ИДЗ 6

Держапольский Юрий Витальевич

1. Исследуйте на непрерывность на указанном множестве.
$$f(y) = \int_1^\infty \frac{\cos x}{4 + x^y} dx \, E = (0; \infty)$$
Рассмотрим функцию $g(x, y) = \frac{\cos x}{4+x^y}$ на $[1; \infty) \times (0; \infty)$. Т.к. $\cos x$ непрерына $\forall x$, а $4 + x^y$ непрерывна и $\neq 0 \, \forall x, y$, то $g(x, y)$ непрерывна как функция двух переменных. Значит $f(y)$ непрерывна на множестве E.

2. Вычислить
$$\int_0^\infty x^{2n+1} e^{-a^2x^2} dx = \left| t = a^2x^2, x = \frac{t^{1/2}}{a}, dx = \frac{1}{2} \frac{dt}{t^{1/2}a} \right| =$$
$$= \int_0^\infty \frac{t^{n+\frac{1}{2}}}{2a^{2n+1}t^{1/2}a} e^{-t} dt = \frac{1}{2a^{2n+2}} \int_0^\infty t^n e^{-t} dt = \frac{1}{2a^{2n+2}} \Gamma(n+1) = \frac{n!}{2a^{2n+2}}$$

3. Вычислить. $(a > 0)$
$$f(a) = \int_0^\infty \frac{\sin ax}{x(1+x^2)} dx$$
$$\frac{df}{da}(a) = \int_0^\infty \frac{x \cos ax}{x(1+x^2)} dx = \int_0^\infty \frac{\cos ax}{1+x^2} dx = \frac{\pi}{2} e^{-a}$$
$$f(a) = \frac{\pi}{2} \int e^{-a} da = -\frac{\pi}{2} e^{-a} + C$$
$$f(0) = 0 = -\frac{\pi}{2} + C \implies C = \frac{\pi}{2} \implies f(a) = \frac{\pi}{2}(1 - e^{-a})$$

4. Вычислить
$$\int_{-\infty}^{+\infty} \cos(ax^2 + 2bx + c) dx = \int_{-\infty}^{+\infty} \cos\left(a\left(x + \frac{b}{a}\right)^2 + \frac{ac - b^2}{a}\right) dx =$$
$$= \int_{-\infty}^{+\infty} \left[\cos\left(a\left(x + \frac{b}{a}\right)^2\right) \cos\frac{ac - b^2}{a} - \sin\left(a\left(x + \frac{b}{a}\right)^2\right) \sin\frac{ac - b^2}{a}\right] dx =$$
$$= \sqrt{\frac{\pi}{a}} \left(\cos\frac{ac - b^2}{a} - \sin\frac{ac - b^2}{a}\right) = \sqrt{\frac{2\pi}{a}} \sin\left(\frac{\pi}{4} - \frac{ac - b^2}{a}\right)$$

5. Вычислить
$$f(a, b, c) = \int_0^\infty \frac{\sin ax}{x} \frac{\sin bx}{x} e^{-cx} dx$$
$$\frac{\partial f}{\partial a} = \int_0^\infty \frac{x \cos ax}{x} \frac{\sin bx}{x} e^{-cx} dx = \int_0^\infty \frac{\sin bx \cos ax}{x} e^{-cx} dx =$$
$$= \frac{1}{2} \left(\int_0^\infty \frac{\sin x(b+a)}{x} e^{-cx} dx + \int_0^\infty \frac{\sin x(b-a)}{x} e^{-cx} dx\right) = I_d$$

Рассмотрим интеграл:

$$g(a,b)=\int_0^\infty \frac{\sin ax}{x}e^{-bx}dx$$

$$\frac{\partial g}{\partial a}=\int_0^\infty \cos ax\, e^{-bx}dx=\frac{b}{a^2+b^2}\implies g(a,b)=\operatorname{arctg}\frac{a}{b}+\phi(b)$$

$$g(0,b)=0=\phi(b)\implies g(a,b)=\operatorname{arctg}\frac{a}{b}$$

$$I_d=\frac{1}{2}\left(\operatorname{arctg}\frac{a+b}{c}+\operatorname{arctg}\frac{b-a}{c}\right)=\frac{1}{2}\left(\operatorname{arctg}\frac{a+b}{c}-\operatorname{arctg}\frac{a-b}{c}\right)$$

