[CLT] The **chi-squared distribution** with parameter n is the distribution of the sum of the squares of n independent standard normal random variables.

Let S_k be the sum of k independent chi-squared random variables with parameter 8. Find the limit as $k \to \infty$ of

$$P(8k \leq S_k \leq 8.01k).$$

$S_1 = X_1^2 + \cdots X_8^2 \quad X_i \sim N(0,1)$

$S_k = k \cdot (X_1^2 + \cdots + X_8^2). \quad k \geq 1$

$E(S_1) = \sum_{i=1}^{8} E(X_i^2) = \sum_{i=1}^{8} Var(X_i) = 8$

$Var(S_1) = 6^2$

According to Central Limit Theorem:

$$\lim_{k \to \infty} \left(\frac{8k - 8k}{6 \cdot \sqrt{k}} < \frac{S_k - n\mu}{6 \cdot \sqrt{k}} < \frac{8.01 - 8k}{6\sqrt{k}} \right)$$

$$= \lim_{k \to \infty} \left(0 < Z < \frac{0.01k}{6\sqrt{k}} \right) \approx \frac{1}{2}$$

The mean of the chi-squared distribution is

$$E[Z_1^2 + \cdots + Z_8^2],$$

where Z_i's are independent standard normals. Applying linearity and using the fact that $E[Z_i^2] = Var Z_i = 1$, we find that the mean of the chi-squared distribution is 8. The variance of the chi-squared distribution is not as straightforward to calculate explicitly; let's call it σ^2.

The sum S_k has mean $8k$ and variance $k\sigma^2$. Therefore, its typical values are close to $8k$, with fluctuations on the order of $\sigma\sqrt{k}$. Since $0.01k$ is much larger than $\sigma\sqrt{k}$ when k is large (and since the normal distribution is symmetric), approximately $\boxed{\frac{1}{2}}$ of the mass is between $8k$ and $8k + 0.01k$.

Theorem (Central Limit theorem)
Suppose that $X_1, X_2, \ldots,$ are independent, identically distributed random variables with mean μ and finite standard deviation σ, and defined the normalized sums $S_n^* = (X_1 + \cdots + X_n - n\mu)/(\sigma\sqrt{n})$ for $n \geq 1$.

For all $-\infty \leq a < b \leq \infty$, we have

$$\lim_{n\to\infty} \mathbb{P}(a < S_n^* < b) = \mathbb{P}(a < Z < b),$$

where $Z \sim \mathcal{N}(0,1)$. In other words, the sequence $S_1^*S_2^* \ldots$ *converges in distribution* to Z.

Write

$$U_n = \sqrt{n}\left(\frac{\overline{Y} - \mu}{\sigma}\right)$$
$$= \frac{1}{\sqrt{n}}\left(\frac{\sum_{i=1}^{n} Y_i - n\mu}{\sigma}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} Z_i, \qquad \text{where } Z_i = \frac{Y_i - \mu}{\sigma}.$$

Because the random variables Y_i's are independent and identically distributed, Z_i, $i = 1, 2, \ldots, n$, are independent, and identically distributed with $E(Z_i) = 0$ and $V(Z_i) = 1$.

Since the moment-generating function of the sum of independent random variables is the product of their individual moment-generating functions,

$$m_{\sum Z_i}(t) = m_{Z_1}(t) \times m_{Z_2}(t) \times \cdots \times m_{Z_n}(t) = [m_{Z_1}(t)]^n$$

and

$$m_{U_n}(t) = m_{\sum Z_i}\left(\frac{t}{\sqrt{n}}\right) = \left[m_{Z_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n.$$

By Taylor's theorem, with remainder (see your *Calculus II* text)

$$m_{Z_1}(t) = m_{Z_1}(0) + m'_{Z_1}(0)t + m''_{Z_1}(\xi)\frac{t^2}{2}, \qquad \text{where } 0 < \xi < t,$$

and because $m_{Z_1}(0) = E(e^{0Z_1}) = E(1) = 1$, and $m'_{Z_1}(0) = E(Z_1) = 0$,

$$m_{Z_1}(t) = 1 + \frac{m''_{Z_1}(\xi)}{2}t^2, \qquad \text{where } 0 < \xi < t.$$

Therefore,

$$m_{U_n}(t) = \left[1 + \frac{m''_{Z_1}(\xi_n)}{2}\left(\frac{t}{\sqrt{n}}\right)^2\right]^n$$
$$= \left[1 + \frac{m''_{Z_1}(\xi_n)t^2/2}{n}\right]^n, \qquad \text{where } 0 < \xi_n < \frac{t}{\sqrt{n}}.$$

Notice that as $n \to \infty$, $\xi_n \to 0$ and $m''_{Z_1}(\xi_n)t^2/2 \to m''_{Z_1}(0)t^2/2 = E(Z_1^2)t^2/2 = t^2/2$ because $E(Z_1^2) = V(Z_1) = 1$. Recall that if

$$\lim_{n\to\infty} b_n = b \quad \text{then} \quad \lim_{n\to\infty}\left(1 + \frac{b_n}{n}\right)^n = e^b.$$

Finally,

$$\lim_{n\to\infty} m_{U_n}(t) = \lim_{n\to\infty}\left[1 + \frac{m''_{Z_1}(\xi_n)t^2/2}{n}\right]^n = e^{t^2/2},$$

®**Exercise**

Define $f_n(x)$ to be n when $0 \leq x \leq 1/n$ and 0 otherwise, and let ν_n be the probability measure with density f_n. Show that ν_n converges to the probability measure ν which puts of all its mass at the origin.

Suppose $I = (a, b)$ is a continous interval. if I contains the origin, then the terms of sequence $\nu_1(I), \nu_2(I) \cdots$ are equal to 1 for large enough n, since all of probability mass of ν_n is in the interval $[0, \frac{1}{n}]$ and eventually $[0, \frac{1}{n}] \in I$, If I does not contain the origin, then the terms of sequence $\nu_1(I), \nu_2(I) \ldots$ are eventually equal to 0. for the same reason.

$\nu_n(I)$ converges to $\nu(I)$. $\Rightarrow$ ν_n converges to ν

Definition (Convergence of probability measures on $\mathbb{R}$)

A sequence $\nu_1, \nu_2, \ldots$ of probability measures on $\mathbb{R}$ converges to a probability measure ν on $\mathbb{R}$ if $\nu_n(I) \to \nu(I)$ whenever I is an interval satisfying $\nu(\{a, b\}) = 0$, where a and b are the endpoints of I.