mathbf{x}}'(t) \times \tilde{\mathbf{x}}''(t)\|}{\|\tilde{\mathbf{x}}'(t)\|^3} \quad \text{and} \quad \tilde{\tau}(t) = \frac{\det[\tilde{\mathbf{x}}'(t), \tilde{\mathbf{x}}''(t), \tilde{\mathbf{x}}'''(t)]}{\|\tilde{\mathbf{x}}'(t) \times \tilde{\mathbf{x}}''(t)\|^2}$$

where $' = d/dt$, and show that

$$\tilde{\kappa}(t) = \kappa(f(t)) \text{ and } \tilde{\tau}(t) = \tau(f(t))$$

and equivalently

$$\tilde{\kappa}(f^{-1}(s)) = \kappa(s) \text{ and } \tilde{\tau}(f^{-1}(s)) = \tau(s)$$

<u>Problem 3.</u> The surface of a 3D ellipsoid aligned with the x,y,z axes can be parameterized as

$$\tilde{\mathbf{x}}(\theta, \phi) = \begin{pmatrix} a\cos\phi\sin\theta \\ b\sin\phi\sin\theta \\ c\cos\theta \end{pmatrix}$$

and its implicit equation can be written as $\phi(\mathbf{x}) = 1$ where

$$\phi(\mathbf{x}) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

a) Show that $\phi(\tilde{\mathbf{x}}) = 1$

b) Compute $G(\theta, \phi)$, $\mathbf{n}(\theta, \phi)$ and $L(\theta, \phi)$ from $\tilde{\mathbf{x}}(\theta, \phi)$

c) Use the results of b) to calculate the Gaussian and mean curvatures at each point

d) Use the formulas

$$k(\mathbf{x}) = \frac{1}{\|\nabla\phi\|^4} \det \begin{bmatrix} \nabla\nabla^T\phi & \nabla\phi \\ \nabla^T\phi & 0 \end{bmatrix}$$

and

$$m(\mathbf{x}) = \frac{\|\nabla\phi\|^2 \operatorname{tr}(\nabla\nabla^T\phi) - (\nabla^T\phi)(\nabla\nabla^T\phi)(\nabla\phi)}{2\|\nabla\phi\|^3} = \nabla \cdot \left(\frac{\nabla\phi}{\|\nabla\phi\|}\right)$$

to compute the Gaussian and mean curvature from the implicit equation
e) Show that as $k(\tilde{\mathbf{x}}(\theta, \phi))$ and $m(\tilde{\mathbf{x}}(\theta, \phi))$ give the same answer as in c).

Possible Class Project: Given n points in 3D, write Matlab programs to compare various methods for fitting surfaces to the points (polynomial splines, left/right pseudo-inverse, etc) to generate a surface patch, and compute G, $\mathbf{n}$, L, and curvatures for any point in the surface patch. Compare with built-in Matlab functions.