Linear Models for Classification

Master's Degree in Bioinformatics and Computational Biology - Machine Learning

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Contents

Exercise ▷ **Questionnaire**

Please, fill in the questionnaire regarding your prior knowledge about this topic.

Introduction

Supervised Learning: Classification (I)

Definition (Classification Problem)

A **classification problem** is a supervised learning problem where the outputs are discrete.

Examples (Classification Problems)

- Predicting if a patient has a certain disease or not depending on medical data.
- Distinguishing the species of captured fish using the data provided by several sensors.
- Discerning the type of object that appears in a picture.

Supervised Learning: Classification (II)

Elements of a Supervised Learning Problem

Data	Set of input–output pairs, $\mathcal{D} = \{(\mathbf{x}_i, y_i)\}_{i=1}^N$.
Features	Vector of attributes (independent/input variables, covariates...), $\mathbf{x}_i \in \mathcal{X}$.
Label	Target (dependent variable, outcome...), $y_i \in \mathcal{Y}$.
Model	Mapping from the input to the output space, $f_{\boldsymbol{\theta}} : \mathcal{X} \to \mathcal{Y}$, with $\boldsymbol{\theta}$ the model parameters.
Learning Algorithm	Procedure to obtain a model based on the data, $\mathcal{A} : \mathcal{D} \to f_{\boldsymbol{\theta}}(\cdot)$.

- In a classification setting $\mathcal{Y} = \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_M\}$.
- In many situations, specially after preprocessing the data, $\mathcal{X} = \mathbb{R}^d$.
- The resultant model assigns to each input a certain class, $f_{\boldsymbol{\theta}} : \mathcal{X} \to \{\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_M\}$.

Binary Classification and Linear Models

- The most important classification scenario is when $M = 2$ (**binary classification**).
  - If $M > 2$, there are encoding techniques to transform the problem into several binary subproblems.
- The classes are usually denoted as $\mathcal{C}_0$ and $\mathcal{C}_1$, and they are represented with a $0/1$ (or $-1/1$) encoding.
  - The labels are transformed to:

$$t_i = \begin{cases} 0 & \text{if } y_i = \mathcal{C}_0, \\ 1 & \text{if } y_i = \mathcal{C}_1. \end{cases}$$

- "Simplest" approaches to classification:
  - Ignore the input: **constant model** (usually, **majority class**).
  - Define the output as a linear combination of the inputs plus a **transformation**: **linear model**.
    - Simple. Robust (small variance). Interpretable. Easy to train. Easy to predict.
    - Limited flexibility. Under-fitting (large bias).

Binary Linear Classification

Binary Linear Classifier

- For simplicity, $\mathcal{X} = \mathbb{R}^d$.
- The data becomes $\mathcal{D} = \{(\mathbf{x}_i, t_i)\}_{i=1}^N$, with $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,d}) \in \mathbb{R}^d$ and $t_i \in \{0, 1\}$.

- The corresponding linear model is a hyperplane, with parameters $\boldsymbol{\theta} = \{b, \mathbf{w}\}$.
  - $b \in \mathbb{R}$ is the intercept or bias term.
  - $\mathbf{w} = (w_1, w_2, \dots, w_d) \in \mathbb{R}^d$ is the normal vector of the hyperplane.
  - The model is defined as:

$$f_{\boldsymbol{\theta}}(\mathbf{x}) = \begin{cases} 0 & \text{if } b + \mathbf{w}^\intercal \mathbf{x} < 0, \\ 1 & \text{if } b + \mathbf{w}^\intercal \mathbf{x} \geq 0. \end{cases}$$

  - The hyperplane divides the space into two halves, one for class $\mathcal{C}_0$ and the other for class $\mathcal{C}_1$.

- The **learning algorithm** will determine b and $\mathbf{w}$ using $\mathcal{D}$.

Binary Linear Classifier - Exercise

Exercise

▷ **Questionnaire**

Given a 2-dimensional binary linear classification model with parameters $\boldsymbol{\theta} = \{b, \mathbf{w}\}$, with $b = 1$ and $\mathbf{w} = (1, 2)^\intercal$.

1. Compute the output of the model for $\mathbf{x}_1 = (1, 1)^\intercal$.
2. Compute the output of the model for $\mathbf{x}_2 = (1, -2)^\intercal$.
3. Compute the output of the model for $\mathbf{x}_3 = (0, 0)^\intercal$.

