Group HW #5

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November 9, 2025

Problem 1

Question:
Suppose that there are animals waiting in a single-file line at a watering hole on a plain in Tanzania. Suppose the line contains baboons, zebras, and elephants. Each baboon takes up one foot; each zebra takes up two feet; and each elephant takes up three feet. Find a recurrence relation and appropriate initial conditions to determine the number of ways an unlimited supply of these animals (identical within each species) can fill an n foot long line. Explain your work in detail. Please also include the first 10 terms, using computational power (such as Wolfram Alpha) if you wish.

Solution:
When looking at our question, we can label our three different options for animals that we will line up in a line:

- Baboon: Takes up one foot
- Zebra: Takes up two feet
- Elephant: Takes up three feet

From this, we can define a_n as the number of ways to fill an n-foot line using these animals. At each position, we can choose to place either:

- A baboon, which leaves $n-1$ feet to fill (a_{n-1} ways),
- A zebra, which leaves $n-2$ feet to fill (a_{n-2} ways),
- An elephant, which leaves $n-3$ feet to fill (a_{n-3} ways).

This can be seen in the figure below:
Adding these together gives our recurrence relation for all $n \geq 3$:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

Now with this we can start looking at solutions starting with some of the initial values.
Initial Conditions:

- $a_0 = 1$ (one way to fill a line of length 0)
- $a_1 = 1$ (only a baboon)
- $a_2 = 2$ (either two baboons or one zebra)

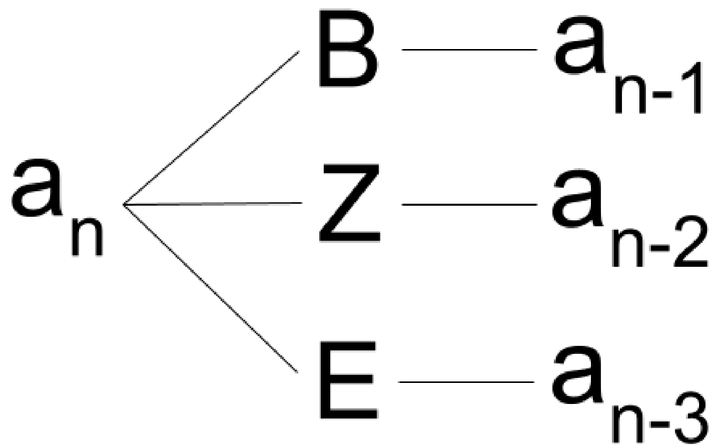

Figure 1: a_n recursion problem layout

- $a_3 = 4$ (combinations: BBB, BZ, ZB, E)

First 11 Terms: We can now use the recurrence to find the first 11 terms.

- $a_0 = 1$
- $a_1 = 1$
- $a_2 = 2$
- $a_3 = 4$
- $a_4 = a_3 + a_2 + a_1 = 4 + 2 + 1 = 7$
- $a_5 = a_4 + a_3 + a_2 = 7 + 4 + 2 = 13$
- $a_6 = a_5 + a_4 + a_3 = 13 + 7 + 4 = 24$
- $a_7 = a_6 + a_5 + a_4 = 24 + 13 + 7 = 44$
- $a_8 = a_7 + a_6 + a_5 = 44 + 24 + 13 = 81$
- $a_9 = a_8 + a_7 + a_6 = 81 + 44 + 24 = 149$
- $a_{10} = a_9 + a_8 + a_7 = 149 + 81 + 44 = 274$

Final Answer:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}, \quad a_0 = 1,\ a_1 = 1,\ a_2 = 2$$

The first 11 terms (from a_0 to a_{10}) are:

$$1,\ 1,\ 2,\ 4,\ 7,\ 13,\ 24,\ 44,\ 81,\ 149,\ 274$$

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.

- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how counting with fixed-size tiles/blocks reduces to adding counts for smaller lengths

Problem 2

Question:
Suppose you are making a new error control code over a quaternary alphabet using 0, 1, 2, 3 as your symbols. Find a recurrence relation and appropriate initial conditions to determine q_n = the number of length n codewords you can make if you must never have three (or more) consecutive 3s appearing. (A codeword is just a string of the symbols, in this case, a string of 0s, 1s, 2s, 3s.) Explain your work in detail. Please also include the first 10 terms, using computational power (such as Wolfram Alpha) if you wish.

Solution:
When looking at our question, we can break down the main conditions that our recursion equation must follow. We can use either 0, 1, 2, or 3 for values in our sequence. The only other condition is that we must not have three or more consecutive 3s in a single sequence.

From this, we can define q_n as the number of ways to make an n-long sequence without having three or more 3s in a row. From the beginning, we have four options for the first value:

- A 0, which leaves $n-1$ positions left to fill and does not lead toward a rule break (q_{n-1}).
- A 1, which leaves $n-1$ positions left to fill and does not lead toward a rule break (q_{n-1}).
- A 2, which leaves $n-1$ positions left to fill and does not lead toward a rule break (q_{n-1}).
- A 3, which is one step toward a rule break, so we must look deeper.

