Report 1

1. Modification

Considering the direction of the generated torque and estimating the stability of the system, I modify the torque from $\tau = k\dot{\theta}^2$ to $\tau = k\theta$.

2. Description

Since we focus on the attitude, but not the translational motion of the unicycle, we choose the translational velocity of the unicycle $\dot{y}$, the deflection angle of the seat θ, and the rotation speed of the seat $\dot{\theta}$ as the states of the system.
Let $x_1 = \dot{y},\ x_2 = \theta,\ x_3 = \dot{\theta}$, then we have,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} \frac{lk}{I} x_2 \\ x_3 \\ \frac{g}{L}\sin x_2 - \frac{kl}{IL} x_2 \cos x_2 - \frac{k}{mL^2} x_2 \end{pmatrix}$$

For simplicity[1], we design a series of reasonable parameters, and the description will be like,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 3\sin x_2 - x_2 \cos x_2 - x_2 \end{pmatrix}$$

Let $\dot{x} = f(x)$ denote the description, the system can be represented by $\Sigma = (T, X, \Phi, D_\Phi)$, where $T = [0, +\infty)$, $X = \{(x_1, x_2, x_3) | -\frac{\pi}{2} < x_2 < \frac{\pi}{2}, x_1, x_2 \in R\} \subset R^3$, $\Phi: R^3 \to R^3$.
This is a continuous-time system. The trajectory starting from (t_0, x_0) is $\Phi(t, x) = \Phi(t_0, x_0) + \int_{t_0}^{t} f(x)dt$. The trajectory varies when the initial value changes.

3. Fixed points

Let $\dot{x} = f(x) = 0$, we find the fixed points satisfy $x_2 = 0$ and $x_3 = 0$. There is no constraint to x_1, i.e., the system will be fixed if and only if $x_2 = 0,\ x_3 = 0$, and arbitrary x_1. We don't except to have periodic points/orbits and we only except the system could be stable around the fixed points, under our self-designed control strategy.

4. Existence of solutions

Since x_1 has no influence on the state, we can focus on $\dot{x_2}$ and $\dot{x_3}$. The projection of the trajectory on x_1 exists if x_2 doesn't blow up, which implies that x_2 in integrable. And of course, x_2 wouldn't blow up, because it is bounded by $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$.

Now we explore the Lipschitz continuity of $(x_2, x_3)^T$:

---

[1] The shape of the unicycle and the material of the torque generator will exactly influence the stability of the unicycle, which means that this kind of simplicity may be modified later.

$\|x-y\|^2 = (x_2-y_2)^2 + (x_3-y_3)^2$
$\|f(x)-f(y)\|^2$
$= (x_3-y_3)^2 + [3(\sin x_2 - \sin y_2) - (x_2\cos x_2 - y_2\cos y_2) - (x_2-y_2)]^2$
$\le (x_3-y_3)^2 + 9(\sin x_2 - \sin y_2)^2 + (x_2\cos x_2 - y_2\cos y_2)^2 + (x_2-y_2)^2$
$\le (x_3-y_3)^2 + 9(x_2-y_2)^2 + (x_2\cos x_2 - y_2\cos y_2)^2 + (x_2-y_2)^2$

Since x_2 is bounded by $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$, then $(x_2\cos x_2 - y_2\cos y_2)^2 \le \frac{\pi^2}{4}(x_2 - y_2)^2$.

Therefore, $\|f(x)-f(y)\|^2 \le 9(x_3-y_3)^2 + (2+\frac{\pi^2}{4})(x_2-y_2)^2 \le 9[(x_3-y_3)^2 + (x_2-y_2)^2]^2 \le 9\|x-y\|^2$.

Since f is Lipschitz continuous with Lip constant $K=3$, and X is an open set, then the trajectory, starting from arbitrary initial value (x_0, t_0), exists locally.

The existence of solutions (trajectories) could be expanded globally, and my guess is based on the followings:

i) This is a real physical system, and we can imagine the conditions: the seat oscillates while the wheel moves, or the seat falls down to the ground. There is limitation in reality to the deflection angle of the seat. The solutions wouldn't blow up, even if it may blow up when there is no limitation to θ;

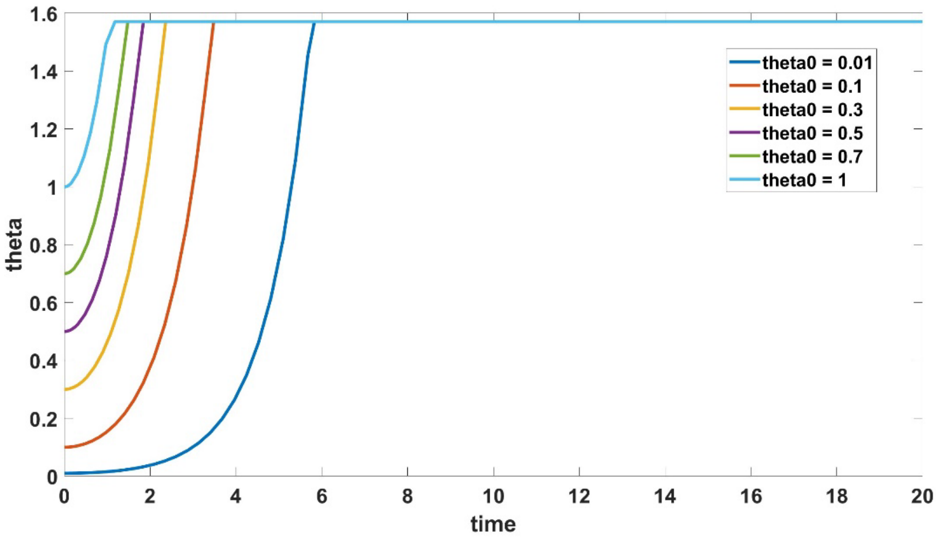
ii) The boundary of the close ball $B(x_0, r)$ can be as adjacent as we want to the boundaries of X, i.e., we can choose the ball $B(x_0, r)$ bound by one side of $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$. Then according to Picard-Lindelöf theorem, there exists a unique C^1-mapping $x: [t_0 - a, t_0 + a] \to B(x_0, r)$ which is the solution. If when $t = t_0 + a$ the system is still in a reasonable state, we can again start from the ending state and generate a new solution in a new time range. Therefore, we get the solutions, starting from any initial value and globally both in state domain and time domain.

5. *Simulations*

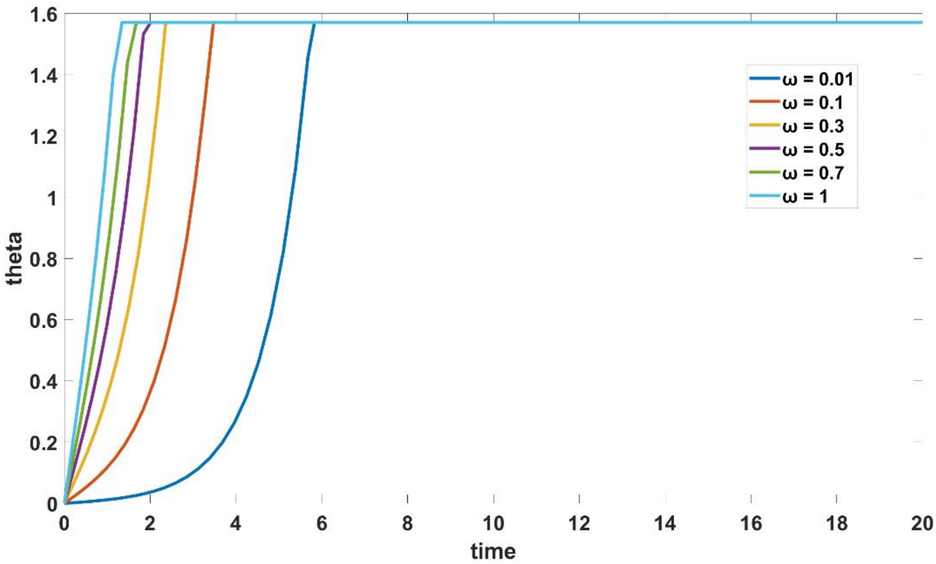
In this part, we observe the relation between the deflection angle of the seat θ and time t, under different initial conditions.