Notebook

Binary Linear Classification: First Example

Quality of the Model

- A procedure is needed to determine the bias b and the vector $\mathbf{w}$, optimizing the **quality** of the model.
- The quality of the model has to be defined. Usually from two points of view:

  Fitness A fitness term $\mathcal{F}_{\mathcal{D}}(\boldsymbol{\theta})$ measures how well the model fits the training data.

  Complexity A regularization term $\mathcal{R}(\boldsymbol{\theta})$ penalizes the complexity of the model.

Fitness Term for a Classification Linear Model

Correct Prediction For the i-th pattern,

$$c_i = \begin{cases} 0 & \text{if } t_i \neq f_{\boldsymbol{\theta}}(\mathbf{x}_i) \\ 1 & \text{if } t_i = f_{\boldsymbol{\theta}}(\mathbf{x}_i) \end{cases} = \begin{cases} 0 & \text{if } (t_i = 0, b + \mathbf{w}^\intercal \mathbf{x}_i \geq 0) \text{ or } (t_i = 1, b + \mathbf{w}^\intercal \mathbf{x}_i < 0), \\ 1 & \text{if } (t_i = 0, b + \mathbf{w}^\intercal \mathbf{x}_i < 0) \text{ or } (t_i = 1, b + \mathbf{w}^\intercal \mathbf{x}_i \geq 0). \end{cases}$$

Accuracy $\mathrm{Acc}(b, \mathbf{w}) = \mathbb{E}[C] \approx \frac{1}{N} \sum_{i=1}^{N} c_i$.

Quality of the Model - Exercise

Exercise

▷ **Questionnaire**

Given a 2-dimensional binary linear classification model with parameters $\boldsymbol{\theta} = \{b, \mathbf{w}\}$, with $b = 1$ and $\mathbf{w} = (1, 2)^\intercal$, and for the following data:

$x_{i,1}$	$x_{i,2}$	t_i
1	1	1
1	-2	0
0	0	0

1. Compute the accuracy.

Notebook

Binary Linear Classification: Quality of the Model

Training a Linear Classifier: Using the Regression Framework

- The most common choice for evaluating the model is the accuracy.
  - It is a sensible and intuitive measure.
  - It is **non-convex**.
  - It is **non-differentiable**.
  - It is **discontinuous**.

- Optimizing the accuracy is a problem that cannot (in general) be tackled directly.

- An alternative idea could be to train a Linear Regression model.
  - Labels $-1/1$.
  - The predicted label is determined by taking the sign of the output.

Notebook

Binary Linear Classification: Training a Regression Linear Model

Training a Linear Classifier: Logistic Regression (I)

- A different quality measure is needed.
  - It should be simpler to optimize than the accuracy.
  - It should not penalize points far from the decision boundary (but on the correct side).

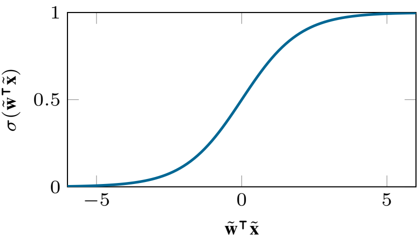
- A probabilistic approach can be helpful.
- In particular, the main framework is **Logistic Regression**.
  - The linear model is used to estimate the posterior probability of one class.
  - A sigmoid transformation is used.

Training a Linear Classifier: Logistic Regression (II)

- Denoting by $\tilde{\mathbf{x}} = [1, \mathbf{x}]$ and by $\tilde{\mathbf{w}} = [b, \mathbf{w}]$, the posterior probabilities are defined as:

$$\mathrm{p}(\mathcal{C}_1 | \tilde{\mathbf{x}}; \tilde{\mathbf{w}}) = \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}) = \frac{1}{1 + e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}},$$

$$\mathrm{p}(\mathcal{C}_0 | \tilde{\mathbf{x}}; \tilde{\mathbf{w}}) = 1 - \mathrm{p}(\mathcal{C}_1 | \tilde{\mathbf{x}}; \tilde{\mathbf{w}}) = 1 - \frac{1}{1 + e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}} = \frac{e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}}{1 + e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}} = \frac{1}{1 + e^{\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}} = \sigma(-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}).$$



- $\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}} < 0 \implies \mathrm{p}(\mathcal{C}_1 | \tilde{\mathbf{x}}; \tilde{\mathbf{w}}) < 0.5$: Class $\mathcal{C}_0$ is predicted.
- $\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}} \geq 0 \implies \mathrm{p}(\mathcal{C}_1 | \tilde{\mathbf{x}}; \tilde{\mathbf{w}}) \geq 0.5$: Class $\mathcal{C}_1$ is predicted.

