Individual Deep Dive #1

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Problems Competed: I completed these problems:

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Problem 1.3

Question:

Suppose that m identical-looking Blue Men are on the elevator which stops at 30 floors, but no Blue Men exit on at least 3 (random not pre-determined) floors, and some Blue Men might stay on the elevator and joyride back down. On each floor, any Blue Men who exit do so in a group, not in an ordered line.

Solution Summary:

The total number of ways Blue Men can exit or joyride the elevator, with at least 3 empty floors: $\binom{m+30}{30} - (\binom{30}{0}\binom{m}{30} + \binom{30}{1}\binom{m}{29} + \binom{30}{2}\binom{m}{28})$

Justification:

To solve this problem we can define set A. A is the set of all combinations of ways that Blue Men can either exit or joyride the elevator. We can represent all the floors as sticks and all the Blue Men as stones. There are 30 floors, which means we only need 29 sticks to form the 30 bins. However, we need an extra bin to account for the option of joyriding the elevator. This means there are m stones and 30 sticks. Now we can solve for $|A|$, by counting the number of ways to arrange 30 sticks amongst $m+30$ sticks and stones: $\binom{m+30}{30}$.

Next we define set A_0. A_0 is the set of all combinations of ways that Blue Men can either exit or joyride the elevator, where there are exactly 0 empty floors. We first need to pick a floor to be empty. This involves choosing 0 floors to be empty out of 30: $\binom{30}{0}$. Now to force exactly 0 floors to be empty, we preload each floor with a Blue Man. Since each Blue Man is identical, there is only 1 way to do this.

Now we can think about there being m stones again. This time we need to account for the preplaced Blue Men, by subtracting 30 from m: $m-30$. We have the same 30 sticks from above, so we pick where the sticks go amongst the stones: $\binom{m-30+30}{30} = \binom{m}{30}$. Now we solve for $|A_0|$ by using the product rule: $|A_0| = \binom{30}{0}\binom{m}{30}$.

But we want there to be at least 3 empty floors. So we must do the same for there being exactly 1 empty floor. We define A_1 as there being exactly 1 empty floor. We pick 1 floor to be empty out of 30: $\binom{30}{1}$. Then we preload the Blue Men, but this time we only preload them on 29 floors, because we need 1 floor to be empty. So we now have $m-29$ stones. We also don't want to sort any other Blue Men to be on that empty floor. To do this, we can just get rid of the stick representing that empty floor. So we now only have $30-1=29$ sticks. Now we pick where the sticks go amongst the stones: $\binom{m-29+29}{29} = \binom{m}{29}$. Using the product rule, we solve again for $|A_1|$: $|A_1| = \binom{30}{1}\binom{m}{29}$

We use this same reasoning for A_2, but we would preload only 28 Blue Men, and have only 28 sticks: $|A_2| = \binom{30}{2}\binom{m-28+28}{28}$

Finally, we can solve for the total number of ways with at least 3 empty floors: $|A| - (|A_0| + |A_1| + |A_2|) = \binom{m+30}{30} - (\binom{30}{0}\binom{m}{30} + \binom{30}{1}\binom{m}{29} + \binom{30}{2}\binom{m}{28})$

Problem 2.2

Question:

After winning chips, players can queue at 20 different cashier stations to trade their chips for cash. The cashier stations are numbered 1 to 20, and they each have a different cashier, so these lines are different. Suppose you have p people who are going to line up themselves in front of the 20 different cashiers, but some cashiers might get no people in line. The lines that do form in front of any cashiers are definitely ordered. Imagine that b of the p people are identical looking Blue Men, c of the p people are identically-costumed and therefore indistinguishable circus artists from Cirque du Soleil, and d of them are regular people who are, of course, distinguishable from each other. To avoid any doubt: All p people must be in some line to cash out, but not all cashier stations must have people lined up in front of them. How many ways can these line(s) form?

Solution Summary:

The number of ways these lines can form is: $\frac{(b+c+d+19)!}{b!c!19!}$

Justification:

We can solve this problem by imagining that everyone is going to stand in one long line. Everyone to the right of a cashier will be in that cashiers line, effectively making each cashier a Divider. We want there to be 20 sections in this line, which means we need to use 19 Dividers (cashiers).

We can count the number of ways to order a line of N items, by doing: $N!$. For now, we can assume that everyone is distinct. The total number of items in this line are the Blue Men, the Circuit Artists, the Regular People and the Dividers: $N = b + c + d + 19$. This means that there are $N! = (b + c + d + 19)!$ ways to order this line.

However, we know that the Blue Men, the Circus Artists and the Cashiers are all identical. We can have a distinct line where the b Blue Men are in specific positions. The above factorial over counts this line by $b!$, where each Blue Man could be in any of the b positions. This applies to every unique pattern in the line, which means we need to divide $N!$ by $b!$: $\frac{N!}{b!}$.

We can use the above reasoning with the Circus Artists and the Cashiers as well: $\frac{N!}{b!c!19!}$

Therefore, the number of ways these lines can form is: $\frac{(b+c+d+19)!}{b!c!19!}$

Problem 3.1

Question:

How many ways can these 1000 distinct-looking Un-Blue Men empty out of the elevator, in groups not ordered lines, at the 30 different floors, where it matters which specific Un-Blue Man gets off on which floor as you can now see the people as distinct, where at least one Un-Blue Man gets off at each floor, and where all Un-Blue Men have exited by the time this elevator ride is over?

