Preprints of the 22nd IFAC World Congress
Yokohama, Japan, July 9-14, 2023

A Maximum Degree related Condition to Asymmetric Game in Weighted Vertex Cover Networks ⋆

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Abstract: Minimum weighted vertex cover (MWVC) is an important optimization problem in the application of complex network theory, and the asymmetric game model sheds light on the essential mechanism of the MWVC. In this paper, a new condition for the effectiveness of the asymmetric game is discovered based on the imbalance of weighted networks. When the new condition satisfies, a Nash equilibrium (NE) becomes a bridge state between vertex cover (VC) state and MWVC state, and any initial state of the network can reach the NE following the feedback-based best response (FBR) algorithm. Numerical experiments verify that this new condition has stronger stability and wider range of applications than the original one. FBR algorithm with the new condition shows superiority over other distributed algorithms. We also qualitatively clarify the network property factors affecting the performance of MWVC to some extent.

Keywords: Optimization and control of large-scale network systems, Modelling and decision making in complex systems, Game theory, Minimum weighted vertex cover, Nash equilibrium

1. INTRODUCTION

Minimum vertex cover (MVC) and minimum weighted vertex cover (MWVC) are important optimization problems in the engineering application of complex network theory. Liao and Lee (2005), Hao et al. (2018), Grout (2005) and others raised a large number of practical planning and optimization problems in the fields including power transmission, transportation and communications which are the concrete representation of these abstract problems. Anderson and Chakrabortty (2012) gave an example on the large-scale power supply and transmission networks of a city. There are a large number of power stations and transmission lines in the networks. In order to reduce environmental pollution caused by power generation, power stations often adopt the peak-load shifting, i.e. only part of the power stations work in a period of time and supply power to the whole city. Therefore, how to cover all the transmission lines to the whole city with the minimum number of working power stations has become an important problem, calling the application of MVC. In reality, the actual operation costs of various power stations are often different. Power grid managers tend to maintain the power supply to the whole city with minimum operation cost. In such cases, we need to replace the simple graph model with weighted graph model, and expand the MVC to a more general MWVC.

Traditional solutions for the optimization problems such as vertex cover can be mainly categorized into centralized optimization, represented by Karakostas (2005), Fomin et al. (2006), Chen et al. (2001) and distributed optimization represented by Fang and Kong (2007), Cardinal and Hoefer (2006), Gairing (2009). In a centralized optimization, the actions of all vertices need to be coordinated by a central administrator, and it needs to know the global information of the whole network for decision making. Centralized solutions have some inherent limitations, where the computation burden will inevitably become too much to bear with the expansion of network scale. Distributed optimizations have attracted more attention to cope with this problem. In a distributed optimization, the action of each vertex in the network is determined by its own analysis based on itself and its neighbors. Each vertex only needs to know some local information, which makes the distributed solution practically scalable. Based on game theory and Nash equilibrium, Gairing (2009) provided a theoretical basis for the solution to max-n-cover. Cardinal and Hoefer (2006) proposed the solution to non-cooperative vertex cover and took distributed server deployment as an example. However, these works have some limitations. First of all, they did not provide the method for selecting game models, and they did not give the performance analysis in applications. More importantly, these optimization theories set some restrictions

⋆ This work was partly supported by the Fundamental Research Funds for the Central Universities and the Natural Science Foundation of China under Grant No. 71731004.

on the scope of their potential applications, which makes it difficult to expand their results from simple graphs to weighted graphs. Srivastava et al. (2005) stated that game theory provides useful tools for modeling and predicting the interaction among independent vertices. Therefore, many researchers, represented by Roshanbin (2014), Agarwal and Ergun (2008), McSweeney et al. (2012), Madeo and Mocenni (2014) paid more attention to game theory to solve graph optimization problems.

Wang et al. (2006) introduced the snowdrift game and memory-based best response (MBR) algorithm, and analyzed the stability of the game results under different parameters. Yang and Li (2013) applied the snowdrift game to solve the MVC problem, and revealed the significance of its Nash equilibrium. Based on the snowdrift game model and MBR algorithm, Tang et al. (2017) creatively put forward the asymmetric game model and feedback-based best response (FBR) algorithm applicable to weighted graphs, and extended their solution to MWVC. Tang et al. (2017) proposed that when the parameters of the game model meet $\Delta_A/\Delta_B > 4\lambda_A\lambda_B$, a Nash equilibrium in the network must be a VC state, and a MWVC state must be a Nash equilibrium. When the FBR is adopted, states of the vertices in the network will be able to reach Nash equilibrium, and the result is closer to the optimal MWVC state using only MBR.

However, the work of Tang et al. (2017) may still be improved. For example, in the process of theoretical demonstration, the authors simplified the weight relations between the focus vertex and its neighbors to $\Delta_A/\Delta_B > 4\lambda_A\lambda_B$, which is independent to the complexity of the involved network topology among the population of players. More close to our interest in this paper, we put more attention to revisit this problem to address such impacts to improve the condition's effectiveness.

