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9.6

① Prove $\overline{c_1 c_2} = \bar{c_1}\,\bar{c_2}$

Soln: Assume $c_1 = a_1 + ib_1$, $c_2 = a_2 + ib_2$

$$c_1 c_2 = (a_1 + ib_1)(a_2 + ib_2)$$
$$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + b_1 a_2)$$
$$\overline{c_1 c_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2) \quad \text{— eqn 1}$$
$$\bar{c_1} = a_1 - ib_1 \qquad \bar{c_2} = a_2 - ib_2$$
$$\bar{c_1}\bar{c_2} = (a_1 - ib_1)(a_2 - ib_2)$$
$$= (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + b_1 a_2) \quad \text{— eqn 2.}$$

It is obvious that

R.H.S of eqn 1 = RHS of eqn 2.

Therefore, $\overline{c_1 c_2} = \bar{c_1}\,\bar{c_2}$. get proved.

② Prove $(AB)C = A(BC)$

Soln: Assume $A \in \mathbb{R}^{m\times n}$ $B \in \mathbb{R}^{n\times p}$ $C \in \mathbb{R}^{p\times q}$

Then we can calculate:

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

and

$$(BC)_{ij} = \sum_{k=1}^{p} b_{ik} c_{kj}.$$

Then calculate $(AB)C$

$$\left((AB)C\right)_{ij} = \sum_{k=1}^{p} (AB)_{ik} (C)_{kj} = \sum_{k=1}^{p}\left(\sum_{l=1}^{n} a_{il} b_{lk}\right) c_{kj} = \sum_{k=1}^{p}\sum_{l=1}^{n} a_{il} b_{lk} c_{kj} \quad \text{— eqn 1}$$

And calculate $A(BC)$

$$\left(A(BC)\right)_{ij} = \sum_{l=1}^{n} (A)_{il} (BC)_{lj} = \sum_{l=1}^{n} a_{il}\left(\sum_{k=1}^{p} b_{lk} c_{kj}\right) = \sum_{l=1}^{n}\sum_{k=1}^{p} a_{il} b_{lk} c_{kj} \quad \text{— eqn 2}$$

Since $\sum_{k=1}^{p}\sum_{l=1}^{n} a_{il} b_{lk} c_{kj} = \sum_{l=1}^{n}\sum_{k=1}^{p} a_{il} b_{lk} c_{kj}$,

RHS of eqn 1 = RHS of eqn 2,

And ~~left~~ LHS of eqn 1 = LHS of eqn 2,

Therefore, $(AB)C = A(BC)$ get proved

③ Define $A^{-T} \doteq (A^{-1})^T$ a) Show $A^{-T} = (A^T)^{-1}$,
b) write $(AB)^{-T}$ in terms of $A^{-T}$ and $B^{-T}$

Soln a). Since $A \cdot A^{-1} = I$.

take transpose $(A \cdot A^{-1})^T = I^T = I$.

then $(A^{-1})^T A^T = I$.

multiple $(A^T)^{-1}$ on right of $A^T$.

$$(A^{-1})^T A^T (A^T)^{-1} = I (A^T)^{-1} \quad \text{— eqn 1}$$

LHS of eqn 1 $= (A^{-1})^T \cdot I = (A^{-1})^T = A^{-T}$

RHS of eqn 1 $= I \cdot (A^T)^{-1} = (A^T)^{-1}$

Therefore, $A^{-T} = (A^T)^{-1}$ get proved.

b) $(AB)^{-T} = ((AB)^{-1})^T$.

since $(AB)^{-1} = B^{-1} A^{-1}$

then, $(AB)^{-T} = (B^{-1} A^{-1})^T = A^{-T} B^{-T}$

therefore, $(AB)^{-T} = A^{-T} B^{-T}$

④ Prove $(f_1 * f_2)(x) = (f_2 * f_1)(x)$.

