A0225282J Sun Zhanhong HW# 12

9

1. a) $I_1(A^{-1}) = tr(A^{-1}) = \sum \lambda(A^{-1})$

$= \sum \frac{1}{\lambda(A)} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}$

$\frac{I_2(A)}{I_3(A)} = \frac{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3}{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}$

Therefore, $I_1(A^{-1}) = \frac{I_2(A)}{I_3(A)}$

b) $I_2(A^{-1}) = \frac{1}{2}\left[(tr(A^{-1}))^2 - tr\left((A^{-1})^2\right)\right]$

$= \frac{1}{2}\left[\left(\sum \frac{1}{\lambda(A)}\right)^2 - \sum \frac{1}{\lambda^2(A)}\right]$

$= \frac{1}{2}\left[\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)^2 - \left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2}\right)\right]$

$= \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3}$

$\frac{I_1(A)}{I_3(A)} = \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1\lambda_2\lambda_3} = \frac{1}{\lambda_1\lambda_2} + \frac{1}{\lambda_1\lambda_3} + \frac{1}{\lambda_2\lambda_3}$

Therefore, $I_2(A^{-1}) = \frac{I_1(A)}{I_3(A)}$

c) $I_3(A^{-1}) = det(A^{-1}) = (det(A))^{-1} = \frac{1}{\lambda_1\lambda_2\lambda_3} = \frac{1}{I_3(A)}$

Therefore, $I_3(A^{-1}) = \frac{1}{I_3(A)}$

d) $L.H.S = \frac{1}{6}\left([tr(A)]^3 - 3tr(A)tr(A^2) + 2tr(A^3)\right)$

$= \frac{1}{6}\left((\lambda_1+\lambda_2+\lambda_3)^3 - 3(\lambda_1+\lambda_2+\lambda_3)(\lambda_1^2+\lambda_2^2+\lambda_3^2) + 2(\lambda_1^3+\lambda_2^3+\lambda_3^3)\right)$

$= \frac{1}{6}[\cancel{\lambda_1^3} + \cancel{\lambda_2^3} + \cancel{\lambda_3^3} + \cancel{3\lambda_1^2\lambda_2} + \cancel{3\lambda_1\lambda_2^2} + \cancel{3\lambda_1^2\lambda_3} + \cancel{3\lambda_1\lambda_3^2} + \cancel{3\lambda_2^2\lambda_3} + \cancel{3\lambda_2\lambda_3^2}$
$+ 6\lambda_1\lambda_2\lambda_3 - \cancel{3\lambda_1^3} - \cancel{3\lambda_2^3} - \cancel{3\lambda_3^3} - \cancel{3\lambda_1\lambda_2^2} - \cancel{3\lambda_1\lambda_3^2} - \cancel{3\lambda_2\lambda_1^2} - \cancel{3\lambda_2\lambda_3^2} - \cancel{3\lambda_3\lambda_1^2}$
$- \cancel{3\lambda_3\lambda_2^2} + \cancel{2\lambda_1^3} + \cancel{2\lambda_2^3} + \cancel{2\lambda_3^3}]$

$= \lambda_1\lambda_2\lambda_3 = I_3(A)$

Therefore, $I_3(A) = \frac{1}{6}\left([tr(A)]^3 - 3tr(A)tr(A^2) + 2tr(A^3) \right)$

e) $A^{-1} = I_3(A^{-1})A^2 - I_2(A^{-1})A + I_1(A^{-1})\mathbb{I}$

$\mathbb{I} = I_3(A^{-1})A^3 - I_2(A^{-1})A^2 + I_1(A^{-1})A$

Characteristic eqn of A:

$P_A(\lambda) = \det(A - \lambda\mathbb{I}) = -\lambda^3 + I_1(A)\lambda^2 - I_2(A)\lambda + I_3(A) = 0$

$\lambda(A^{-1}) = \frac{1}{\lambda(A)}$

Then,

$P_{A^{-1}}(\lambda) = \det(A^{-1} - \lambda\mathbb{I}) = -\frac{1}{\lambda^3} + I_1(A^{-1})\frac{1}{\lambda^2} - I_2(A^{-1})\frac{1}{\lambda} + I_3(A^{-1})$

$= 0$

$\Rightarrow 1 = I_1(A^{-1})\lambda - I_2(A^{-1})\lambda^2 + I_3(A^{-1})\lambda^3$

According to thm,

We can substitude the eigenvalue λ in the eqn with A, the eqn still stands.

