Group HW #3

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Problem 1

Question:

How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1,000 common elements, each triple of sets has 100 common elements, every four of the sets has 10 common elements, and there is 1 common element in all five set?

Solution:

Let there be sets: S_1, S_2, S_3, S_4, S_5, where $|S_i| = 10,000$ for $i = 1, 2, ..., 5$. We want to find $|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5|$. To do this, we can start by defining set A, where $A = \{S_1, S_2, S_3, S_4, S_5\}$. We know that $|A| = 5 \cdot 10,000 = 50,000$, since each set S_i containts 10,000 elements.

Now, we must account for overcounting. By combining all the sets together, we have overcounted any elements in at least 2 sets. We can define subset A_1, such that it counts all the common elements between set S_1 and set S_2. From the problem definition, we know that $|A_1| = 1,000$, since each pair of sets has 1,000 common elements. We must do this for all possible combinations of the five sets: $S_1, ..., S_5$. To do this, we need to figure out the number of ways to pick two sets from five, which can be expressed as $\binom{5}{2}$. For whatever pair of sets we pick, the cardinality will always be 1,000. We can write this as $1,000 \cdot \binom{5}{2}$. We can now subtract this value from the cardinality of A: $50,000 - 1,000 \cdot \binom{5}{2}$ to account for overcounting

That subtraction, oversubtracts elements that are in at least 3 sets. We can define subset A_2, such that it counts all the commont elements between sets $S_1, S_2 and S_3$. From the problem definition, we know that $|A_2| = 100$, since each triple of sets has 100 common elements. We must do this for all possible cominations of the five sets: $S_1, ..., S_5$. To do this, we need to figure out the number of ways to pick three sets from five, which can be expressed as $\binom{5}{3}$. For whatever triple of sets we pick, the cardinality will always be 100. We can write this as $100 \cdot \binom{5}{3}$. We now add this value to the previous expression: $50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3}$

We use a similar process for elements that are in at least 4 sets. Define subset A_3 for elements in sets $S_1, ..., S_4$. We know $|A_3| = 10$, since every four sets have 10 common elements. We need to pick four sets out of five: $\binom{5}{4}$ and multiply by the cardinality: $10 \cdot \binom{5}{4}$. Since we overcounted elements in 4 sets, we need to subtract this value from the above expression: $50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3} - 10\binom{5}{4}$

We again use a similar process for elements that are in at least 5 sets. Define subset A_4 for elements in sets $S_1, ..., S_5$. We know $|A_4| = 1$, since all five sets have 1 common element. Since we oversubtracted elements in 5 sets, we need to add this value to the above expression:

$50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3} - 10\binom{5}{4} + 1$

We can express the above expression in summation notation to be more concise:

$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5| = \sum_{i=1}^{5}(-1)^{i-1}\binom{5}{i} \cdot 10^{5-i}$

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to correct for overcounting and undercounting when counting the number of elements in the union of sets.

Problem 2

Question:

Provide a Jeopardy-style combinatorial proof of the following equation by asking one question and then answering that one question in two different ways:

$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \frac{36!}{9!9!9!9!}$

Jeopardy Answer: How many ways can you make 4 distinct teams of 9 from 36 people?

LHS:

Since the order within each team doesn't matter, we can make teams by "choosing" k people from n people: $\binom{n}{k}$. To make the first team, we choose 9 people from 36: $\binom{36}{9}$. Now to form the second team, we only have $36 - 9$ people to choose from: $\binom{36-9}{9} = \binom{27}{9}$. For the third team, we only have $27 - 9$ people to chose from: $\binom{27-9}{9} = \binom{18}{9}$. And finally for the fourth team, we only have the remaining 9 people to chose from: $\binom{9}{9}$. Because we have framed these choices as independent tasks, we can use the product rule to determine the number of ways to form the teams: $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}$

RHS:

We can also think about this problem as forming an ordered line of all n people, picking the first k people, and then unordering that selection, and unordering the rest of the line. We can show this mathametically as: $\frac{n!}{k!(n-k)!}$. So to form the first team of 9, we line up all 36 people, pick the first 9, and unorder the selection as well as the 27 people not selected: $\frac{36!}{9!27!}$. Then to form the second team of 9, we line up everyone not selected last time, pick the first 9, and unorder the selection and the 18 people not selected: $\frac{27!}{9!18!}$. The third team is picked the same way. Line up the 18 people not selected last time, pick the first 9 and unorder the selection and the 9 poeple not selected: $\frac{18!}{9!9!}$. Finally the for the last team, we order all 9 people and unorder all 9 of the on the team: $\frac{9!}{9!0!}$.

Because we framed all of the choices as independent tasks, we can use the product rule to detemrine the number of ways to form the teams: $\frac{36!}{9!27!}\frac{27!}{9!18!}\frac{18!}{9!9!}\frac{9!}{9!0!}$. If we simplify that expression, we get: $\frac{36!}{9!9!9!9!}$

Proof:

Since both the LHS and the RHS count exactly the same number, they must be equal. Therefore: $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \frac{36!}{9!9!9!9!}$

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand the the choose function has a fundamental combinatorial background.

Problem 3

For this problem, our goal is to determine the amount of possible outcomes there are from rolling a dice 13 times results in a sum of the rolls being 32 or less.

Before we start to solve this problem, we can think of the outcome of the roll of the dice as "points" (ex: Rolling a 6 is 6 points).

We can say that we are finding the cardinality of outcomes that have a sum of 32 points or fewer.

Since we have 13 dice and at most 32 points, we can employ the sticks and stones method to help us solve this problem. First, we must realize that any roll will have a value of at least 1. This means that our absolute minimum outcomes is going to be 13. This means that we can pre-load each bucket (dice outcome) with 1.

This means that we have at most (32-13 =) 19 available points to distribute among the dice, or the buckets. Since we are dealing with a problem in which we do not need to use all 32 points, we need to add an extra bucket that accounts for unused points. This means that we have 13 + 1 - 1 sticks in our equations (13 dice, 1 extra bucket for unused points, and remove one so that we have the proper amount of buckets).

Now we see that we are left with 19 points and 13 sticks. This means that the total number of ways we can arrange these sticks and stones is

$$\binom{32}{13}$$

But, this does not take into account when buckets, or dice, get more than 6 points assigned to them. We can use use inclusion and exclusion to remove impossible outcomes.

To eliminate outcomes where a dice has 7 or points within its bucket, we can pre load one of the dice with 6 more points (7 total now as we already pre-loaded each dice bucket with 1). This will help us find all of the outcomes that need to be removed for when a specific dice has an impossible scenario.

For example, lets say that in our A1, we are determining how many scenearios there are where the bucket of dice 1 has 7 or more points. We will still have the same amount of sticks, but since we are pre-loading one of the buckets with 6 more points, we will have (19-6 =) 13 points, or stones. This means that for our A1 scenario, there are

$$\binom{26}{13}$$

outcomes.

We can do this for every bucket (for now we will do this for the "not-used" bucket as well and account for these outcomes later), meaning that we have to remove $14 \cdot \binom{26}{13}$ outcomes from our total.

Now we need to account for all the outcomes where 2 dice have 7 or more points in their buckets. Our A1 for this situation is when dice 1 and 2 both have 7 or more points. Again, we will account remove 6 additional points, or stones, from our remaining, leaving us with (13-6 =7) points. Our sticks will stay the same at 13, so the total arrangements we can make in this scenario is

$$\binom{20}{13}.$$

Since we need to account for all combinations of two bucket pairings, which is $\binom{14}{2}$, combinations (we will still include the "not-used" bucket for now). This means that there are

$$\binom{14}{2} \cdot \binom{20}{13}$$

outcomes where 2 buckets have 7 or more points.

$$\binom{14}{2} \cdot \binom{20}{13}$$

These outcomes were accounted for twice each when we looked at the situation where at least 1 bucket had at least 7 points, so we need to add back the situations where at least 2 buckets have at least 7 points.

