Aprendizaje Automático

Review of Linear Algebra

Máster en Bioinformática y Biología Computacional

General Notation

❒ **Vector** – We note $x \in \mathbb{R}^n$ a vector with n entries, where $x_i \in \mathbb{R}$ is the i^{th} entry:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Another notation: **x**

❒ **Matrix** – We note $A \in \mathbb{R}^{m \times n}$ a matrix with n rows and m, where $A_{i,j} \in \mathbb{R}$ is the entry located in the i^{th} row and j^{th} column:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} \in \mathbb{R}^{m \times n}$$

Another notation: **A**

Remark: the vector x defined above can be viewed as a $n \times 1$ matrix and is more particularly called a column-vector.

General Notation

❒ **Identity matrix** – The identity matrix $I \in \mathbb{R}^{n\times n}$ is a square matrix with ones in its diagonal and zero everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Remark: for all matrices $A \in \mathbb{R}^{n\times n}$, we have $A \times I = I \times A = A$.

Matrix Operations

❒ **Vector-vector multiplication** – There are two types of vector-vector products:

- inner product: for $x, y \in \mathbb{R}^n$, we have:

$$\boxed{x^T y = \sum_{i=1}^{n} x_i y_i \in \mathbb{R}}$$

- outer product: for $x \in \mathbb{R}^m, y \in \mathbb{R}^n$, we have:

$$\boxed{xy^T = \begin{pmatrix} x_1 y_1 & \cdots & x_1 y_n \\ \vdots & & \vdots \\ x_m y_1 & \cdots & x_m y_n \end{pmatrix} \in \mathbb{R}^{m \times n}}$$

Matrix Operations

❒ **Matrix-vector multiplication** – The product of matrix $A \in \mathbb{R}^{m \times n}$ and vector $x \in \mathbb{R}^{n}$ is a vector of size $\mathbb{R}^{m}$, such that:

$$\underset{A}{\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}} \times \underset{x}{\begin{bmatrix} \; \\ \\ \\ \end{bmatrix}} = \underset{y}{\begin{bmatrix} \; \\ \\ \end{bmatrix}}$$

m x n matrix (m rows, n columns)

n x 1 matrix (n-dimensional vector)

m-dimensional vector

Matrix Operations

❒ **Matrix-matrix multiplication** – The product of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ is a matrix of size $\mathbb{R}^{m\times p}$, such that:

$$A \times B = C$$

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \times \begin{bmatrix} \quad & \quad \\ \quad & \quad \\ \quad & \quad \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

m x n matrix
(m rows,
n columns)

n x p matrix
(n rows,
o columns)

m x p
matrix

Matrix Multiplication Properties

Let A and B be matrices. Then in general, $A \times B \neq B \times A.$ (not commutative.)

E.g. $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}$

Matrix Multiplication Properties

❒ **Identity matrix** – The identity matrix $I \in \mathbb{R}^{n\times n}$ is a square matrix with ones in its diagonal and zero everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

Remark: for all matrices $A \in \mathbb{R}^{n\times n}$, we have $A \times I = I \times A = A$.

Matrix Multiplication Properties

Matrix inverse:

If A is an m x m matrix, and if it has an inverse, then:

$$AA^{-1} = A^{-1}A = I.$$

Matrix Properties

Matrix Transpose

Let A be an m x n matrix, and let $B = A^T$.

Then B is an n x m matrix, and $B_{ij} = A_{ji}$.

Example:

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 5 & 9 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 5 \\ 0 & 9 \end{bmatrix}$$

Exercise

Given 3 samples of information: $x^{(1)}, x^{(2)}, x^{(3)} \in \mathbb{R}^5$,create matrix X:

$$x^{(1)} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} 2 \\ 3 \\ 4 \\ 6 \\ 5 \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 3 \\ 1 \end{bmatrix}$$

$$X = \begin{bmatrix} -x^{(1)T} & - \\ -x^{(2)T} & - \\ -x^{(3)T} & - \end{bmatrix} \in \mathbb{R}^{3x5}$$

Using Python, compute the mean and standard deviation of X, using the equations:

$$\mu_j = \frac{1}{m} \sum_{i=1}^{m} x_j^{(i)}$$

$$\sigma_j = \sqrt{\frac{1}{(m)} \sum_{i=1}^{m} (x_i^{(i)} - \mu_j)^2}$$