$\Rightarrow \mathbb{I} = I_3(A^{-1})A^3 - I_2(A^{-1})A^2 + I_1(A^{-1})A$

$\times A^{-1}$ on both sides $\Rightarrow A^{-1} = I_3(A^{-1})A^2 - I_2(A^{-1})A + I_1(A^{-1})$

$\times I_3(A)$ on both sides $\Rightarrow I_3(A)A^{-1} = A^2 - I_1(A)A + I_2(A)\mathbb{I}$

2. 1) $\epsilon_{ijk}\delta_{jk} = \epsilon_{ikk} = 0$

2) $\epsilon_{ijk}\epsilon_{mjk}\delta_{im} = (\epsilon_{ijk})^2$ $= 6$

3) $\epsilon_{ijk}\delta_{km}\delta_{jn} = \epsilon_{inm}$

4) $\epsilon_{ijk}\epsilon_{imn}\delta_{jm} = \epsilon_{ijk}\epsilon_{ijn}$

$= 2\delta_{kn}$

3. $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)^T$

1) $\nabla \times \nabla \phi = \begin{vmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ \partial\phi/\partial x_1 & \partial\phi/\partial x_2 & \partial\phi/\partial x_3 \end{vmatrix}$

$= 0$

2) RHS $= (\nabla f) g + (\nabla g) f$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} g_1 + \frac{\partial f_1}{\partial x_2} g_2 + \frac{\partial f_1}{\partial x_3} g_3 + \frac{\partial g_1}{\partial x_1} f_1 + \frac{\partial g_1}{\partial x_2} f_2 + \frac{\partial g_1}{\partial x_3} f_3 \\ \frac{\partial f_2}{\partial x_1} g_1 + \frac{\partial f_2}{\partial x_2} g_2 + \frac{\partial f_2}{\partial x_3} g_3 + \frac{\partial g_2}{\partial x_1} f_1 + \frac{\partial g_2}{\partial x_2} f_2 + \frac{\partial g_2}{\partial x_3} f_3 \\ \frac{\partial f_3}{\partial x_1} g_1 + \frac{\partial f_3}{\partial x_2} g_2 + \frac{\partial f_3}{\partial x_3} g_3 + \frac{\partial g_3}{\partial x_1} f_1 + \frac{\partial g_3}{\partial x_2} f_2 + \frac{\partial g_3}{\partial x_3} f_3 \end{pmatrix}$$

LHS $= \nabla (f \cdot g) = \nabla (f_1 g_1 + f_2 g_2 + f_3 g_3)$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1} g_1 + \frac{\partial g_1}{\partial x_1} f_1 + \frac{\partial f_2}{\partial x_1} g_2 + \frac{\partial g_2}{\partial x_1} f_2 + \frac{\partial f_3}{\partial x_1} g_3 + \frac{\partial g_3}{\partial x_1} f_3 \\ \frac{\partial f_1}{\partial x_2} g_1 + \frac{\partial g_1}{\partial x_2} f_1 + \frac{\partial f_2}{\partial x_2} g_2 + \frac{\partial g_2}{\partial x_2} f_2 + \frac{\partial f_3}{\partial x_3} g_3 + \frac{\partial g_3}{\partial x_3} f_3 \\ \frac{\partial f_1}{\partial x_3} g_1 + \frac{\partial g_1}{\partial x_3} f_1 + \frac{\partial f_2}{\partial x_3} g_2 + \frac{\partial g_2}{\partial x_3} f_2 + \frac{\partial f_3}{\partial x_3} g_3 + \frac{\partial g_3}{\partial x_3} f_3 \end{pmatrix}$$

Therefore, $\nabla (f \cdot g) \neq (\nabla f) g + (\nabla g) f$.