Solution Summary:

The number of ways these Un-Blue Men empty out of the elevator: $\sum_{i=0}^{30} (-1)^i \binom{30}{i} (30 - i)^{1000}$

Justification:

To solve this problem, we can start by defining set A: A is the set of all ways the Un-Blue men can leave the elevator. Since we know that the elevator is empty at the end, each person has 30 choices of floor to get off. This means $|A| = 30 \cdot 30 \cdot ... \cdot 30 = 30^{1000}$

However, we know that no floor can be empty. This means we must subtract the bad cases where a floor is empty to get the final answer. We can define set A_1 as the set where at least one floor is empty. If at least one floor is empty, then each person now only has 29 choices of floor to get off: 29^{1000}. But we also have to pick the empty floor. This means we have to choose 1 floor to be empty out of 30: $\binom{30}{1}$. Combining these, we get $|A_1| = \binom{30}{1}29^{1000}$

We can't simply just subtract the above value from $|A|$, since we are over counting scenarios where more than two floors are empty. To correct for this, we can define set A_2 as the set where at least two floors are empty. Each person now only has 28 choices of floor to get off: 28^{1000}. We again have to pick two floors to be empty out of 30: $\binom{30}{2}$. Combining these, we get $|A_2| = \binom{30}{2}28^{1000}$ This time, we would add this value to the above subtraction, to account for the over counting.

We keep doing this process, for all 30 floors. The final case, A_{30}, is when all the floors are empty. Using the above pattern, $|A_{30}| = \binom{30}{30}0^{1000} = 0$, which makes complete sense. If all the Un-Blue Men must get off at a floor, then there should be 0 combinations where all the floors are empty.

Combining all the above using Sum Rule: $|A| - |A_1| + |A_2| - ... + |A_{30}| = \binom{30}{0}30^{1000} - \binom{30}{1}29^{1000} + \binom{30}{2}28^{1000} - ... + 0$

We can write the above as a summation: $\sum_{i=0}^{30} (-1)^i \binom{30}{i}(30-i)^{1000}$

Problem 4.3

Question:

Provide a TRIPLE Jeopardy-style combinatorial proof of the following equation:

$\sum_{k=0}^{m} \binom{m}{k}(x-1)^k = x^m = \sum_{k=0}^{m} \binom{m}{k}(-1)^k(x+1)^{m-k}$

Jeopardy Answer:

How many unique strings of length m can be made with x letters.

Solution 1:

We are trying to make a string of length m from x letters. This means that each position in the string has x options: $x \cdot x \cdot ... \cdot x$. We can rewrite this as: x^m. So there are x^m unique strings of length m with x distinct letters.

Solution 2:

We can reframe this problem by thinking about the number of unique strings that have non-'A' characters in exactly 2 positions. To count the number of strings we can make this way, we can think about choosing 2 positions from the entire length m to be some character other than 'A': $\binom{m}{2}$. Then for each of those positions, there is only $x-1$ options of letters to pick from, since we already accounted for the 'A's: $(x-1)^2$. Now we count the number of ways we can put the 'A's in, which has to be 1, since there are $m-2$ positions left in the string and the 'A's are identical.

We can use the above analogy to count all the total number of unique strings of length m with x letters. We know that non-'A' characters can appear anywhere from 0 to m times in the string. This just means that to count the total number of strings, we need to use the sum rule to add up the strings with exactly 0 non-'A's, exactly 1 non-'A', ... exactly m non-'A's: $\binom{m}{0}(x-1)^0 + \binom{m}{1}(x-1)^1 + ... + \binom{m}{m}(x-1)^m$. We can see that in the first term, where there are exactly 0 non-'A's (all 'A's), we get: $\binom{m}{0}(x-1)^0$, which simplifies to 1. This makes sense, as there is exactly 1 string of length m, with m 'A's.

We can rewrite the above as a summation: $\sum_{k=0}^{m} \binom{m}{k}(x-1)^k$

Solution 3:

We can again reframe this problem by imagining that the letter 'A' is an invalid character, not counted in the x letters. If we make strings of length m using x characters plus the invalid character 'A', then we get $(x+1)^m$ ways to make the string, since each position now has $x+1$ options.

Since we want to only count the number of strings that have the original x characters, we have to subtract the bad cases. The first bad case is when there is at least one 'A' in the string. We can count the number of strings that have at least one 'A' by choosing 1 position out of m to have the 'A': $\binom{m}{1}$. This guarantees that there is an 'A', so the rest of the string has $x+1$ options for each position: $(x+1)^{m-1}$. We would subtract this value from the total number of strings: $(x+1)^m - \binom{m}{1}(x+1)^{m-1}$

Just subtracting the above two values is incorrect, since we counted strings with more than 1 'A' in them twice. So we must add those back to the above subtraction. We use a similar process as above, but this time we have to pick 2 positions: $\binom{m}{2}(x+1)^{m-2}$. This addition would overcount any strings with more than 2 'A's, so we must repeat this process in an alternating pattern to get the total number of strings. If we use the above pattern, the last value would be subtracting (or adding) 1: $\binom{m}{m}(x+1)^{m-m} = 1(x+1)^0 = 1$. This makes sense, since there is only one way to have a string with all 'A's.

We can write out the expression: $\binom{m}{0}(x+1)^{m-0} - \binom{m}{1}(x+1)^{m-1} + \binom{m}{2}(x+1)^{m-2} - ... + \binom{m}{m}(x+1)^{m-m}$

As a summation: $\binom{m}{k}(-1)^k(x+1)^{m-k}$

Conclusion:

Since all 3 of the above count the exact same number of things, they must be equivalent.

Bonus:

Best Advice: Doing something imperfectly is better than not doing it at all