Training a Linear Classifier: Logistic Regression - Exercise

Exercise

▷ **Questionnaire**

Given a 2-dimensional binary linear classification model with parameters $\boldsymbol{\theta} = \{b, \mathbf{w}\}$, with $b = 1$ and $\mathbf{w} = (1, 2)^\intercal$.

1. Compute the probability of $\mathbf{x}_1$ belonging to class $\mathcal{C}_1$ for $\mathbf{x}_1 = (1, 1)^\intercal$.
2. Compute the probability of $\mathbf{x}_2$ belonging to class $\mathcal{C}_1$ for $\mathbf{x}_2 = (1, -2)^\intercal$.
3. Compute the probability of $\mathbf{x}_3$ belonging to class $\mathcal{C}_0$ for $\mathbf{x}_3 = (0, 0)^\intercal$.

Training a Linear Classifier: Maximum Likelihood (I)

- The probabilistic interpretation can help to define a quality measure.

Exercise

▷ **Questionnaire**

- In the following dataset, both $\boxed{\clubsuit}$ and $\boxed{\heartsuit}$ have been randomly generated with probabilities π_1 and $1-\pi_1$, respectively:

$$\boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\heartsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\heartsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}\ \boxed{\clubsuit}$$

Between the following options, what is the most likely value of π_1? Why?

- 0 %.
- 5 %.
- 50 %.
- 95 %.
- 100 %.

Training a Linear Classifier: Maximum Likelihood (II)

- The **likelihood** of the data is a common choice to quantify the quality of a probabilistic model:

$$\mathcal{L}(\mathcal{D}; \tilde{\mathbf{w}}) = \prod_{i=1}^{N} \mathrm{p}(t_i | \tilde{\mathbf{x}}_i; \tilde{\mathbf{w}}) = \prod_{i=1}^{N} \underbrace{\mathrm{p}(\mathcal{C}_0 | \tilde{\mathbf{x}}_i; \tilde{\mathbf{w}})^{1-t_i} \, \mathrm{p}(\mathcal{C}_1 | \tilde{\mathbf{x}}_i; \tilde{\mathbf{w}})^{t_i}}_{\begin{cases} \mathrm{p}(\mathcal{C}_0 | \tilde{\mathbf{x}}_i; \tilde{\mathbf{w}}) & \text{if } t_i = 0, \\ \mathrm{p}(\mathcal{C}_1 | \tilde{\mathbf{x}}_i; \tilde{\mathbf{w}}) & \text{if } t_i = 1. \end{cases}}.$$

- The **Cross Entropy (CE)** error is defined as the minus log-likelihood:

$$\begin{aligned} \mathrm{CE}(\tilde{\mathbf{w}}) &= -\log \mathcal{L}(\mathcal{D}; \tilde{\mathbf{w}}) \\ &= \sum_{i=1}^{N} \left(-(1 - t_i) \log(\mathrm{p}(\mathcal{C}_0 | \tilde{\mathbf{x}}_i; \tilde{\mathbf{w}})) - t_i \log(\mathrm{p}(\mathcal{C}_1 | \tilde{\mathbf{x}}_i; \tilde{\mathbf{w}})) \right) \\ &= \sum_{i=1}^{N} \left(-(1 - t_i) \log(1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) - t_i \log(\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) \right). \end{aligned}$$

Training a Linear Classifier: Maximum Likelihood - Exercise

Exercise

▷ **Questionnaire**

Given a 2-dimensional binary linear classification model with parameters $\boldsymbol{\theta} = \{b, \mathbf{w}\}$, with $b = 1$ and $\mathbf{w} = (1, 2)^\intercal$, and for the following data:

$x_{i,1}$	$x_{i,2}$	t_i
1	1	1
1	-2	0
0	0	0

1. Compute the likelihood of this model.

Training a Linear Classifier: Maximum Likelihood (III)

- The minimizer of $\mathrm{CE}(\tilde{\mathbf{w}})$ is the maximizer of $\mathcal{L}(\mathcal{D}; \tilde{\mathbf{w}})$.
- The learning algorithm for training a Linear Logistic Regression model consists in solving the problem:

$$\min_{\tilde{\mathbf{w}} \in \mathbb{R}^{d+1}} \{\mathrm{CE}(\tilde{\mathbf{w}})\} = \min_{\tilde{\mathbf{w}} \in \mathbb{R}^{d+1}} \left\{ \sum_{i=1}^{N} \left(-(1 - t_i) \log(1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) - t_i \log(\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) \right) \right\}.$$

- How is this problem solved?
  - It is **convex**: there are no local minima.
  - It is **differentiable**: the optima are characterized by the zeros of the gradient.