3) RHS $= (\nabla f) \times g - (\nabla g) \times f$

$$= \begin{pmatrix} \nabla f_1^T \\ \nabla f_2^T \\ \nabla f_3^T \end{pmatrix} \times \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} - \begin{pmatrix} \nabla g_1^T \\ \nabla g_2^T \\ \nabla g_3^T \end{pmatrix} \times \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}$$

$$= \begin{pmatrix} g_3 \nabla f_2^T - g_2 \nabla f_3^T \\ g_3 \nabla f_1^T - g_1 \nabla f_3^T \\ g_2 \nabla f_1^T - g_1 \nabla f_2^T \end{pmatrix} - \begin{pmatrix} f_3 \nabla g_2^T - f_2 \nabla g_3^T \\ f_3 \nabla g_1^T - f_1 \nabla g_3^T \\ f_2 \nabla g_1^T - f_1 \nabla g_2^T \end{pmatrix}$$

$$= \begin{pmatrix} g_3 \nabla f_2^T - g_2 \nabla f_3^T - f_3 \nabla g_2^T + f_2 \nabla g_3^T \\ g_3 \nabla f_1^T - g_1 \nabla f_3^T - f_3 \nabla g_1^T + f_1 \nabla g_3^T \\ g_2 \nabla f_1^T - g_1 \nabla f_2^T - f_2 \nabla g_1^T + f_1 \nabla g_2^T \end{pmatrix}$$

$$LHS = \nabla (f \times g) = \nabla \cdot \begin{pmatrix} f_2 g_3 - g_2 f_3 \\ f_1 g_3 - g_1 f_3 \\ f_1 g_2 - g_1 f_2 \end{pmatrix}$$

$$= \begin{pmatrix} g_3 \nabla f_2^T - g_2 \nabla f_3^T - f_3 \nabla g_2^T + f_2 \nabla g_3^T \\ g_3 \nabla f_1^T - g_1 \nabla f_3^T - f_3 \nabla g_1^T + f_1 \nabla g_3^T \\ g_2 \nabla f_1^T - g_1 \nabla f_2^T - f_2 \nabla g_1^T + f_1 \nabla g_2^T \end{pmatrix}$$

$$LHS = RHS.$$

4\. a) Since A is linear in X (only relate to t)

$\Rightarrow \frac{\partial y}{\partial x} = A$.

And $A \in SL \Rightarrow \det(A) = 1$

$\Rightarrow \det\left(\frac{\partial y}{\partial x}\right) = 1$

Therefore this deformation is locally volume preserving.

b) $\vec{y} = A(t)\vec{x} = \vec{x} + (A - \mathbb{I})\vec{x}$

$\vec{u}(\vec{x}) = (A - \mathbb{I})\vec{x}$ since A is constant matrix,

$\nabla\vec{u}(\vec{x}) = (A - \mathbb{I})\nabla\vec{x}$

$\nabla\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbb{I}$

$\Rightarrow \nabla\vec{u}(\vec{x}) = A - \mathbb{I}$

$\varepsilon = \frac{1}{2}(\nabla\vec{u}(\vec{x}))^T + \nabla\vec{u}(\vec{x})$

$= \frac{1}{2}(A^T - I) + A - I$

$= A + \frac{1}{2}A^T - \frac{3}{2}I$

the answer sample is wrong

C) $\vec{x} = x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3}$

$A\vec{x}$ is to perform a linear transform at $\vec{x}$.

And $A\vec{x}$ can still be represented as its basis.

Thus, $A\vec{x} = \sigma_1 x_1 \vec{u_1} + \sigma_2 x_2 \vec{u_2} + \sigma_3 x_3 \vec{u_3}$

σ_i are singular values of A

u_i are new basic of $A\vec{x}$.

we can let, $w_i = \sigma_i x_i$

Then, since $\vec{x}$ is on $\|\vec{x}\| \le 1$

$$\Rightarrow \quad \sum x_i^2 \le 1$$

we have new function:

$$\frac{w_1^2}{\sigma_1^2} + \frac{w_2^2}{\sigma_2^2} + \frac{w_3^2}{\sigma_3^2} = \sum x_i^2 \le 1$$

Therefore, this is a ellipsoid

(oval disk)

d) $dE = \frac{1}{2} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl}$

For this problem, strain is uniform.

$$\varepsilon = A + \frac{1}{2} A^T - \frac{3}{2} I$$

Since this is linear deformation,

$\sigma \propto \varepsilon \Rightarrow$ Each entry of C is the same.