This article's contributions are summarized as:

(1) A more rigorous condition for the asymmetric game model to achieve more effective optimization is given. The relations among different states and the effectiveness of the FBR algorithm are demonstrated;
(2) Simulation experiments verify the superiority of this new condition compared to the original ones, and they show the superiority of FBR algorithm compared to other distributed MWVC algorithms.

The rest of this paper is arranged as follows: *Section 2* introduces the preliminaries, including the mathematical description of MWVC, the asymmetric game model, Nash equilibrium and the FBR algorithm. *Section 3* gives the strict parameter constraint condition for the asymmetric game model as well as its proof. In *Section 4*, simulation experiments are demonstrated, and *Section 5* concludes this paper.

2. PRELIMINARIES

2.1 Minimum wighted vertex cover

Given a weighted undirected graph $G = (V, E, W)$, where $V = \{v_i \mid i = 1, 2, ..., n\}$, $W = \{w_i \mid i = 1, 2, ...n\}$, and $E = \{e_{ij} \mid v_i, v_j \in V\}$. If there exists a vertex subset $V_{sub} \subseteq V$ such that $\forall e_{ij} \in E$, $v_i \in V_{sub}$ or $v_j \in V_{sub}$, then

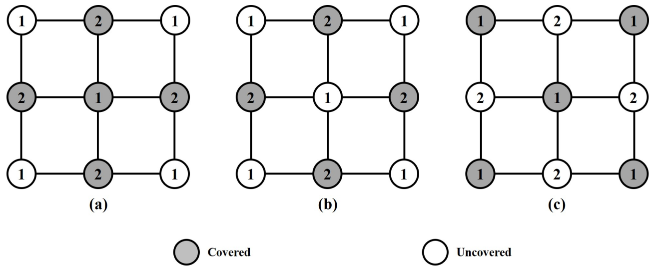

Fig. 1. Illustration of the VC state, the WVC state and the MWVC state. (a) represents the VC state and at least one vertex of each edge is covered, marked gray. (b) represents the MVC state and the number of vertices covered has reached the minimum 4, but when it comes to the weights, it is obvious that it can decrease continuously. (c) represents the MWVC state and the sum weight of covered vertices has reached the minimum.

V_{sub} is called a vertex cover set (V_C). Let $\{V_C\}$ denote the vertex cover sets and define the sum weight of a vertex set V_k as $\mid V_k \mid = \sum_{v_i \in V_k} w_i$, then the set of MWVC satisfies $V_{MWVC} \in \{V_C\}$ and $\mid V_{MWVC} \mid = min \mid V_C \mid$. Fig.1 shows the relations between VC, MVC, and MWVC states.

2.2 Asymmetric game model

Extending the snowdrift game model to the game between two adjacent nodes with different weights, Topp (1983) and Grafen (1979) proposed asymmetric game model. It is widely used in dealing with the optimization problems with differences among the game players, take Accinelli and Carrera (2011), Smilovitch and Lachman (2019), and Cardaliaguet (2007) for examples. There are two choices in the strategy space of each vertex, namely *Covered (C)* and *Uncovered (N)*.

The snowdrift game model is applicable to unweighted graphs. Obviously, it is the local optimal solution of the MVC problem that one of the two vertices adopts C strategy and the other one adopts N strategy. Because the optimization aims at reducing the number of C vertices, the vertex with C strategy gets smaller payoff, and the vertex with N strategy gets larger payoff. The model may also achieve vertex cover where both vertices adopt C strategy, but it is a suboptimal solution. If both vertices adopt N strategy, it is an unacceptable failure solution. Based on the demonstration above, the payoff matrix of snowdrift game is specified as:

$$\begin{array}{c c} & \begin{array}{cc} C & N \end{array} \\ \begin{array}{c} C \\ N \end{array} & \begin{pmatrix} 1 & 1-r \\ 1+r & 0 \end{pmatrix} \end{array}$$

where $0 < r < 1$.

The model will become asymmetric when the weights are considered. The situation where two vertices adopt C and N strategy respectively needs to be divided into two different cases, because the two vertices have asymmetric weights. The optimization aims at reducing the sum weight of C vertices. Therefore, when the low weight (L) vertex adopts C strategy, the payoff of the corresponding high weight (H) vertex will increase. When the L vertex adopts

