Double Pendulum EOMs

Locations of pendulum bobs:

$$\begin{cases} x_1 = l_1 \sin \alpha_1 \\ x_2 = l_1 \sin \alpha_1 + l_2 \sin \alpha_2 \\ y_1 = -l_1 \cos \alpha_1 \\ y_2 = -l_1 \cos \alpha_1 - l_2 \cos \alpha_2 \end{cases} \tag{1}$$

Lagrangian of pendulum bob positions:

$$\begin{cases} T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2)}{2} + \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} \\ V = m_1 g y_1 + m_2 g y_2 \end{cases} \tag{2}$$

$$L = T - V = \frac{m_1(\dot{x}_1^2 + \dot{y}_1^2)}{2} + \frac{m_2(\dot{x}_2^2 + \dot{y}_2^2)}{2} - m_1 g y_1 - m_2 g y_2 \tag{3}$$

Lagrangian in generalized coordinates:

$$\begin{cases} \dot{x}_1^2 = l_1 \cos \alpha_1 \cdot \dot{\alpha}_1 \\ \dot{x}_2^2 = l_1 \cos \alpha_1 \cdot \dot{\alpha}_1 + l_2 \cos \alpha_2 \cdot \dot{\alpha}_2 \\ \dot{y}_1^2 = l_1 \sin \alpha_1 \cdot \dot{\alpha}_1 \\ \dot{y}_2^2 = l_1 \sin \alpha_1 \cdot \dot{\alpha}_1 + l_2 \sin \alpha_2 \cdot \dot{\alpha}_2 \end{cases} \tag{4}$$

$$\begin{aligned} L = T - V = & \frac{(m_1 + m_2) l_1^2 \dot{\alpha}_1^2 + m_2 l_2^2 \dot{\alpha}_2^2}{2} \\ & + m_2 l_1 l_2 \dot{\alpha}_1^2 \dot{\alpha}_2^2 \cos(\alpha_1 - \alpha_2) \\ & + (m_1 + m_2) g l_1 \cos \alpha_1 \\ & + m_2 g l_2 \cos \alpha_2 \end{aligned} \tag{5}$$

Lagrangian equations of motion

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\alpha}_i} - \frac{\partial L}{\partial \alpha_i} = \tau_i, \quad i = 1, 2 \tag{6}$$

$$\begin{cases} \frac{\partial L}{\partial \dot{\alpha}_1} = (m_1 + m_2) l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) \\ \frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - (m_1 + m_2) g l_1 \sin \alpha_1 \\ \frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1 \cos(\alpha_1 - \alpha_2) \\ \frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - m_2 g l_2 \sin \alpha_2 \end{cases} \tag{7}$$

From (6) and (7), we can get the 2 equations of motion for double pendulum:

$$\begin{cases} (m_1 + m_2) l_1^2 \ddot{\alpha}_1 + m_2 l_1 l_2 \ddot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + m_2 l_1 l_2 \dot{\alpha}_2^2 \sin(\alpha_1 - \alpha_2) + (m_1 + m_2) g l_1 \sin \alpha_1 = \tau_1 \\ m_2 l_2^2 \ddot{\alpha}_2 + m_2 l_1 l_2 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - m_2 l_1 l_2 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + m_2 g l_2 \sin \alpha_2 = \tau_2 \end{cases} \tag{8}$$

Substitute equations into each other, we get:

$$\begin{cases} \ddot{\alpha}_1 = \frac{A}{m_1 l_1 + m_2 l_1 \sin^2(\alpha_1 - \alpha_2)} + B, \quad \text{where} \\ A = -m_2 l_1 \sin(\alpha_1 - \alpha_2) \cos(\alpha_1 - \alpha_2) \dot{\alpha}_1^2 + m_2 g \cos(\alpha_1 - \alpha_2) \sin \alpha_2 \\ \qquad - m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1 - \alpha_2) - (m_1 + m_2) g \sin \alpha_1, \\ B = \frac{m_2 l_2 \cos(\alpha_1 - \alpha_2) \tau_1 - (m_1 + m_2) l_1 \tau_2}{(m_1 + m_2) l_1^2 l_2 - m_2 \cos^2(\alpha_1 - \alpha_2) l_1^2 l_2}. \\ \ddot{\alpha}_2 = \frac{C}{m_1 l_2 + m_2 l_2 \sin^2(\alpha_1 - \alpha_2)} + D, \quad \text{where} \\ C = (m_1 l_1 + m_2 l_1) \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + m_2 l_2 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) \cos(\alpha_1 - \alpha_2) \\ \qquad + (m_1 + m_2) g \sin \alpha_1 \cos(\alpha_1 - \alpha_2) - (m_1 + m_2) g \sin \alpha_2 \\ D = \frac{m_2 l_2 \cos(\alpha_1 - \alpha_2) \tau_1 - (m_1 + m_2) l_1 \tau_2}{m_1 m_2 \cos^2(\alpha_1 - \alpha_2) l_1 l_2^2 - (m_1 + m_2) m_2 l_1 l_2^2}. \end{cases} \tag{9}$$