We observe[2] the influence of position perturbance and angular velocity perturbance separately.

i) Position perturbance



ii) Angular velocity perturbance



We can verify that the trajectory exists globally through these rough simulations. Beyond this, we discover that the fixed point isn't stable totally. However, the stability of this system varies when the parameters change.

[2] Since the differential equations are unlikely to have analytical solutions, then I mainly use the 'ode45' function in MATLAB to implement the simulations.

Report 2: Regularity and Stability

Deyi Wang

February 26, 2023

0. Modification

As mentioned in Report 1, the stability of the system would change when the parameters vary. According to the simulations in Report 1, the original system is not stable, which would make this report meaningless. Therefore, I apply two modifications to this system:

1. Ignore the x_1 dimension and only focus on x_2 and x_3 (the rationality of this ignorance has been illustrated in Report 1). The space domain $X = \{(x_2, x_3)| -\pi/2 < x_2 < \pi/2, x_3 \in \mathbb{R}\}$,and the time domain T remains.
2. Change the $\dot{x} = f(x)$ to,

$$\begin{pmatrix} \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ \sin x_2 - x_2 \cos x_2 - x_2 \end{pmatrix}$$

Through these modifications, there exists only one fixed point $(0, 0)$ in this system (See Fig.1), which make the Lyapunov theory possibly appliable. These modifications didn't change the Lipschitz continuity of the system, but with a new Lip constant $K = \sqrt{2 + \frac{\pi^2}{4}}$, which could be easily derived from the proof of Lipschitz continuity in Report 1.

1. Regularity

Here I will focus on the C^0-regularity. Let $x(t)$ and $z(t)$ be solutions to the IVPs:

$$\dot{x} = f(x), x(t_0) = x_0$$

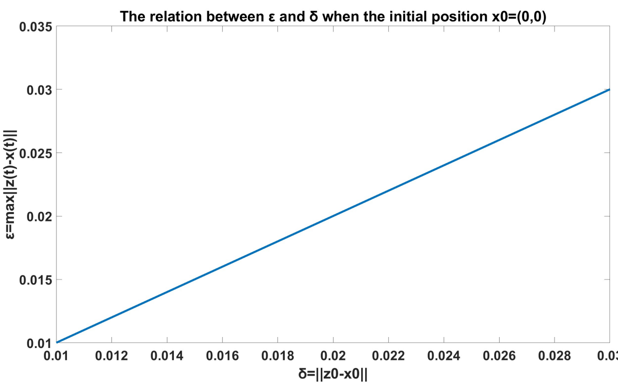
$$\dot{z} = f(z), z(t_0) = z_0$$

Since the fucntion $f : X \to \mathbb{R}^2$ is Lipschitz continuous with Lip constant $K = \sqrt{2 + \frac{\pi^2}{4}}$, and X is an open and connected set, then based on **Theorem 3.2** in *Nonlinear Dynamics and Control*, the following inequality holds,

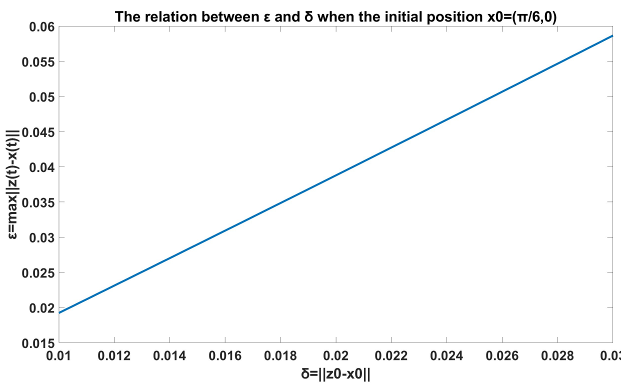
$$\|x(t) - z(t)\| \leq \|x_0 - z_0\| e^{K(t-t_0)}$$

when $t \geq t_0$. For an arbitrary time domain $T_1 = [t_0, t_1]$, if we hope the change of final position to be restricted in $\|x(t) - z(t)\| \leq \epsilon$, we can choose $\|x(t_0) - z(t_0)\| \leq \delta$, where $\delta = \frac{\epsilon}{e^{K(t_1-t_0)}}$. Therefore, the system is continuous with respect to its initial condition, i.e., C^0-regularity, regardless of the choice of the initial condition.

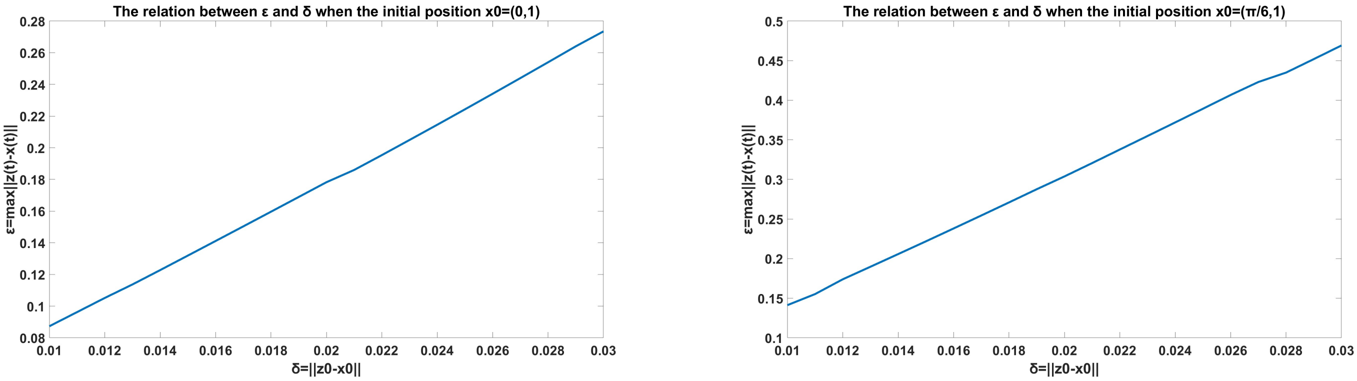
I also did 4 simulations with different types of initial positions to show that the C^0-regularity doesn't depend on the initial condition (See Figure 1).



(a)



(b)



(c) (d)

Figure 1: The relation between ϵ and δ with 4 different types of initial conditions

Here, we find that $\epsilon(\delta)$ is bounded regardless of the initial conditions, which roughly implies that the system's continuity with respect of the initial condition is independent from the choice of initial conditions.

In the next section of invraiance and stability, we will show that the system will do periodic motion, and we can easily observe the C^0-regularity of the system from the flow sketch picture.

2. Invariance and Stability

Figure 2 shows a sketch picture of the flow of this system. Since the sketch is derived from "ode45", a approximate function to solve ODEs, the line of the same flow may not coincide perfectly.