$$f(a,b,c)=\frac{1}{2}\int\left(\operatorname{arctg}\frac{a+b}{c}-\operatorname{arctg}\frac{a-b}{c}\right)da=\frac{a+b}{2c}\operatorname{arctg}\frac{a+b}{c}-$$

$$-\frac{1}{4}\ln\left(\left(\frac{a+b}{c}\right)^2+1\right)-\frac{a-b}{2c}\operatorname{arctg}\frac{a-b}{c}+\frac{1}{4}\ln\left(\left(\frac{a-b}{c}\right)^2+1\right)+\phi(b,c)$$

$$f(0,b,c)=0=\phi(b,c)\implies f(a,b,c)=$$

$$=\frac{a+b}{2c}\operatorname{arctg}\frac{a+b}{c}-\frac{1}{4}\ln\left(\left(\frac{a+b}{c}\right)^2+1\right)-\frac{a-b}{2c}\operatorname{arctg}\frac{a-b}{c}+\frac{1}{4}\ln\left(\left(\frac{a-b}{c}\right)^2+1\right)$$

6. Вычислить

$$\int_0^\infty \frac{e^{-ax}\sin^3 bx}{x^2}dx=f(a,b)$$

$$\frac{\partial f}{\partial a}=\int_0^\infty \frac{-xe^{-ax}\sin^3 bx}{x^2}dx=\int_0^\infty \frac{-e^{-ax}\sin^3 bx}{x}dx$$

$$\frac{\partial^2 f}{\partial a^2}=\int_0^\infty \frac{xe^{-ax}\sin^3 bx}{x}dx=\int_0^\infty e^{-ax}\sin^3 bx dx=\frac{1}{4}\int_0^\infty \left(e^{-ax}\sin bx+e^{-ax}\sin 3bx\right)dx=$$

$$=\frac{1}{4}\left(\frac{b}{a^2+b^2}-\frac{3b}{a^2+9b^2}\right)$$

$$\frac{\partial f}{\partial a}=\frac{1}{4}\left(\operatorname{arctg}\frac{a}{b}+\operatorname{arctg}\frac{a}{3b}\right)+\phi(b)$$

$$\frac{\partial f}{\partial a}(0,b)=\int_0^\infty \frac{-\sin^3 bx}{x}dx=-\int_0^\infty \frac{3\sin bx-\sin 3bx}{4x}dx=-\frac{3\pi}{4*2}\operatorname{sgn}(b)+\frac{\pi}{2}\operatorname{sgn}(3b)=$$

$$=-\frac{\pi}{4}\operatorname{sgn}(b)==0+\phi(b)$$

$$\frac{\partial f}{\partial a}=\frac{1}{4}\left(arctg\frac{a}{b}+\operatorname{arctg}\frac{a}{3b}-\pi\operatorname{sgn}(b)\right)$$

$$f(a,b)=\frac{1}{4}\left(a\operatorname{arctg}\frac{a}{b}-\frac{b}{2}\ln(a^2+b^2)+a\operatorname{arctg}\frac{a}{3b}-\frac{3b}{2}\ln(a^2+9b^2)-\pi a\operatorname{sgn}(b)\right)+\psi(b)$$

$$\lim_{b\to+0}f(a,b)=0=\frac{1}{4}\left(a\pi-a\pi\right)+\psi(b)\implies\psi(b)=0$$

$$\lim_{b\to-0}f(a,b)=0=\frac{1}{4}\left(-a\pi+a\pi\right)+\psi(b)\implies\psi(b)=0$$

$$f(a,b)=\frac{1}{4}\left(a\left(\operatorname{arctg}\frac{a}{b}+\operatorname{arctg}\frac{a}{3b}\right)-\frac{b}{2}\ln\left((a^2+b^2)(a^2+9b^2)\right)-a\pi\operatorname{sgn}(b)\right)$$

