$$I_{ss}(x,y,t) = I(x,y) * G_{\sqrt{t}}(x,y). \quad ①$$

$$G_b(x,y) = \frac{1}{2\pi b^2} e^{-(x^2+y^2)/2b^2} \quad ②$$

$$\therefore \quad G_{\sqrt{t}}(x,y) = \frac{1}{2\pi t} e^{-(x^2+y^2)/2t} \quad ③$$

From ①: $\frac{\partial}{\partial t} I_{ss}(x,y,t) = \frac{\partial}{\partial t}\left[I(x,y) * G_{\sqrt{t}}(x,y)\right]$

From the convolutional differential property,

$$\frac{d^n}{dt^n}\left[f_1(t) * f_2(t)\right] = \frac{d^n}{dt^n} f_1(t) * f_2(t) = f_1(t) * \frac{d^n}{dt^n} f_2(t) \quad ④$$

$$\frac{\partial}{\partial t}\left[I(x,y) * G_{\sqrt{t}}(x,y)\right] = I(x,y) * \frac{\partial}{\partial t}\left(G_{\sqrt{t}}(x,y)\right) \quad ⑤$$

From ③:

$$\frac{\partial}{\partial t} G_{\sqrt{t}}(x,y) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi t} e^{-(x^2+y^2)/2t}\right)$$

$$= -\frac{1}{2\pi} t^{-2} \cdot e^{-(x^2+y^2)/2t} + \frac{1}{2\pi t}\left(\frac{x^2+y^2}{2t^2}\right) \cdot e^{-(x^2+y^2)/2t}$$

$$= \left(-\frac{1}{t}\right) \cdot \frac{1}{2\pi t} e^{-(x^2+y^2)/2t} + \left(\frac{x^2+y^2}{2t^2}\right) \cdot \frac{1}{2\pi t} e^{-(x^2+y^2)/2t}$$

$$\frac{\partial}{\partial t}\left(G_{\sqrt{t}}(x,y)\right) = -\frac{1}{t} \cdot G_{\sqrt{t}}(x,y) + \frac{x^2+y^2}{2t^2} G_{\sqrt{t}}(x,y)$$

$$= \frac{x^2+y^2-2t}{2t} \cdot G_{\sqrt{t}}(x,y) \quad ⑥$$

From ①, ④, ⑤, ⑥ can get:

$$\frac{\partial}{\partial t} I_{ss}(x,y,t) = I(x,y) * \left[\frac{x^2+y^2-2t}{2t} \cdot G_{\sqrt{t}}(x,y)\right] \quad ⑦$$

Then, calculate $\frac{1}{2}\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right) I_{ss}(x,y,t)$. (Δ)

So, $\frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) I_{ss}(x,y,t)$

$$= \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} I_{ss}(x,y,t) + \frac{\partial^2}{\partial y^2} I_{ss}(x,y,t)\right) \quad ⑧$$

From: ①, ④:

$$\frac{\partial^2}{\partial x^2} I_{ss}(x,y,t) = I(x,y) \cdot \frac{\partial^2}{\partial x^2} G_{\sqrt{t}}(x,y)$$

$$\frac{\partial^2}{\partial x^2} G_{\sqrt{t}}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} \frac{1}{2\pi t} e^{-(x^2+y^2)/2t}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{1}{2\pi t} \cdot \frac{x}{t} \cdot e^{-(x^2+y^2)/2t}\right).$$

$$\frac{\partial^2}{\partial x^2} L_{GS}(x,y,t) = \left(-\frac{1}{2\pi t^2}\right)\cdot\left(1-\frac{x^2}{t}\right)\cdot e^{-(x^2+y^2)/2t}$$

$$= -\frac{t-x^2}{t^2}\cdot\left(\frac{1}{2\pi t}\right)\cdot e^{-(x^2+y^2)/2t}$$

From ③ $$= \frac{x^2-t}{t^2}\cdot G_{\sqrt{t}}(x,y)\cdot \quad ⑨$$

The same reasoning lead to

$$\frac{\partial^2}{\partial y} L_{GS}(x,y,t) = \frac{y^2-t}{t^2} G_{\sqrt{t}}(x,y) \quad ⑩$$

From ⑧, ⑨, ⑩

$$\frac{1}{2}\Delta L_{GS}(x,y,t)$$

$$= \frac{1}{2}\cdot I(x,y) * \left[\left(\frac{x^2-t}{t^2}+\frac{y^2-t}{t^2}\right)G_{\sqrt{t}}(x,y)\right]$$

$$= I(x,y) * \left[\frac{x^2+y^2-2t}{2t} G_{\sqrt{t}}(x,y)\right]$$

From ⑦ : $$= \frac{\partial}{\partial t} L_{GS}(x,y,t)$$

Q.E.D.

$$\frac{\partial}{\partial t} L_{GS}(x,y,t) = \frac{1}{2}\Delta L_{GS}(x,y,t)$$