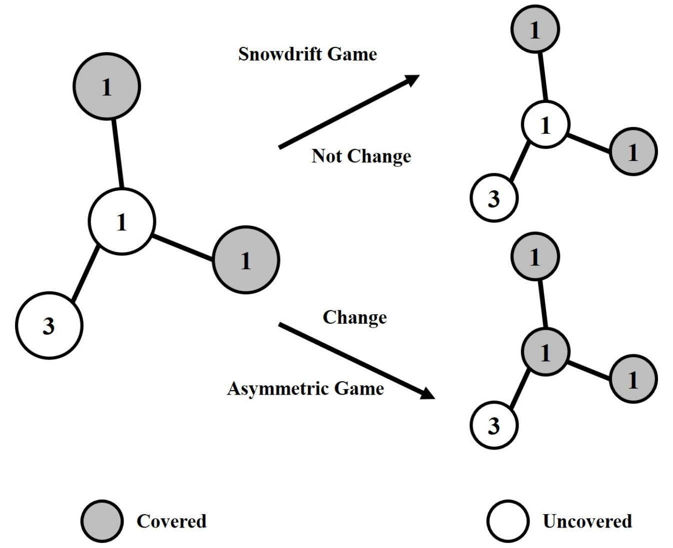

Fig. 2. The differences between the two game models. Assume $r = 0.5$. For the focal vertex, when playing SG, $u(C) = 3 - r = 2.5$ and $u(N) = 2 + 2r = 3$, so the focus vertex will not change its current strategy. When playing AG, $u(C) = 4$ and $u(D) = 2 + 2r = 3$, so the focus vertex will switch into C strategy.

N strategy, the payoff of the corresponding H vertex will reduce. When the H vertex adopts C strategy, the payoff of the corresponding L vertex will reduce. When the H vertex adopts N strategy, the payoff of the corresponding L vertex will increase. Define the asymmetric degree between v_i and v_j as:

$$\lambda_{ij} = \lambda_{ji} = \frac{max\{w_i, w_j\}}{w_i + w_j}$$

where $\lambda_{ij} \in (0, 1)$.

Apply the asymmetric degree λ_{ij} to the game and the payoff matrix between v_i and v_j is given as:

$$\begin{array}{c} \\ H_C \\ H_N \\ L_C \\ L_N \end{array} \begin{array}{c} \begin{array}{cccc} H_C & H_N & L_C & L_N \end{array} \\ \begin{pmatrix} 1 & 1-r & 2\lambda_{ij} & \frac{1-r}{2\lambda_{ij}} \\ 1+r & 0 & 2\lambda_{ij}(1+r) & 0 \\ 2(1-\lambda_{ij}) & \frac{1}{2(1-\lambda_{ij})} & 1 & 1-r \\ 2(1-\lambda_{ij})(1+r) & 0 & 1+r & 0 \end{pmatrix} \end{array}$$

where $0 < r < 1$. The difference between snowdrift game and asymmetric game is shown in Fig.2.

2.3 Nash equilibrium

Nash equilibrium proposed by Nash (1950) and Nash (1951) is a critical state in game theory. It means a balance in the game process. Given an n-person game as $\mathbb{G} = \langle N, S_i, u_i \rangle$, where S_i represents strategy space and u_i represents payoff function. Let $s^* = (s_1^*, s_2^*, ..., s_n^*)$ be a set of strategies of all vertices, and $s_{-i}^* = (s_1^*, s_2^*, ..., s_{i-1}^*, s_{i+1}^*, ..., s_n^*)$ is the set of strategies excluding vertex v_i. Nash equilibrium satisfies that $\forall i \in N, \forall s_i \in S_i, u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$, i.e., for each player, its current strategy is its optimal strategy in reaction to other players' strategies. If the inequality above holds strict when $s_i \neq s_i^*$, the Nash equilibrium is a strict Nash equilibrium (SNE).

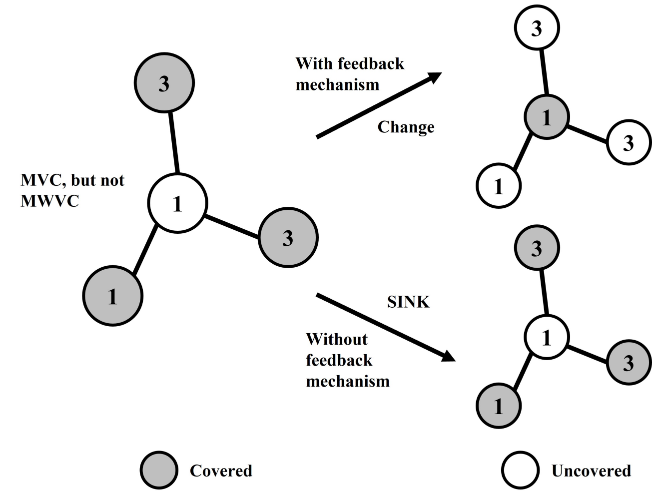

Fig. 3. Example of feedback mechanism. Because the weights and the number of vertices are different, states of the graph may sink into MVC and may not reach MWVC without the feedback mechanism. The feedback mechanism increase the communication between vertices.

2.4 Feedback-based best response algorithm

FBR guides the weighted graph to reach SNE state through multiple rounds of asymmetric game from the initial state. The algorithm has two mechanisms: one is that each vertex can hold the memory of its optimal strategies selected in the previous finite rounds, and the other is that it can optimize the strategy selection of the high weight vertices.

Steps of the algorithm are summarized as follows:

(1) Each vertex in the graph plays an asymmetric game with its neighbors, using its current strategy and the opposite strategy, respectively.
FeedbackMechanism : If the weight of this vertex is larger than its neighbors' who take strategy N, the vertex should play an asymmetric game assuming that all its neighbors take strategy C. Fig.3 shows an example of the feedback mechanism.
(2) By comparing the payoff with two opposite strategy, Each vertex stores the optimal strategy obtained in step 1 into its memory and removes the oldest strategy.
(3) A new strategy is selected randomly form the strategy memory of each vertex.