7. Вычислить

$$\int_0^1 \frac{\ln(1+a^2x^2)}{\sqrt{1-x^2}}dx = f(a)$$

$$f'(a) = \int_0^1 \frac{2ax^2}{(1+a^2x^2)\sqrt{1-x^2}} = \Big| x = \sin t, dx = \cos t dt \Big| = \int_0^1 \frac{2a\sin^2 t\cos t dt}{(1+a^2\sin^2 t)\cos t} =$$

$$= \int_0^1 \frac{2a\sin^2 t dt}{(1+a^2\sin^2 t)} = \Big| \operatorname{tg} t = y, t = \operatorname{arctg} y, dt = \frac{dy}{1+y^2} \Big| = \int_0^1 \frac{2ay^2 dy}{(1+\frac{a^2y^2}{1+y^2})(1+y^2)} =$$

$$= \int_0^1 \frac{2ay^2dy}{(1+y^2(1+a^2))(1+y^2)} = \frac{2}{a}\int_0^1 \left(\frac{1}{1+y^2} - \frac{1}{1+y^2(1+a^2)}\right)dy$$

$$= \frac{2}{a}\left(\operatorname{arctg} y\Big|_0^\infty - \frac{1}{\sqrt{1+a^2}}\operatorname{arctg}\frac{y}{\sqrt{1+a^2}}\Big|_0^\infty\right) = \frac{\pi}{a}\left(1 - \frac{1}{\sqrt{1+a^2}}\right)$$

$$f(a) = \pi\left(\ln a - \int \frac{da}{a^2\sqrt{\frac{1}{a^2}+1}}\right) = \pi\ln a + \pi\int\frac{d\frac{1}{a}}{\sqrt{\frac{1}{a^2}+1}}$$

$$= \pi\ln a + \pi\ln\left(\frac{1}{a} + \sqrt{\frac{1}{a^2}+1}\right) + C = \pi\ln\left(1+\sqrt{1+a^2}\right) + C$$

$$f(0) = 0 = \pi\ln 2 + C \implies C = -\pi\ln 2$$

$$f(a) = \pi\ln\left(1+\sqrt{1+a^2}\right) - \pi\ln 2$$

8. Вычислить

$$\int_0^\infty \frac{\sin ax\sin bx}{x^2}dx = f(a,b)$$

$$\frac{\partial f}{\partial a} = \int_0^\infty \frac{\cos ax\sin bx}{x}dx = \frac{1}{2}\int_0^\infty\left(\frac{\sin(b-a)x}{x} + \frac{\sin(b+a)x}{x}\right)dx =$$

$$= \frac{\pi}{4}(\operatorname{sgn}(b-a) + sgn(b+a)) = \frac{\pi}{4}(\operatorname{sgn}(a+b) - sgn(a-b))$$

$$f(a,b) = \frac{\pi}{4}(|a+b| - |a-b|) + \phi(b)$$

$$f(0,b) = 0 == \frac{\pi}{4}(|0+b| - |0-b|) + \phi(b) = \phi(b)$$

$$f(a,b) = \frac{\pi}{4}(|a+b| - |a-b|)$$

9. Вычислить

$$\int_0^{\pi/2} \sin^6 x\cos^4 x dx = \Big| \operatorname{tg}^2 x = t, x = \operatorname{arctg}\sqrt{t}, dx = \frac{dt}{2t^{\frac{1}{2}}(1+t)} \Big| =$$

$$= \int_0^\infty \left(\frac{t}{1+t}\right)^6\left(\frac{1}{1+t}\right)^4\frac{dt}{2t^{\frac{1}{2}}(1+t)} = \int_0^\infty \frac{t^{6-\frac{1}{2}}dt}{2(1+t)^{11}} = \frac{1}{2}B\left(6+\frac{1}{2}, 4+\frac{1}{2}\right) =$$

$$= \frac{1}{2}B\left(\frac{13}{2}, \frac{9}{2}\right) = \frac{1}{2}\frac{\Gamma(\frac{13}{2})\Gamma(\frac{9}{2})}{\Gamma(11)} = \frac{\frac{11!!}{2^6}\ \frac{7!!}{2^4}}{2*10!} = \frac{11!!\,7!!}{2^{11}10!}$$

10. Вычислить

$$\int_0^\infty \frac{dx}{\sqrt{1+x^3}} = \Big| x^3 = t, x = t^{\frac{1}{3}}, dx = \frac{1}{3}t^{\frac{-2}{3}}dt \Big| = \frac{1}{3}\int_0^\infty \frac{t^{-\frac{2}{3}}dt}{(1+t)^{\frac{1}{2}}} = \frac{1}{3}B\left(\frac{1}{3}, \frac{1}{6}\right)$$