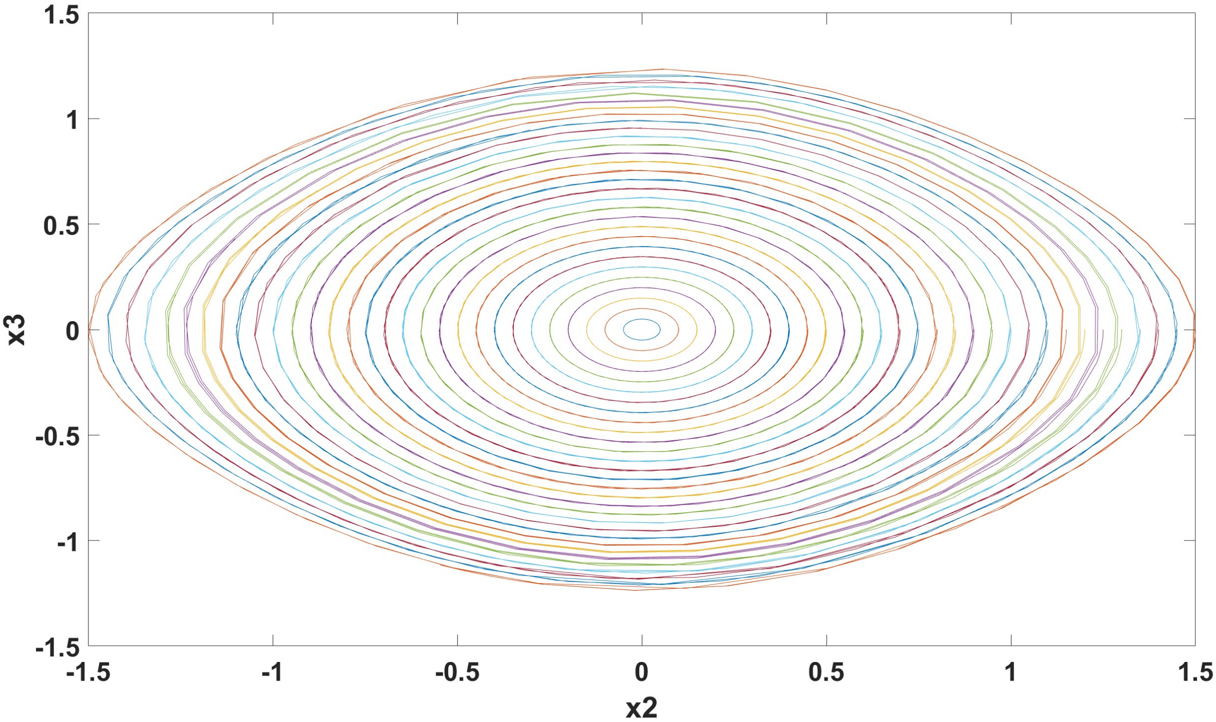


Figure 2: The sketch pic of the flow of the system

The flow picture reveals that the autonomous system will do periodic motion, and each flow with the different initial condition form a approximately closed ellipse in the space. Therefore, we can conclude that each ellipse in the space corresponding to an arbitrary flow is a (forward) invariant set, i.e., any flow starting from the initial condition inside the ellipse will also form a smaller ellipse inside it and will not break out.

What's more, we can also conclude that the unique fixed point $(0,0)$ of this system is stable but not asympototically stable.

Proof: Let $\Phi(x_0, t_0, t) = x_0 + \int_{t_0}^{t} f(x(t))dt$ denote the flow starting from x_0. Define that $a(\Phi) = \max\{\|\Phi\|\}$, and $b(\Phi) =$

$\min\{\|\Phi\|\}$.

For $\forall\epsilon$, find the initial positions whose flows are inside the euclidean ball $\{x|\|x\| \leq \epsilon\}$: $\{x_\epsilon\} = \{x|a(\Phi(x, t_0, t)) \leq \epsilon\}$. $\{x_\epsilon\}$ will cover an approximately ellipse range in the space. Let $\delta = \min_{x\in\{x_\epsilon\}} \|x\|$, then for $\|x_0 - 0\| \leq \delta$, we can derive that $\|x(t)\| = \|\Phi(x_0, t_0, t)\| \leq \epsilon$. Therefore, the fixed point $(0, 0)$ is stable.

However, since the system will do periodic oscillation, then for $\forall x_0 \neq 0$, $\lim_{t\to\infty} x(t) \neq 0$. Therefore, the fixed point is not asympototically stable.

Finally, to answer the question in the outline, it is obvious that in this system, there is no other limit points than this fixed point.

3. Lyapunov Theory

Since there exists a unique fixed point $(0, 0)$ in the system, Lyapunov theory is appliable.

I could have used a easiler conservation-of-energy method to identify the Lyapunov function. Since the energy of this system is conserved, then we could use the function representing the total mechanical energy of the system as the Lyapunov function $\mathcal{V}(x)$. Obviously, $\mathcal{V}(x)$ will be nonnegative and $\dot{\mathcal{V}}(x) = 0$.

However, since I have made some simplification to the system, the conservation-of-energy method is now invalid. Therefore, I turn to try the method of balance.

Firstly,

$$\frac{d}{dt}x_3^2 = 2x_3\dot{x_3} = 2x_3(\sin x_2 - x_2\cos x_2 - x_2)$$

For $-2x_3x_2$, since $\frac{d}{dt}x_2^2 = 2x_2x_3$, we can use x_2^2 to balance it.

For $2x_3(\sin x_2 - x_2\cos x_2)$, since $\frac{d}{dt}2f(x_2) = 2\dot{x_2}f'(x_2) = 2x_3f'(x_2)$, we can form that,

$$f(x) = \int(x\cos x - \sin x)dx = x\sin x + 2\cos x + C$$

Since we need $f(0) = 0$, then $f(x) = x\sin x + 2(\cos x - 1)$ Therefore, the Lyapunov function,

$$\mathcal{V}(x) = x_2^2 + 2f(x_2) + x_3^2 = x_2^2 + 2x_2\sin x_2 + 4(\cos x_2 - 1) + x_3^2$$

Figure 3 reveals the nonnegativity of $\mathcal{V}(x)$,

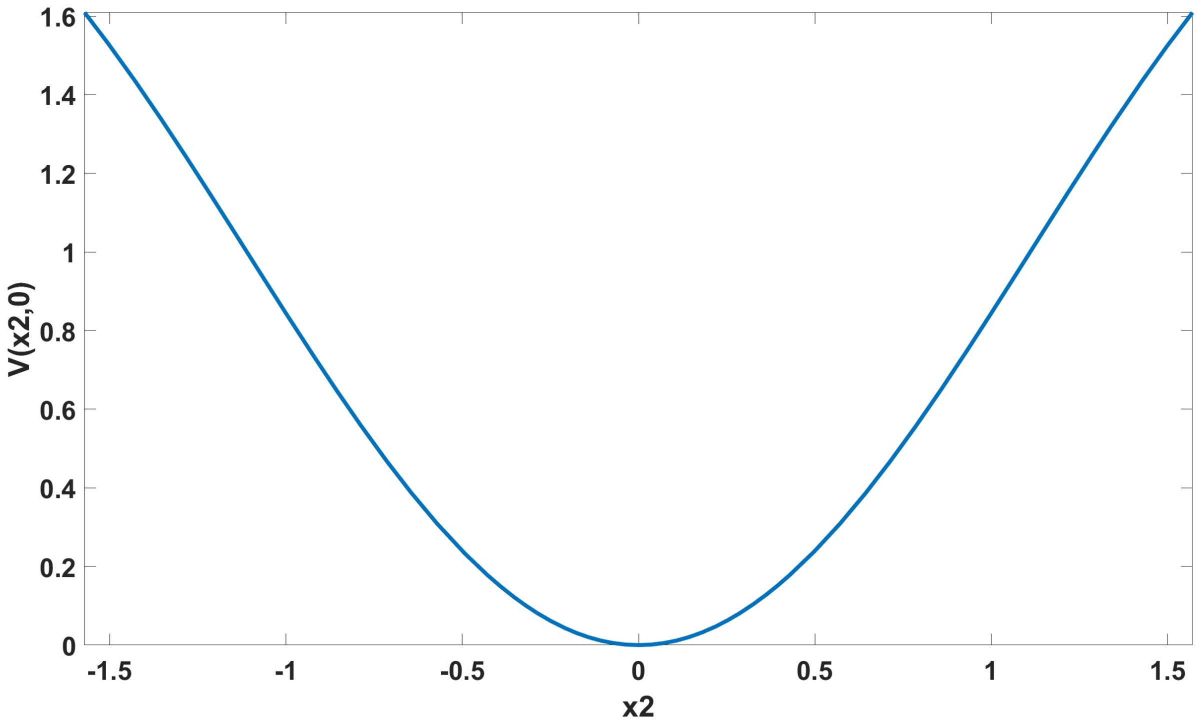


Figure 3: The plot on $\mathcal{V}(x_2, 0)$ versus x_2 in the range of $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$

