Uniqueness of Limits for Sequences and Subsequences in Metric Spaces

Let (X, d) be a metric space. Let (x_n) be a sequence in X that converges to $x \in X$, and let (x_{n_k}) be a subsequence of (x_n) that converges to $y \in X$. We want to prove that $x = y$.

Let $\varepsilon > 0$ be arbitrary. We aim to show that $d(x, y) < \varepsilon$.

By the triangle inequality,

$$d(x, y) \leq d(x, x_n) + d(x_n, y),$$

and again by the triangle inequality,

$$d(x_n, y) \leq d(x_n, x_{n_k}) + d(x_{n_k}, y).$$

Since $x_n \to x$, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$,

$$d(x, x_n) < \frac{\varepsilon}{3}.$$

Since $x_{n_k} \to y$, there exists $N_2 \in \mathbb{N}$ such that for all $k > N_2$,

$$d(x_{n_k}, y) < \frac{\varepsilon}{3}.$$

Because (x_n) converges, it is a Cauchy sequence. Therefore, there exists $N_3 \in \mathbb{N}$ such that for all $n, k > N_3$,

$$d(x_n, x_{n_k}) < \frac{\varepsilon}{3}.$$

Let

$$N = \max\{N_1, N_2, N_3\}.$$

Then for all $n, k > N$, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, x_{n_k}) + d(x_{n_k}, y) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $d(x, y) = 0$, and hence

$$x = y.$$

$\square$