Sequentially Compact and Lebesgue Number

Let (X, d) be a metric space.

Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover of X, i.e.

$$X = \bigcup_{\alpha \in A} U_\alpha.$$

X is sequentially compact. That means that for every sequence (x_n) in X, there exists a convergent subsequence (x_{n_k}). The limit of (x_{n_k}) belongs to X.

I want to prove that X has a Lebesgue number. That means that there exists $r > 0$ such that for all $x \in X$ there exists $\alpha \in A$ such that

$$B(x, r) \subset U_\alpha.$$

Assume such an r does not exist. Then for any $r > 0$, there exists $x \in X$ such that for every $\alpha \in A$,

$$B(x, r) \not\subset U_\alpha.$$

In particular, for $r = 1$ choose x_1 such that for every $\alpha \in A$,

$$B(x_1, 1) \not\subset U_\alpha.$$

For $r = \frac{1}{2}$ choose x_2 such that for every $\alpha \in A$,

$$B(x_2, \tfrac{1}{2}) \not\subset U_\alpha.$$

In general, for each $n \in \mathbb{N}$ choose x_n such that for every $\alpha \in A$,

$$B(x_n, \tfrac{1}{n}) \not\subset U_\alpha.$$

Thus we have constructed a sequence (x_n) in X. Since X is sequentially compact, there exists a convergent subsequence (x_{n_k}) with limit $x \in X$.

Since $\mathcal{U}$ is an open cover of X, there exists $\alpha \in A$ such that

$$x \in U_\alpha.$$

Because U_α is open, there exists $r > 0$ such that

$$B(x, r) \subset U_\alpha.$$

Since $x_{n_k} \to x$, there exists K such that for all $k > K$,

$$d(x_{n_k}, x) < \tfrac{r}{2}.$$

Choose k large enough so that $\frac{1}{n_k} < \frac{r}{2}$. Then, for any $y \in B\left(x_{n_k}, \frac{1}{n_k}\right)$,

$$d(y,x) \leq d(y, x_{n_k}) + d(x_{n_k}, x) < \tfrac{r}{2} + \tfrac{r}{2} = r.$$

Hence

$$B\left(x_{n_k}, \tfrac{1}{n_k}\right) \subset B(x,r) \subset U_\alpha,$$

which contradicts the construction of x_{n_k}.

$\square$