GRADE 12 EXAMINATION
NOVEMBER 2017

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

MARKING GUIDELINES

Time: 2 hours | 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 (a) $(\ln x)^2 + 2\ln x - 3 = 0$

$k = \ln x$

$(k+3)(k-1) = 0$

$\ln x = -3 \qquad \ln x = 1$

$x = e^{-3} = 0{,}0498 \qquad x = e = 2{,}718$

(b) $|x+p| = \ln q$

$x \geq -p: \qquad x + p = \ln q$

$\qquad\qquad x = \ln q - p$

$x \leq -p: \qquad -x - p = \ln q$

$\qquad\qquad x = -\ln q - p$

1.2 (a) $(0;\ 3)$

(b) $0 = x^2 + |2x-3|$

$x^2 \neq -|2x-3|$

since LHS and RHS cannot both be zero simultaneously

(c) $\left(\frac{3}{2}; \frac{9}{4}\right)$ (Critical point is at zero of abs value term)

(d) $y = x^2 - 2x + 3$ (Note that other branch does not contain tp)

$0 = 2x - 2$

$\therefore x = 1;\ y = 2$

QUESTION 2

2.1 $889 = Ae^{15k}$

$596 = Ae^{5k}$

$\therefore e^{10k} = \frac{889}{596}$

$\therefore k = 0{,}04$

$\therefore A = 488$

2.2 $6000 = 488e^{0{,}04t}$

$\therefore t = 62{,}72$

$\therefore$ *year* 2032 (*or accept* 2033)

QUESTION 3

3.1 $x = \dfrac{-p \pm \sqrt{p^2 - 4p}}{2p}$

$p(p-4) < 0$

0 4

$p = 2$

3.2 $x = -2i$ is also a solution

$\therefore x^2 + 4$ is a factor

$\left(x^2+4\right)\left(x^2-2x+5\right) = x^4 - 2x^3 + px^2 - 8x + 20$

$\therefore x = 1 \pm 2i$ and $p = 9$

3.3 Note that $i + i^2 + i^3 + i^4 = 0$

Therefore: $i + i^2 + i^3 + i^4 + ... + i^{2016} = 0$

Therefore answer = i

QUESTION 4

Prove true for n = 2:

LHS = $1 - \dfrac{1}{4} = \dfrac{3}{4}$ RHS = $\dfrac{2+1}{2(2)} = \dfrac{3}{4}$

Assume true for $n = k$:

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)..........\left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$$

Prove true for $n = k + 1$

$$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{9}\right)..........\left(1-\frac{1}{k^2}\right)\left(1-\frac{1}{(k+1)^2}\right)$$

$= \left(\dfrac{k+1}{2k}\right)\left(1-\dfrac{1}{(k+1)^2}\right)$ by assumption

$$= \left(\frac{k+1}{2k}\right)\left(\frac{(k+1)^2-1}{(k+1)^2}\right)$$

$$= \frac{k(k+2)}{2k(k+1)}$$

$$= \frac{k+2}{2(k+1)}$$

But this is the formula with $n = k + 1$. Therefore, we have proved by PMI that the result is true for all natural values of n.

QUESTION 5

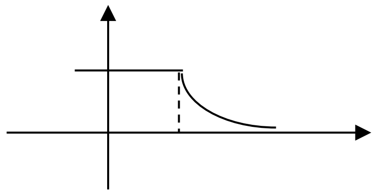
5.1 (a) $\lim_{x\to1^-} f(x) = \lim_{x\to1} 4 = 4$

$\lim_{x\to1^+} f(x) = \lim_{x\to1} \frac{4}{x} = 4$

$f(1) = 4$

Therefore, continuous at x = 1.

Not differentiable due to sudden change of gradient.



(b) $\lim_{x\to2^-} f'(x) = \lim_{x\to2^+} f'(x)$

$\frac{-4}{x^2} = a$

$\therefore a = -1$

$\frac{4}{x} = -x + b$

$\therefore b = 4$

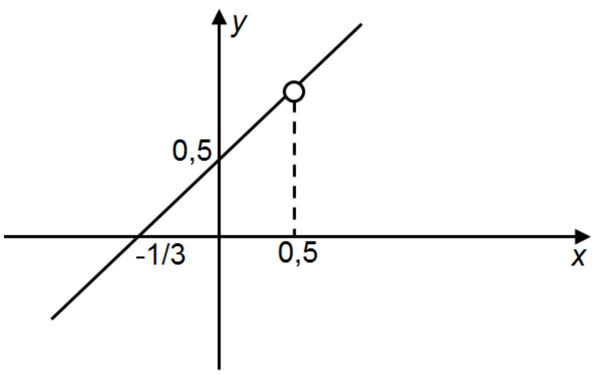
5.2 (a) $6x^2 - x - 1 = (3x-2)(2x+1) + k$

$p = 3$

(b) (i) $f(x) = \frac{(3x+1)(2x-1)}{2(2x-1)}$

Removable discontinuity at x = 0,5. Factor cancels out.