3. PARAMETER CONDITIONS FOR ASYMMETRIC GAME MODEL

3.1 Relations among SNE, VC, and MWVC

Theorem 1. Given a weighted undirected graph $G = (V, E, W)$. For an SNE state $s^* = (s_1^*, s_2^*, ..., s_n^*)$ and its corresponding set of covered vertexes $V_{SNE} = \{v_i \mid s_i^* = C, v_i \in V\}$, when the parameter r in asymmetric game satisfies $\frac{1-r}{r} > 4\lambda_{max}^2 k_{max}$, where $\lambda_{max} = max\{\lambda_{ij}\}$ and

k_{max} is the maximum vertex degree, then $\{V_{MWVC}\} \subseteq \{V_{SNE}\} \subseteq \{V_{VC}\}$.

Proof. We divide the proof into two steps. First we prove $\{V_{MWVC}\} \subseteq \{V_{SNE}\}$, then we prove $\{V_{SNE}\} \subseteq \{V_{VC}\}$.

$\{\mathbf{V_{MWVC}}\} \subseteq \{\mathbf{V_{SNE}}\}$: Arbitrarily given an MWVC state and its corresponding strategy combination s_{MWVC}. Demonstrate the focus vertex in different cases:

Case 1: The focus vertex i performs C strategy, then its neighbors' strategies can be C or N, i.e., the focus vertex has $p(p \geq 1)$ strategy N neighbors with a set of weight $\{w_{N1}, w_{N2}, ..., w_{Np}\}$ and q strategy C neighbors with a set of weight $\{w_{C1}, w_{C2}, ..., w_{Cq}\}$. Then the degree of the focus vertex $D_i = p + q$. Let w_i be the weight of the focus vertex.

Without the loss of generality, let $w_{N1} \leq w_{N2} \leq ... \leq w_{Nr} \leq w_i \leq w_{N(r+1)} \leq ... \leq w_{Np}$, and $w_{C1} \leq w_{C2} \leq ... \leq w_{Cs} \leq w_i \leq w_{C(s+1)} \leq ... \leq w_{Cp}$.

According to the payoff matrix of asymmetric game, the payoff functions of the focus vertex with current and inverse strategies are:

$$u(C) = \sum_{j=N1}^{Nr} \frac{1-r}{2\lambda_{ij}} + \sum_{j=N(r+1)}^{Np} \frac{1-r}{2(1-\lambda_{ij})} + \sum_{C1}^{Cs} 2\lambda_{ij} + \sum_{C(s+1)}^{Cq} 2(1-\lambda_{ij})$$

$$u(N) = \sum_{j=N1}^{Nr} 0 + \sum_{j=N(r+1)}^{Np} 0 + \sum_{C1}^{Cs} 2\lambda_{ij}(1+r) + \sum_{C(s+1)}^{Cq} 2(1-\lambda_{ij})(1+r)$$

Compare the two functions:

$$u(C) - u(N) = (1-r)\left(\sum_{j=N1}^{Nr} \frac{1}{2\lambda_{ij}} + \sum_{j=N(r+1)}^{Np} \frac{1}{2(1-\lambda_{ij})}\right) - r\left(\sum_{C1}^{Cs} 2\lambda_{ij} + \sum_{C(s+1)}^{Cq} 2(1-\lambda_{ij})\right)$$

for $\forall j \in \{N_1, N_2, ..., N_p, C_1, C_2, ..., C_q\}$, $\lambda_{max} \geq \lambda_{ij}$ and $\lambda_{max} \geq 1 - \lambda_{ij}$. Then, $u(C) - u(N) \geq (1-r)\frac{p}{2\lambda_{max}} - 2rq\lambda_{max} = \frac{pr}{2\lambda_{max}}(\frac{1-r}{r} - 4\lambda_{max}^2 \frac{q}{p})$.

Because $\frac{q}{p} < D_i \leq k_{max}$, $u(C) - u(N) \geq \frac{pr}{2\lambda_{max}}(\frac{1-r}{r} - 4\lambda_{max}^2 k_{max})$.

For $\frac{1-r}{r} > 4\lambda_{max}^2 k_{max}$, $u(C) > u(N)$, and strategy C is the best choice of the focus vertex. This conclusion implies that the current strategy is the best choice and the focus vertex do not need to change its strategy.

Case 2: The focus vertex i adopts N strategy, then its neighbors' strategies can only be C, i.e., vertex i is surrounded by the covered neighbors with a set of weights $\{w_{C1}, w_{C2}, ..., w_{Cq}\}$. Consider the most general case: $w_{C1} \leq w_{C2} \leq ... \leq w_{Cs} \leq w_i \leq w_{C(s+1)} \leq ... \leq w_{Cp}$.

$$u(C) = \sum_{j=C1}^{Cs} 2\lambda_{ij} + \sum_{j=C(s+1)}^{Cq} 2(1-\lambda_{ij})$$

$$u(N) = \sum_{j=C1}^{Cs} 2\lambda_{ij}(1+r) + \sum_{j=C(s+1)}^{Cq} 2(1-\lambda_{ij})(1+r)$$

Because $1 + r > 1$, $u(C) < u(D)$.The focus vertex do not need to change its strategy.