Since $x_3^2 \geq 0$, and $\mathcal{V}(x)$ have the unique zero point $(0, 0)$, then we have,

- $\mathcal{V}(0) = 0$;
- $\mathcal{V}(x) > 0$ for $\forall x \neq 0$;
- $\dot{\mathcal{V}}(x) \leq 0$ (in fact $\dot{\mathcal{V}}(x) \equiv 0$).

Therefore, we prove that the fixed point is stable via Lyapunov theory. Since the third inequality doesn't strictly hold, we cannot prove the asympototically stability via this Lyapunov function. Luckily, we have proven that the fixed point is not asympototically stable, so we don't need to find a new Lyapunov function again!

Report 3: Input - Output System

Deyi Wang

March 13, 2023

1. Input Description

In my system (see Figure 1), the input $u \in \mathbb{R}$ is a force applied to the mass point. Its direction is perpendicular to the rod connecting the mass point and the roller wheel. The controller generating the input could be a PID controller[1], a feedback linearization controller, etc.

Involving this input u, the kinetic equations will be,

$$\frac{I}{l}\ddot{y} = k\dot{\theta}^2$$
$$mL^2\ddot{\theta} = mgL\sin\theta - m\ddot{y}L\cos\theta - k\theta - Lu$$

With the same modifications as in Report 2, my system will be like,

$$\dot{x} = f(x) + g(x)u$$

where $x = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$, $f(x) = \begin{pmatrix} x_3 \\ \sin x_2 - x_2\cos x_2 - x_2 \end{pmatrix}$, $g(x) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$.

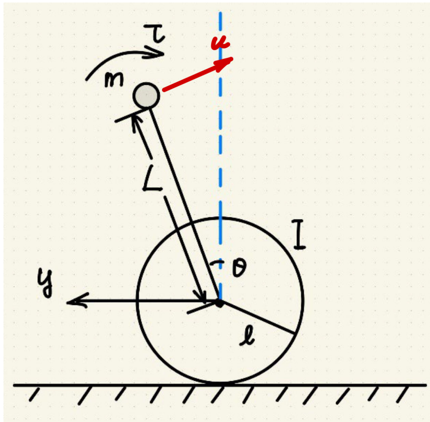


Figure 1: The unicycle system involving the input.

The input could be a function of state, or a function of both state and time. When the input is generated by a feedback linearization controller, it is a function of state; when the input is generated by a PID controller, it is a function of both state and time.

Since the $f(x)$ and $g(x)$ are both elementary functions, and both PID controllers and feedback linearization controllers are common continuous controllers, I can expect the input to have enough regularity, i.e., $u \in \mathbb{C}^k$ with k large enough.

2. Output Description

The output of my system is the state,

$$h(x) = x$$

The outcome of my project is to make the fixed point $x^\star = (0,0)^T$ asympototically stable (In Report 2 I have proven that the fixed point is stable but not asympototically stable).

Furthermore, if applying feedback linearization controllers, I will further expect "input-output feedback linearization".

3. Simulations

I will simulate the two types of controllers mentioned above: PID controller and feedback linearization controller.

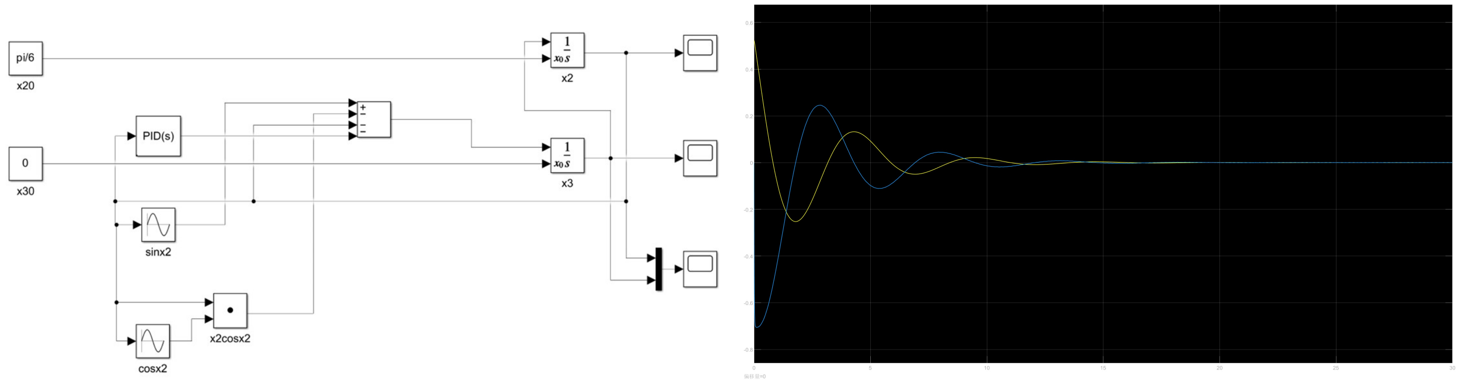
a. PID controller

Generated by a PID controller, the input will be the function of state and time, and it will feedback the state of x_2 to state of x_3,

$$u(x,t) = K_p x_2(t) + K_i \int_0^t x_2(t)dt + K_d \dot{x_2}(t)$$

In the simulation, I set the parameters as $K_p = 1$, $K_i = 1$ and $K_d = 1.3$, and the initial state as $x(0) = (\frac{\pi}{6}, 0)^T$ (see Figure 2(a)).

[1] The PID controller here is not a standard one. This PID controller's error input is the angle offset, and the output will be the force applied not directly to the angle, but to the angular velocity



(a) (b)

Figure 2: (a) is the simulink module; (b) is the figure of the scope. The yellow plot represent the state x_2, and the blue plot represent the state x_3.

