[BAYES]

(a) Suppose that the conditional probability of an email (chosen uniformly at random from a large collection of emails) containing the phrase "additional income", given that the email is spam, is 14%. Suppose that the conditional probability of an email being spam, given that it contains the phrase "additional income", is 88%. Find the ratio of the probability that an email is spam to the probability that an email contains the phrase "additional income".

(b) We flip a weighted coin that has probability $\frac{3}{4}$ of turning up heads. If we get heads, we roll a six-sided die, and otherwise we roll an eight-sided die. Given that the die turns up 4, what is the conditional probability that the coin turned up heads?

(a) A: contains the phrase "additional Income"

B: email is spam

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = 14\%$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = 88\%$$

$$\frac{P(B)}{P(A)} = \frac{P(B|A)}{P(A|B)} = \frac{44}{7}$$

(b) A: coin turn up head. E: die turns up 4

$$P(A|E) = \frac{P(E|A)\cdot P(A)}{P(E)} = \frac{P(E|A)\cdot P(A)}{P(E|A)\cdot P(A) + P(E|A^c)\cdot P(A^c)}$$

$P(E|A) = \frac{1}{6}$ $P(E|A^c) = \frac{1}{8}$. $P(A) = \frac{3}{4}$. $P(A^c) = \frac{1}{4}$

$$P(A|E) = \frac{1}{6}\cdot\frac{3}{4} \Big/ \left(\frac{1}{6}\cdot\frac{3}{4} + \frac{1}{8}\cdot\frac{1}{4}\right) = 0.8$$

Definition
Given a probability space $(\Omega, \mathbb{P})$ and an event $E \subset \Omega$ whose probability is positive, the *conditional probability mass function* given E, written as $\omega \mapsto m(\omega|E)$ is defined by

$$m(\omega|E) = \begin{cases} \frac{m(\omega)}{P(E)} & \text{if } \omega \in E \\ 0 & \text{otherwise.} \end{cases}$$

The conditional probability measure given E is the measure associated to $\omega \mapsto m(\omega|E)$: for all events F, we have

$$\mathbb{P}(F|E) = \frac{\mathbb{P}(F \cap E)}{\mathbb{P}(E)}.$$

Bayes' Theorem

Bayes' theorem tells us how to update beliefs in light of new evidence. It relates the conditional probabilities $\mathbb{P}(A|E)$ and $\mathbb{P}(E|A)$:

$$\mathbb{P}(A|E) = \frac{\mathbb{P}(E|A)\mathbb{P}(A)}{\mathbb{P}(E)} = \frac{\mathbb{P}(E|A)\mathbb{P}(A)}{\mathbb{P}(E|A)\mathbb{P}(A) + \mathbb{P}(E|A^c)\mathbb{P}(A^c)}.$$

The last step follows from writing out $\mathbb{P}(E)$ as $\mathbb{P}(E \cap A) + \mathbb{P}(E \cap A^c)$.

The term $\mathbb{P}(A)$ is going to be called the **prior**. This is the probability of A with no additional information.

The term $\mathbb{P}(A \mid E)$ is going to be called the **posterior**. This is our updated probability *after* gaining the knowledge that E occured.

The term $\mathbb{P}(E \mid A)$ is going to be called the **likelihood**. Even though we're going to find out that E occurred, this factor allows us to say how likely E was *a priori*, given that the event A occurs.

Exercise

Two objects are submerged in a deep and murky body of water. The objects are chosen to be both positively buoyant with probability $\frac{1}{4}$, both are negatively buoyant with probability $\frac{1}{4}$, and with probability $\frac{1}{2}$ the objects have opposite buoyancy. The objects, if they float, rise in the water at different rates, but they are visually indistinguishable.

After the objects are released, an observer sees one of them emerge at the water's surface. What is the conditional probability, given the observed information, that the second object will emerge?

$$P(\text{both positively buoyant}) = \frac{1}{4}$$

$$P(\text{opposite buoyancy}) = \frac{1}{2}$$

$$P(\text{both positively buoyant} \mid \text{one is pos buoyant})$$

$$= \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{2}} = \frac{1}{3}$$

Solution. Let's use the given sample space:

$$\Omega = \{\text{both positive, opposite buoyancy, both negative}\}$$

The emergence of the object tells us precisely that the event

$$E = \{\text{both positive, opposite buoyancy}\}$$

occurs. The conditional probability of the event $\{\text{both positive}\}$ given E is

$$\frac{\mathbb{P}(\{\text{both positive}\} \cap E)}{\mathbb{P}(E)} = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{1}{2}} = \frac{1}{3}.$$