Homework 11: ME5701, 2 November 2020, Due 9 November 2020

Problem 1. The moment-of-inertia matrix of a rigid body in 3D in its body fixed frame attached to its center of mass is defined as the following volume integral

$$\mathcal{I} \doteq \int_{Vol} \left\{(\mathbf{x}\cdot\mathbf{x})\mathbb{I} - \mathbf{x}\mathbf{x}^T\right\} \rho(\mathbf{x})\, dVol$$

where $\mathbb{I}$ is the 3×3 identity matrix, $\mathbf{x} = [x, y, z]^T$ is a position vector relative to the body fixed frame, $\rho(\mathbf{x})$ is the mass density defined over the 3D volume of the body, and $dVol = dx\, dy\, dz$.

An alternative kind of moment-of-inertia can be defined as

$$\mathcal{K} \doteq \int_{Vol} \mathbf{x}\mathbf{x}^T \rho(\mathbf{x})\, dVol\,.$$

This is like a covariance matrix, and if the total mass of the body $\int_{Vol} \rho(\mathbf{x}) dVol = 1$, then $\mathcal{K}$ is exactly covariance (with mass density becoming probability density).

Show that

$$\mathcal{I} = \mathrm{tr}[\mathcal{K}]\mathbb{I} - \mathcal{K}$$

and solve for $\mathcal{K}$ in terms of $\mathcal{I}$.

Problem 2. Consider a solid ellipsoid parameterized as

$$\mathbf{x}(r, \phi, \theta) = \begin{pmatrix} ar\sin\theta\cos\phi \\ br\sin\theta\sin\phi \\ cr\cos\theta \end{pmatrix}$$

where $r \in [0, 1]$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ and a, b, c are constants called the semi-axis lengths of the ellipsoid. The surface defined by $r = 1$ satisfies $(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$.

a) If the solid ellipsoid has uniform mass density, $\rho(\mathbf{x}) = \rho_0$, compute the volume integral

$$Vol = \int_{Vol} dVol = \int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \left|\det\left[\frac{\partial \mathbf{x}}{\partial r}, \frac{\partial \mathbf{x}}{\partial \phi}, \frac{\partial \mathbf{x}}{\partial \theta}\right]\right| d\phi d\theta dr$$

to obtain the mass, $m = \rho_0 Vol$.

b) For this solid ellipsoid compute the matrix $\mathcal{K}$ defined in problem 1, and from it compute $\mathcal{I}$.

Problem 3. We learned in class that if $\mathcal{I}$ is computed in a body-fixed reference frame at the center of mass that the kinetic energy of a rigid body is

$$T = \frac{1}{2}\omega_b^T \mathcal{I} \omega_b$$

where $\omega_b = \text{vect}(R^T \dot{R})$ is the body-fixed description of angular velocity. Show that this same kinetic energy can be written in terms of $\mathcal{K}$ as

$$T = \frac{1}{2}\text{tr}\left[\dot{R}\mathcal{K}\dot{R}^T\right] .$$

<u>Problem 4.</u> The kinetic energy due to rotation of a rigid body using the reference frame attached to its center of mass and aligned with principal moments of inertia (i.e., the eigenvectors of the moment of inertia) is

$$T_{rot} = \frac{1}{2}\left\{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right\} .$$

If the orientation of this body is described in ZXZ Euler angles as $R(\alpha, \beta, \gamma) = R_3(\alpha)R_1(\beta)R_3(\gamma)$ then we know that the body-fixed description of angular velocity is

$$\begin{aligned}
\omega_1 &= \dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma \\
\omega_2 &= \dot{\alpha}\sin\beta\cos\gamma - \dot{\beta}\sin\gamma \\
\omega_3 &= \dot{\alpha}\cos\beta + \dot{\gamma} .
\end{aligned}$$

a) Consider a spinning top with mass m and principal moments of inertia $I_1 = I_2$ and I_3 about its center of mass. That is

$$\mathcal{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} .$$

When $I_1 = I_2$, as in this case, show that the expression for T_{rot} simplifies, and write it.

b) If the center of mass is a distance of L from the pivot along the body-fixed z axis, then the position of the center of mass as seen in the space-fixed reference frame will be

$$\mathbf{x}_{cm} = R(\alpha, \beta, \gamma)(L\mathbf{e}_3) .$$

Using this fact, compute the kinetic energy due to translation of the center of mass

$$T_{trans} = \frac{1}{2}m\dot{\mathbf{x}}_{cm} \cdot \dot{\mathbf{x}}_{cm}$$

and the potential energy due to gravity, $V = mgh$, where h is the height to the center of mass relative to the x-y plane.

c) Using Lagrange's equations, write the equations of motion for this system where V from part b is used, and the total kinetic energy is $T = T_{rot} + T_{trans}$.

d) Evaluate the above equations for the case of stable precession, defined by the conditions that β is constant and all second derivatives of all Euler angles are equal to zero. In that case, what must the relationship between $\dot{\alpha}$ and $\dot{\gamma}$ be ?