Group HW #3

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Problem 1

Question:

How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1,000 common elements, each triple of sets has 100 common elements, every four of the sets has 10 common elements, and there is 1 common element in all five set?

Solution:

Let there be sets: S_1, S_2, S_3, S_4, S_5, where $|S_i| = 10,000$ for $i = 1, 2, ..., 5$. We want to find $|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5|$. To do this, we can start by defining set A, where $A = \{S_1, S_2, S_3, S_4, S_5\}$. We know that $|A| = 5 \cdot 10,000 = 50,000$, since each set S_i containts 10,000 elements.

Now, we must account for overcounting. By combining all the sets together, we have overcounted any elements in at least 2 sets. We can define subset A_1, such that it counts all the common elements between set S_1 and set S_2. From the problem definition, we know that $|A_1| = 1,000$, since each pair of sets has 1,000 common elements. We must do this for all possible combinations of the five sets: $S_1, ..., S_5$. To do this, we need to figure out the number of ways to pick two sets from five, which can be expressed as $\binom{5}{2}$. For whatever pair of sets we pick, the cardinality will always be 1,000. We can write this as $1,000 \cdot \binom{5}{2}$. We can now subtract this value from the cardinality of A: $50,000 - 1,000 \cdot \binom{5}{2}$ to account for overcounting

That subtraction, oversubtracts elements that are in at least 3 sets. We can define subset A_2, such that it counts all the commont elements between sets $S_1, S_2 and S_3$. From the problem definition, we know that $|A_2| = 100$, since each triple of sets has 100 common elements. We must do this for all possible cominations of the five sets: $S_1, ..., S_5$. To do this, we need to figure out the number of ways to pick three sets from five, which can be expressed as $\binom{5}{3}$. For whatever triple of sets we pick, the cardinality will always be 100. We can write this as $100 \cdot \binom{5}{3}$. We now add this value to the previous expression: $50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3}$

We use a similar process for elements that are in at least 4 sets. Define subset A_3 for elements in sets $S_1, ..., S_4$. We know $|A_3| = 10$, since every four sets have 10 common elements. We need to pick four sets out of five: $\binom{5}{4}$ and multiply by the cardinality: $10 \cdot \binom{5}{4}$. Since we overcounted elements in 4 sets, we need to subtract this value from the above expression: $50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3} - 10\binom{5}{4}$

We again use a similar process for elements that are in at least 5 sets. Define subset A_4 for elements in sets $S_1, ..., S_5$. We know $|A_4| = 1$, since all five sets have 1 common element. Since we oversubtracted elements in 5 sets, we need to add this value to the above expression:

$50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3} - 10\binom{5}{4} + 1$

We can express the above expression in summation notation to be more concise:

$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5| = \sum_{i=1}^{5} (-1)^{i-1} \binom{5}{i} \cdot 10^{5-i}$

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to correct for overcounting and undercounting when counting the number of elements in the union of sets.

Problem 2

Question:

Provide a Jeopardy-style combinatorial proof of the following equation by asking one question and then answering that one question in two different ways:

$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \frac{36!}{9!9!9!9!}$

Jeopardy Answer: How many ways can you make 4 distinct teams of 9 from 36 people?

LHS:

Since the order within each team doesn't matter, we can make teams by "choosing" k people from n people: $\binom{n}{k}$. To make the first team, we choose 9 people from 36: $\binom{36}{9}$. Now to form the second team, we only have $36 - 9$ people to choose from: $\binom{36-9}{9} = \binom{27}{9}$. For the third team, we only have $27 - 9$ people to chose from: $\binom{27-9}{9} = \binom{18}{9}$. And finally for the fourth team, we only have the remaining 9 people to chose from: $\binom{9}{9}$. Because we have framed these choices as independent tasks, we can use the product rule to determine the number of ways to form the teams: $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}$

RHS:

We can also think about this problem as forming an ordered line of all n people, picking the first k people, and then unordering that selection, and unordering the rest of the line. We can show this mathametically as: $\frac{n!}{k!(n-k)!}$. So to form the first team of 9, we line up all 36 people, pick the first 9, and unorder the selection as well as the 27 people not selected: $\frac{36!}{9!27!}$. Then to form the second team of 9, we line up everyone not selected last time, pick the first 9, and unorder the selection and the 18 people not selected: $\frac{27!}{9!18!}$. The third team is picked the same way. Line up the 18 people not selected last time, pick the first 9 and unorder the selection and the 9 poeple not selected: $\frac{18!}{9!9!}$. Finally the for the last team, we order all 9 people and unorder all 9 of the on the team: $\frac{9!}{9!0!}$.

Because we framed all of the choices as independent tasks, we can use the product rule to detemrine the number of ways to form the teams: $\frac{36!}{9!27!}\frac{27!}{9!18!}\frac{18!}{9!9!}\frac{9!}{9!0!}$. If we simplify that expression, we get: $\frac{36!}{9!9!9!9!}$

Proof:

Since both the LHS and the RHS count exactly the same number, they must be equal. Therefore: $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \frac{36!}{9!9!9!9!}$

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand the the choose function has a fundamental combinatorial background.

Problem 3

For this problem, our goal is to determine the amount of possible outcomes there are from rolling a dice 13 times results in a sum of the rolls being 32 or less.

Before we start to solve this problem, we can think of the outcome of the roll of the dice as "points" (ex: Rolling a 6 is 6 points).

We can say that we are finding the cardinality of outcomes that have a sum of 32 points or fewer.

Since we have 13 dice and at most 32 points, we can employ the sticks and stones method to help us solve this problem. First, we must realize that any roll will have a value of at least 1. This means that our absolute minimum outcomes is going to be 13. This means that we can pre-load each bucket (dice outcome) with 1.

This means that we have at most (32-13 =) 19 available points to distribute among the dice, or the buckets. Since we are dealing with a problem in which we do not need to use all 32 points, we need to add an extra bucket that accounts for unused points. This means that we have 13 + 1 - 1 sticks in our equations (13 dice, 1 extra bucket for unused points, and remove one so that we have the proper amount of buckets).

Now we see that we are left with 19 points and 13 sticks. This means that the total number of ways we can arrange these sticks and stones is

$$\binom{32}{13}$$

But, this does not take into account when buckets, or dice, get more than 6 points assigned to them. We can use use inclusion and exclusion to remove impossible outcomes.

To eliminate outcomes where a dice has 7 or points within its bucket, we can pre load one of the dice with 6 more points (7 total now as we already pre-loaded each dice bucket with 1). This will help us find all of the outcomes that need to be removed for when a specific dice has an impossible scenario.

For example, lets say that in our A1, we are determining how many scenarios there are where the bucket of dice 1 has 7 or more points. We will still have the same amount of sticks, but since we are pre-loading one of the buckets with 6 more points, we will have (19-6 =) 13 points, or stones. This means that for our A1 scenario, there are

$$\binom{26}{13}$$

outcomes.

We can do this for every bucket (for now we will do this for the "not-used" bucket as well and account for these outcomes later), meaning that we have to remove $14 \cdot \binom{26}{13}$ outcomes from our total.

Now we need to account for all the outcomes where 2 dice have 7 or more points in their buckets. Our A1 for this situation is when dice 1 and 2 both have 7 or more points. Again, we will account remove 6 additional points, or stones, from our remaining, leaving us with (13-6 =7) points. Our sticks will stay the same at 13, so the total arrangements we can make in this scenario is

$$\binom{20}{13}.$$

Since we need to account for all combinations of two bucket pairings, which is $\binom{14}{2}$, combinations (we will still include the "not-used" bucket for now). This means that there are

$$\binom{14}{2} \cdot \binom{20}{13}$$

outcomes where 2 buckets have 7 or more points.

$$\binom{14}{2} \cdot \binom{20}{13}$$

These outcomes were accounted for twice each when we looked at the situation where at least 1 bucket had at least 7 points, so we need to add back the situations where at least 2 buckets have at least 7 points.

