Group HW #3

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Problem 1

Question:

How many elements are in the union of five sets if the sets contain 10,000 elements each, each pair of sets has 1,000 common elements, each triple of sets has 100 common elements, every four of the sets has 10 common elements, and there is 1 common element in all five set?

Solution:

Let there be sets: S_1, S_2, S_3, S_4, S_5, where $|S_i| = 10,000$ for $i = 1, 2, ..., 5$. We want to find $|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5|$. To do this, we can start by defining set A, where $A = \{S_1, S_2, S_3, S_4, S_5\}$. We know that $|A| = 5 \cdot 10,000 = 50,000$, since each set S_i containts 10,000 elements.

Now, we must account for overcounting. By combining all the sets together, we have overcounted any elements in at least 2 sets. We can define subset A_1, such that it counts all the common elements between set S_1 and set S_2. From the problem definition, we know that $|A_1| = 1,000$, since each pair of sets has 1,000 common elements. We must do this for all possible combinations of the five sets: $S_1, ..., S_5$. To do this, we need to figure out the number of ways to pick two sets from five, which can be expressed as $\binom{5}{2}$. For whatever pair of sets we pick, the cardinality will always be 1,000. We can write this as $1,000 \cdot \binom{5}{2}$. We can now subtract this value from the cardinality of A: $50,000 - 1,000 \cdot \binom{5}{2}$ to account for overcounting

That subtraction, oversubtracts elements that are in at least 3 sets. We can define subset A_2, such that it counts all the commont elements between sets $S_1, S_2 and S_3$. From the problem definition, we know that $|A_2| = 100$, since each triple of sets has 100 common elements. We must do this for all possible cominations of the five sets: $S_1, ..., S_5$. To do this, we need to figure out the number of ways to pick three sets from five, which can be expressed as $\binom{5}{3}$. For whatever triple of sets we pick, the cardinality will always be 100. We can write this as $100 \cdot \binom{5}{3}$. We now add this value to the previous expression: $50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3}$

We use a similar process for elements that are in at least 4 sets. Define subset A_3 for elements in sets $S_1, ..., S_4$. We know $|A_3| = 10$, since every four sets have 10 common elements. We need to pick four sets out of five: $\binom{5}{4}$ and multiply by the cardinality: $10 \cdot \binom{4}{5}$. Since we overcounted elements in 4 sets, we need to subtract this value from the above expression: $50,000 - 1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3} - 10\binom{5}{4}$

We again use a similar process for elements that are in at least 5 sets. Define subset A_4 for elements in sets $S_1, ..., S_5$. We know $|A_4| = 1$, since all five sets have 1 common element. Since we oversubtracted elements in 5 sets, we need to add this value from the above expression: $50,000 -$

$1,000 \cdot \binom{5}{2} + 100 \cdot \binom{5}{3} - 10\binom{5}{4} + 1$

We can express the above expression in summation notation to be more concise:

$|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5| = \sum_{i=1}^{5}(-1)^{i-1}\binom{5}{i} \cdot 10^{5-i}$

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to correct for overcounting and undercounting when counting the number of elements in the union of sets.

Problem 2

Question:

Provide a Jeopardy-style combinatorial proof of the following equation by asking one question and then answering that one question in two different ways:

$\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \frac{36!}{9!9!9!9!}$

Jeopardy Answer: How many ways can you make 4 distinct teams of 9 from 36 people?

LHS:

Since the order within each team doesn't matter, we can make teams by "choosing" k people from n people: $\binom{n}{k}$. To make the first team, we choose 9 people from 36: $\binom{36}{9}$. Now to form the second team, we only have $36 - 9$ people to choose from: $\binom{36-9}{9} = \binom{27}{9}$. For the third team, we only have $27 - 9$ people to chose from: $\binom{27-9}{9} = \binom{18}{9}$. And finally for the fourth team, we only have the remaining 9 people to chose from: $\binom{9}{9}$. Because we have framed these choices as independent tasks, we can use the product rule to determine the number of ways to form the teams: $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9}$

RHS:

We can also think about this problem as forming an ordered line of all n people, picking the first k people, and then unordering that selection, and unordering the rest of the line. We can show this mathematically as: $\frac{n!}{k!(n-k)!}$. So to form the first team of 9, we line up all 36 people, pick the first 9, and unorder the selection as well as the 27 people not selected: $\frac{36!}{9!27!}$. Then to form the second team of 9, we line up everyone not selected last time, pick the first 9, and unorder the selection and the 18 people not selected: $\frac{27!}{9!18!}$. The third team is picked the same way. Line up the 18 people not selected last time, pick the first 9 and unorder the selection and the 9 poeple not selected: $\frac{18!}{9!9!}$. Finally the for the last team, we order all 9 people and unorder all 9 of the on the team: $\frac{9!}{9!0!}$.

Because we framed all of the choices as independent tasks, we can use the product rule to detemrine the number of ways to form the teams: $\frac{36!}{9!27!}\frac{27!}{9!18!}\frac{18!}{9!9!}\frac{9!}{9!0!}$. If we simplify that expression, we get: $\frac{36!}{9!9!9!9!}$

Proof:

Since both the LHS and the RHS count exactly the same number, they must be equal. Therefore: $\binom{36}{9}\binom{27}{9}\binom{18}{9}\binom{9}{9} = \frac{36!}{9!9!9!9!}$

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- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand the the choose function has a fundamental combinatorial background.