Soln $(f_1 * f_2)(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_1(k) \hat{f}_2(k) e^{2\pi i k x / L}$ — eqn 1

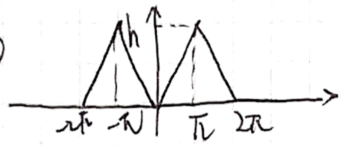
$(f_2 * f_1)(x) = \sum_{k=-\infty}^{+\infty} \hat{f}_2(k) \hat{f}_1(k) e^{2\pi i k x / L}$ — eqn 2

For each $k$, $\hat{f}_1(k) \hat{f}_2(k) = \hat{f}_2(k) \hat{f}_1(k)$

therefore, RHS of eqn 1 = RHS of eqn 2.

therefore, $(f_1 * f_2)(x) = (f_2 * f_1)(x)$ get proved.

?

⑤ 

$L=2\pi$, period $=2\pi$

$\Rightarrow$ fourier expansion?

soln: Useful formulas:

$$f(x) = \sum_{k=-\infty}^{+\infty} \hat{f}(k)\, e^{2\pi i x/L}$$

for this problem, fourier expansion is:

$$f(x) = \sum_{k=-B}^{+B} \hat{f}(k)\, e^{ix},$$

where

$$\hat{f}(k) = \frac{1}{L}\int_0^L f(x)\, e^{-2\pi i k x/L}\, dx$$

for this problem,

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-ikx}\, dx.$$

In one period $(0, 2\pi]$,

$$f(x) = \begin{cases} \frac{h}{\pi}x, & 0 < x \leq \pi \\ -\frac{h}{\pi}x + 2h, & \pi < x \leq 2\pi. \end{cases}$$

Therefore, when $k \neq 0$,

$$\hat{f}(k) = \frac{1}{2\pi}\left[\int_0^{\pi} \frac{h}{\pi}x\, e^{-ikx}\, dx + \int_{\pi}^{2\pi}\left(-\frac{h}{\pi}x + 2h\right)e^{-ikx}\, dx\right]$$

Now we calculate the integral seperately.

$$\int x e^{ax}\, dx = \frac{1}{a^2}(ax-1)e^{ax}, \text{ so } \int x e^{-ikx}\, dx = \frac{-1}{k^2}(-ikx-1)e^{-ikx}$$

Therefore

$$\hat{f}(k) = \frac{1}{2\pi}\left[\frac{-h}{\pi k^2}(-ikx-1)e^{-ikx}\Big|_{x=0}^{\pi} + \frac{h}{\pi k^2}(-ikx-1)e^{-ikx}\Big|_{x=\pi}^{2\pi} + \frac{2hi}{k}e^{-ikx}\Big|_{x=\pi}^{2\pi}\right]$$

$$\hat{f}(k) = \frac{1}{2\pi}\left[\frac{h}{\pi k^2}(ik\pi+1)e^{-ik\pi} - \frac{h}{\pi k^2} + \frac{h}{\pi k^2}(-ik2\pi-1)e^{-ik2\pi} - \frac{h}{\pi k^2}(-ik\pi-1)e^{-ik\pi} + \frac{2hi}{k}e^{-ik2\pi} - \frac{2hi}{k}e^{-ik\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2h}{\pi k^2}e^{-ik\pi} - \frac{h}{\pi k^2}e^{-ik2\pi} - \frac{h}{\pi k^2}\right]$$

$$= \frac{h}{2\pi^2 k^2}\left[2e^{-ik\pi} - \left(e^{-ik\pi}\right)^2 - 1\right]$$

when $k=0$,

$$\hat{f}(0) = \frac{1}{2\pi}\int_0^{\pi} \frac{h}{\pi}x\,dx + \int_{\pi}^{2\pi} (2h-\frac{h}{\pi}x)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{h}{2\pi}x^2\Big|_{x=0}^{\pi} + (2hx-\frac{h}{2\pi}x^2)\Big|_{x=\pi}^{2\pi}\right]$$

$$= \frac{h}{2}$$

Therefore, $\hat{f}(k) = \begin{cases} \frac{h}{2} & \text{, when } k=0 \\ \frac{h}{2\pi^2k^2}[2e^{-ik\pi} - (e^{-ik\pi})^2 - 1] & \text{, when } k=\pm1,\pm2,\dots,\pm B \end{cases}$

And hense the fourier expansion is:

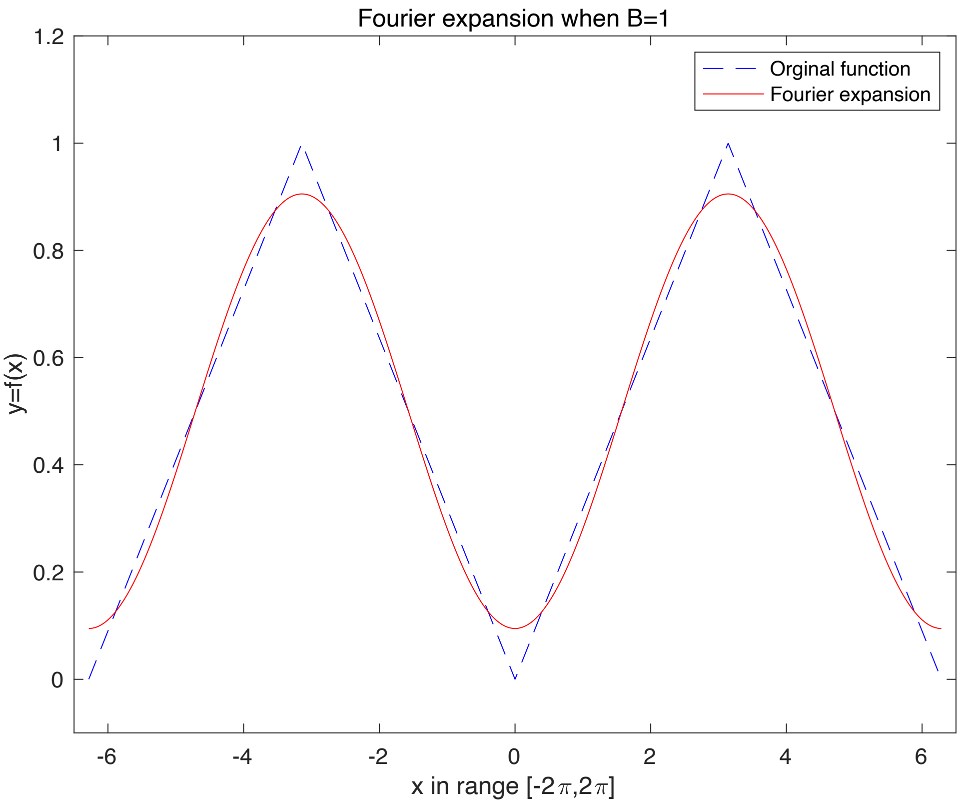
$$f(x) = \sum_{k=-B}^{B} \hat{f}(k)\,e^{ikx}$$

for $\hat{f}(k) = \begin{cases} \frac{h}{2} & \text{when } k=0 \\ \frac{h}{2\pi^2k^2}[2e^{-ik\pi} - (e^{-ik\pi})^2 - 1] & \text{when } k=\pm1,\pm2,\dots,\pm B \end{cases}$

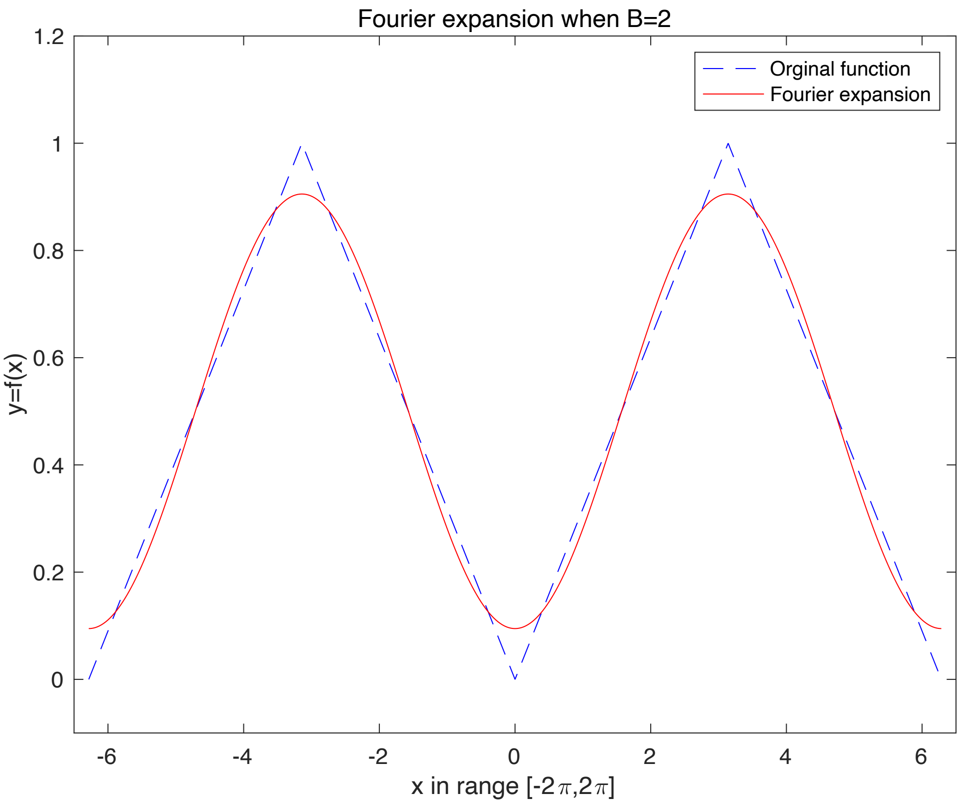
But actually I think my answer should be right...

