Group HW #2

Michael Ku, Dhvan Shah, Pranav Bonthu

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Problem 1

Question:

Suppose that f is a function from a finite set A to a finite set B where $|A| = a$ and $|B| = b$, and $a > b$. Use the Pigeon-hole Principle and the definition of 1-1 to prove that f cannot be 1-1

Proof:

A function f from $S_1 \to S_2$ is 1-1, if for any 2 elements $a, b \in S_1$: if $f(a) = f(b)$ then $a = b$. In the above case, we have 2 finite sets: A, B, where $|A| > |B|$. We can think of each element a in A as the pigeons, and each element b in B as the pigeon-holes. The act of placing the pigeon in a hole, is what the function f is doing. This means that for f to be 1-1, each hole can only have 1 pigeon. But we know that the number pigeons is greater than the number of holes: $a > b$. This means that there must be at least 2 pigeons in the same hole. Mathematically, this means that there exists $a_1, a_2 \in A$ where $a_1 \neq a_2$, BUT $f(a_1) = f(a_2)$. This proves that the function f that maps A to B CANNOT be 1-1.

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: The textbook
- **Main points:** Based on group discussions, we felt that the main point of this problem was to prove the fact that functions that map from a bigger set to a smaller set cannot exist.

Problem 2

Question:

Give a Jeopardy-style combinatorial proof of: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

Jeopardy:

How many ways can you form a non-empty team from n people, where one person in the team must be a captain.

LHS:

$\binom{n}{k} \equiv \frac{n!}{k!(n-k)!}$ tells us the number of ways to form teams of k from n people. There are $n!$ ways to line up n people. To count all teams of k, we can pick the first k people from each distinct line.

This leads to major overcounting, since the same group of k people could be a different order, and thus we would overcount the same team. For each distinct line, there are $k!$ ways the team could be ordered, and $(n-k)!$ ways the people left out could be ordered. Since this is for each distinct line, we have to divide $n!$ by $k!(n-k)!$ to 'unorder' the line, which gives us the number of ways to form teams of k from n people.

Now we must pick a captain from the team of k. This is exactly k, since only one person can be captain out of the entire team.

To count all possible teams we can form from n people, we can change k, and use Sum Rule to add up all the different possiblities. We set $k = 1$, since the team MUST be non-empty:

$k = 1 \rightarrow$ Team of 1 person $\rightarrow 1 \cdot \binom{n}{1} +$

$k = 2 \rightarrow$ Team of 2 people $\rightarrow 2 \cdot \binom{n}{2} +$

$\ldots$

$k = n \rightarrow$ Team of n people $\rightarrow n \cdot \binom{n}{n}$

We can rewrite this as a sum: $\sum_{k=1}^{n} k\binom{n}{k}$

RHS:
Another way to approach this problem, is to first pick the captain first and then decide the team. We can do this because we know that the team MUST be non-empty, and there MUST be a captain. Because of this, there are always n ways to pick a captain from n people, no matter the team size.

Each person (aside from the captain) has 2 options: ON the team, OFF the team. Since each person has an independent choice, we can use the Product Rule to represent this: $2 \cdot 2 \cdot 2 \cdot \ \ldots \ \cdot 2$ The captain only has 1 option: On the team, so the above multiplication is only repeated $n-1$ times.

Picking a captain and forming teams (excluding the captain) are also independent choices, so we can once again use the Product Rule to simplify the expression to: $n2^{n-1}$

Proof: Since the LHS and RHS both count the exact same thing, they MUST be equal

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** Based on group discussions, we felt that the main point of this problem was to develop familiarity with Jeopardy-style proofs. We had to understand how to set up a question to prove LHS = RHS

Problem 3

$$(B - A) \cup (C - A) \ = \ (B \cup C) - A$$

We can prove that the above equation is true. To start, we can say that on the right side, we are combining sets B and C. We are then removing everything from

$$(B \cup C)$$

, that is within the set of A. This means that we are left with a set of everything in set B that is not in set A plus everything in the set C that is not also in A.