(ii) $\therefore y = \frac{3}{2}x + \frac{1}{2};\quad x \neq \frac{1}{2}$



(c) $f'(x) = \frac{(3x-2)(12x-1)-(6x^2-x-1)(3)}{(3x-2)^2}$

$\therefore 0 = 18x^2 - 24x + 5 \qquad i.e.\ \Delta > 0$

QUESTION 6

6.1 Using the cosine rule:

$10^2 = 10^2 + 8^2 - 2(10)(8)\cos\hat{O}$

$\therefore \hat{O} = 1{,}159$ radians (4)

6.2 Area of sector = $\frac{1}{2}(10)^2(1{,}159) = 57{,}96$ units^2

Area of triangle = $\frac{1}{2}(8)(10)\sin 1{,}159 = 36{,}656$

Shaded area = 21,3 units^2

QUESTION 7

7.1 $y = -\left(4x+3\right)^{-\frac{1}{2}}$

$$\therefore \frac{dy}{dx} = \frac{1}{2}\left(4x+3\right)^{-\frac{3}{2}}(4)$$

$$= \frac{2}{\left(4x+3\right)^{\frac{3}{2}}}$$

$m = 2$ and $n = \frac{3}{2}$

7.2 (a) $\cos y\dfrac{dy}{dx} - \sin x = 0$

$$\therefore \frac{dy}{dx} = \frac{\sin x}{\cos y}$$

(b) $x = \dfrac{\pi}{3}$:

$$\sin y + \frac{1}{2} = 1$$

$$\sin y = \frac{1}{2}$$

$$y = \frac{\pi}{6}$$

$$\therefore \frac{dy}{dx} = \frac{\sin\frac{\pi}{3}}{\cos\frac{\pi}{6}} = 1$$

7.3 (a) $x_{r+1} = x_r - \dfrac{\tan x + x^2 + 1}{\dfrac{1}{\cos^2 x} + 2x}$

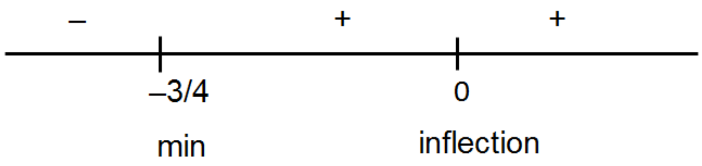
$x_0 = -1$
$x1 = -1{,}31047803...$
$x2 = -1{,}227348576...$
$x3 = -1{,}181802226...$
$x4 = -1{,}172412988...$

(b) $x5 = -1{,}172093968...$
$x6 = -1{,}172093617...$
$x7 = -1{,}1720936...$

QUESTION 8

8.1 $4x^3 + 3x^2 = 0$

$\therefore x^2(4x+3) = 0$



$g''(x) = 12x^2 + 6x$

$0 = 6x(2x+1)$

$x = 0$ or $x = -\frac{1}{2}$

Therefore x = 0 is stationary (noting sign of gradient does not change) and $x = -\frac{1}{2}$ is non-stationary.

8.2 $y = \int 4x^3 + 3x^2 dx$

$y = x^4 + x^3 + C$

$4 = 1 + 1 + C$

$y = x^4 + x^3 + 2$

8.3 For a cubic, f''(x) is linear and hence f''(x) = 0 always has a solution.
For a quartic, f''(x) is quadratic and hence f''(x) = 0 may not have real solutions.

QUESTION 9

9.1 (a) $RHS = \sec^2\theta\left(\tan^2\theta + 1\right)$

$$= \sec^2\theta\left(\sec^2\theta\right)$$

$$= \sec^4\theta$$

(b) $\int \sec^2\theta.\tan^2\theta + \sec^2\theta \, d\theta$

$$= \frac{\tan^3\theta}{3} + \tan\theta + C$$

9.2 (a) $\int \sin^2 x + \cos^2 x + 2\sin x\cos x \; dx$

$$= \int 1 + \sin 2x \, dx$$

$$= x - \frac{\cos 2x}{2} + C$$

(b) $\frac{1}{6}\int 6\left(x-2\right)\left(3x^2 - 12x + 5\right)^{\frac{1}{2}} dx$

$$= \frac{1}{9}\left(3x^2 - 12x + 5\right)^{\frac{3}{2}} + C$$

QUESTION 10

10.1 The turning point of the graph is (2; 4). At the turning point, the rectangles change from under-approximating to over-approximating so the error cancels out to some extent.

10.2 (a) $2 \times 2 = 4$

(b) $h(-x) = \frac{3}{2}h(x)$

$\frac{3}{2} \times 2 + 2 = 5$ units

(c) $2 - 2 = 0$

10.3 (a) $y = a\left(x - \frac{p}{2}\right)^2 + \frac{1}{p}$

$0 = a\left(\frac{p^2}{4}\right) + \frac{1}{p}$

$a = -\dfrac{4}{p^3}$

(b) Area = $\displaystyle\int_0^p -\frac{4}{p^3}\left(x - \tfrac{p}{2}\right)^2 + \frac{1}{p}\,dx$

$$= \left[-\frac{4}{p^3}\frac{\left(x - \frac{p}{2}\right)^3}{3} + \frac{1}{p}x\right]_0^p$$

$$= \frac{2}{3}$$

Total: 200 marks