Now, we will account for situations where 3 buckets have at least 7 points. So again, we will pre-load a third bucket, and remove 6 points, or stones, for a total of 1 more remaining stone (still 13 sticks). For this situation, our A1 will be where buckets for rolls 1,2, and 3, all have at least 7 points. There is

$$\binom{14}{13}$$

ways to do this. Now, we need to do this for every combination of 3 buckets, which is expressed as $\binom{14}{3}$. Since we are doing this for every combination of groups of 3 dice, we will need to remove

$$\binom{14}{3} \cdot \binom{14}{13}$$

from our total. We are removing these because we we account for them in both the at least 1 dice had 7 points as well the scenarios where we had at least 2 dice with 7 points.

So far, we have removed every scenario in which the "not-used" bucket has 7 or more points. But, this bucket is the only bucket that CAN have more than 7 points, so we will re-evaluate these scenarios.

For the scenario where there was at least 1 bucket with at least 7 points, which is

$$\binom{26}{13}$$

outcomes. This is because this is the total amount of combination possible when we preload the "not used bucket".

For the scenario where at least 2 buckets have at least 7 points, there are

$$\binom{20}{13} \cdot 13$$

cases where the "not-used" bucket. This is because in every sceneario where the "not-used" bucket is pre-loaded, there are $\binom{20}{13}$ and there are 13 scenarios where the "not-used" bucket is used (one combination with each other bucket). Since these cases are not distinct, we multiply them.

For the scenario where at least 3 buckets have at least 7 points, the "not-used" bucket is preloaded in

$$\binom{14}{13} \cdot \binom{2}{13}$$

outcomes. This is because there are $\binom{2}{13}$ ways to group the "no-outcome" bucket with 2 other buckets and each grouping has $\binom{14}{13}$ Arrays. Since these cases are not distinct, we multiply them.

This leaves us with the final equation:

$$\binom{32}{13} - (14 \cdot \binom{26}{13} - \binom{26}{13}) + (\binom{20}{13} \cdot \binom{14}{2} - 13 \cdot \binom{20}{13}) - (\binom{14}{13} \cdot \binom{14}{3} - \binom{14}{13} \cdot \binom{13}{2})$$

Which simplifies too:

$$\binom{32}{13} - (13 \cdot \binom{26}{13}) + (\binom{20}{13}(\binom{14}{2} - 13)) - (\binom{14}{13}(\binom{14}{3} - \binom{13}{2}))$$

- **Who Contributed:** Pranav Bonthu was the main contributor, doing the formal write-up of the solution. Dhvan Shah and Michael Ku compared their solutions to Pranav's and provided feedback and edits.
- **Resources**: AI was used to help get an outside perspective on the problem
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to use sticks and stones in conjunctions with inclusion/exclusion

Problem 4

In this problem, we need to determine how many possible ways there are to distribute 9 distinct apples and 8 identical mangos among 5 kids, with each kid getting at least one mango.

To start, we can split this problem into 2 parts, one part for how the mangos are distributed and the second being how the apples are distributed.

When looking at how the mangos can be distributed, we can use sticks and stones. The buckets that the sticks create will represent the children, and the stones will be the mangos.

This mean we will have 5 - 1 = 4 sticks (5 children and remove one as n-1 sticks are needed to create n buckets) and 8 (as there are 8 mangos)

Since we know that each child needs to get at least 1 mango, we can preload each bucket (representing a child) with 1 mango. This leaves us with 8 - 5 = 3 mangos (or stones) left to distribute.