Now, we will account for situations where 3 buckets have at least 7 points. So again, we will pre-load a third bucket, and remove 6 points, or stones, for a total of 1 more remaining stone (still 13 sticks). For this situation, our A1 will be where buckets for rolls 1,2, and 3, all have at least 7 points. There is

$$\binom{14}{13}$$

ways to do this. Now, we need to do this for every combination of 3 buckets, which is expressed as $\binom{14}{3}$. Since we are doing this for every combination of groups of 3 dice, we will need to remove

$$\binom{14}{3} \cdot \binom{14}{13}$$

from our total. We are removing these because we we account for them in both the at least 1 dice had 7 points as well the scenarios where we had at least 2 dice with 7 points.

So far, we have removed every scenario in which the "not-used" bucket has 7 or more points. But, this bucket is the only bucket that CAN have more than 7 points, so we will re-evaluate these scenarios.

For the scenario where there was at least 1 bucket with at least 7 points, which is

$$\binom{26}{13}$$

outcomes. This is because this is the total amount of combination possible when we preload the ”not used bucket”.

For the scenario where at least 2 buckets have at least 7 points, there are

$$\binom{20}{13} \cdot 13$$

cases where the ”not-used” bucket. This is because in every sceneario where the ”not-used” bucket is pre-loaded, there are $\binom{20}{13}$ and there are 13 scenarios where the ”not-used” bucket is used (one combination with each other bucket). Since these cases are not distinct, we multiply them.

For the scenario where at least 3 buckets have at least 7 points, the ”not-used” bucket is preloaded in

$$\binom{14}{13} \cdot \binom{2}{13}$$

outcomes. This is because there are $\binom{2}{13}$ ways to group the ”no-outcome” bucket with 2 other buckets and each grouping has $\binom{14}{13}$ Arrays. Since these cases are not distinct, we multiply them.

This leaves us with the final equation:

$$\binom{32}{13} - (14 \cdot \binom{26}{13} - \binom{26}{13}) + (\binom{20}{13} \cdot \binom{14}{2} - 13 \cdot \binom{20}{13}) - (\binom{14}{13} \cdot \binom{14}{3} - \binom{14}{13} \cdot \binom{13}{2})$$

Which simplifies too:

$$\binom{32}{13} - (13 \cdot \binom{26}{13}) + (\binom{20}{13}(\binom{14}{2} - 13)) - (\binom{14}{13}(\binom{14}{3} - \binom{13}{2}))$$

- **Who Contributed:** Pranav Bonthu was the main contributor, doing the formal write-up of the solution. Dhvan Shah and Michael Ku compared their solutions to Pranav’s and provided feedback and edits.
- **Resources**: Textbook, and previous homeworks
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to use sticks and stones in conjunctions with inclusion/exclusion

Problem 4

In this problem, we need to determine how many possible ways there are to distribute 9 distinct apples and 8 identical mangos among 5 kids, with each kid getting at least one mango.

To start, we can split this problem into 2 parts, one part for how the mangos are distributed and the second being how the apples are distributed.

When looking at how the mangos can be distributed, we can use sticks and stones. The buckets that the sticks create will represent the children, and the stones will be the mangos.

This mean we will have 5 - 1 = 4 sticks (5 children and remove one as n-1 sticks are needed to create n buckets) and 8 (as there are 8 mangos)

Since we know that each child needs to get at least 1 mango, we can preload each bucket (representing a child) with 1 mango. This leaves us with 8 - 5 = 3 mangos (or stones) left to distribute.

We now have 4 sticks and 3 stones remaining. So, the amount of ways we can arrange the sticks (the total amount of outcomes to distribute the mangos), is equal to $\binom{7}{4}$

Now we will look at how many ways we can distribute the apples. Each apple is distinct, meaning none are the same.

