Solution to HW# 11.

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9.5 ?

1. Soln.

Define: $I_{in} = (\vec{x}\cdot\vec{x})\mathbb{I} - \vec{x}\vec{x}^T$

$K_{in} = \vec{x}\vec{x}^T$

Because it's matrix integration,

If $I_{in} = tr[K_{in}]\mathbb{I} - K_{in}$,

then $I = Tr[K]\mathbb{I} - K$ is also satisfied

Set $\vec{x} = [x_1, x_2, x_3]^T$.

$\Rightarrow \vec{x}\cdot\vec{x} = x_1^2 + x_2^2 + x_3^2$

$$\vec{x}\vec{x}^T = \begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix}$$

$\Rightarrow tr[\vec{x}\vec{x}^T] = x_1^2 + x_2^2 + x_3^2$ $\Rightarrow Tr[K_{in}] = \vec{x}\cdot\vec{x}$

Therefore, $I_{in} = tr[K_{in}]\mathbb{I} - K_{in}$

then, $I = tr[K]\mathbb{I} - K$ get proved. ✓

With $I = Tr[K]\mathbb{I} - K$, for each entry on the diagonal of I, It remains the trace of K except for the corresponding one.

So, $Tr[I] = 2\, tr[K]$

$\Rightarrow K = \frac{1}{2} tr[I]\mathbb{I} - I$.

2. $\frac{\partial \vec{x}}{\partial r} = (a\, s\theta\, c\phi,\ b\, s\theta\, s\phi,\ c\, c\theta)^T$

$\frac{\partial \vec{x}}{\partial \phi} = (-ar\, s\theta\, s\phi,\ br\, s\theta\, c\phi,\ 0)^T$

$\frac{\partial \vec{x}}{\partial \theta} = (ar\, c\theta\, c\phi,\ br\, c\theta\, s\phi,\ -cr\, s\theta)^T$

Then, $\left| \det \left[\frac{\partial \vec{x}}{\partial r}, \frac{\partial \vec{x}}{\partial \phi}, \frac{\partial \vec{x}}{\partial \theta} \right] \right|$

$$= \left| \det \begin{bmatrix} a s\theta c\phi & b s\theta s\phi & c c\theta \\ -ar s\theta s\phi & br s\theta c\phi & 0 \\ ar c\theta c\phi & br c\theta s\phi & -cr s\theta \end{bmatrix}^T \right|$$

$$= \left| -c c\theta (abr^2 s\theta c\theta s^2\phi + abr^2 s\theta c\theta c^2\phi) - cr s\theta (abr s^2\theta c^2\phi + abr s^2\theta s^2\phi) \right|$$

$$= \left| -abc r^2 s\theta c^2\theta - abc r^2 s^3\theta \right|$$

Since $\theta \in [0, \pi]$, $\sin\theta \geq 0$.

$$= abc\, r^2 \sin\theta$$

$$Vol = \int_{Vol} dVol = \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left| \det \left[\frac{\partial \mathbf{x}}{\partial r}, \frac{\partial \mathbf{x}}{\partial \phi}, \frac{\partial \mathbf{x}}{\partial \theta} \right] \right| d\phi\, d\theta\, dr$$

$$= abc \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 \sin\theta \, d\phi\, d\theta\, dr$$

$$= \frac{4}{3}\pi abc$$

Therefore, $m = \rho \cdot Vol = \frac{4}{3}\pi abc\, \rho_0$

$$\mathcal{K} \doteq \int_{Vol} \mathbf{x}\mathbf{x}^T \rho(\mathbf{x})\, dVol\,.$$

$$= \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \vec{x}\,\vec{x}^T \rho_0 \left| \det \left[\frac{\partial \vec{x}}{\partial r}, \frac{\partial \vec{x}}{\partial \theta}, \frac{\partial \vec{x}}{\partial \phi} \right] \right| d\phi\, d\theta\, dr$$

$$\vec{x}\,\vec{x}^T = \begin{pmatrix} ar s\theta c\phi \\ br s\theta s\phi \\ cr c\theta \end{pmatrix} \left(ar s\theta\, c\phi,\ br\, s\theta c\phi,\ cr c\theta \right)$$

$$\Rightarrow K = abc\rho_0 \int_{r=0}^{1} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \begin{pmatrix} a^2 r^4 s^3\theta c^2\phi & ab r^4 s^3\theta s\phi c\phi & ac r^4 s^2\theta c\theta c\phi \\ ab r^4 s^3\theta s\phi c\phi & b^2 r^4 s^3\theta s^2\phi & bc r^4 s^2\theta c\theta s\phi \\ ac r^4 s^2\theta c\theta c\phi & bc r^4 s^2\theta c\theta s\phi & c^2 r^4 c^2\theta s\theta \end{pmatrix} d\phi\, d\theta\, dr$$

$$= \frac{4}{15}\pi abc \rho_0 \begin{pmatrix} a^2 & & 0 \\ & b^2 & \\ 0 & & c^2 \end{pmatrix}$$