We now have 4 sticks and 3 stones remaining. So, the amount of ways we can arrange the sticks (the total amount of outcomes to distribute the mangos), is equal to $\binom{7}{4}$

Now we will look at how many ways we can distribute the apples. Each apple is distinct, meaning none are the same.

Each of the 9 apples can go to any one of the 5 children. This means that an individual apple has 5 possible outcomes. Since each apple has 5 outcomes, there are 9 apples, and the outcomes are not distinct, there are 5^9 total outcomes for how the apples can be distributed.

Since the outcomes for the mango distribution and the distribution of the apples are not distinct, we can multiply both total outcomes to get the total outcomes of the problem.

This means our final answer for the total possible outcomes is:

$$\binom{7}{4} \cdot 5^9$$

- **Who Contributed:** Pranav Bonthu was the main contributor, doing the formal write-up of the solution. Dhvan Shah and Michael Ku compared their solutions to Pranav's and provided feedback and edits.
- **Resources**: No resources used
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to define 2 different scenarios and how to combine them in the end

Problem 5

Question:
Give a Jeopardy-style combinatorial proof of

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Ask one question; answer it two ways; be careful to justify every step of your answers!

Solution:

Made up question:
Take a group of m people and a group of n people. Out of all the people in those two groups, how many ways can you make teams of r people can you make?

Solution 1:
When approaching classic team-selection problems, it is often most straightforward to use the binomial coefficient:

$$\binom{n}{k}$$

as this directly counts the number of ways to choose k objects from n items. In this problem, the total number of people available to choose from is n+m, and the number we are selecting is r. Therefore, the total number of possible selections is given by

$$\binom{m+n}{r}$$

Solution 2:
When approaching this problem, we can think of it as selecting players from each team individually and then combining the chosen players. For example, let k represent the number of people selected from team n. We can choose these k people from team n using

$$\binom{n}{k}$$

these k individuals make up part of the total r people we need. We then select the remaining r-k people from the other team (with m members) using

$$\binom{m}{r-k}$$

Because k can vary ranging from 0 to r we must sum over all possible values of k to cover every possible combination. This leads to the final expression:

$$\sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$$

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** For our group, the central ideas of this problem were understanding combinatorial proofs, formulating a question from a solution, and recognizing how two different solutions can yield the same result.

Problem 6

Question:
Suppose you have 32 identical candies. How many ways can you distribute some, but not necessarily all, of these 32 candies to 13 children if each child must get at least one candy? Fully justify and explain your reasoning/work. (Your solution should somehow allow for every case, ranging from distributing a max of all 32 candies to a min of only distributing 13 candies.)

Solution:
In this problem, we are asked to distribute 32 candies among 13 children under certain conditions. Specifically, each child must receive at least one candy. Because of this restriction, we can use the sticks and stones method. For 13 children, we need 12 dividers (sticks) to create 13 “buckets,” one

for each child.

Since each child is guaranteed at least one candy, we can preload one candy into each bucket.

$$*|*|*|*|*|*|*|*|*|*|*|*|*$$

This immediately uses up 13 candies, leaving us with $32 - 13 = 19$ candies to distribute freely among the 13 children. The number of ways to distribute these remaining candies is:

$$\binom{19+13-1}{13-1} = \binom{31}{12}$$

Which represents the 19 candies plus the the 12 sticks that would be used to divide the candy between all the children. This would be the answer if all 32 candies had to be used. However, the question states that not all candies must be used. To account for this, we introduce an additional "discard" bucket to represent unused candies. Adding this bucket increases the total number of "buckets" to 14, which requires 13 dividers. We then have 32 candies and 14 buckets (13 children + 1 discard). The total number of distributions, including the possibility of leftover candies, is:

$$\binom{19+14-1}{14-1}$$

Thus, the final answer is:

$$\binom{32}{13}$$

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** For our group, the central ideas of this problem were understanding the sticks-and-stones method and examining how different conditions, such as every child receiving at least one candy or not all the candy being used, affect the answer.