In other words, we are saying that we will end up with a set that contains element from (B - A) and (C - A) . In set notation, this would be

$$(B - A) \cup (C - A)$$

. This means that

$$(B - A) \cup (C - A) \; = \; (B \cup C) - A$$

Example

Let

$$B = \{1, 2, 3, 4\}, \quad C = \{5, 6, 7, 8\}, \quad A = \{1, 2, 5, 6\}.$$

Left-hand side:

$$B - A = \{3, 4\}, \quad C - A = \{7, 8\}.$$

Thus,

$$(B - A) \cup (C - A) = \{3, 4, 7, 8\}.$$

Right-hand side:

$$B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\},$$

so

$$(B \cup C) - A = \{3, 4, 7, 8\}.$$

Since both sides equal $\{3, 4, 7, 8\}$, the identity holds in this example.

- **Who Contributed:** Pranav Bonthu was the main contributor, doing the formal write-up of the solution. Dhvan Shah and Michael Ku compared their solutions to Pranav's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** Based on group discussions, we felt that the main point of this probelm was to understand set proofs, and different ways to approach solving them. The solution we ended up going with was the easiest and most concise.

Problem 4

In order to determine how many outcomes of this race are possible, we can look at how many outcomes there are for each situations.

We can define the situations as:

1. No ties.
2. A tie between 2 horses.
3. Two separate ties (each between 2 horses).

4. A tie between 3 of the horses.

5. A tie between all 4 horses.

The possible outcomes for the first situation, no ties, can be expressed as 4!. This is because we have four horses, and if they each finish at separate times, we have 4 options for which horse finishes first, 3 options for second, 2 for third, and 1 for fourth, or 4 factorial.

$$\text{No ties} = 4!$$

In situation two, we have a singular tie between 2 horses. To start, we found how many combinations of pairs of horses there are. Since we are picking 2 horses from a set of 4, we can do

$$\binom{4}{2}$$

to find the total number of possible pairings. We also need to determine how the race outcome ends, as in, which horses finish at what times. Since we have two horses finishing simultaneously and finishing in the same place (first, second, or third), we can treat this tie as an individual "entity". This means we have three of these "entities", the two separate horses and one tie.

The total number of possible outcomes for these entities to be placed can be defined as 3 factorial: three options for first, two for second, and one for third.

So, in order to find the total amount of possible outcomes for this situation, we need to do:

$$\left(\binom{4}{2}\right) \cdot 3!$$

We are multiplying the number of possible pairings and the outcomes of how the race can finish as the two are not mutually exclusive.

Moving to situation three, we have two distinct ties, each between two horses. Just like in situation two, we will do

$$\binom{4}{2}$$

4 choose 2 will not only make all possible combination of pairs, but also gives us the outcome of the race. This is because when a pair to tie is picked, you can say that pairing will go first, and the other second. Since we only have 2 pairings, or two entities, there will only be first and second. Since 4 choose 2 determines both the pairs as well as who goes first and second, the possible outcomes for this situation is

$$\binom{4}{2}$$

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Situation four has one tie between 3 horses. To determine the total combination of horses that are within the tie, we can do

$$\binom{4}{3}$$

(4 choose 1 could also work as you could pick which horse is NOT within the tie),

In this situation, we are left with two entities, the group of 3 tying horse and the horse that finishes on its own. The amount of ways these two groups can be ordered is 2!, as we have two options for which entity finishes first, and one for finishing second.

Since the grouping are outcomes for the tie and the order in which the entities finish are not mutually exclusive, we can say that the the total possible outcome for this situation is

$$\left(\binom{4}{3}\right) \cdot 2!$$

Situation five is where all four horses tie together. In this situation, we only have one possible way of 4 horses tying (as well as only one possible way to order this entity), so there is only 1 possible outcome.

