Group HW #4

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November 2, 2025

Problem 4

Part A:

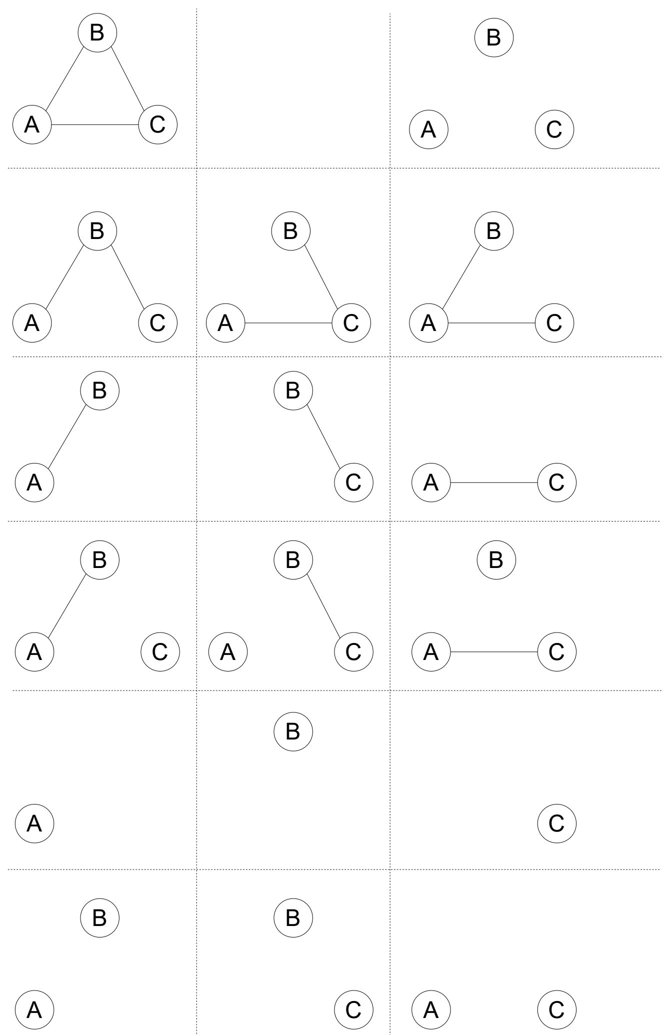

Figure 1: The 17 subgraphs of K_3

Part C:
In order to determine how many subgraphs with at least 1 vertex from a K_n complete graph with n vertices, we can start by looking at how many possible vertices can be included in each subgraph.

Since there are n vertices in the graph, and each subgraph must have at least 1 vertex, a subgraph can have n, $n-1$, $n-2$, $\ldots$, 2, or 1 vertices.

To find the total amount of subgraphs of the K_n complete graph, we must find how many subgraphs can be made of each combination of vertices.

For example, the subgraphs of a K_3 graph may have 3, 2, or 1 vertices. To find the total subgraphs of K_3, we need to find how many subgraphs can be made from 3 vertices, how many from 2, and how many from 1. We would then be able to find the sum by adding up each of these totals.

We need to know how many ways there are to make a subgraph, given a certain amount of vertices. To do this, we can determine how many possible edges there are, which is represented by $\binom{i}{2}$, where i is the number of vertices. This represents the number of edges since an edge is a connection between 2 vertices, so the total combinations of 2 vertices are equal to the number of edges.

We also know that in each subgraph, the edges can either be present or not present. Since there are 2 options for each edge, we can say that the total possible combinations of edges is $2^{\binom{i}{2}}$.

Next, since each vertex is named, they are distinguishable. This means we need to find the combinations possible for picking i vertices. This can be represented by $\binom{n}{i}$, where n is the number of total vertices in the set, and i is the amount of vertices in the subgraph.

This leaves us with the equation $(2^{\binom{i}{2}}) \times \binom{n}{i}$ for the amount of subgraphs that can be made with i vertices.

To find the total number of subgraphs, we need to do this with every possible value between 1 and n and add up the total. This gives us the equation

$$\sum_{i=1}^{n} \binom{n}{i} 2^{\binom{i}{2}}.$$

- **Who Contributed:** Pranav Bonth and Dhvan Shah were the main contributor, doing the formal write-up of the solution. Michael Ku compared his solutions to theirs and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** The main goal of this question was to relate graphs back to combinatorics problems, and help us understand how to connect the two concepts together.

