Compactness implies Sequential Compactness in Metric Spaces

In metric spaces, compactness is equivalent to sequential compactness.

I would like to construct a convergent subsequence from an arbitrary sequence in X.

Let (X, d) be a metric space. Let $(x_n) \subset X$ be a sequence.

I want to find a convergent subsequence of (x_n) whose limit belongs to X.

The first thing to observe is that if some value of the sequence appears infinitely often, then we are done.

So, let us assume that this is not the case. This implies that

$$A = \{x_n : n \in \mathbb{N}\}$$

is an infinite set.

Since X is compact, every closed subset of X is also compact. Therefore, the adherence (closure) $\overline{A} \subset X$ is compact.

This allows us to apply the finite subcover property to any open cover of $\overline{A}$.

Let us define an open cover of $\overline{A}$ by the family of open balls

$$\{B(x_n, 1)\}_{n \in \mathbb{N}}.$$

By compactness of $\overline{A}$, there exists a finite subcover. Hence,

$$\overline{A} \subset B(x_{n_1}, 1) \cup \cdots \cup B(x_{n_k}, 1).$$

Since A is infinite, at least one of these balls contains infinitely many points of A. Denote such a ball by $B_1 = B(x_{n_1}, 1)$.

Choose $x_{m_1} \in A \cap B_1$ as the first element of the subsequence.

Now consider the compact set $\overline{A} \cap B_1$. Cover it with open balls of radius $1/2$ centered at points of A. By compactness, we can extract a finite subcover, and again at least one of these balls contains infinitely many points of A. Denote this ball by $B_2 \subset B_1$.

Choose $x_{m_2} \in A \cap B_2$ with $m_2 > m_1$.

Proceeding inductively, we construct a nested sequence of open balls

$$B_1 \supset B_2 \supset B_3 \supset \cdots$$

such that the radius of B_k is $1/k$ and each B_k contains infinitely many points of A.

At each step, we choose an element $x_{m_k} \in A \cap B_k$ with $m_k > m_{k-1}$, thereby constructing a subsequence (x_{m_k}).

Since the balls are nested and their radii tend to zero, the subsequence (x_{m_k}) is Cauchy. Because $\overline{A}$ is compact (and hence complete), the subsequence converges to some point $x \in \overline{A} \subset X$.

Therefore, every sequence in X admits a convergent subsequence with limit in X, proving that X is sequentially compact.

$\square$

I would like to finish the document by describing systematically the steps of the proof:

1. We define the set $A = \{x_n : n \in \mathbb{N}\}$.

2. We assume that no element of A appears infinitely many times in the sequence. Otherwise, the sequence has a constant convergent subsequence and we are done.

3. We consider the closure $\overline{A}$. Since $\overline{A}$ is a closed subset of the compact space X, it is compact.

4. We define an open cover of $\overline{A}$ using open balls of radius 1 centered at points of A.

5. By compactness, we extract a finite subcover. At least one of these balls contains infinitely many points of A. From this ball $B(x_i, 1)$, we pick one element to begin constructing the subsequence.

6. We define a new open cover of $\overline{A} \cap B(x_i, 1)$ using open balls of radius $1/2$, and repeat the argument. Iterating this process, we obtain a nested sequence of open balls with radii tending to zero.

7. From each ball in the nested sequence, we select one element, forming a subsequence. This subsequence is Cauchy because the balls are nested and their radii go to zero. Since $\overline{A}$ is compact (and hence complete), the subsequence converges to some point $x \in X$.