Sequential Compactness implies Compactness in Metric Spaces

In metric spaces, compactness is equivalent to sequential compactness.

In this document, we prove that sequential compactness implies compactness.

Let (X, d) be a metric space, and let X be sequentially compact.

We argue by contradiction.

Assume that X is not compact. Then there exists an open cover of X that admits no finite subcover.

Since metric spaces are Lindelöf, we may assume without loss of generality that this open cover is countable. That is, there exists a family of open sets

$$\{U_n\}_{n\in\mathbb{N}}$$

such that

$$X \subset \bigcup_{n=1}^{\infty} U_n,$$

but for every $N \in \mathbb{N}$,

$$X \not\subseteq \bigcup_{n=1}^{N} U_n.$$

For each $N \in \mathbb{N}$, since the first N sets do not cover X, we may choose a point

$$x_N \in X \setminus \bigcup_{n=1}^{N} U_n.$$

This defines a sequence $(x_N) \subset X$.

By construction, for every fixed $k \in \mathbb{N}$, if $N \geq k$ then

$$x_N \notin U_k.$$

Since X is sequentially compact, the sequence (x_N) admits a convergent subsequence (x_{N_j}) such that

$$x_{N_j} \to x \in X.$$

Because $\{U_n\}_{n\in\mathbb{N}}$ is an open cover of X, there exists some $k \in \mathbb{N}$ such that

$$x \in U_k.$$

Since U_k is open and $x_{N_j} \to x$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$,

$$x_{N_j} \in U_k.$$

On the other hand, since $N_j \to \infty$, there exists $j_1 \in \mathbb{N}$ such that for all $j \geq j_1$,

$$N_j \geq k,$$

which implies

$$x_{N_j} \notin U_k.$$

This is a contradiction.

Therefore, our assumption that X is not compact is false. Hence, X is compact.

$\square$

We conclude by summarizing the steps of the proof:

1. We assume that X is sequentially compact but not compact.

2. From non-compactness, we obtain a countable open cover $\{U_n\}_{n \in \mathbb{N}}$ with no finite subcover.

3. For each N, we choose a point $x_N \in X$ that is not covered by the first N open sets.

4. The resulting sequence (x_N) has a convergent subsequence by sequential compactness.

5. The limit of this subsequence belongs to some open set U_k in the cover.

6. Openness of U_k forces the tail of the subsequence to lie in U_k, contradicting the construction of the sequence.

7. This contradiction shows that a finite subcover must exist, proving that X is compact.