Each of the 9 apples can go to any one of the 5 children. This means that an individual apple has 5 possible outcomes. Since each apple has 5 outcomes, there are 9 apples, and the outcomes are not distinct, there are 5^9 total outcomes for how the apples can be distributed.

Since the outcomes for the mango distribution and the distribution of the apples are not distinct, we can multiply both total outcomes to get the total outcomes of the problem.

This means our final answer for the total possible outcomes is:

$$\binom{7}{4} \cdot 5^9$$

- **Who Contributed:** Pranav Bonthu was the main contributor, doing the formal write-up of the solution. Dhvan Shah and Michael Ku compared their solutions to Pranav's and provided feedback and edits.
- **Resources**: No resources used
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to define 2 different scenarios and how to combine them in the end

Problem 5

Question:
How many ways can you distribute identical \$1 bills to bribe your four professors under the following conditions? Ben must get at least one, Alessandra must get at least two, Sam must get at least four, and Steve must get at least ten. No professor will get twenty or more, since that might seem overboard. You have sixty bills to distribute, but you don't need to distribute all of your bills. Explain all models, reasoning, and work; do not just use formulas without explaining the logic.

Solution:

To solve this question, we have to look at our main conditions:

- We have 60 bills that are identical.
- Ben has at least 1 ($B \geq 1$)
- Alessandra has at least 2 ($A \geq 2$)
- Sam has at least 4 ($S \geq 4$)
- Steve has at least 10 ($T \geq 10$)
- No professor gets 20 or more ($B, A, S, T \leq 19$)

- Not all bills need to be distributed

With this we can begin to figure things out. The first thing we should do is preload the minimum amounts, as that removes many impossible outcomes. We preload 1 for Ben, 2 for Alessandra, 4 for Sam and 10 for Steve. This leaves us with $60 - 17 = 43$ bills left to distribute. Next we use the "sticks and stones" method to count all the ways the 43 remaining bills can be divided between the four people if not all the bills are used. We create four buckets for the professors and one extra bucket for the bills that are not used. With these five total buckets, we have 4 bars to be distributed between the 43 bills, making the total number of possibilities

$$\binom{43+4}{4} = \binom{47}{4}.$$

However this is not the end of the question. Since teachers cannot get more than 19 bills, we have to remove all possibilities where this occurs, while also taking overlap into account. To do this we use inclusion–exclusion. Inclusion–exclusion says we take the total possibilities and remove the cases that don't work, while adding back overlaps. Below are the possible violations:

- one professor has over 19 ($B \geq 20$, $A \geq 20$, $S \geq 20$, or $T \geq 20$)
- two professors each have over 19 (e.g. $B \geq 20$ and $A \geq 20$)

No more than two professors can receive over 19 bills, because there are only 60 bills in total and 17 are already allocated to meet the minimums, leaving 43 remaining. To give three professors at least 20 bills each would require at least $18 + 16 + 10 = 44$ extra bills (for example if $T = 20,\ S = 20,\ A = 20$), but since only 43 remain, this situation is impossible.
From here we can begin to compute the counts we must subtract. For example, for $A \geq 20$ we already gave Alessandra 2, so $x_A = A - 2 \geq 18$. Preloading those extra 18 into bucket A leaves $43 - 18 = 25$ bills to distribute into the 5 buckets, so we get $\binom{25+4}{4} = \binom{29}{4}$.

