A0225282J . Sun Zhanhong , ME5701 HW# 10.

9.5

Homework 10: ME5701, 25 October 2020. Due 1 November 2020

Problem 1.

An *orthogonal matrix*, R, is a matrix with the property that $R^T R = RR^T = \mathbb{I}$. A *rotation matrix* is a "special" orthogonal matrix with the extra condition that its determinant is equal to $+1$. The set of all $n \times n$ matrices with these two conditions is called $SO(n)$.

(a) Show by direct calculation that the following matrices are rotation matrices:

$$R_3(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_2(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

R_1,R_2, and R_3 given above have the physical interpretation of rotations by the angle ϕ about the x, y, and z axes respectively. Based on physical intuition or mathematical calculation, determine if the following are true or false for $\phi \in (0, \pi)$ (i.e., the open interval not containing 0 or π):

(b) $R_1(\phi)R_2(\phi) = R_2(\phi)R_1(\phi)$
(c) $R_1(\phi)R_3(\phi) = R_3(\phi)R_1(\phi)$
(d) $R_2(\phi)R_3(\phi) = R_3(\phi)R_2(\phi)$
(e) $R_1(\phi)R_1(\phi) = R_1(2\phi)$
(f) $R_2(\phi)R_2(-\phi) = \mathbb{I}$

NOTE: A rotation matrix which *rotates* a vector with respect to a fixed reference frame is the same as the matrix which *describes* the rotated vector with respect to the fixed reference frame. That is, if the vector $\mathbf{x}_0$ is rotated relative to a fixed frame by the matrix R, then the rotated vector is represented as $\mathbf{x} = R\mathbf{x}_0$ in the fixed reference frame.

Problem 2.

Let $x_1, x_2, x_3 \in \mathbb{R}$ and

$$X = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} = -X^T.$$

The set of all such matrices forms a vectorspace (called $so(3)$). Exponentiating matrices $X \in so(3)$ produces matrices $\exp(X) \in SO(3)$. If the $\vee$ operator is defined such that

$$X^\vee = \mathbf{x}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

prove for any $R \in SO(3)$, regardless of whether or not $R = \exp(X)$, that

$$(RXR^T)^\vee = R\mathbf{x}.$$

Hint: divide R and R^T into columns and rows.

<u>Problem 3.</u>

a) If $\mathbf{x}, \mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$, prove or disprove the following:

$$\|\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}\| = \|\mathbf{n} \times \mathbf{x}\|$$

b) If $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$, and $N = -N^T \in \mathbb{R}^{3\times 3}$ is defined by the condition $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^3$, let

$$R(\theta, \mathbf{n}) = I + \sin\theta N + (1 - \cos\theta)N^2$$

be the rotation matrix from the Rodrigues formula. Show that

$$R(\theta, \mathbf{n}) = Q\, R_3(\theta)\, Q^T$$

where $Q = [\mathbf{a}, \mathbf{b}, \mathbf{n}]$ and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary except for the fact that $Q \in SO(3)$.

<u>Problem 4.</u>

Recall the notation $N^\vee = \mathbf{n}$. Given a rotation matrix $R(\mathbf{q})$ parameterized with $\mathbf{q} = [q_1, q_2, q_3]^T$, the 'left' and 'right' Jacobians for $SO(3)$ are

$$J_l = \left[\left(\frac{\partial g}{\partial q_1} g^{-1} \right)^\vee, \left(\frac{\partial g}{\partial q_2} g^{-1} \right)^\vee, \left(\frac{\partial g}{\partial q_3} g^{-1} \right)^\vee \right]$$

and

$$J_r = \left[\left(g^{-1} \frac{\partial g}{\partial q_1} \right)^\vee, \left(g^{-1} \frac{\partial g}{\partial q_2} \right)^\vee, \left(g^{-1} \frac{\partial g}{\partial q_3} \right)^\vee \right].$$

Show that in general

$$J_l = RJ_r$$

and in the special case when $[q_1, q_2, q_3] = [\alpha, \beta, \gamma]$ are the ZXZ Euler angles, explicitly compute J_l and J_r.

1.