Q5#2) Plot using MATLAB for B=1,2,5,10

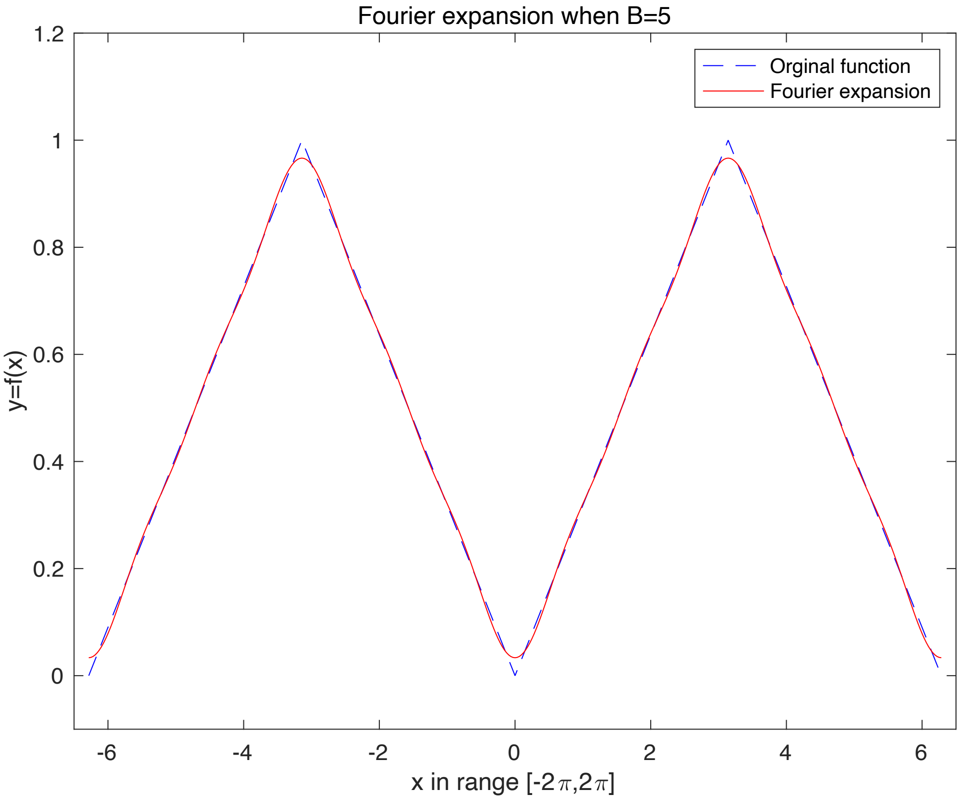
When B=1,



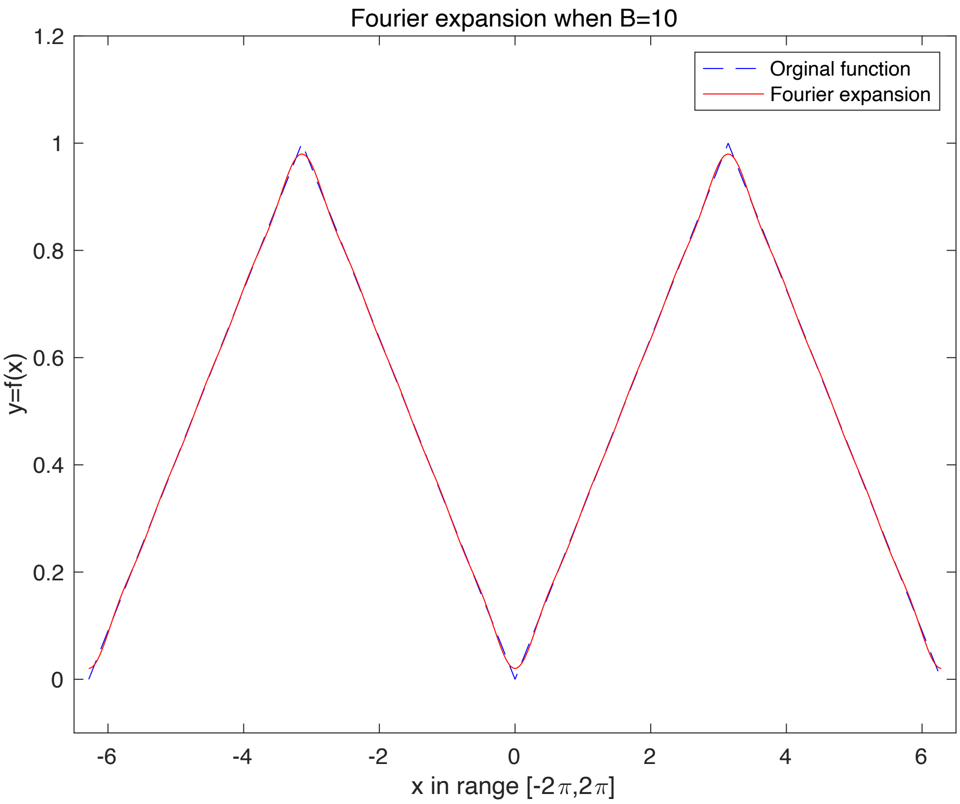

When B=2,



When B=5,

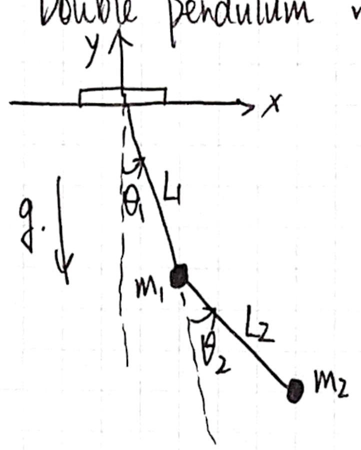


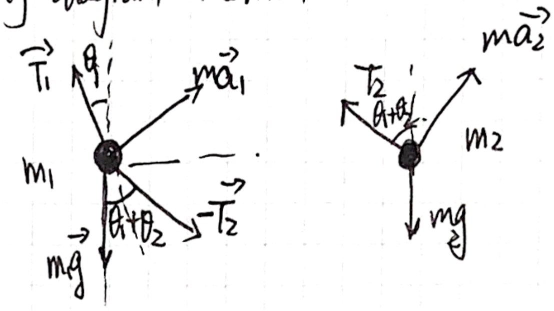

When B=10,




It is obvious that the larger B is, the more accurate the expansion is.

(b). Double pendulum with free-body-diagram method.




Soln: Again, compute absolute positions.

$$\vec{X_1} = \begin{pmatrix} L_1 \sin\theta_1 \\ -L_1 \cos\theta_1 \end{pmatrix}, \quad \vec{X_2} = \begin{pmatrix} L_1 \sin\theta_1 + L_2 \sin(\theta_1+\theta_2) \\ -L_1 \cos\theta_1 - L_2 \cos(\theta_1+\theta_2) \end{pmatrix}$$

Then compute velocities, which is the time derivative of position.

$$\dot{\vec{X_1}} = \begin{pmatrix} L_1 \dot\theta_1 \cos\theta_1 \\ L_1 \dot\theta_1 \sin\theta_1 \end{pmatrix} \quad \dot{\vec{X_2}} = \begin{pmatrix} L_1 \dot\theta_1 \cos\theta_1 + L_2(\dot\theta_1 + \dot\theta_2) \cos(\theta_1+\theta_2) \\ L_1 \dot\theta_1 \sin\theta_1 + L_2(\dot\theta_1 + \dot\theta_2) \sin(\theta_1+\theta_2) \end{pmatrix}$$