Therefore, we prove $\{V_{MWVC}\} \subseteq \{V_{SNE}\}$, which means that if the graph reaches an MWVC state, it must have also reached an SNE state.

$\{\mathbf{V_{SNE}}\} \subseteq \{\mathbf{V_{VC}}\}$:Arbitrarily given an SNE state and its corresponding strategy combination s_{SNE}. For each focus vertex, we need to prove that if there exists at least one N vertex in its neighbors, the focus vertex must choose C strategy under the rules of SNE.

Assume the focus vertex i has $p(p \geq 1)$ strategy N neighbors with a set of weights $\{w_{N1}, w_{N2}, ..., w_{Np}\}$ and q strategy C neighbors with a set of weights $\{w_{C1}, w_{C2}, ..., w_{Cq}\}$.Then the degree of vertex v_i is $D_i = p + q$. Let w_i be the weight of v_i.

We can discover the similarity between this case and the previous *Case 1* in the proof of $\{\mathbf{V_{MWVC}}\} \subseteq \{\mathbf{V_{SNE}}\}$, so we can get the conclusion straightway that $u(C) > u(N)$ for the focus vertex v_i, and it has to choose strategy C to fit the definition of Nash equilibrium.

Therefore, we have proven $\{V_{SNE}\} \subseteq \{V_{VC}\}$, which means that if the graph reaches an SNE state, it must have also reached a VC state. (**QED**)

3.2 FBR Algorithm

Theorem 2. Given a weighted undirected graph $G = (V, E, W)$ and s^0 is its initial strategies. When the parameter r in asymmetric game satisfies $\frac{1-r}{r} > 4\lambda_{max}^2 k_{max}$, where $\lambda_{max} = max\{\lambda_{ij}\}$ and k_{max} is the maximum vertex degree, s^0 can always transfer into stable SNE state with FBR algorithm, whose memory space length $m > 1$.

Let $M_i \in \mathbb{S}^m, \mathbb{S} = \{C, N\}$ denote the memory space of vertex v_i in G. Let $\{(s_1, \ldots, s_n)|s_i \in M_i\}$ denote the set of strategy combination in G. We will divide the proof of *Theorem 2* into proving the following two lemmas.

Lemma 3. If there always exists an SNE strategy combination in the memory spaces of all vertices, the graph state can always transfer into a stable SNE state.

Proof. Consider an arbitrary set of memory spaces $M = \{M_1, \ldots, M_n\}$ and the existing SNE strategy combination $(s_1^*, \ldots, s_n^*)$. Obviously, it can be chosen with a positive probability p_1. If chosen, since it's an SNE state, it will be stored into memory space set. Then it can be chosen again with a positive probability p_2, and so on. Therefore, there is a positive probability $p = p_1 p_2 \ldots p_m$ that it can be chosen m times, which is the length of the memory space. If so, the memory space set will be filled with this only one strategy combination and the state of gragh will be stable at an SNE state.

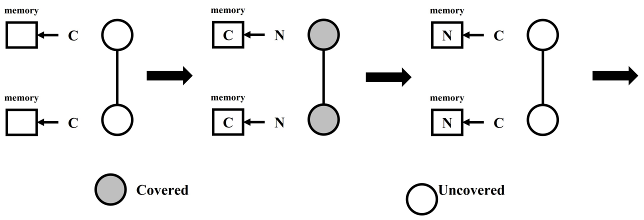

Fig. 4. Illustration of potential oscillation between $N-N$ and $C-C$

Therefore, under the condition mentioned in *Lemma 3*, a graph with any memory space set has a positive probability to converge to a stable SNE state. As the FBR algorithm runs continuously, the probability that convergence never happens will settle to 0, meaning the graph state can always transfer into a stable SNE state. (**QED**)

Lemma 4. If there exists a set of strategy combination containing no SNE strategy combination, there is a positive probability that an SNE strategy combination occurs in finite running time with FBR.

Proof. There must exist at least one of the following scenarios in the graph with non-SNE state:

(1) Two adjacent vertices v_i and v_j both have strategy N.
(2) One center vertex v_i and all its neighbor $\forall v_j$ vertices have C strategy.

For the first scenario, the possible SNE states between them are $C-N$, $N-C$, $C-C$, and they will both store strategy C into their memory.

As is shown in Fig.4, if the length of memory space $m=1$, these two vertices may sink into the oscillation between $N-N$ and $C-C$, and unable to transfer into $C-N$ or $N-C$.

However, with $m>1$, said oscillation would not occur. If there are all N in memory spaces, the newly added C will make it possible that $C-N$, $N-C$, $C-C$ can be chosen. If there are all C in memory spaces, then they will choose $C-C$ the next first time. If $C-C$ is not an SNE state, N will be stored in the next second time and it will be possible that $C-N$ or $N-C$ can be chosen.