As shown in Figure 2(b), it is obvious that the fixed point is asympototically stable.

b. feedback linearization controller

Since,

$$\mathcal{L}_f h(x) = (1,1)\begin{pmatrix} x_3 \\ sinx_2 - x_2\cos x_2 - x_2 \end{pmatrix} = x_3 + sinx_2 - x_2\cos x_2 - x_2$$

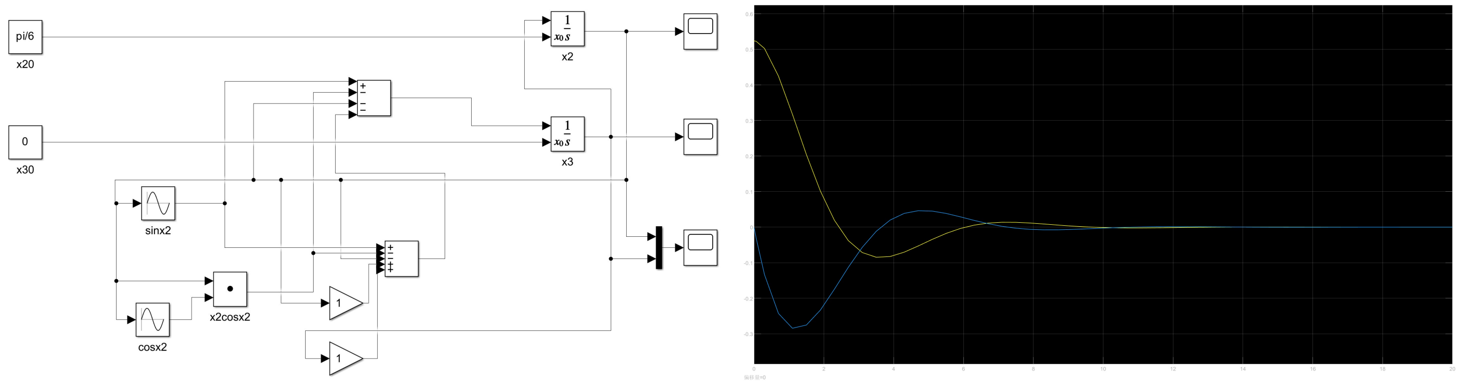
$$\mathcal{L}_g h(x) = (1,1)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1$$

then, $u = \mathcal{L}_g h(x)(-\mathcal{L}_f h(x) + v) = x_3 + sinx_2 - x_2\cos x_2 - x_2 - v$

Set $v = k_2 x_2$, $k_2 < 0$, then the system will be like,

$$\begin{pmatrix} \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ k_2 & -1 \end{pmatrix}\begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$

which is a linear stable system. In the simulation, I set the parameters as $k_2 = -1$, and the initial state as $x(0) = (\frac{\pi}{6}, 0)^T$. The settting and the result are shown in Figure 3.



(a) (b)

Figure 3: (a) is the simulink module; (b) is the figure of the scope. The yellow plot represent the state x_2, and the blue plot represent the state x_3.

As shown in Figure 3(b), the fixed point is asympototically stable.

4. Control Lyapunov Function

In this section, I will identify a control Lyapunov function and find a corresponding input.

The control Lyapunov function may be,

$$\mathcal{V} = \frac{1}{2}(x_2+x_3)^2 + x_2^2 + \frac{1}{2}x_3^2 + x_2\sin x_2 + 2(\cos x_2 - 1)$$

then,

$$\mathcal{L}_f\mathcal{V} = (3x_2+x_3-\sin x_2 + x_2\cos x_2, x_2+2x_3)\begin{pmatrix} x_3 \\ \sin x_2 - x_2\cos x_2 - x_2 \end{pmatrix} = (x_2+x_3)(\sin x_2 - x_2\cos x_2) + x_2x_3$$
$$\mathcal{L}_g\mathcal{V} = (3x_2+x_3-\sin x_2 + x_2\cos x_2, x_2+2x_3)\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -x_2 - 2x_3$$

then if $x_2+2x_3 \neq 0$, there exists $u = \frac{(x_2+x_3)(\sin x_2 - x_2\cos x_2)+x_2x_3+1}{x_2+2x_3}$, which makes $\mathcal{L}_f\mathcal{V} + u\mathcal{L}_g\mathcal{V} = -1 < 0$;
if $x_2+2x_3 = 0$, then $\mathcal{L}_f\mathcal{V} + u\mathcal{L}_g\mathcal{V} = -\frac{1}{2}x_2^2 + \frac{1}{2}x_2(\sin x_2 - x_2\cos x_2)$. Figure 4 shows that when $x_2 \neq 0$, i.e., $x \neq 0$, $-\frac{1}{2}x_2^2 + \frac{1}{2}x_2(\sin x_2 - x_2\cos x_2) < 0$.

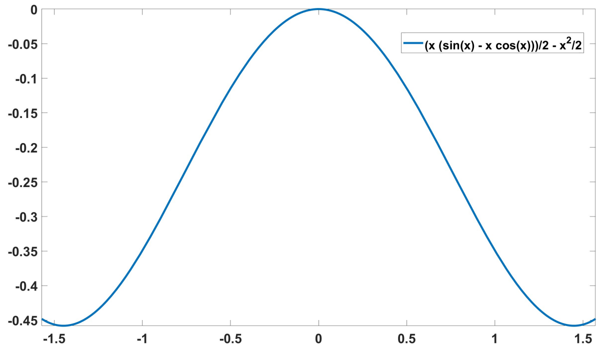


Figure 4: The plot of $y = -\frac{1}{2}x^2 + \frac{1}{2}x(\sin x - x\cos x)$

Therefore, I prove that $\mathcal{V}$ is a control Lyapunov function.
According to *E. D. Sontag Theorem*,

$$u = \begin{cases} \dfrac{(x_2+x_3)(\sin x_2 - x_2\cos x_2) + x_2x_3 + \sqrt{((x_2+x_3)(\sin x_2 - x_2\cos x_2)+x_2x_3)^2 + (x_2+2x_3)^4}}{x_2+2x_3} & x_2+2x_3 \neq 0 \\ 0 & x_2+2x_3 = 0 \end{cases}$$

Unluckily, this input is not continuous. However, I successfully generate a simple input by applying *E. D. Sontag Theorem* to a relaxed control Lyapunov function[2], and I prove the asympototical stability using *LaSalle's invariance principle*[3].
The function,

$$\tilde{\mathcal{V}} = \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_2\sin x_2 + 2(\cos x_2 - 1)$$

is the Lyapunov function I found in Report 2, then,

$$\mathcal{L}_f\tilde{\mathcal{V}} = 0$$
$$\mathcal{L}_g\tilde{\mathcal{V}} = -x_3$$

then $\mathcal{L}_f\tilde{\mathcal{V}} + u\mathcal{L}_g\tilde{\mathcal{V}} = -ux_3$. If $x_3 = 0$, there doesn't exist a u which could make $-ux_3 < 0$. However, $\forall x \neq 0$, there exists a u which could make $-ux_3 \leq 0$.
Applying *E. D. Sontag Theorem* to $\tilde{\mathcal{V}}$, then,

$$u = \begin{cases} -\dfrac{0+\sqrt{x_3^4}}{-x_3} = x_3 & x_3 \neq 0 \\ 0 & x_3 = 0 \end{cases}$$

[2]A relaxed control Lyapunov function, which is my self-created definition, needs only a negative semi-definite $\dot{V}(x,u)$

[3]Here, since I applied *E. D. Sontag Theorem* to a function which is not a CLF, I cannot guarantee that the generated input would stablize the system. Therefore, I further prove the stability using another method.

that is, $u = x_3$.