Then compute accelerations, which is the time derivative of velocity.

$$\ddot{\vec{X_1}} = \begin{pmatrix} L_1 \ddot\theta_1 \cos\theta_1 - L_1 \dot\theta_1^2 \sin\theta_1 \\ L_1 \ddot\theta_1 \sin\theta_1 + L_1 \dot\theta_1^2 \cos\theta_1 \end{pmatrix}$$

$$\ddot{\vec{X_2}} = \begin{pmatrix} L_1 \ddot\theta_1 \cos\theta_1 - L_1 \dot\theta_1^2 \sin\theta_1 + L_2(\ddot\theta_1 + \ddot\theta_2) \cos(\theta_1+\theta_2) - L_2 (\dot\theta_1 + \dot\theta_2)^2 \sin(\theta_1+\theta_2) \\ L_1 \ddot\theta_1 \sin\theta_1 + L_1 \dot\theta_1^2 \cos\theta_1 + L_2 (\ddot\theta_1 + \ddot\theta_2) \sin(\theta_1+\theta_2) + L_2 (\dot\theta_1 + \dot\theta_2)^2 \cos(\theta_1+\theta_2) \end{pmatrix}.$$

~~The acceleration~~

According to the free body diagram of $m_1$.

$$m\vec{a_1} = \vec{T_1} + (-\vec{T_2}) + m_1\vec{g}, \quad \text{and } \vec{a_1} = \ddot{\vec{X_1}}$$

Set the magnitude of $\vec{T_1}$ is $T_1$

and that of $\vec{T_2}$ is $T_2$.

$$\vec{T_1} = T_1 \cdot (-\vec{X_1}) / \|\vec{X_1}\| =$$

and similarly,

$$\vec{T_2} = T_2 (\vec{X_1} - \vec{X_2}) / \|\vec{X_1} - \vec{X_2}\|$$

Therefore, equilibrium of $m_1$ is:

$$m_1 \ddot{\vec{X_1}} = -T_1 \frac{\vec{X_1}}{\|\vec{X_1}\|} + T_2 \frac{\vec{X_2} - \vec{X_1}}{\|\vec{X_2} - \vec{X_1}\|} + m_1\vec{g}$$

$$= -\frac{T_1}{L_1} \vec{X_1} + \frac{T_2}{L_2} (\vec{X_2} - \vec{X_1}) + m_1\vec{g}$$

Then we expand it , Set $\vec{g} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$

$$m_1 \begin{pmatrix} L_1\ddot{\theta}_1\cos\theta_1 - L_1\dot{\theta}_1^2\sin\theta_1 \\ L_1\ddot{\theta}_1\sin\theta_1 + L_1\dot{\theta}_1^2\cos\theta_1 \end{pmatrix} = -\frac{T_1}{L_1}\begin{pmatrix} L_1\sin\theta_1 \\ -L_1\cos\theta_1 \end{pmatrix} + \frac{T_2}{L_2}\begin{pmatrix} L_2\sin(\theta_1+\theta_2) \\ -L_2\cos(\theta_1+\theta_2) \end{pmatrix} + m_1\begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\begin{pmatrix} m_1L_1\ddot{\theta}_1\cos\theta_1 - m_1L_1\dot{\theta}_1^2\sin\theta_1 \\ m_1L_1\ddot{\theta}_1\sin\theta_1 + m_1L_1\dot{\theta}_1^2\cos\theta_1 \end{pmatrix} = \begin{pmatrix} -T_1\sin\theta_1 + T_2\sin(\theta_1+\theta_2) \\ T_1\cos\theta_1 - T_2\cos(\theta_1+\theta_2) - m_1 g \end{pmatrix} \quad \begin{matrix} \text{— eqn 1} \\ \text{— eqn 2.} \end{matrix}$$