Then, $I = tr(K)\,\mathbb{I} - K$

$$= \frac{4}{15}\pi abc\, \rho_0 \begin{pmatrix} b^2+c^2 & & 0 \\ & a^2+c^2 & \\ 0 & & a^2+b^2 \end{pmatrix}$$

3. According to thm,

$$\omega_b = \mathrm{vect}\,(R^T \dot{R}),$$

$$\lambda(RMR^T) = \lambda(M)$$

So,

$$
\begin{aligned}
\mathrm{tr}[\dot{R} K \dot{R}^T] &= \sum \lambda(\dot{R} K \dot{R}^T) \\
&= \sum \lambda(\underbrace{R^T \dot{R}} K \underbrace{\dot{R}^T R}) \\
&= \sum \lambda(\Omega K \Omega^T), \quad \text{where } \Omega = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \\
&= \mathrm{tr}\,[\Omega K \Omega^T] \\
&= \mathrm{tr}\,[\Omega (\tfrac{1}{2}\mathrm{tr}[I]\mathbb{1} - I)\Omega^T]
\end{aligned}
$$

$$\Omega(\tfrac{1}{2}\mathrm{tr}[I]\mathbb{1} - I)\Omega^T$$

$$
= \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}
\begin{pmatrix} \frac{1}{2}(I_{yy}+I_{zz}-I_{xx}) & -I_{xy} & -I_{xz} \\ -I_{xy} & \frac{1}{2}(I_{xx}+I_{zz}-I_{yy}) & -I_{yz} \\ -I_{xz} & -I_{yz} & \frac{1}{2}(I_{xx}+I_{yy}-I_{zz}) \end{pmatrix}
\begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix}
$$

$$= [A_1, A_2, A_3].$$

By matlab,

$A_1 =$

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[ w2*(iyz*w3 + w2*(ixx/2 + iyy/2 - izz/2)) + w3*(iyz*w2 + w3*(ixx/2 - iyy/2 + izz/2))
                w3*(ixy*w3 - iyz*w1) - w2*(ixz*w3 + w1*(ixx/2 + iyy/2 - izz/2))
                w2*(ixz*w2 - iyz*w1) - w3*(ixy*w2 + w1*(ixx/2 - iyy/2 + izz/2)) ]
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$A_2 =$

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[           w3*(ixy*w3 - ixz*w2) - w1*(iyz*w3 + w2*(ixx/2 + iyy/2 - izz/2)),
 w1*(ixz*w3 + w1*(ixx/2 + iyy/2 - izz/2)) + w3*(ixz*w1 + w3*(iyy/2 - ixx/2 + izz/2)),
          - w3*(ixy*w1 + w2*(iyy/2 - ixx/2 + izz/2)) - w1*(ixz*w2 - iyz*w1), ]
```

$A_3 =$

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[        - w1*(iyz*w2 + w3*(ixx/2 - iyy/2 + izz/2)) - w2*(ixy*w3 - ixz*w2)
         - w2*(ixz*w1 + w3*(iyy/2 - ixx/2 + izz/2)) - w1*(ixy*w3 - iyz*w1)
 w1*(ixy*w2 + w1*(ixx/2 - iyy/2 + izz/2)) + w2*(ixy*w1 + w2*(iyy/2 - ixx/2 + izz/2)) ]
```

Therefore, $tr[\dot{R}K\dot{R}^T]$

$$= \frac{1}{2} I_{xx} w_1^2 + \frac{1}{2} I_{yy} w_2^2 + \frac{1}{2} I_{zz} w_3^2 + I_{xy} w_1 w_2 + I_{xz} w_1 w_3 + I_{yz} w_2 w_3$$

And, $w_b^T I w_b$

$$= \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

$$= \frac{1}{2} I_{xx} w_1^2 + \frac{1}{2} I_{yy} w_2^2 + \frac{1}{2} I_{zz} w_3^2 + I_{xy} w_1 w_2 + I_{xz} w_1 w_3 + I_{yz} w_2 w_3$$

$$= tr[\dot{R}K\dot{R}^T]$$

Therefore, $T = \frac{1}{2} w_b^T I w_b$

$$= \frac{1}{2} tr[\dot{R}K\dot{R}^T]$$

4. a) $T_{rot} = \frac{1}{2}\{ I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 \}$

$$= \frac{1}{2}\{ I_1(\dot\alpha s\beta s\gamma + \dot\beta c\gamma)^2 + I_1(\dot\alpha s\beta c\gamma - \dot\beta s\gamma)^2 + I_3(\dot\alpha c\beta + \dot\gamma)^2 \}$$

$$= \frac{1}{2}\{ I_1(\dot\alpha^2 \sin^2\beta + \dot\beta^2) + I_3(\dot\alpha\cos\beta + \dot\gamma)^2 \}$$

b) $\vec{X}_{cm} = R(\alpha,\beta,\gamma)(L\vec{e}_3)$

where $L\vec{e}_3$ is a fixed vector (constant)

