Exercise 1: Supplementary Document

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Navigation Equations for Local-Navigation-Frame (NED – North, East, Down frame)

Description:

This document provides a short overview of all equations necessary to implement the Strapdown navigation algorithm. The equations are presented in the order you will need them for the implementation of your algorithm. Certain simplifications are made and discussed for the purpose of Exercise 1 of the module "MGE-MSR-01 Sensors and State Estimation". In addition, at the beginning of the document there is an explanation about how to define initial values for the purposes of this specific exercise.

Reference:

- Groves, P. D. (2013). *Principles of GNSS, Inertial, and Multisensor Integrated Navigation Systems*, 2nd edition. Artech House, Boston, London, ISBN: 978-1-60807-005-3, Chapter 5.3.
- Groves, P.D. (2015) *Navigation Using Inertial Sensors*. IEEE A&E Systems Magazine, February 2015, Part II.
- MGE-MSR-01 – Sensors and State Estimation (Lectures L2 – L4)

Navigational Equations:

1. *Defining initial values*

Initial Attitude/Coordinate Transformation Matrix

The kinematic multi sensor system (MSS) used in this exercise is an aviation-grade navigational system (Groves 2015, Ch. 2.c) - high measurement quality. It is capable of estimating roll (ϕ) and pitch angles (θ) relative to the local gravity vector using the process known as *Accelerometer Leveling* (Groves 2015, Ch. V.a). Additionally, it is capable of estimating the yaw angle (ψ) with respect to the Earth using the process known as *Gyrocompassing* (Groves 2015, Ch. V.b). Therefore, this sensor can determine the absolute initial orientation of our MSS (our body or b frame) with respect to the local navigation frame (navigation or n frame). The orientation is described by a vector containing Euler angles – roll, pitch and yaw angles $[\phi \quad \theta \quad \psi]'$. Using these values it is possible to estimate the initial coordinate transformation matrix (CTM) $\boldsymbol{C}_b^n$, which is used for vector transformations from *b* to *n* frame.

$$\boldsymbol{C}_b^n(t_0) = \begin{bmatrix} \cos\theta\cos\psi & -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix} \quad (1)$$

Initial Velocity

In Exercise 1, at the start of the measuring process, our MSS was in a steady state. Therefore, initial velocities describing the velocity of the body frame b with respect to Earth centred Earth fixed reference frame e along the axes of the local navigation frame n can be described with the null vector.

$$\boldsymbol{v}_{eb}^n(t_0) = \begin{bmatrix} v_{eb,N}^n \\ v_{eb,E}^n \\ v_{eb,D}^n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (2)$$

Initial Position

The MSS used in Exercise 1 contains a GNSS receiver that observes the position of the b frame with respect to the e frame. This position can be expressed by latitude (L_b), longitude (λ_b) and ellipsoidal height (h_b). The values, which the sensor collected at the very beginning of the measurement process, can be used as initial values for the navigation equations: $[L_b\ \lambda_b\ h_b]'_{(t_0)}$.

2. Attitude update

When making the attitude update in the local navigation frame n we have to consider: the angular rates measured by the gyroscopes, contained in matrix ($\boldsymbol{\Omega}_{ib}^b$), the influence on the measurements of the gyroscopes due to the Earth's rotation rate, contained in matrix ($\boldsymbol{\Omega}_{ie}^n$), as well as the transport rate, contained in matrix ($\boldsymbol{\Omega}_{en}^n$). The attitude update can be done using the following equation.

$$\boldsymbol{C}_b^n(t_i) = \boldsymbol{C}_b^n(t_0)\left(\mathbf{I}_3 + \boldsymbol{\Omega}_{ib}^b\tau_i\right) - \left(\boldsymbol{\Omega}_{ie}^n(t_0) + \boldsymbol{\Omega}_{en}^n(t_0)\right)\boldsymbol{C}_b^n(t_0)\tau_i \tag{3}$$

where t_0 denotes the time stamp of the previous epoch (old values, before the update), t_i denotes the time stamp of the actual epoch (new values, after the update), τ_i denotes the length of the time interval between t_0 and t_i (new – old), and $\mathbf{I}_3$ is the identity matrix with dimensions of 3x3. The complete derivation of this equation can be found in Groves (2013), Ch. 5.3.1.