Training a Linear Classifier: Optimization (I)

$$\min_{\tilde{\mathbf{w}} \in \mathbb{R}^{d+1}} \{\mathrm{CE}(\tilde{\mathbf{w}})\} = \min_{\tilde{\mathbf{w}} \in \mathbb{R}^{d+1}} \left\{ \sum_{i=1}^{N} \left(-(1 - t_i) \log(1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) - t_i \log(\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) \right) \right\}.$$

$$\begin{aligned}
\nabla_{\tilde{\mathbf{w}}} \mathrm{CE}(\tilde{\mathbf{w}}) &= \sum_{i=1}^{N} \left(-(1 - t_i) \nabla_{\tilde{\mathbf{w}}} \log(1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) - t_i \nabla_{\tilde{\mathbf{w}}} \log(\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) \right) \\
&= \sum_{i=1}^{N} \left((1 - t_i) \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i) \tilde{\mathbf{x}}_i - t_i (1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) \tilde{\mathbf{x}}_i \right) \\
&= \sum_{i=1}^{N} \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i) \tilde{\mathbf{x}}_i - t_i \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i) \tilde{\mathbf{x}}_i - t_i \tilde{\mathbf{x}}_i + t_i \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i) \tilde{\mathbf{x}}_i \\
&= \sum_{i=1}^{N} (\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i) - t_i) \tilde{\mathbf{x}}_i .
\end{aligned}$$

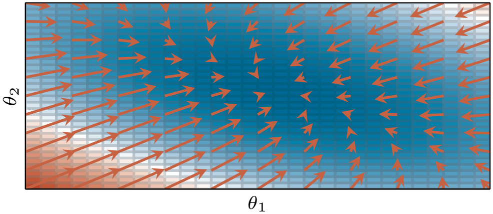
More detail in the appendix **Gradient of the Sigmoid Transformation**.

Gradient-Based Optimization: Iterative Methods

- Can the problem $\nabla_{\boldsymbol{\theta}} \mathcal{F}(\boldsymbol{\theta}^\star) = \mathbf{0}$ be directly solved?
  - Only for simple cases.
  - In several dimensions, it implies solving a system of d equations (one for each partial derivative).
  - This condition does not imply minimum (valley), it may be a maximum or a saddle point.

- But the gradient points in the direction of greatest increase:
  - $-\nabla_{\boldsymbol{\theta}} \mathcal{F}$ points in the direction of **steepest descent**.
  - This can be used to define where to go next.

$$\mathcal{F}(\boldsymbol{\theta} + \boldsymbol{\epsilon}) \approx \mathcal{F}(\boldsymbol{\theta}) + \nabla_{\boldsymbol{\theta}} \mathcal{F}(\boldsymbol{\theta})^\intercal \boldsymbol{\epsilon} \implies \mathcal{F}(\boldsymbol{\theta} - \eta \nabla_{\mathbf{x}} \mathcal{F}(\boldsymbol{\theta})) \approx \mathcal{F}(\boldsymbol{\theta}) - \eta \|\nabla_{\boldsymbol{\theta}} \mathcal{F}(\boldsymbol{\theta})\|_2^2 \leq \mathcal{F}(\boldsymbol{\theta}).$$



Gradient Descent

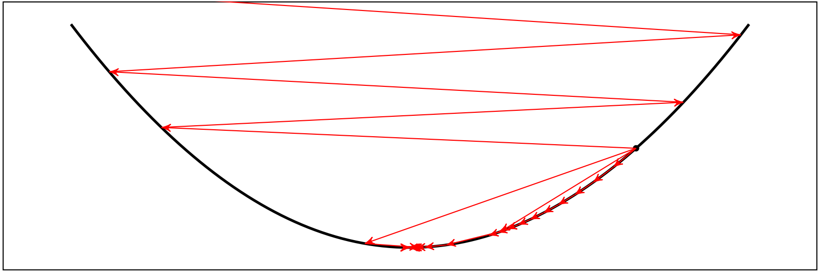
- It is a simple (yet useful) optimization algorithm that is often used in Machine Learning (ML) to find the local minimum.