For the second scenario, possible SNE states among them are: 1)v_i with C and a part of(not all) v_j with C; 2)v_i with N and $\forall v_j$ with C. v_i will store N into its memory.

As is shown in Fig.5, if v_i chooses N the next second time, $\forall v_j$ may store C in their memory and it is possible that the possible SNE 2) can be chosen.

Using proof by contradiction, to avoid choosing the possible SNE 1), we can only assume v_i with memory of all C, and v_j with memory of all C or all N. If v_j with memory of all C, v_i will choose N and $\forall v_j$ will choose C the next second time. If this is not SNE, some v_j will store N into its memory and the next third time, $N-N$ has a positive probability to be chosen, and the situation returns to scenario (1).

If v_j with memory of all N, $N-N$ will occur in the next time, and the situation returns to scenario (1).

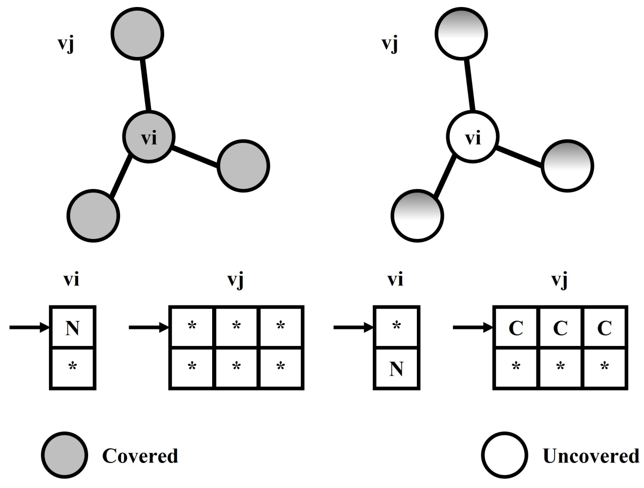

Fig. 5. A possible following state of "all-C" state

As analyzed above, graph with no SNE strategy combination always has an positive probability to obtain an SNE strategy combination in finite time. (**QED**)

With *Lemma 3* and *Lemma 4*, we can conclude that an SNE state is attainable, starting from any arbitrary state with any arbitrary memory space by conducting FBR algorithm. *Lemma 4* guaranteed that by conducting FBR algorithm, an SNE state could always have a positive probability to occur in the memory space, and *Lemma 3* guaranteed the convergence of any initial states to an SNE state. The condition $\frac{1-r}{r} > 4\lambda_{max}^2 k_{max}$ made sure that each SNE state must be a VC state, and each MWVC state must be an SNE state (recall *Theorem 1*). *Theorem 2* is equivalent to the combination of *Lemma 3*, *Lemma 4* and the condition $\frac{1-r}{r} > 4\lambda_{max}^2 k_{max}$. Therefore, we proved *Theorem 2*.

4. SIMULATIONS

In this section we set up two groups of simulations. In the first one, we verify the superiority of condition $\frac{1-r}{r} > 4\lambda_{max}^2 k_{max}$ compared to the original condition $\frac{1-r}{r} > 4\lambda_{max}^2$. In the second one, we compare the performance of FBR algorithm to the original MBR algorithm and WMA algorithm based on potential game proposed by Sun et al. (2020).

4.1 Performance over conditions with different upper bounds

Three undirected weighted graph networks, scale-free network proposed by Barrat et al. (2004) (BBV for short), lattice network and an actual aviation network, are set up as the operation environment of the algorithm. Hao et al. (2009) stated that in a BBV scale-free network, the degrees and weights of vertices approximately follow the power-law distribution, with a significant degree imbalance. Goh et al. (2005) stated that the BBV model is similar to the popular Barabasi-Albert model, but the BBV model applied the preferential attachment rule to the nodes' weight, instead of the nodes' degree, which is more suitable to weighted vertex cover problem. In a lattice network, the degree of each vertex is the same, and the weight of each

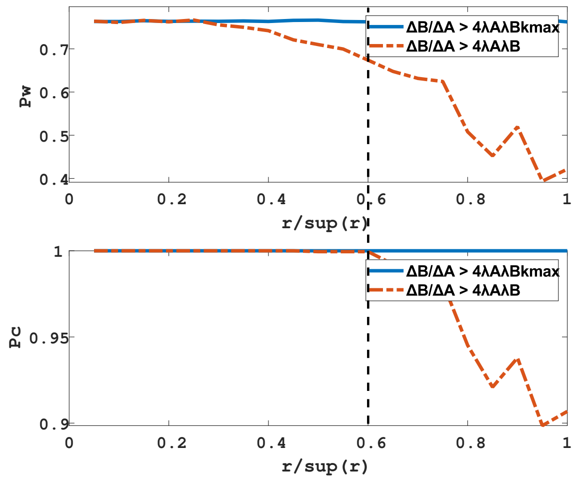

Fig. 6. The performance of FBR algorithm with different upper bounds of r in BBV scale-free network. The scale-free network is generated from a complete graph $\mathbf{K_8}$ and it has 1024 vertices.