Measurements made with a gyroscope are recorded as angular rates of the b frame with respect to the inertial frame i along the axes of the b frame (i.e. along the axes of our measurement device). Angular rates are usually expressed in radians per second ($rad\ s^{-1}$), and they can be represented by the measurement vector $\boldsymbol{\omega}_{ib}^b$ ($\boldsymbol{\omega}_{ib}^b = [(\omega_{ib,x}^b \quad \omega_{ib,y}^b \quad \omega_{ib,z}^b]'$). For further calculations this vector has to be transformed into the skew symmetric matrix $\boldsymbol{\Omega}_{ib}^b$:

$$\boldsymbol{\Omega}_{ib}^b = \begin{bmatrix} 0 & -\omega_{ib,z}^b & \omega_{ib,y}^b \\ \omega_{ib,z}^b & 0 & -\omega_{ib,x}^b \\ -\omega_{ib,y}^b & \omega_{ib,x}^b & 0 \end{bmatrix}. \tag{4}$$

The matrix $\boldsymbol{\Omega}_{ie}^n$ is defined by the latitude of the body frame L_b and the Earth's rotational speed ω_{ie} as:

$$\boldsymbol{\Omega}_{ie}^n = \omega_{ie}\begin{bmatrix} 0 & \sin L_b & 0 \\ -\sin L_b & 0 & -\cos L_b \\ 0 & \cos L_b & 0 \end{bmatrix}. \tag{5}$$

The matrix $\boldsymbol{\Omega}_{en}^n$ is defined as a skew symmetric matrix of the angular rates of the navigation frame n with respect to the Earth centred Earth fixed frame e, along the axes of the n frame as:

$$\boldsymbol{\Omega}_{en}^n = \begin{bmatrix} 0 & -\omega_{en,z}^n & \omega_{en,y}^n \\ \omega_{en,z}^n & 0 & -\omega_{en,x}^n \\ -\omega_{en,y}^n & \omega_{en,x}^n & 0 \end{bmatrix}, \tag{6}$$

The vector $\boldsymbol{\omega}_{en}^n$ is defined by the current velocity of the body frame $\boldsymbol{v}_{eb}^n$, the latitude L_b and the height h_b of the body frame, as well as by the curvature radii R_N and R_E:

$$\boldsymbol{\omega}_{en}^n = \begin{bmatrix} \omega_{en,x}^n \\ \omega_{en,y}^n \\ \omega_{en,z}^n \end{bmatrix} = \begin{bmatrix} v_{eb,E}^n/(R_E(L_b) + h_b) \\ -v_{eb,N}^n/(R_N(L_b) + h_b) \\ -v_{eb,E}^n tan\, L_b/(R_E(L_b) + h_b) \end{bmatrix}. \tag{7}$$

Finally, the curvature radii R_N and R_E are a function of the latitude L_b. These radii are defined as:

$$R_E = \frac{a}{\sqrt{1 - e^2 \sin^2 L_b}} \tag{8}$$

$$R_N = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 L_b)^{3/2}} \tag{9}$$

where a is the Earth's ellipsoid radius at the equator, and e is the Earth's ellipsoid eccentricity.

3. Specific force frame transformation

The accelerometers measure the specific force $\boldsymbol{f}_{ib}^b$ ($\boldsymbol{f}_{ib}^b = [f_{ib,x}^b \quad f_{ib,y}^b \quad f_{ib,z}^b]'$) of the b frame with respect to the inertial frame i along the axes of t
he b frame. This specific force needs to be transformed into the vector $\boldsymbol{f}_{ib}^n$ describing the specific force in the local navigation frame n. In the case of accelerometer and gyroscope measurements we expect high measurement rates, and therefore we can presume a linear change in the body frame state. Hence, the required transformation can simply be done using the average of the coordinate transformation matrix $\boldsymbol{C}_b^n(t)$ before and after the update:

$$\boldsymbol{f}_{ib}^n = \frac{1}{2}(\boldsymbol{C}_b^n(t_0) + \boldsymbol{C}_b^n(t_i))\boldsymbol{f}_{ib}^b. \tag{10}$$

4. Velocity Update

For the velocity update we use the integrated values of the accelerometer measurements. However, this process is not straightforward because we need to account for several factors influencing the accelerometer readings such as: gravity (gravitational + centrifugal acceleration), Coriolis acceleration and Euler acceleration. After rearranging the equation (Groves 2013, Ch. 5.3.2) the final relation for the velocity update in the local navigation frame is given as:

$$\boldsymbol{v}_{eb}^n(t_i) = \boldsymbol{v}_{eb}^n(t_0) + \left[\boldsymbol{f}_{ib}^n + \boldsymbol{g}_b^n\left(L_b(t_0),\, h_b(t_0)\right) - (\,\boldsymbol{\Omega}_{en}^n(t_0) + 2\,\boldsymbol{\Omega}_{ie}^n(t_0))\boldsymbol{v}_{eb}^n(t_0)\right]\tau_i \tag{11}$$

