# Equivalence of Norms on $\mathbb{R}^n$ Using the $\ell^\infty$ Norm

In finite-dimensional vector spaces, all norms are equivalent.

In this document, we prove that any two norms on $\mathbb{R}^n$ are equivalent, and we do so by using the $\ell^\infty$ norm as the fixed reference norm (instead of the Euclidean norm).

**Notation.** For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, define

$$\|x\|_\infty := \max_{1 \le i \le n} |x_i|.$$

Its closed unit ball is the cube

$$\overline{B_\infty(0,1)} = \{x : \|x\|_\infty \le 1\} = [-1,1]^n.$$

**Theorem.** Let $\|\cdot\|_a$ and $\|\cdot\|_b$ be two norms on $\mathbb{R}^n$. Then there exist constants $c, C > 0$ such that for all $x \in \mathbb{R}^n$,

$$c\|x\|_a \le \|x\|_b \le C\|x\|_a.$$

Consequently, the metrics $d_a(x,y) = \|x-y\|_a$ and $d_b(x,y) = \|x-y\|_b$ are bi-Lipschitz equivalent.

**Proof.**

We will prove the result in three logically separate parts:

**(I) A compact set for $\|\cdot\|_\infty$.**

Consider the $\|\cdot\|_\infty$-unit sphere

$$S_\infty := \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}.$$

We claim that $S_\infty$ is compact.

First, $S_\infty$ is closed because $x \mapsto \|x\|_\infty$ is continuous and

$$S_\infty = \|\cdot\|_\infty^{-1}(\{1\}).$$

Second, $S_\infty \subset [-1,1]^n$, and $[-1,1]^n$ is compact (finite product of compact intervals). Therefore, $S_\infty$ is a closed subset of a compact set, hence compact.

**(II) A uniform comparison between $\|\cdot\|_a$ and $\|\cdot\|_\infty$.**

Define the continuous function

$$g : S_\infty \to \mathbb{R}, \qquad g(u) = \|u\|_a.$$

Since $S_\infty$ is compact, $g$ attains its minimum:

$$m_\infty := \min_{u \in S_\infty} \|u\|_a.$$

We have $m_\infty > 0$, because $\|u\|_a = 0$ implies $u = 0$, but $0 \notin S_\infty$.

Now take any $x \in \mathbb{R}^n$, $x \neq 0$. Set

$$u := \frac{x}{\|x\|_\infty}.$$

Then $\|u\|_\infty = 1$, hence $u \in S_\infty$. By homogeneity of the norm $\|\cdot\|_a$,

$$\|x\|_a = \|\|x\|_\infty\, u\|_a = \|x\|_\infty\, \|u\|_a \geq \|x\|_\infty\, m_\infty.$$

Thus for all $x \in \mathbb{R}^n$,

$$\|x\|_\infty \leq \frac{1}{m_\infty}\|x\|_a. \tag{$*$}$$

**(III) Compactness of the $\|\cdot\|_a$-unit sphere and comparison with $\|\cdot\|_b$.**

Let

$$S_a := \{x \in \mathbb{R}^n : \|x\|_a = 1\}.$$

We claim that $S_a$ is compact.

First, $S_a$ is closed since $x \mapsto \|x\|_a$ is continuous and

$$S_a = \|\ \cdot\ \|_a^{-1}(\{1\}).$$

Second, $S_a$ is bounded in $\|\cdot\|_\infty$. Indeed, if $x \in S_a$, then $\|x\|_a = 1$, and by $(*)$,

$$\|x\|_\infty \leq \frac{1}{m_\infty}\|x\|_a = \frac{1}{m_\infty}.$$

Hence

$$S_a \subset \left[-\frac{1}{m_\infty}, \frac{1}{m_\infty}\right]^n,$$

and the cube on the right-hand side is compact. Since $S_a$ is closed and contained in a compact set, $S_a$ is compact.

Now define the continuous function

$$f : S_a \to \mathbb{R}, \qquad f(u) = \|u\|_b.$$

Since $S_a$ is compact, $f$ attains a minimum and a maximum:

$$m := \min_{u \in S_a} \|u\|_b, \qquad M := \max_{u \in S_a} \|u\|_b.$$

We have $M < \infty$, and also $m > 0$ because $\|u\|_b = 0 \Rightarrow u = 0$ and $0 \notin S_a$.

Therefore, for all $u \in S_a$,

$$m \leq \|u\|_b \leq M. \tag{$\dagger$}$$

Finally, take any $x \in \mathbb{R}^n$, $x \neq 0$, and normalize it with respect to $\|\cdot\|_a$:

$$u := \frac{x}{\|x\|_a}.$$

Then $\|u\|_a = 1$, hence $u \in S_a$. By homogeneity of $\|\cdot\|_b$,

$$\|x\|_b = \|\|x\|_a\, u\|_b = \|x\|_a\, \|u\|_b.$$

Using (†),

$$m\|x\|_a \le \|x\|_b \le M\|x\|_a.$$

For $x = 0$, the inequality is trivial.

This proves the equivalence of $\|\cdot\|_a$ and $\|\cdot\|_b$.

□

**Corollary (Bi-Lipschitz equivalence of induced metrics).** Let $d_a(x, y) = \|x - y\|_a$ and $d_b(x, y) = \|x - y\|_b$. Then for all $x, y \in \mathbb{R}^n$,

$$d_b(x, y) \le M\, d_a(x, y), \qquad d_a(x, y) \le \frac{1}{m}\, d_b(x, y),$$

so the identity map is bi-Lipschitz between $(\mathbb{R}^n, d_a)$ and $(\mathbb{R}^n, d_b)$.

**Summary of the logical structure:**

1. Use $\|\cdot\|_\infty$ to build a compact sphere $S_\infty$.

2. On $S_\infty$, the continuous function $u \mapsto \|u\|_a$ has a positive minimum $m_\infty > 0$.

3. This gives the global inequality $\|x\|_\infty \le \frac{1}{m_\infty}\|x\|_a$ and implies $S_a$ is contained in a compact cube.

4. Hence $S_a$ is compact.

5. On $S_a$, the continuous function $u \mapsto \|u\|_b$ has a positive minimum and finite maximum, yielding $m\|x\|_a \le \|x\|_b \le M\|x\|_a$.