After placing a single 3, we have four options for what can happen next:

- A 0, which leaves $n-2$ positions left to fill and ends the possible rule break (resets) (q_{n-2}).
- A 1, which leaves $n-2$ positions left to fill and ends the possible rule break (resets) (q_{n-2}).
- A 2, which leaves $n-2$ positions left to fill and ends the possible rule break (resets) (q_{n-2}).
- A 3, which is a second step toward a rule break, so we must look deeper.

Finally, after placing a second 3 in a row, the next symbol must reset:

- A 0, which leaves $n-3$ positions left to fill and ends the possible rule break (resets) (q_{n-3}).
- A 1, which leaves $n-3$ positions left to fill and ends the possible rule break (resets) (q_{n-3}).
- A 2, which leaves $n-3$ positions left to fill and ends the possible rule break (resets) (q_{n-3}).

And that is it. We cannot have a third 3 in a row, as that is a clear rule break. Summing all of those possibilities, we get our final equation below. This can also be seen in Figure 1.

Final Equation:

$$q_n \;=\; 3q_{n-1} \;+\; 3q_{n-2} \;+\; 3q_{n-3} \qquad (n \geq 3).$$

From here, we solve for our initial conditions, which will be used to calculate the rest of the terms in the recurrence.

Initial Conditions:

- $q_0 = 1$ (empty string)
- $q_1 = 4$ (0, 1, 2, 3)
- $q_2 = 16$ (4^2, since 33 is allowed)
- $q_3 = 63$ ($4^3 - 1$, exclude only 333)

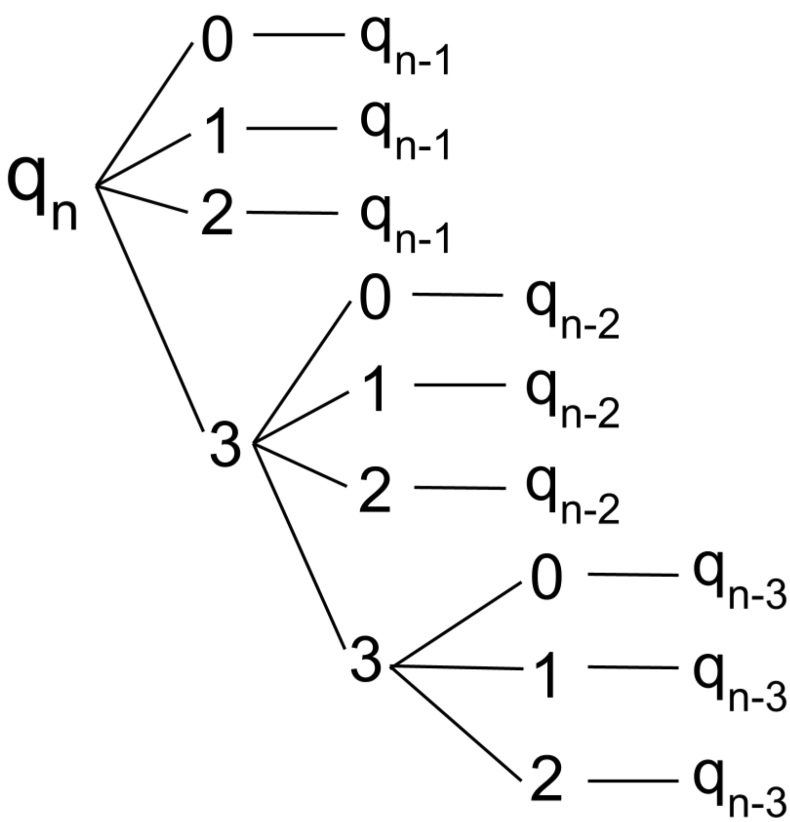

Figure 2: q_n recursion problem layout

We can now use the recurrence to find the first 11 terms.
First 11 terms (from $n = 0$ to $n = 10$):

- $q_0 = 1$
- $q_1 = 4$

- $q_2 = 16$
- $q_3 = 63$
- $q_4 = 3(q_3 + q_2 + q_1) = 3(63 + 16 + 4) = 249$
- $q_5 = 3(q_4 + q_3 + q_2) = 3(249 + 63 + 16) = 984$
- $q_6 = 3(q_5 + q_4 + q_3) = 3(984 + 249 + 63) = 3888$
- $q_7 = 3(q_6 + q_5 + q_4) = 3(3888 + 984 + 249) = 15363$
- $q_8 = 3(q_7 + q_6 + q_5) = 3(15363 + 3888 + 984) = 60705$
- $q_9 = 3(q_8 + q_7 + q_6) = 3(60705 + 15363 + 3888) = 239868$
- $q_{10} = 3(q_9 + q_8 + q_7) = 3(239868 + 60705 + 15363) = 947808$

Final Answer:

$$q_n = 3q_{n-1} + 3q_{n-2} + 3q_{n-3}, \quad q_0 = 1,\ q_1 = 4,\ q_2 = 16,\ q_3 = 63$$

The first 11 terms (from q_0 to q_{10}) are:

$$1,\ 4,\ 16,\ 63,\ 249,\ 984,\ 3888,\ 15363,\ 60705,\ 239868,\ 947808$$

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to understand how to build forbidden pattern avoidance recurrences by short prefix cases

1 Problem 3

In order to solve this problem, we need to consider what each symbol (0, 1, 2, 3) represents. When a zero is appended to our codeword, the number of 0s switches from even to odd or from odd to even. In this problem, we are only counting codewords that contain an even number of 0s. When a 1, 2, or 3 is appended to the codeword, the state of being even or odd does not change.