We can fill this out for the other cases:

- if $B \geq 20$, preload 19, leaving $43 - 19 = 24$: $\binom{28}{4}$
- if $A \geq 20$, preload 18, leaving $43 - 18 = 25$: $\binom{29}{4}$
- if $S \geq 20$, preload 16, leaving $43 - 16 = 27$: $\binom{31}{4}$
- if $T \geq 20$, preload 10, leaving $43 - 10 = 33$: $\binom{37}{4}$

Now we have to add back all the possibilities where two professors each get more than 19, as those were double-removed. Using the same idea:

- $B \geq 20$ and $A \geq 20$: $43 - (19 + 18) = 6$: $\binom{10}{4}$
- $B \geq 20$ and $S \geq 20$: $43 - (19 + 16) = 8$: $\binom{12}{4}$
- $B \geq 20$ and $T \geq 20$: $43 - (19 + 10) = 14$: $\binom{18}{4}$
- $A \geq 20$ and $S \geq 20$: $43 - (18 + 16) = 9$: $\binom{13}{4}$
- $A \geq 20$ and $T \geq 20$: $43 - (18 + 10) = 15$: $\binom{19}{4}$
- $S \geq 20$ and $T \geq 20$: $43 - (16 + 10) = 17$: $\binom{21}{4}$

With this, to get our final answer, we take the total number of solutions with no max restrictions, subtract all the cases where one professor goes over, and add back the cases where two professors go over:

$$\binom{47}{4} - [\binom{28}{4} + \binom{29}{4} + \binom{31}{4} + \binom{37}{4}] + [\binom{10}{4} + \binom{12}{4} + \binom{18}{4} + \binom{13}{4} + \binom{19}{4} + \binom{21}{4}].$$

Evaluating gives 50,970 ways.

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** We felt that the main point of this question was applying the sticks-and-stones method while also understanding how different conditions modify the way the strategy is used, as well as how to combine it with inclusion–exclusion.

Problem 6

Question:
Give a Jeopardy-style combinatorial proof of the given equation. You know several interpretations of C(n,k), which we have been writing as (first introduced in Section 4.3), and you can revisit Section 6.6 to see the definition of derangements D_n as the number of permutations of distinct objects that leave no object in its original position. Ask one question that can be answered twice, once for each side of the given equation. Do not manipulate either side with algebra. Use well-explained sentences to justify all aspects of your answers!

$$n! = C(n,0)D_n + C(n,1)D_{n-1} + C(n,2)D_{n-2} + \cdots + C(n,n-1)D_1 + C(n,n)D_0$$

Jeopardy Style Question:
How many ways are there to organize n distinct books on a bookshelf?

Solution 1:
We solve this problem by classifying permutations according to how many books remain in their original positions and then summing over all such cases. If exactly k books remain fixed, we must first choose which k books stay in place. This can be done in $\binom{n}{k}$ ways.
The remaining $n-k$ books must all move to positions different from their originals. The number of ways to do this is D_{n-k}, the derangement number of size $n-k$ (the number of permutations of $n-k$ objects with no fixed points).
Thus, the total number of permutations with exactly k fixed books is

$$\binom{n}{k} D_{n-k}.$$

Since every permutation of n books has some specific number k of fixed books, summing over all possible values of k from 0 to n counts every permutation exactly once:

$$\sum_{k=0}^{n} \binom{n}{k} D_{n-k}.$$

In particular, when $k = 0$ no books remain in their original positions; the term

$$\binom{n}{0} D_n$$

counts exactly the number of complete derangements of all n books (i.e. permutations where none of the books is in its original position).

Solution 2:
A well-known way to solve this problem is to use the factorial definition. When we place the first book out of the n books on the shelf, we have n options for which book goes into the first slot. After that, we are left with $n-1$ options for the second slot, then $n-2$ for the next, and so on until all the slots have been filled. Applying the product rule (multiplying the number of possibilities at each step), we obtain the pattern

$$n \times (n-1) \times (n-2) \times (n-3) \cdots \times (n-(n-1)).$$

This is exactly the definition of a factorial: we multiply by each successive integer one less than the previous until we reach 1. Therefore, $n!$ gives the total number of ways to arrange n books.

- **Who Contributed:** MMichael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: Textbook (section 6.6)
- **Main points:** The main goal of this question is to examine a solution and understand how it can be applied to solving a similar problem. In addition, it encourages a deeper look into the definition of a derangement and how it can be used to approach problems in a different way.