Therefore, $\dot{\vec{X}}_{cm} = \dot{R}(\alpha,\beta,\gamma)(L\vec{e}_3)$

Then, $T_{trans} = \frac{1}{2} m \dot{\vec{X}}_{cm} \cdot \dot{\vec{X}}_{cm}$

$$= \frac{1}{2} m L^2 \dot{R}_3 \cdot \dot{R}_3$$

$$\dot{R}_3 = \begin{pmatrix} \dot\beta\cos\beta\sin\alpha + \dot\alpha\sin\beta\cos\alpha \\ \dot\alpha\sin\alpha\sin\beta - \dot\beta\cos\alpha\cos\beta \\ -\dot\beta\sin\beta \end{pmatrix}$$

$$T_{trans} = \frac{1}{2} m L^2 (\dot\alpha^2 \sin^2\beta + \dot\beta^2)$$

$$V = mgh = mg[X_{cm}]_3 = mgL\cos\beta$$

c) $V = Mg(X_{cm} \cdot \vec{e_3}) = mgL\cos\beta$

According to previous sections,

$\frac{1}{2}(I_1\dot\beta + 2mL^2\dot\beta)$

$T = T_{rot} + T_{trans}$

$= \frac{1}{2}\{ I_1(\dot\alpha^2\sin^2\beta + \dot\beta^2) + I_3(\dot\alpha\cos\beta + \dot\gamma)^2 \}$

$+ \frac{1}{2}mL^2(\dot\alpha^2\sin^2\beta + \dot\beta^2)$

Lagrange's eqn:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\Delta_i}\right) - \frac{\partial T}{\partial \Delta_i} + \frac{\partial V}{\partial \Delta_i} = 0 \quad \text{for } \Delta_i = \alpha, \beta, \gamma,$$

① $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\alpha}\right) - \frac{\partial T}{\partial \alpha} + \frac{\partial V}{\partial \alpha}$

$\Rightarrow (mL^2 + I_1)(\ddot\alpha\sin^2\beta + \dot\alpha\dot\beta\sin 2\beta) + I_3(\ddot\alpha\cos^2\beta - \dot\alpha\dot\beta\sin 2\beta + \ddot\gamma\cos\beta - \dot\gamma\dot\beta\sin\beta)$
$= 0$

② $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\beta}\right) - \frac{\partial T}{\partial \beta} + \frac{\partial V}{\partial \beta}$

$\Rightarrow (4mL^2 + 2I_1)\ddot\beta - I_1\dot\alpha^2\sin 2\beta + 2I_3(\dot\alpha^2\sin 2\beta - 2\dot\alpha\dot\gamma\sin\beta)\, 2mgL\sin\beta$
$+ mL^2\dot\alpha^2\sin 2\beta = 0$

③ $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\gamma}\right) - \frac{\partial T}{\partial \gamma} + \frac{\partial V}{\partial \gamma}$

$\Rightarrow I_3(\ddot\gamma + \ddot\alpha\cos\beta - \dot\alpha\dot\beta\sin\beta) = 0$

$(I_1 + mL^2)\ddot\beta + \dot\alpha^2 s\beta c\beta(I_3 - (I_1 + mL^2))$
$+ I_3\dot\alpha\dot\gamma s\beta - mgLs\beta = 0$

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d) Set angular momentum

$$\vec{L} = [L_1, L_2, L_3]^T \quad . \quad \|\vec{L}\| = \sqrt{L_1^2 + L_2^2 + L_3^2} = L$$

For body-fixed principle axis,

$$\vec{L} = I_{xx} w_x \vec{i} + I_{yy} w_y \vec{j} + I_{zz} w_z \vec{k}$$

And in xyz coordinate,

$$\vec{L} = L(\sin\beta \sin\gamma \, \vec{i} + \sin\beta \cos\gamma \, \vec{j} + \cos\beta \, \vec{k})$$

Then,

$$\begin{cases} L \sin\beta \sin\gamma = I_{xx} W_x = I_1 (\dot{\alpha} \sin\beta \sin\gamma + \dot{\beta} \cos\gamma) \\ L \sin\beta \cos\gamma = I_{yy} W_y = I_1 (\dot{\alpha} \sin\beta \cos\gamma - \dot{\beta} \sin\gamma) \\ L \cos\beta = I_{zz} W_z = I_3 (\dot{\alpha} \cos\beta + \dot{\gamma}) \end{cases}$$

We can solve for $\dot{\alpha}$, $\dot{\beta}$, $\dot{\gamma}$ as:

$$\begin{cases} \dot{\alpha} = \frac{L}{I_1} \\ \dot{\beta} = 0 \\ \dot{\gamma} = L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos\beta \end{cases}$$

Set equation with L.

$$L = \dot{\alpha} I_1 = \frac{\dot{\gamma}}{\cos\beta} \cdot \frac{I_1 I_3}{I_1 - I_3}$$

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Therefore, $$\frac{\dot{\alpha}}{\dot{\gamma}} = \frac{I_3}{I_1 - I_3} \frac{1}{\cos\beta}.$$