Applying $u = x_3$, the system will be like,

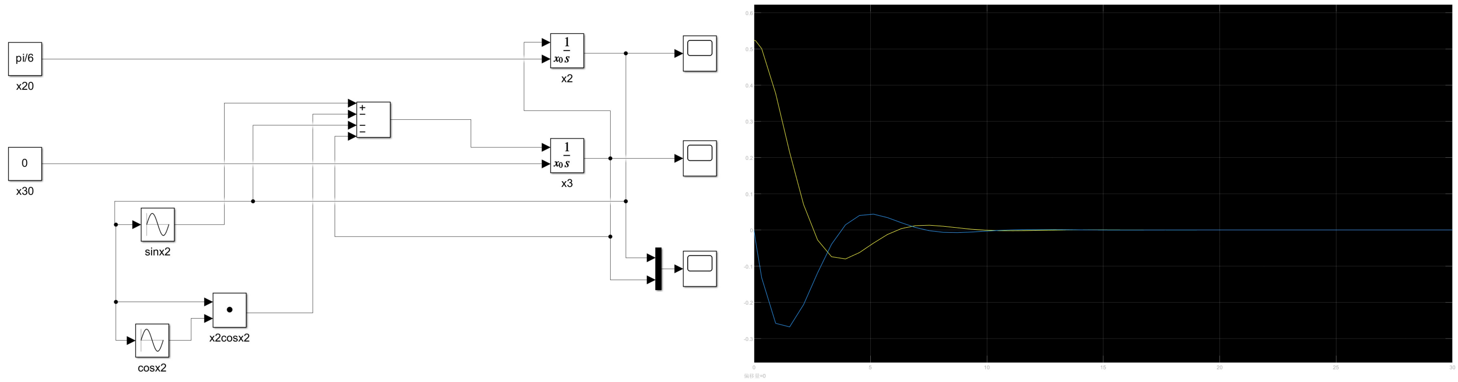
$$\dot{x} = \tilde{f}(x)$$

where $\tilde{f}(x) = \begin{pmatrix} x_3 \\ sinx_2 - x_2\cos x_2 - x_2 - x_3 \end{pmatrix}$. The fixed point remains $x^* = (0,0)^T$. Then,

$$\mathcal{L}_{\tilde{f}}\tilde{\mathcal{V}}(x) = -x_3^2$$

Except for $x \in \mathcal{I} = \{x | x_3 = 0\}$, $\forall x$ satisfies $\mathcal{L}_{\tilde{f}}\tilde{\mathcal{V}}(x) < 0$. $\forall x = (x_2, 0)$, $x_2 \neq 0$, then $\dot{x_3} \neq 0$, which means that $\mathcal{I}$ doesn't contain any trajectories. According to *LaSalle's invariance principle*, the system is asympototically stable.

For verification, I implement a simulation with the initial state $x(0) = (\frac{\pi}{6}, 0)^T$. Figure 5 shows the simulink module and the result.



(a) (b)

Figure 5: (a) is the simulink module; (b) is the figure of the scope. The yellow plot represent the state x_2, and the blue plot represent the state x_3.

4. Input to State Stability

The definition of ISS is that there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{L}$ and $a \in \mathbb{R}_{>0}$, s.t. $\forall t \geq 0$, we have,

$$||x(0)|| \leq a \Longrightarrow ||x(t)|| \leq \beta(||x(0)||, t) + \gamma(\sup_{0 \leq \tau \leq t} ||u(\tau)||)$$

However, since the original system is stable, but not asympototically stable, and it is an oscillation system (we have proven it in Report 2), there doesn't exist such a β satisfying the definition above.

Therefore, I decide to first stablize the system, and then analyze the ISS property. I will stablize the system in three ways in which we previously did the simulations.

a. PID controller

With the PID controller, the system becomes a time-variant system. Since it is difficult to analyze such a system theoratically, I choose to implement simulations.

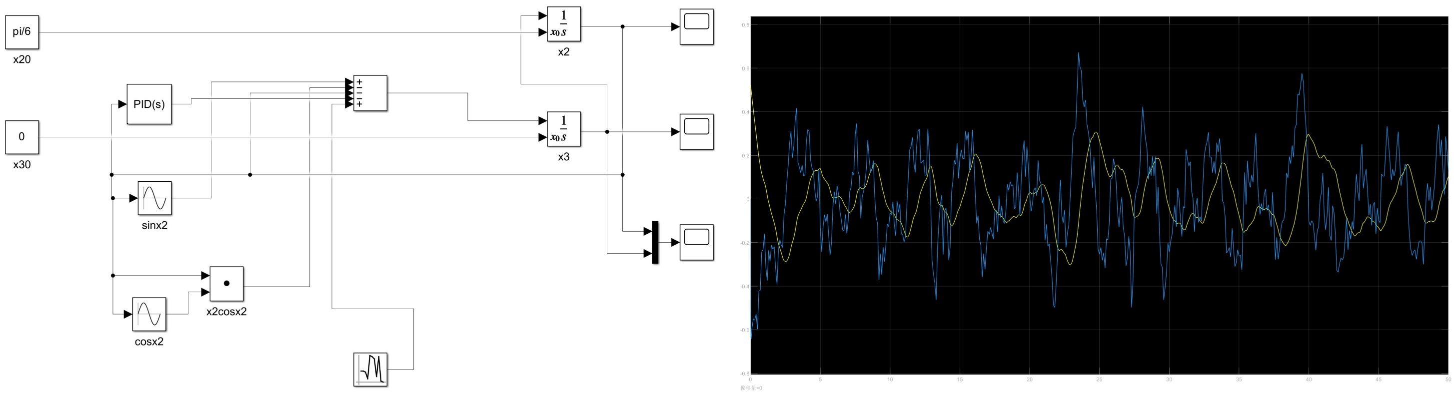
In addition to the designed PID controller input $u(x,t) = K_p x_2(t) + K_i \int_0^t x_2(t)dt + K_d \dot{x_2}(t)$, I give pertubations to the input, which follows Gaussian distribution $\mathcal{N}(0,1)$. Figure 6 shows the simulink setting and the result.

As is shown in Figure 5, with the bounded input pertubation, the output is also bounded, which could roughly illustrate that the system, stablized by the PID controller, has ISS property.

b. feedback linearization controller

With the feedback linearization controller, the system becomes a linear system as follow,

$$\begin{pmatrix} \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$$



(a) (b)

Figure 6: (a) is the simulink module involving random pertubations of the input, the pertubations are random signals following $\mathcal{N}(0,1)$; (b) is the scope of the output with the PID controller.

Adding to the previously designed input, let Δu denote the pertubation, then,

$$\begin{pmatrix} \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Delta u$$

We choose a Lyapunov function to this system,

$$\mathcal{V} = x_2^2 + \frac{1}{2}x_3^2 + \frac{1}{2}(x_2 + x_3)^2$$

then we have,

$$\begin{aligned} \mathcal{V} &\leq 2x_2^2 + \frac{3}{2}x_3^2 \leq 2\|x\|^2 \\ \mathcal{V} &\geq x_2^2 + \frac{1}{2}x_3^2 \geq \frac{1}{2}\|x\|^2 \\ \dot{\mathcal{V}} &= -x_2^2 - x_3^2 - x_2 u - 2x_3 u \leq -\|x\|^2 + 2\|x\||u| \end{aligned}$$

For $|u| \leq \frac{1}{4}\|x\|$, we have,

$$\dot{\mathcal{V}} \leq -\frac{1}{2}\|x\|^2$$

Hence this Lyapunov function is an ISS Lyapunov fucntion, and therefore, the system, stablized by a feedback linearization controller, has ISS property.

c. relaxed Sontag controller

With the controller and the additional pertubation Δu, the system will be like,

$$\begin{pmatrix} \dot{x_2} \\ \dot{x_3} \end{pmatrix} = \begin{pmatrix} x_3 \\ sinx_2 - x_2 \cos x_2 - x_2 - x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \Delta u$$

To find the ISS Lyapunov function, we start from the Lyapunov function we found in Report 2,

$$\mathcal{V}_1 = \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 + x_2 \sin x_2 + 2(\cos x_2 - 1)$$

then,

$$\dot{\mathcal{V}_1} = -x_3^2 - ux_3$$

Here we notice that,

$$\frac{\partial}{\partial t}(\frac{1}{2}x_3^2) = x_3 \sin x_2 - x_2 x_3 \cos x_2 - x_2 x_3 - x_3^2$$

And we continuously notice that,

$$\frac{\partial}{\partial t}(\frac{1}{2}(x_2+x_3)^2) = \ldots x_3 \sin x_2 - x_2 x_3 \cos x_2 - x_2 x_3 \ldots$$