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NOTE: A rotation matrix which *rotates* a vector with respect to a fixed reference frame is the same as the matrix which *describes* the rotated vector with respect to the fixed reference frame. That is, if the vector $\mathbf{x}_0$ is rotated relative to a fixed frame by the matrix R, then the rotated vector is represented as $\mathbf{x} = R\mathbf{x}_0$ in the fixed reference frame.

Soln:

$$R_3(\phi) R_3^T(\phi) = \begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$R_3^T(\phi) R_3(\phi) = \begin{pmatrix} c\phi & s\phi & 0 \\ -s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\det(R_3(\phi)) = \begin{vmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} c\phi & -s\phi \\ s\phi & c\phi \end{vmatrix} = c^2\phi + s^2\phi = 1$$

Therefore $R_3(\phi)$ is a rotation matrix

$$R_2(\phi) R_2^T(\phi) = \begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{pmatrix}\begin{pmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$R_2^T(\phi) R_2(\phi) = \begin{pmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{pmatrix}\begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\det(R_2(\phi)) = \begin{vmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{vmatrix} = 1$$

Therefore $R_2(\phi)$ is a rotation matrix

$$R_1(\phi)\,R_1^T(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$R_1^T(\phi)\,R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

$$\det(R_1(\phi)) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & c\phi & s\phi \\ 0 & -s\phi & c\phi \end{vmatrix} = 1$$

Therefore $R_1(\phi)$ is a rotation matrix

(b) $R_1(\phi)R_2(\phi) = R_2(\phi)R_1(\phi)$
(c) $R_1(\phi)R_3(\phi) = R_3(\phi)R_1(\phi)$
(d) $R_2(\phi)R_3(\phi) = R_3(\phi)R_2(\phi)$

b). $$R_1(\phi)\,R_2(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix}\begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{pmatrix} = \begin{pmatrix} c\phi & 0 & s\phi \\ s^2\phi & c\phi & -s\phi c\phi \\ -s\phi c\phi & s\phi & c^2\phi \end{pmatrix}$$

$$R_2(\phi)\,R_1(\phi) = \begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix} = \begin{pmatrix} c\phi & s^2\phi & s\phi c\phi \\ 0 & c\phi & -s\phi \\ -s\phi & c\phi s\phi & c^2\phi \end{pmatrix} \neq R_1(\phi)\,R_2(\phi)$$

c) $$R_1(\phi)\,R_3(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix}\begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c\phi & -s\phi & 0 \\ c\phi s\phi & c^2\phi & -s\phi \\ s^2\phi & s\phi c\phi & c\phi \end{pmatrix}$$

$$R_3(\phi)\,R_1(\phi) = \begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix} = \begin{pmatrix} c\phi & -s\phi c\phi & s^2\phi \\ s\phi & c^2\phi & -s\phi c\phi \\ 0 & s\phi & c\phi \end{pmatrix} \neq R_1(\phi)\,R_3(\phi)$$

d) $$R_2(\phi)\,R_3(\phi) = \begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{pmatrix}\begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} c^2\phi & -s\phi c\phi & s\phi \\ s\phi & c\phi & 0 \\ -s\phi c\phi & s^2\phi & c\phi \end{pmatrix}$$

$$R_3(\phi)\,R_2(\phi) = \begin{pmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{pmatrix} = \begin{pmatrix} c^2\phi & -s\phi & s\phi c\phi \\ s\phi c\phi & c\phi & s^2\phi \\ -s\phi & 0 & c\phi \end{pmatrix} \neq R_2(\phi)\,R_3(\phi)$$

Physically, it is totally not the same to change the sequence of rotation if the axis are fixed. So for questions b, c, and d, the expressions are not equal.

(e) $R_1(\phi)R_1(\phi) = R_1(2\phi)$
(f) $R_2(\phi)R_2(-\phi) = \mathbb{I}$

(e) $$R_1(\phi)R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c^2\phi - s^2\phi & -2s\phi c\phi \\ 0 & 2c\phi s\phi & c^2\phi - s^2\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c2\phi & -s2\phi \\ 0 & s2\phi & c2\phi \end{pmatrix}$$

$$R_1(2\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c2\phi & -s2\phi \\ 0 & s2\phi & c2\phi \end{pmatrix} = R_1(\phi)R_1(\phi)$$

Physically, both expressions on the two sides of equation means that the object rotates 2ϕ by x-axis. The L.H.S means it rotates ϕ and then another ϕ. The R.H.S means it rotates 2ϕ directly. They have the same meaning.