When we append a nonzero number to a codeword that is even and has a length of $n-1$, it will remain even. Since there are three possible options for nonzero numbers, it can be said that there are $3q_{n-1}$ codewords that can be formed from a codeword of length n in this way.

As we know, if instead of adding 1, 2, or 3 we append a zero, the codeword changes from odd to even or from even to odd. When a codeword becomes odd, we do not include it in our total. When it becomes even, we can look at a string that is odd and has a length of $n-1$. To find the total number of odd strings of length $n-1$, we note that since the even ones are represented by q_{n-1}, we can subtract the number of even strings from the total possible codewords.

Since there are 4 options for each digit in the codeword, there are 4^{n-1} total arrangements. This means that there are $4^{n-1} - q_{n-1}$ odd codewords, which become even after appending a 0.

Therefore, when finding the total number of codewords with an even number of 0s, we have:

$$q_n = 3q_{n-1} + \left(4^{n-1} - q_{n-1}\right),$$

which simplifies to

$$q_n = 2q_{n-1} + 4^{n-1}.$$

It must also be noted that there is only one possible way to have a code string of length 0 that is even, so

$$q_0 = 1.$$

Our first 10 terms are:

$$\begin{aligned}
n &= 0 : 1, \\
n &= 1 : 3, \\
n &= 2 : 10, \\
n &= 3 : 36, \\
n &= 4 : 136, \\
n &= 5 : 528, \\
n &= 6 : 2080, \\
n &= 7 : 8256, \\
n &= 8 : 32896, \\
n &= 9 : 131328.
\end{aligned}$$

- **Who Contributed:** Pranav Bonthu was the main contributor, doing the formal write-up of the solution. Dhvan Shah and Michael Ku compared their solutions to Pranav's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to use state counting (even/odd) to convert a global constraint into a simple recurrence

Problem 4

Question:

Prove using mathematical induction that:

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

Whenever n is a positive integer

Setup:

Let $P(n)$ be the proposition that $1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$. We want to show that $P(n)$ is true for all $n \geq 1$

Base Cases:

We can check that $P(1)$ is true, since it is a trivial base case:

$$1(1+1) = 1(2) = 2$$
$$\frac{1(1+1)(1+2)}{3} = \frac{(2)(3)}{3} = 2$$

Inductive Hypothesis:

Assume that $P(k)$ is true for some $k \geq 1$, which means that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + k(k+1) = \frac{k(k+1)(k+2)}{3}$$

This is a valid assumption, since we just proved it to be true for $k = 1$.

Inductive Step:

Consider $P(k+1)$. We want to show that $P(k+1)$ is true, which would mean that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)((k+1)+1) = \frac{(k+1)((k+1)+1)((k+1)+2)}{3}$$

$$= 1 \cdot 2 + 2 \cdot 3 + \cdots + (k+1)(k+2) = \frac{(k+1)(k+2)(k+3)}{3}$$

If we write out the second-to-last term on the left-hand side, we would get

$$1 \cdot 2 + 2 \cdot 3 + \cdots + (k)(k+1) + (k+1)(k+2)$$

Notice that up until that second-to-last term, the sum is the exact same as $P(k)$. This means we can rewrite the left-hand side as

$$P(k) + (k+1)(k+2)$$

We can substitute in the definition of $P(k)$ to get

$$\frac{k(k+1)(k+2)}{3} + (k+1)(k+2)$$

Combine these fractions

$$\frac{k(k+1)(k+2) + 3(k+1)(k+2)}{3}$$

Factor out the common $(k+1)(k+2)$ in the numerator

$$\frac{(k+1)(k+2)(k+3)}{3}$$

$$LHS = RHS$$

Conclusion:

We have shown that $P(1)$ is true, and we have shown that IF $P(k)$ is true for some $k \geq 1$, THEN $P(k+1)$ is true. Thus, by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$, and we can conclude that

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

for all $n \geq 1$

- **Who Contributed:** Dhvan Shah was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Michael Ku compared their solutions to Dhvan's and provided feedback and edits.
- **Resources**: The textbook, class notes
- **Main points:** Based on group discussions, we felt that the main point of this problem was to prove polynomial sum identities by induction

Improvement Goal: We didn't have an improvement goal for this week
Next Week: Next week our improvement goal is to be done before Saturday night.