No to determine the total possible outcomes for the race, we will add up all the possible outcomes from each situation. We are adding since each situation is mutually exclusive. We also have no overlapping outcomes, so we do not need to subtract any possible outcomes.

This leaves us with:

$$\text{Situation}_1 + \text{Situation}_2 + \text{Situation}_3 + \text{Situation}_4 + \text{Situation}_5$$

which is equal too (with all 'choose' being simplified)

$$4! + 6 \cdot 3! + 6 + 4 \cdot 2! + 1$$

which gives us a final total of:

$$75$$

- **Who Contributed:** Pranav Bonthu was the main contributor, doing the formal write-up of the solution. Dhvan Shah and Michael Ku compared their solutions to Pranav's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** Based on group discussions, we felt that the main point of this problem was understanding how to solve counting problems by breaking down the problem into simpler, mutually exclusive cases. Then using sum rule to add up all the possibilities.

Problem 5

Question:
Give a Jeopardy-style combinatorial proof of

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Ask one question; answer it two ways; be careful to justify every step of your answers!

Solution:

Made up question:
Take a group of m people and a group of n people. Out of all the people in those two groups, how many ways can you make teams of r people can you make?

Solution 1:
When approaching classic team-selection problems, it is often most straightforward to use the binomial coefficient:

$$\binom{n}{k}$$

as this directly counts the number of ways to choose k objects from n items. In this problem, the total number of people available to choose from is n+m, and the number we are selecting is r. Therefore, the total number of possible selections is given by

$$\binom{m+n}{r}$$

Solution 2:
When approaching this problem, we can think of it as selecting players from each team individually and then combining the chosen players. For example, let k represent the number of people selected from team n. We can choose these k people from team n using

$$\binom{n}{k}$$

these k individuals make up part of the total r people we need. We then select the remaining r-k people from the other team (with m members) using

$$\binom{m}{r-k}$$

Because k can vary ranging from 0 to r we must sum over all possible values of k to cover every possible combination. This leads to the final expression:

$$\sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** For our group, the central ideas of this problem were understanding combinatorial proofs, formulating a question from a solution, and recognizing how two different solutions can yield the same result.

Problem 6

Question:
Suppose you have 32 identical candies. How many ways can you distribute some, but not necessarily all, of these 32 candies to 13 children if each child must get at least one candy? Fully justify and explain your reasoning/work. (Your solution should somehow allow for every case, ranging from distributing a max of all 32 candies to a min of only distributing 13 candies.)

Solution:
In this problem, we are asked to distribute 32 candies among 13 children under certain conditions. Specifically, each child must receive at least one candy. Because of this restriction, we can use the sticks and stones method. For 13 children, we need 12 dividers (sticks) to create 13 "buckets," one for each child.

Since each child is guaranteed at least one candy, we can preload one candy into each bucket.

$$*|*|*|*|*|*|*|*|*|*|*|*|*$$

This immediately uses up 13 candies, leaving us with $32 - 13 = 19$ candies to distribute freely among the 13 children. The number of ways to distribute these remaining candies is:

$$\binom{19+13-1}{13-1} = \binom{31}{12}$$

Which represents the 19 candies plus the the 12 sticks that would be used to divide the candy between all the children. This would be the answer if all 32 candies had to be used. However, the question states that not all candies must be used. To account for this, we introduce an additional "discard" bucket to represent unused candies. Adding this bucket increases the total number of "buckets" to 14, which requires 13 dividers. We then have 32 candies and 14 buckets (13 children + 1 discard). The total number of distributions, including the possibility of leftover candies, is:

$$\binom{19+14-1}{14-1}$$

Thus, the final answer is:

$$\binom{32}{13}$$

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** For our group, the central ideas of this problem were understanding the sticks-and-stones method and examining how different conditions, such as every child receiving at least one candy or not all the candy being used, affect the answer.