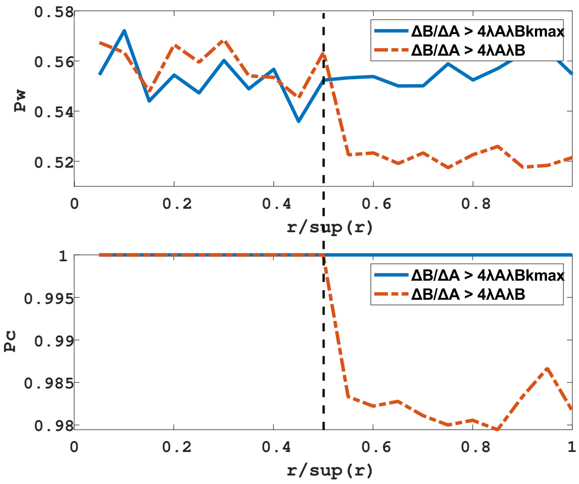

Fig. 7. The performance of FBR algorithm with different upper bounds of r in Lattice network. There are 900 vertices in this lattice network and the degree of each vertex is 4. Vertex weights are uniformly distributed.

vertex in the experiment follows a uniform distribution, which makes the difference between vertices in the network little. Both the BBV scale-free network and the lattice network are artificially generated under specific rules. An aviation network generated from real world data completes the experiments with empirical analysis.

In each type of networks, two groups of experiments are carried out. Each group of experiments let the vertices play an asymmetric game with implemented the FBR algorithm. Performances of the FBR algorithm under different value ranges of the parameter r will be examined. One group of experiments sets the value range $r \in (0, sup_1(r))$, where $sup_1(r) = r \mid_{\frac{1-r}{r}=4\lambda_{max}^2 k_{max}}$. The other sets the range as $r \in (0, sup_2(r))$, where $sup_2(r) = r \mid_{\frac{1-r}{r}=4\lambda_{max}^2}$. The experiments will examine the performance and ef-

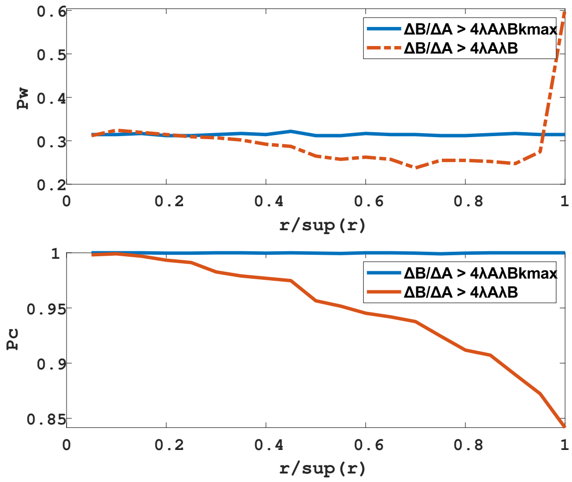

Fig. 8. The performance of FBR algorithm with different upper bounds of r in an actual aviation network. There are 404 vertices in this aviation network and vertex weights are uniformly distributed.

fectiveness of the FBR algorithm when $\mu = \frac{r}{sup(r)}$ takes different values.

We presents two indicators to characterize the performance of the FBR algorithm under different game models, that are, Weight Coverage P_w and Edge Coverage P_c. Define Weight Coverage P_w as the proportion of the sum of weights of the vertices using C strategy to the sum of weights of all vertexes in the network, after reaching a stable state:

$$P_w = \frac{\sum_i^{s_i=C} w_i}{\sum_i w_i}$$

By definition, when the network reaches a VC state, the smaller P_w is, the closer the state is to MWVC. Define Edge Coverage P_c as the proportion of number of covered edges to the number of all edges, after reaching a stable state:

$$P_c = \frac{\sum_{i,j}^{s_i|s_j=C} l_{i,j}}{\sum_{i,j} l_{i,j}}$$

where $l_{i,j}$ represents the weight of edges. By definition, when $P_c = 1$, the network reaches a VC state. The smaller P_c is, the farther away the network state is from VC.

In each of the three networks, play the game in two different value ranges of r, respectively. In each group, take $0.05 \leq \mu \leq 1$ and play game 100 times independently with each value of μ. The result of the experiment is the average value of P_w and P_c corresponding to the value of μ.

In Fig.6, the graph line of P_c with μ shows that, when $sup_1(r)$, $P_c = 1$ in spite of the value of μ. When $sup_2(r)$, P_c will not remain 1 when $\mu > 0.6$. This is because the value of r is so large that the vertices tend to choose N strategy when some of their neighbors choose C strategy. The wide range of $r \in (0, sup_2(r))$ has a particularly obvious impact on the vertex with high weight and large degree. Therefore, $sup_1(r) = r \mid_{\frac{1-r}{r}=4\lambda_{max}^2 k_{max}}$ has stronger stability.