Gradient Descent: Algorithm

Require: Objective function $\mathcal{F}$, starting point $\boldsymbol{\theta}^{(0)}$ ▷ Start at a random point.
Ensure: $\boldsymbol{\theta}^{(t-1)} \in \mathbb{R}^d$ an approximate local minimum of $\mathcal{F}(\boldsymbol{\theta})$ ▷ Arrive to a valley.
  for $t = 1, 2, \cdots$ **do** ▷ Several steps.
    $\mathbf{g} \leftarrow \nabla_{\boldsymbol{\theta}} \mathcal{F}\left(\boldsymbol{\theta}^{(t-1)}\right)$ ▷ Stop, look for the direction most downhill.
    if $\mathbf{g} \approx \mathbf{0}$ **then**
      return $\boldsymbol{\theta}^{(t-1)}$ ▷ If there is not direction downhill, stop.
    end if
    $\boldsymbol{\theta}^{(t)} \leftarrow \boldsymbol{\theta}^{(t-1)} - \eta^{(t)} \mathbf{g}$ ▷ Take a step in that direction.
  end for

Gradient Descent: Step-Size

- The step-size $\eta^{(t)}$ has to be set at each iteration t.
- This is a crucial issue:
  1. If the step-size is too small, the algorithm will require too many epochs (iterations) to converge and can become trapped in local minima more easily.
  2. If the step-size is large, the convergence will also be slow.
  3. If the step-size is too large, gradient descent will overshoot the minima an diverge.
- In some cases, optimal step-sizes can be computed.
- There are heuristics that guarantee convergence, but only to **local minima**, and usually slowly and zigzagging.



Training a Linear Classifier: Optimization (II)

- In summary, the Linear Logistic Regression Model is the solution of the following problem:

$$\min_{\tilde{\mathbf{w}} \in \mathbb{R}^{d+1}} \left\{ \sum_{i=1}^{N} \left(-(1 - t_i) \log(1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) - t_i \log(\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i)) \right) \right\}.$$

- There is not closed-form solution to the resultant equation for the stationary points:

$$\nabla_{\tilde{\mathbf{w}}} \mathrm{CE}(\tilde{\mathbf{w}}) = \sum_{i=1}^{N} (\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}_i) - t_i) \tilde{\mathbf{x}}_i = \mathbf{0}.$$

- An iterative algorithm, such as **gradient descent**, should be used.

Linear Logistic Regression Model

- The model can be trained iteratively by updating the weights as:

$$\tilde{\mathbf{w}}^{(s+1)} = \tilde{\mathbf{w}}^{(s)} - \eta^{(s)} \sum_{i=1}^{N} \left(\sigma\left(\left(\tilde{\mathbf{w}}^{(s)} \right)^\intercal \tilde{\mathbf{x}}_i \right) - t_i \right) \tilde{\mathbf{x}}_i.$$

Notebook

Binary Linear Classification: Optimization

Summary

Linear Models for Classification: Summary

- A **classification** problem is a supervised problem with discrete targets.

- A simple yet useful classification model is the **linear model**.
  - The decision boundary is a hyperplane dividing the space.

- In order to train the linear model, an **optimization problem** is usually solved.
- The **accuracy**, the more natural choice, is hard to optimize in practice.

- The **likelihood** is used instead, leading to the **Logistic Regression** model.
  - The resultant problem can be solved iteratively using gradient descent.

Linear Models for Classification

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Additional Material

Gradient of the Sigmoid Transformation

Expressions for the Gradient of the Sigmoid Transformation

- The linear model with sigmoid transformation satisfies the following equations:

$$
\begin{aligned}
\nabla_{\tilde{\mathbf{w}}} \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}) &= \nabla_{\tilde{\mathbf{w}}} \frac{1}{1 + e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}} = \frac{1}{\left(1 + e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}\right)^2} e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}} \tilde{\mathbf{x}} = \frac{1}{1 + e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}} \frac{e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}}{1 + e^{-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}}} \tilde{\mathbf{x}} \\
&= \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}})(1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}))\tilde{\mathbf{x}}; \\
\nabla_{\tilde{\mathbf{w}}} \log(\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}})) &= \frac{1}{\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}})} \nabla_{\tilde{\mathbf{w}}} \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}) = (1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}))\tilde{\mathbf{x}}; \\
\nabla_{\tilde{\mathbf{w}}} \log(1 - \sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}})) &= \nabla_{\tilde{\mathbf{w}}} \log(\sigma(-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}})) = -(1 - \sigma(-\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}}))\tilde{\mathbf{x}} = -\sigma(\tilde{\mathbf{w}}^\intercal \tilde{\mathbf{x}})\tilde{\mathbf{x}}.
\end{aligned}
$$

- These properties are one of the reasons why this function is so commonly used.