(f) $$R_2(\phi)R_2(-\phi) = \begin{pmatrix} c\phi & 0 & s\phi \\ 0 & 1 & 0 \\ -s\phi & 0 & c\phi \end{pmatrix}\begin{pmatrix} c\phi & 0 & -s\phi \\ 0 & 1 & 0 \\ s\phi & 0 & c\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$$

Physically, it means that an object rotates ϕ by y axis and then rotates back. So obviously the resultant matrix is $\mathbb{I}$.

$R = (R_1, R_2, R_3)$ $\qquad$ $R^T = (R_1^T, R_2^T, R_3^T)^T$

(2) $R X R^T = (R_1, R_2, R_3)\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}\begin{pmatrix} R_1^T \\ R_2^T \\ R_3^T \end{pmatrix}$

$= (x_3R_2 - x_2R_3, x_1R_3 - x_3R_1, x_2R_1 - x_1R_2)\begin{pmatrix} R_1^T \\ R_2^T \\ R_3^T \end{pmatrix}$

$= x_3\vec{R_2}\vec{R_1}^T - x_2\vec{R_3}\vec{R_1}^T + x_1\vec{R_3}\vec{R_2}^T - x_3\vec{R_1}\vec{R_2}^T + x_2\vec{R_1}\vec{R_3}^T - x_1\vec{R_2}\vec{R_3}^T = \begin{matrix} +0 & -x_3R_1R_2^T & +x_2R_1R_3^T \\ +x_3R_2R_1^T & +0 & -x_1R_2R_3^T \\ -x_2R_3R_1^T & +x_1R_3R_2^T & +0 \end{matrix}$

$Rx = (R_1, R_2, R_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1\vec{R_1} + x_2\vec{R_2} + x_3\vec{R_3} \end{pmatrix}$

Since $R \in SO(3)$, R_1, R_2, R_3 are independent unit vectors, and they can form a new coordinate system: $[R_1, R_2, R_3]$

Then if we view from the R-system:

$(RXR^T)_R = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$, $(Rx)_R = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Since the transformation from Cartesian to R space is linear,

$\Rightarrow (RXR^T)^\vee = Rx$ ✓

Problem 3.

a) If $\mathbf{x}, \mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$, prove or disprove the following:

$$\|\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}\| = \|\mathbf{n} \times \mathbf{x}\|$$

b) If $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$, and $N = -N^T \in \mathbb{R}^{3\times 3}$ is defined by the condition $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^3$, let

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where $Q = [\mathbf{a}, \mathbf{b}, \mathbf{n}]$ and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary except for the fact that $Q \in SO(3)$.

Problem 4.

Recall the notation $N^\vee = \mathbf{n}$. Given a rotation matrix $R(\mathbf{q})$ parameterized with $\mathbf{q} = [q_1, q_2, q_3]^T$, the 'left' and 'right' Jacobians for $SO(3)$ are

$$J_l = \left[\left(\frac{\partial g}{\partial q_1} g^{-1}\right)^\vee, \left(\frac{\partial g}{\partial q_2} g^{-1}\right)^\vee, \left(\frac{\partial g}{\partial q_3} g^{-1}\right)^\vee \right]$$

and

$$J_r = \left[\left(g^{-1}\frac{\partial g}{\partial q_1}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial q_2}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial q_3}\right)^\vee \right].$$

Show that in general

$$J_l = RJ_r$$

and in the special case when $[q_1, q_2, q_3] = [\alpha, \beta, \gamma]$ are the ZXZ Euler angles, explicitly compute J_l and J_r.