The graph line of P_w with μ shows the limitations of $sup_1(r)$. When $\mu < 0.6$, the P_w to $sup_2(r)$ is smaller than P_w to $sup_1(r)$, and P_c keeps at 1. This shows that to a certain extent, $sup_1(r) = r \mid_{\frac{1-r}{r}=4\lambda_{max}^2 k_{max}}$ is an overly strict upper bound, which sacrifices the potential of FBR to reach the MWVC while maintaining the stability.

Fig.7 shows that the stability $sup_1(r)$ gives is maintained in the lattice network. Different from the BBV scale-free network, the limitation of $sup_1(r)$ hardly exists in a lattice network. The performances of P_w with two different upper bounds are almost the same when $\mu < 0.5$. It means that the limitation can only be shown in specific networks with large degree imbalance and obvious clustering among vertices with similar degrees.

In Fig.8,the limitation mentioned above disappears completely. In the BBV scale-free network and lattice network, although $sup_2(r)$ lacks stability as an upper bound, it has certain potential in making the game result closer to the MWVC. However, in this actual aviation network, P_w shows obvious instability with $sup_2(r)$.

4.2 Comparison among FBR, MBR and WMA

Three types of undirected weighted graphs, BBV scale-free network, lattice network and WS small-world network proposed by Watts and Strogatz (1998) are involved to test the performance of FBR algorithm under the new condition, compared to the original MBR algorithm(without Feedback mechanism) and a newly proposed weighted memory based algorithm (WMA for short) proposed by Sun et al. (2020). We chose the WMA algorithm, which is also a memory based algorithm, proposed recently and performing well to verify the superiority of the FBR algorithm under the new condition.

Table 1. Experimental parameter setting table

Graph	n	r_{FBR}	r_{MBR}	λ_{WMA}
BBV(8)	1024	0.001	0.001	$5\max\{w_i\}$
BBV(8)	2048	0.001	0.001	$5\max\{w_i\}$
Lattice	900	0.001	0.001	$5\max\{w_i\}$
Lattice	1600	0.001	0.001	$5\max\{w_i\}$
WS(2,0.1)	200	0.05	0.05	$5\max\{w_i\}$
WS(4,0.1)	200	0.05	0.05	$5\max\{w_i\}$

BBV(k) is a network generated from k-complete graph, and K and p in WS(K,p) represent initial degree of vertex and probability of connection.

We set the length of memory space to 50 across all the three algorithms, but with different r for FBR and MBR algorithms, λ for WMA algorithm will be set to different graghs.

We generated 2 BBV scale-free networks, 2 lattice networks and 2 small world networks with different parameters. For different networks, we implement algorithms with different parameters. See Table1 for details.

We did the three MWVC generation algorithms 200 times for each network and examined the average times of iterations and the Weight Coverage P_w when the algorithms finish. Details in Table2 & 3:

Table 2. P_w comparison among FBR, MBR and WMA

Graph	n	FBR	MBR	WMA
BBV(8)	1024	0.6424	0.7595	0.8847
BBV(8)	2048	0.6442	0.7590	0.8893
Lattice	900	0.6192	0.5570	NaN*
Lattice	1600	0.6263	0.5605	NaN*
WS(2,0.1)	200	0.4637	0.5712	0.6934
WS(4,0.1)	200	0.6607	0.6920	0.7644

Table 3. Iterations comparison among FBR, MBR and WMA

Graph	n	FBR	MBR	WMA
BBV(8)	1024	539	2108	366
BBV(8)	2048	371	2347	372
Lattice	900	321	678	NaN*
Lattice	1600	476	1073	NaN*
WS(2,0.1)	200	142	403	74
WS(4,0.1)	200	288	916	179

** in Table 2 and Table 3 NaN means that in long enough iterations, WMA algorithm didn't converge to a stable state.*

In scale-free networks and small world networks, the FBR algorithm performs well in both optimality and convergence speed. Although it converges more rapidly, the WMA algorithm fails to reach the optimal VC state.

In lattice networks which are less usual in real world, the FBR algorithm can converge more rapidly at the expense of optimality, compared to the MBR algorithm. Since the FBR algorithm is more appliable to imbalanced network structures, where the feedback mechanism would be unlikely to sink to local optimum, the MBR algorithm performed better in finding the optimum in lattice networks. The WMA fails to converge, which means it cannot reach an SNE state in a reasonable time frame.

To sum up, FBR algorithm performs well in both optimality and convergence speed in most types of networks, and it can reach a reasonable fine result even in some unusual situations. It has some superiority compared to its origin MBR algorithm and recently proposed MWA algorithm based on potential game.

5. CONCLUSIONS

In this paper, we have investigated the asymmetric game solution to the MWVC problem. A more rigorous parameter condition $\frac{1-r}{r} > 4\lambda_{max}^2 k_{max}$ for the asymmetric game model to achieve more effective and optimal solutions is discovered. The relations among different states and the effectiveness of the FBR algorithm have been discussed. We demonstrate the superiority of this condition compared to the original condition with supporting simulation experiments. In addition, we have shown the comparison among FBR algorithm with our condition, the MBR algorithm, and the WMA algorithm.

Future studies may further investigate the essential mechanism behind multi-player games and derive the analytical solution to the actual effective range.

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