Problem 2

Part A:
Question:
How many non-isomorphic (simple) graphs can you draw on four vertices? ***From the provided solutions, you know the answer is 11. Draw one representative from each isomorphism class to show that you know where the 11 comes from. You can hand-draw these and submit a photo of them!***

Solution:

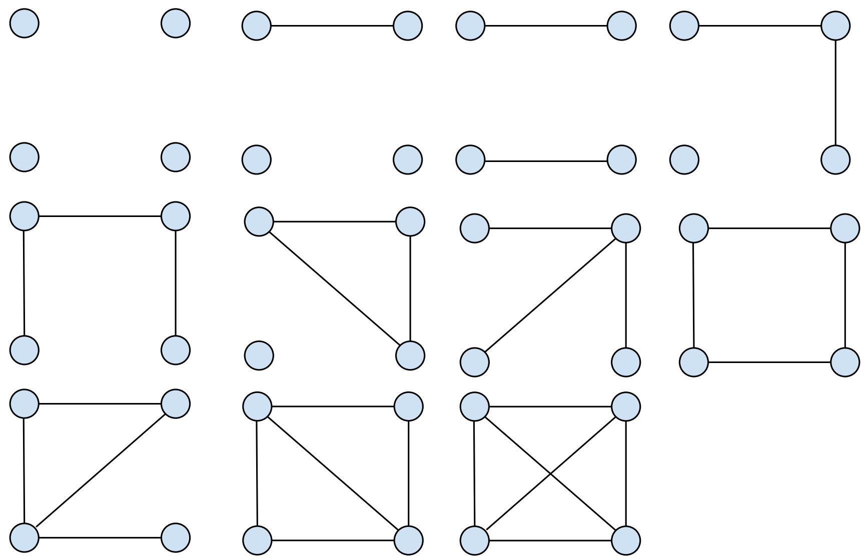

Figure 2: The 11 Non-isomorphic graphs that can be made with four vertices

Part B:
Question:
Suppose you have a set A whose elements are graphs on four vertices. Suppose $|A| = 25$. Without finding them, can you prove that there are three graphs in A that are pairwise isomorphic (meaning every pair of the three graphs are isomorphic)? In other words, show that there are three graphs $G_1, G_2, G_3 \in A$ such that $G_1 \cong G_2$, $G_1 \cong G_3$, and $G_2 \cong G_3$ *(without finding the actual graphs).*

Solution:
We want to prove that there exist three graphs in A that are pairwise isomorphic. To do this, we apply the Pigeonhole Principle.
From previous results, there are exactly 11 non-isomorphic graphs on four vertices. This means that all possible graphs on four vertices can be grouped into 11 isomorphism classes. Each graph in A must belong to one of these classes, where all graphs within the same class are isomorphic to each other.
We can treat the isomorphism classes as the *holes* and the 25 graphs in A as the *pigeons*. If each of the 11 holes contained at most two pigeons, the maximum number of graphs we could have without any three being isomorphic would be

$$11 \times 2 = 22.$$

However, since $|A| = 25 > 22$, by the Pigeonhole Principle, at least one class must contain at least three graphs. Therefore, there exist three graphs $G_1, G_2, G_3 \in A$ such that

$$G_1 \cong G_2, \quad G_1 \cong G_3, \quad \text{and} \quad G_2 \cong G_3.$$

Hence, there are at least three graphs in A that are pairwise isomorphic.

- **Who Contributed:** Michael Ku was the main contributor, doing the formal write-up of the solution. Pranav Bonthu and Dhvan Shah compared their solutions to Michael's and provided feedback and edits.
- **Resources**: No resources were used.
- **Main points:** The main goal of this question was to get comfortable identifying which graphs are isomorphic to each other and which are not. Additionally, it demonstrates how other techniques, such as the pigeonhole principle, can be applied in this context.

Improvement Goal: We didn't have an improvement goal for this week
Next Week: Next week our improvement goal is to be done before Saturday night.