3. a) $\|x - (x \cdot n)n\| = \|n \times x\|$ for $\|n\| = 1$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - (x_1 n_1 + x_2 n_2 + x_3 n_3)\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$

$$\left\| \begin{matrix} n_2 x_3 - n_3 x_2 \\ n_3 x_1 - n_1 x_3 \\ n_1 x_2 - n_2 x_1 \end{matrix} \right\|$$

$$\begin{matrix} i & j & k \\ n_1 & n_2 & n_3 \\ x_1 & x_2 & x_3 \end{matrix}$$

$n_2 x_3 - n_3 x_2)$

$$= \left\| \begin{matrix} -(n_1^2 - 1)x_1 - n_1 n_2 x_2 - n_1 n_3 x_3 \\ -n_1 n_2 x_1 - (n_2^2 - 1)x_2 - n_2 n_3 x_3 \\ -n_1 n_3 x_1 - n_2 n_3 x_2 - (n_3^2 - 1)x_3 \end{matrix} \right\| \quad //$$

$$\Rightarrow [(n_1^2 - 1)x_1 + n_1 n_2 x_2 + n_1 n_3 x_3]^2 +$$

$$(n_1^2 - 1)^2 x_1^2 + n_1^2 n_2^2 x_2^2 + n_1^2 n_3^2 x_3^2$$

$$+ 2(n_1^2 - 1)n_1 n_2 x_1 x_2 + 2(n_1^2 - 1)n_1 n_3 x_1 x_3 + 2n_1^2 n_2 n_3 x_2 x_3$$

$$+ n_1^2 n_2^2 x_1^2 + (n_2^2 - 1)^2 x_2^2 + n_2^2 n_3^2 x_3^2$$

$$+ 2(n_2^2 - 1)n_1 n_2 x_1 x_2 + 2(n_2^2 - 1)n_2 n_3 x_2 x_3 + 2 n_1 n_2^2 n_3 x_1 x_3$$

$$+ n_1^2 n_3^2 x_1^2 + n_2^2 n_3^2 x_2^2 + (n_3^2 - 1)^2 x_3^2$$

$$+ 2 n_1 n_2 n_3^2 x_1 x_2 + 2(n_3^2 - 1)n_1 n_3 x_1 x_3 + 2(n_3^2 - 1)n_2 n_3 x_2 x_3$$

$$= x_1^2 \left(n_1^4 + 1 - 2n_1^2 + n_1^2 n_2^2 + n_1^2 n_3^2\right) + x_2^2 \left(n_2^4 - 2n_2^2 + 1 + n_1^2 n_2^2 + n^2 n_3^2\right)$$

$$+ x_3^2 \left(n_3^4 - 2n_3^2 + 1 + n_2^2 n_3^2 + n_1^2 n_3^2\right.$$

$$+ x_1 x_2 \left(2(n_1^2 - 1) n_1 n_2 + 2(n_2^2 - 1) n_1 n_2 + 2 n_1 n_2 n_3^2\right)$$

$$+ x_1 x_3 \left(2(n_1^2 - 1) n_1 n_3 + 2 n_1 n_2^2 n_3 + 2(n_3^2 - 1) n_1 n_3\right)$$

$$+ x_2 x_3 \left(2 n_1^2 n_2 n_3 + 2(n_2^2 - 1) n_2 n_3 + 2(n_3^2 - 1) n_2 n_3\right)$$

$$= x_1^2 \left(1 - n_1^2\right) + x_2^2 \left(1 - n_2^2\right) + x_3^2 \left(1 - n_3^2\right)$$

$$+ x_1 x_2 \left(-2 n_1 n_2\right) + x_2 x_3 \left(-2 n_2 n_3\right) + x_1 x_3 \left(-2 n_1 n_3\right)$$

$$= x_1^2 + x_2^2 + x_3^2 - \left(n_1 x_1 + n_2 x_2 + n_3 x_3\right)^2$$

$$\begin{pmatrix} n_2 x_3 - n_3 x_2 \\ n_3 x_1 - n_1 x_3 \\ n_1 x_2 - n_2 x_1 \end{pmatrix} = \begin{aligned} & n_2^2 x_3^2 + n_3^2 x_2^2 - 2 n_2 n_3 x_2 x_3 \\ & + n_3^2 x_1^2 + n_1^2 x_3^2 - 2 n_1 n_3 x_1 x_3 \\ & + n_1^2 x_2^2 + n_2^2 x_1^2 - 2 n_1 n_2 x_1 x_2 \end{aligned}$$