Set in this case, $C_{ijkl} = c$.

In all, ε, C has nothing to do with the position of such sample point.

Then, $E = \int dE = \int \frac{1}{2} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \, dV = \frac{1}{2} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} \cdot V$

Since it is locally volume preserving deformation,

$$V = \int dy_1 dy_2 dy_3 = \int dx_1 dx_2 dx_3 = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$$

Thus, $E = \frac{2}{3}\pi \varepsilon_{ij} C_{ijkl} \varepsilon_{kl}$, where $C_{ijkl} = c$

and $\varepsilon = A + \frac{1}{2} A^T - \frac{3}{2} I$

5. a) $\nabla_x \cdot \vec{v} = \nabla_x \cdot A(t)\vec{x}$

take out

Cannot $= \boxed{A(t)}\ \nabla_x \cdot \vec{x}$

$= A(t)\cdot(1,1,1)^T \neq 0$

$\nabla_x \times \vec{v} = \boxed{A(t)} \cdot \nabla_x \times \vec{x}$

cannot take out $= A(t)\cdot \begin{vmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$

$= 0$

Therefore, $\vec{v}(\vec{x},t) = A(t)\vec{x}$ is irrotational.

b) $\nabla_x \cdot \vec{v} = \frac{\partial}{\partial x_1} a(t)x_2 + \frac{\partial}{\partial x_2} b(t)x_3 + \frac{\partial}{\partial x_3} c(t)x_3$

$= 0+0+0$

$= 0$

$\nabla_x \times \vec{v} = \begin{vmatrix} i & j & k \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ a(t)x_2 & b(t)x_3 & c(t)x_1 \end{vmatrix}$

$= -b(t)\,\vec{i} + c(t)\,\vec{j} - a(t)\,\vec{k}$

$\neq 0$

Therefore $\vec{v}(\vec{x},t) = [a(t)x_2, b(t)x_3, c(t)x_1]^T$

is incompressible.

6. 1) ① $\begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2a_1 & 2a_2+a_1a_3 \\ 0 & 1 & 2a_3 \\ 0 & 0 & 1 \end{pmatrix}$

② For matrix multiplication, $A(AA) = (AA)A = A^3$

③ $I \cdot A = A \cdot I = A$

④ $A^{-1} = \begin{pmatrix} 1 & -a_1 & a_1a_3 - a_2 \\ 0 & 1 & -a_3 \\ 0 & 0 & 1 \end{pmatrix}$

Therefore, this structed matrix forms a group.

2) $\begin{pmatrix} 1 & a_1 & a_2 \\ 0 & e^b & a_3 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_1 & a_2 \\ 0 & e^b & a_3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & (e^b+1)a_1 & 2a_2+a_1a_3 \\ 0 & e^{2b} & (e^b+1)a_3 \\ 0 & 0 & 1 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} 1 & -a_1e^{-b} & a_1a_3e^{-b} - a_2 \\ 0 & e^{-b} & -a_3e^{-b} \\ 0 & 0 & 1 \end{pmatrix}$ ← is this the same form?

A^{-1} and A does not have the same form, thus A does not form a group.

?

0

3) $\begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_1a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\det(A) = 0$, therefore A^{-1} does not exist

So, $\begin{pmatrix} 0 & a_1 & a_2 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{pmatrix}$ does not form a group

4)
$$\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} = \begin{pmatrix} -a_2^2-a_3^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & -a_1^2-a_3^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & -a_1^2-a_2^2 \end{pmatrix}$$

$\det(A) = 0$, therefore A^{-1} does not exist.

so, $\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$ does not form a group