We find that $\frac{\partial}{\partial t}(\frac{1}{2}x_3^2)$ consists of two parts: one part is the same cross terms as in $\frac{\partial}{\partial t}(\frac{1}{2}(x_2+x_3)^2)$, the other is an isolate square term $-x_3^2$. Therefore, we choose a new function,

$$\mathcal{V}_2 = \frac{1}{2}x_2^2 + \frac{1}{2}(x_2+x_3)^2 + x_2 \sin x_2 + 2(\cos x_2 - 1)$$

Here, we will prove that $\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2$ is an ISS Lyapunov function.

$$\mathcal{V} = x_2^2 + \frac{1}{2}(x_2+x_3)^2 + \frac{1}{2}x_3^2 + 2x_2 \sin x_2 + 4(\cos x_2 - 1)$$

We have, $\frac{1}{2}(x_2+x_3)^2 \leq x_2^2 + x_3^2$ and $2x_2 \sin x_2 + 4(\cos x_2 - 1) \leq 0$ (recall that the domain of the system limits that $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$), then we have,

$$\mathcal{V} \leq 2x_2^2 + \frac{3}{2}x_3^2 \leq 2\|x\|^2$$

Figure 7 compares $x_2^2 + 2x_2 \sin x_2 + 4(\cos x_2 - 1)$ with $\frac{1}{4}x_2^2$. We get that $x_2^2 + 2x_2 \sin x_2 + 4(\cos x_2 - 1) \geq \frac{1}{4}x_2^2$.

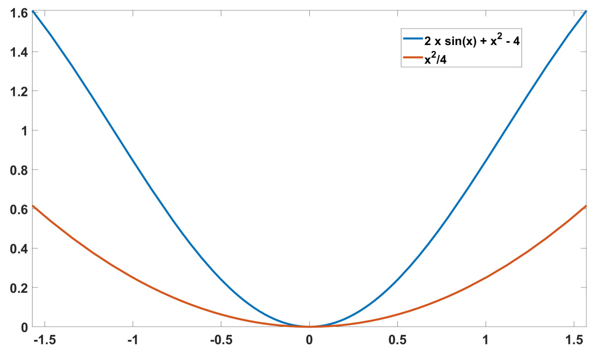


Figure 7: The comparsion between two plots.

Hence we have,

$$\mathcal{V} \geq \frac{5}{4}x_2^2 + \frac{1}{2}x_3^2 \geq \frac{1}{2}\|x\|^2$$

Furthermore, we have,

$$\begin{aligned}
\dot{\mathcal{V}} &= x_2 \sin x_2 - x_2^2 \cos x_2 - x_2^2 - x_3^2 - ux_2 - 2ux_3 \\
&\leq -x_2^2 - x_3^2 - ux_2 - 2ux_3 \\
&\leq -\|x\|^2 + 2\|x\||u|
\end{aligned}$$

For $|u| \leq \frac{1}{4}\|x\|$, we have,

$$\dot{\mathcal{V}} \leq -\frac{1}{2}\|x\|^2$$

Hence this Lyapunov function is an ISS Lyapunov fucntion, and therefore, the system, stablized by a relaxed Sontag controller, has ISS property.

Report 4: Control Design

Deyi Wang

March 28, 2023

1. System Description

The model of my system derived from a unicycle. The details are as follow (recall previous reports):

State: $x = (x_1, x_2)^T \in \{(x_1, x_2)^T | -\frac{\pi}{2} < x_1 < \frac{\pi}{2}, x_2 \in \mathbb{R}\} \subseteq \mathbb{R}^2$

In the state, x_1 represents the angle and x_2 represents the angle velocity. We have $x_2 = \dot{x_1}$.

Output: $h(x) = x_1 \in (-\frac{\pi}{2}, \frac{\pi}{2}) \subseteq \mathbb{R}$

We can easily observe and measure the angle, and we also hope to stablize it.

Control Input: $u \in \mathbb{R}$, $g(x) = (0, 1)^T$

The control is a force applied to changing the angle velocity. (*We cannot expect the control input in more than one dimension, since the system is in only two dimensions and a two-dimension control input will make the system trivial.*)

The dynamics of the system will be like,

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ h(x) &= x_1 \end{aligned}$$

where $f(x) = \begin{pmatrix} x_2 \\ \sin x_1 - x_1 \cos x_1 - x_1 \end{pmatrix}$.

2. Feedback Linearization

$$\begin{aligned} \mathcal{L}_f h(x) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 - x_1 \cos x_1 - x_1 \end{pmatrix} = x_2 \\ \mathcal{L}_g h(x) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0 \\ \mathcal{L}_f^2 h(x) &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 - x_1 \cos x_1 - x_1 \end{pmatrix} = \sin x_1 - x_1 \cos x_1 - x_1 \\ \mathcal{L}_g \mathcal{L}_f h(x) &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \neq 0 \end{aligned}$$

Therefore, $z = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x$, and $u = -(\sin x_1 - x_1 \cos x_1 - x_1) + v$.

Let $v = -k_1 x_1 - k_2 x_2$, $k_1, k_2 > 0$, we get controllable canonical form,

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -k_1 & -k_2 \end{pmatrix} x$$

We successfully applied feedback linearization to the system.

Analysis

1. Since $\mathcal{L}_g h(x) = 0$ and $\mathcal{L}_g \mathcal{L}_f h(x) \neq 0$, the relative degree is 2;
2. Since $y = h(x) = x_1$ and $\dot{y} = \mathcal{L}_f h(x) = x_2$, both states are fully observable from the output of the system, thus this system doesn't include zero dynamics.

However, if we modified the $g(x)$ with $\tilde{g}(x) = (1, 0)^T$, then,

$\mathcal{L}_f h(x) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ \sin x_1 - x_1 \cos x_1 - x_1 \end{pmatrix} = x_2$

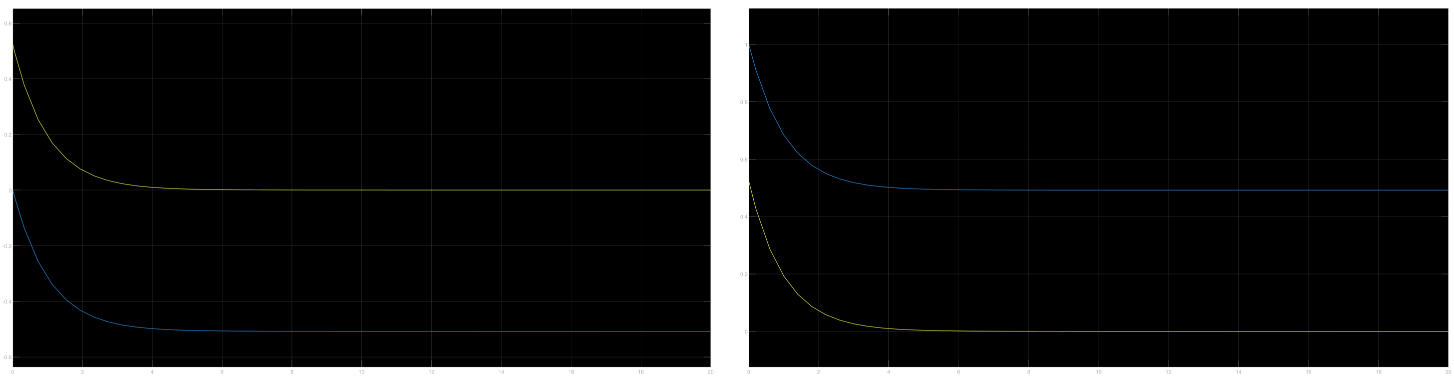
$\mathcal{L}_{\tilde{g}} h(x) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \neq 0$

Therefore, $z = h(x) = x_1$, and $u = -x_2 + v$. Let $v = -kz$, we get controllable canonical form,

$$\dot{z} = -kz$$

Analysis

1. This is a trivial system. We used the feedback linearization to seperate the connection between x_1 and x_2, and focus only on x_1. I analyzed this system just for illustrating the zero dynamics;
2. Since $\mathcal{L}_g h(x) \neq 0$, the system has relative degree 1;
3. Since the relative degree 1 is less than the system's dimension which is 2, there exists zero dynamics. x_2 is not observable from the output;
4. When z, i.e., x_1 asympototically approaches to 0, the dynamics of x_2 is $\dot{x_2} = 0$, which means x_2 is unstable zero dynamics. The final position of x_2 depends on the initial state and x_2 has no fixede points. Figure 1 has shown the different state conditions with different initial states.