Similarly, the equilibrium of $m_2$ is:

$$m_2\ddot{\vec{x_2}} = \vec{T_2} + m_2\vec{g}$$

$$= \frac{T_2}{L_2}(\vec{x_1} - \vec{x_2}) + m_2 g.$$

$$\begin{pmatrix} m_2(L_1\ddot{\theta}_1\cos\theta_1 - L_1\dot{\theta}_1^2\sin\theta_1 + L_2(\ddot{\theta}_1+\ddot{\theta}_2)\cos(\theta_1+\theta_2) - L_2(\dot{\theta}_1+\dot{\theta}_2)^2\sin(\theta_1+\theta_2)) \\ m_2(L_1\ddot{\theta}_1\sin\theta_1 + L_1\dot{\theta}_1^2\cos\theta_1 + L_2(\ddot{\theta}_1+\ddot{\theta}_2)\sin(\theta_1+\theta_2) + L_2(\dot{\theta}_1+\dot{\theta}_2)^2\cos(\theta_1+\theta_2)) \end{pmatrix}$$

$$= \begin{pmatrix} -T_2\sin(\theta_1+\theta_2) \\ T_2\cos(\theta_1+\theta_2) - m_2 g \end{pmatrix} \quad \begin{matrix} \text{— eqn 3} \\ \text{— eqn 4} \end{matrix}$$

~~With eqn 1-4, we now have 4 eqns with 4 unknowns: $\theta_1, \theta_2, T_1, T_2$~~

Our goal is to estimate $T_1$ and $T_2$, to get equs of $\theta_1$ and $\theta_2$.

~~For~~ For eqn 3 and 4., we put $T_2$ on ~~RHS~~ LHS of eqns.

$$T_2 = \frac{m_2}{-\sin(\theta_1+\theta_2)}\left(l_1\ddot{\theta}_1\cos\theta_1 - l_1\dot{\theta}_1^2\sin\theta_1 + l_2(\ddot{\theta}_1+\ddot{\theta}_2)\cos(\theta_1+\theta_2) - l_2(\dot{\theta}_1+\dot{\theta}_2)^2\sin(\theta_1+\theta_2)\right)$$

$$= \frac{m_2}{\cos(\theta_1+\theta_2)}\left(l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1 + l_2(\ddot{\theta}_1+\ddot{\theta}_2)\sin(\theta_1+\theta_2) + l_2(\dot{\theta}_1+\dot{\theta}_2)^2\cos(\theta_1+\theta_2) + g\right) \quad \text{— eqn 5}$$

simplify eqn 5, we get

$$-\cos(\theta_1+\theta_2)\left[l_1\ddot{\theta}_1\cos\theta_1 - l_1\dot{\theta}_1^2\sin\theta_1 + l_2(\ddot{\theta}_1+\ddot{\theta}_2)\cos(\theta_1+\theta_2) - l_2(\dot{\theta}_1+\dot{\theta}_2)^2\sin(\theta_1+\theta_2)\right)$$

$$= \sin(\theta_1+\theta_2)\left[l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1 + l_2(\ddot{\theta}_1+\ddot{\theta}_2)\sin(\theta_1+\theta_2) + l_2(\dot{\theta}_1+\dot{\theta}_2)^2\cos(\theta_1+\theta_2) + g\right).$$

Move L.H.S to the ~~R~~ right and simplify with trigonometric relations.

$$0 = l_1\ddot{\theta}_1\cos\theta_2 + l_1\dot{\theta}_1^2\sin\theta_2 + l_2(\ddot{\theta}_1+\ddot{\theta}_2)\cancel{+l_2(\dot{\theta}_1+\dot{\theta}_2)^2} + g\sin(\theta_1+\theta_2) \quad \text{— eqn 6.}$$

For eqn 1 and 2, we can substitude $T_2\sin(\theta_1+\theta_2)$ and $T_2\cos(\theta_1+\theta_2)$ and put $T_1$ on LHS of eqns.