$$= x_1^2 (1 - n_1^2) + x_2^2 (1 - n_2^2) + x_3^2 (1 - n_3^2)$$

$$+ x_1 x_2 (-2 n_1 n_2) + x_2 x_3 (-2 n_2 n_3) + x_1 x_3 (-2 n_1 n_3)$$

$$= x_1^2 + x_2^2 + x_3^2 - (n_1 x_1 + n_2 x_2 + n_3 x_3)^2 \Rightarrow$$

LHS = RHS ✓

b) If $\mathbf{n} \in \mathbb{R}^3$ and $\|\mathbf{n}\| = 1$, and $N = -N^T \in \mathbb{R}^{3\times 3}$ is defined by the condition $N\mathbf{x} = \mathbf{n} \times \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^3$, let

$$R(\theta, \mathbf{n}) = I + \sin\theta N + (1 - \cos\theta)N^2$$

be the rotation matrix from the Rodrigues formula. Show that

$$R(\theta, \mathbf{n}) = Q\, R_3(\theta)\, Q^T$$

where $Q = [\mathbf{a}, \mathbf{b}, \mathbf{n}]$ and $\mathbf{a}$ and $\mathbf{b}$ are arbitrary except for the fact that $Q \in SO(3)$.

proof:

In stead of proving $R(\theta, n) = Q R_3(\theta) Q^T$,

we can prove that $R(\theta, n) Q = Q R_3(\theta)$

physically,

$Q R_3(\theta)$ means:

First rotating from the identity $I = [e_1, e_2, e_3]$ fixed in space to Q and then rotating relative to Q by $R_3(\theta)$.

On the other hands, a rotation about the vector $e_3^Q = \vec{n}$ as viewed in the fixed frame is $R(Q, n)$.

Hence, shifting the frame of reference Q by multiplying on the left by $R(\theta, n)$ has the same effect as $QR_3(\theta)$

Therefore, $R(\theta, \vec{n}) Q = Q R_3(\theta)$

Furthermore, $R(\theta, \vec{n}) = Q R_3(\theta) Q^T$

Problem 4.

Recall the notation $N^\vee = \mathbf{n}$. Given a rotation matrix $R(\mathbf{q})$ parameterized with $\mathbf{q} = [q_1, q_2, q_3]^T$, the 'left' and 'right' Jacobians for $SO(3)$ are

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and

$$J_r = \left[\left(g^{-1}\frac{\partial g}{\partial q_1}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial q_2}\right)^\vee, \left(g^{-1}\frac{\partial g}{\partial q_3}\right)^\vee \right].$$

Show that in general

$$J_l = RJ_r$$

and in the special case when $[q_1, q_2, q_3] = [\alpha, \beta, \gamma]$ are the ZXZ Euler angles, explicitly compute J_l and J_r.

soln: ① In $SO(3)$ case,

$g = R(\vec{q})$

② $\vec{q}^\wedge = Q$

③ We know that $Ad\,X = gXg^{-1}$

Since $\left(g \left(g^{-1} \frac{\partial g}{\partial q_i}\right) g^{-1} \right)^\vee = \left(\frac{\partial g}{\partial x_i} g^{-1} \right)^\vee$,

$\Rightarrow J_l = [Ad(g)] J_r$

④ We know that in $SO(3)$,

$Ad(R) = R$.

$\Rightarrow J_l = R J_r$. ✓

✓

$$R_{zxz}(\alpha, \beta, \gamma) = R_3(\alpha) R_1(\beta) R_3(\gamma)$$

$$J_l(\alpha, \beta, \gamma) = [e_3, R_3(\alpha) e_1, R_3(\alpha) R_1(\beta) e_3]$$

$$= \begin{bmatrix} 0 & \cos\alpha & \sin\alpha \sin\beta \\ 0 & \sin\alpha & -\cos\alpha \sin\beta \\ 1 & 0 & \cos\beta \end{bmatrix}$$ ✓

$$J_r = R^T J_l = [R_3(-\gamma) R_1(-\beta) e_3, R_3(-\gamma) e_1, e_3]$$

$$= \begin{pmatrix} s\beta s\gamma & c\gamma & 0 \\ s\beta c\gamma & -s\gamma & 0 \\ c\beta & 0 & 1 \end{pmatrix}$$ ✓