(a) (b)

Figure 1: (a) shows the state starting from the initial state $(x_1, x_2) = (\frac{\pi}{6}, 0)$; (b) shows the state starting from the initial state $(x_1, x_2) = (\frac{\pi}{6}, 1)$.

3. Single-Integrator Backstepping

In the lecture, we were taught a theorem saying that for a system with a CLF and a reasonable controller (may come from *E.D. Sontag Theorem*), we can use the backstepping to find a new controller for the new system with a single-integrator.

However, when it comes to my system, we didn't find a CLF from which a resonable controller derived (recall Report 3, we did find a CLF, but the corresponding *Sontag* controller was noncontinuous.) Instead, we found a relaxed CLF from which a reasonable controller derived, and we also proved the stability of the system with this controller.

Recall Report 3, for our system $\dot{x} = f(x) + g(x)u$, if $u = -x_2$, the system will be asympototically stable. In this chapter, I will desgin a controller, based on the $u = -x_2$ to stablize the following system which is a cascade connection of an integrator with the x subsystem,

$$\begin{aligned} \dot{x} &= f(x) + g(x)z_1 \\ \dot{z_1} &= u_1 \end{aligned}$$

We firstly rewrite the system in an equivalent way,

$$\begin{aligned} \dot{x} &= f(x) + g(x)z_1 + g(x)(-x_2) - g(x)(-x_2) \\ \dot{z_1} &= u_1 \end{aligned}$$

Let $e_1 \triangleq z_1 + x_2$, and let $v_1 \triangleq u_1 + \dot{x_2}$ be the new input, we have,

$$\begin{aligned} \dot{x} &= f(x) + g(x)(-x_2) + g(x)e_1 \\ \dot{e_1} &= v_1 \end{aligned}$$

Recall Report 3, for the system $\dot{x} = \tilde{f}(x)$, where $\tilde{f}(x) = f(x) + g(x)(-x_2)$, we have a Lyapunov function,

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_1 \sin x_1 + 2(\cos x_1 - 1)$$

which can be used to prove the stability of the system, with *LaSalle's invariance principle*.
Design v_1 in the following way,

$$v_1 = -\mathcal{L}_g V - re_1 = -x_2 - re_1$$

where $r > 0$. Then the system becomes,

$$\begin{aligned} \dot{x} &= f(x) + g(x)(-x_2) + g(x)e_1 \\ \dot{e_1} &= -x_2 - re_1 \end{aligned}$$

which is,

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{e_1} \end{pmatrix} = \begin{pmatrix} x_2 \\ \sin x_1 - x_1 \cos x_1 - x_1 - x_2 + e_1 \\ -x_2 - re_1 \end{pmatrix}$$

We can prove the stability of this (x, e_1) system based on a new Lyapunov function,

$$\tilde{V} = V + \frac{1}{2}e_1^2$$

It is easy to know that $\dot{\tilde{V}} = -x_2^2 - re_1^2$. Here we can use *LaSalle's invariance principle* again.
Except for $\begin{pmatrix} x \\ e_1 \end{pmatrix} \in \mathcal{I} = \{ \begin{pmatrix} x_1 \\ x_2 \\ e_1 \end{pmatrix} | x_1 = 0\}$, $\forall \begin{pmatrix} x \\ e_1 \end{pmatrix}$ satisfies $\dot{\tilde{V}} < 0$.
$\forall \begin{pmatrix} x \\ e_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix}$, $x_1 \neq 0$, then since $\dot{x_2} = \sin x_1 - x_1 \cos x_1 - x_1 - x_2 + e_1$, $\dot{x_2} \neq 0$ (see Figure 2).

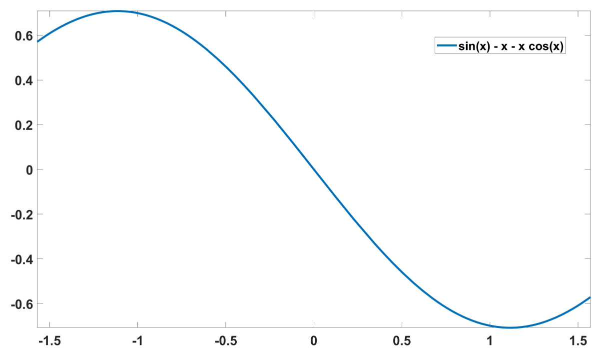


Figure 2: The plot of $y = \sin x - x\cos x - x$

This means $\mathcal{I}$ contains no trajectories. According to *LaSalle's invariance principle*, this system is asympototically stable.
Finally, we didn't find a CLF for this new (x, e_1) system. Instead, we found a input making the system stable, which had the equivalent meaning of finding a CLF.

Summary

The project focused on a classic example of nonlinear systems, the unicycle. There are two problems about the unicycle, one is to make the system stable and not roll over, the other is to allow the player to control the speed. My project focused on the first problem.

In Report 1, I described my system systematically and mathematically, and then I proved the Lipschitz continuity of my system, which ensured the solvability of my system. In Report 2, I showed the C^0 regularity, i.e., the continuity with respect to the state of my system. Then I found the invariant sets. Finally, based on Lyapunov theory, I found a Lyapunov function of my system and proved its stability. In Report 3, I introduced an input and defined an output to my system. I found a CLF and a relaxed CLF to my system. Based on the relaxed CLF, I derived a simple *Sontag* input $u = -x_2$ which could stablize the system. Based on different inputs, such as the PID controller, feedback linearization controller and *Sontag* input controller, I showed the asympototical stability of my system, and proved the ISS property of my system. In the final report, with two types of control design strategies, feedback linearization and backstepping, I designed two inputs to my system, both of which were valid.

The most successful part of my project was the proposition of the Lyapunov function where the proposition of CLF, the proof of ISS and the control desgin via backstepping based in the following reports. After discussing with my classmates, I'm certain that the Lyapunov function is essential, making my system obtain the potential to be analyzed under the framework of the given lecture. The method I used to find the Lyapunov function was a combination of sum of square, conservation of energy and balance match method.

The weak point of my report was the bounded input design. I didn't succeed in find a CLF in good quality and its corresponding Sontag bounded input which should be smooth enough. I compromised by finding a relaxed CLF which corresponded to a simple input $u = -x_2$. Unluckily, this input still scaled poorly with the size of x.

Nonlinear dynamics and control was perfectly appliable to my project, especially the several types of control design method. *E.D. Sontag Theorem*, feedback linearization and backstepping methods all provided remakeable ways to stablize the system. The ISS analysis also provided insight into the requirements of such a product, that is the system needs to tolerate inaccuary.

When it comes to an open-ended question, I hope to design a controller which scales well with the state x. The biggest challenge to this question is to find a CLF with high quality, which is difficult. If I have enough time, I will try to tackle this question by bypass the difficulty of finding a CLF and learn more types of controllers, such as LQR and so on. Maybe some certain types of controllers could satisfy the requirements.