~~$T_1 =$~~ ~~$m_1$~~

$$T_1 = -\frac{1}{\sin\theta_1}\Big[m_1l_1\ddot{\theta}_1\cos\theta_1 - m_1l_1\dot{\theta}_1^2\sin\theta_1 + m_2l_1\ddot{\theta}_1\cos\theta_1 - m_2l_1\dot{\theta}_1^2\sin\theta_1 + \overset{m_2}{l_2}(\ddot{\theta}_1+\ddot{\theta}_2)\cos(\theta_1+\theta_2) - m_2l_2(\dot{\theta}_1+\dot{\theta}_2)^2\sin(\theta_1+\theta_2)\Big].$$

$$= \frac{1}{\cos\theta_1}\Big[m_1g + m_1l_1\ddot{\theta}_1\sin\theta_1 + m_1l_1\dot{\theta}_1^2\cos\theta_1 + m_2g + m_2l_1\ddot{\theta}_1\sin\theta_1 + m_2l_1\dot{\theta}_1^2\cos\theta_1 + m_2l_2(\ddot{\theta}_1+\ddot{\theta}_2)\sin(\theta_1+\theta_2) + \underset{m_2}{l_2}(\dot{\theta}_1+\dot{\theta}_2)^2\cos(\theta_1+\theta_2)\Big]. \quad \text{— eqn 7.}$$

Simplify eqn 7.

$$-\cos\theta_1\left[m_1l_1\ddot{\theta}_1\cos\theta_1 - m_1l_1\dot{\theta}_1^2\sin\theta_1 + m_2l_1\ddot{\theta}_1\cos\theta_1 - m_2l_1\dot{\theta}_1^2\sin\theta_1 + m_2l_2(\ddot{\theta}_1+\ddot{\theta}_2)\cos(\theta_1+\theta_2) - m_2l_2(\dot{\theta}_1+\dot{\theta}_2)^2\sin(\theta_1+\theta_2)\right]$$

$$= \sin\theta_1\left[m_1l_1\ddot{\theta}_1\sin\theta_1 + m_1l_1\dot{\theta}_1^2\cos\theta_1 + m_2l_1\ddot{\theta}_1\sin\theta_1 + m_2l_1\dot{\theta}_1^2\cos\theta_1 + m_2l_2(\ddot{\theta}_1+\ddot{\theta}_2)\sin(\theta_1+\theta_2) + m_2l_2(\dot{\theta}_1+\dot{\theta}_2)^2\cos(\theta_1+\theta_2) + m_1g + m_2g\right]$$

Move L.H.S to the right and simplify with trigonometric relations.

$$0 = m_1l_1\ddot{\theta}_1 + m_2l_1\ddot{\theta}_1 + m_2l_2(\ddot{\theta}_1+\ddot{\theta}_2)\cos\theta_2 - m_2l_2(\dot{\theta}_1+\dot{\theta}_2)^2\sin\theta_2 + (m_1+m_2)g\sin\theta_1 \quad \text{— eqn 8}$$

We then can get eqns of motions as simplified from eqn 6 and 8

$$\begin{cases} (l_1\cos\theta_2 + l_2)\ddot{\theta}_1 + l_2\ddot{\theta}_2 + l_1\sin\theta_2\,\dot{\theta}_1^{2} + g\sin(\theta_1+\theta_2) = 0 \\ \left[(m_1+m_2)l_1 + m_2 l_2\cos\theta_2\right]\ddot{\theta}_1 + m_2 l_2\cos\theta_2\,\ddot{\theta}_2 - m_2 l_2\sin\theta_2\,(\dot{\theta}_1+\dot{\theta}_2)^2 + (m_1+m_2)g\sin\theta_1 = 0 \end{cases}$$

Appendix: MATLAB code for Q5#2

```
h = 1;
B = 10;
i = 1i;

breaks = [-2*pi,-pi,0,pi,2*pi];
coefs = [h/pi 0; -h/pi h; h/pi 0; -h/pi h];
pp = mkpp(breaks, coefs);

xq = -2*pi:pi/100:2*pi;
v = ppval(pp,xq);
figure;
plot(xq,v,'b--')
hold on

sums = h/2;
for k = -B:1:-1
   sums = sums + h/(2*pi^2*k^2)*(2*exp(-i*k*pi)-(exp(-i*k*pi))^2-
1)*exp(i*k*xq);
end
for k = 1:1:B
   sums = sums + h/(2*pi^2*k^2)*(2*exp(-i*k*pi)-(exp(-i*k*pi))^2-
1)*exp(i*k*xq);
end

plot(xq,sums,'r');
axis([-6.5 6.5 -0.1 1.2])
hold off
legend('Orginal function','Fourier expansion')
xlabel('x in range [-2\pi,2\pi]')
ylabel('y=f(x)')
title('Fourier expansion when B=10')
```