where the only new variable is the vector $\boldsymbol{g}_b^n$,which expresses the acceleration due to gravity in the origin of the body frame b along the axes of the local navigation frame n. The vector $\boldsymbol{g}_b^n$ is a function of the longitude L_b, ellipsoid height h_b and the Earth's ellipsoid parameters (constants):

$$\boldsymbol{g}_b^n = \begin{bmatrix} g_{b,N}^n \\ g_{b,E}^n \\ g_{b,D}^n \end{bmatrix}, \tag{12}$$

$$g_{b,E}^{n} = 0, \tag{13}$$

$$g_{b,N}^{n} = -8.08 \times 10^{-9} h_b \sin 2\, L_b \tag{14}$$

$$g_{b,D}^{n} = g_0(\, L_b) \left\{ 1 - \frac{2}{a} \left[1 + f(1 - 2\sin^2 L_b) + \frac{{\omega_{ie}}^2 a^2 b}{\mu} \right] h_b + \frac{3}{a^2}\, {h_b}^2 \right\} \tag{15}$$

$$g_0 = 9.7803253359 \frac{(1 + 0.001931853\, \sin^2 L_b)}{\sqrt{1 - e^2 \sin^2 L_b}} \tag{15}$$

More about these equations can be found in Groves (2013), Ch. 2.3.

REMARK: There is an alternative way for computing the value $g_{b,D}^{n}$. Instead of using the mathematical formulation based on the known longitude L_b and ellipsoid height h_b, it is possible to use accelerometer measurements at steady state while the vehicle is still not moving (e.g. at the beginning of the trajectory). In this period the only acceleration that occurs is due to gravity. It can be calculated as:

$$g_{b,D}^{n} = \sqrt{{\omega_{ib,x}^{b}}^2 + {\omega_{ib,y}^{b}}^2 {\omega_{ib,z}^{b}}^2} \tag{16}$$

For getting meaningful results it is necessary to average several hundreds of observations to reduce the noise level.

5. Position Update

Finally, the position update is based on the estimated velocity values and MSS position. If a high measurement frequency and a linear change of velocity are presumed, the integral functions can be simplified to the following three equations for latitude, longitude and ellipsoid height:

$$h_b(t_i) = h_b(t_0) - \frac{\tau_i}{2} (v_{eb,D}^{n}(t_0) + v_{eb,D}^{n}(t_i)) \tag{17}$$

$$L_b(t_i) = L_b(t_0) + \frac{\tau_i}{2} \left(\frac{v_{eb,N}^{n}(t_0)}{R_N(L_b(t_0)) + h_b(t_0)} + \frac{v_{eb,N}^{n}(t_i)}{R_N(L_b(t_0)) + h_b(t_i)} \right) \tag{18}$$

$$\lambda_b(t_i) = \lambda_b(t_0) + \frac{\tau_i}{2} \left(\frac{v_{eb,E}^{n}(t_0)}{(R_E(L_b(t_0)) + h_b(t_0)) cos L_b(t_0)} + \frac{v_{eb,E}^{n}(t_i)}{(R_E(L_b(t_i)) + h_b(t_i)) cos L_b(t_i)} \right) \tag{19}$$

All variables used in these equations have already been described in the previous steps. More about the equations can be found in Groves (2013) Ch. 5.3.4. It is important to note that these functions have to be calculated in the presented order (first update of height, than latitude, then longitude)!

6. Simplification and summary

The presented set of equations (1-17) takes into account all relevant factors influencing accelerometer and gyroscope measurements. To compute the solution of Exercise 1 you should use the simplified set of equations below. The equations should be simplified by ignoring the terms related to the transport rate ($\boldsymbol{\Omega}_{en}^{n} = \mathbf{0}$) and the Earth's gravity influence in the north direction $(g_{b,N}^{n} = 0)$, which is generally small. This results into the following set of equations:

$$\boldsymbol{C}_{b}^{n}(t_i) = \boldsymbol{C}_{b}^{n}(t_0)\left(\mathbf{I}_3 + \boldsymbol{\Omega}_{ib}^{b}\tau_i\right) - \boldsymbol{\Omega}_{ie}^{n}(t_0)\boldsymbol{C}_{b}^{n}(t_0)\tau_i \tag{3*}$$

$$\boldsymbol{\Omega}_{ib}^{b} = \begin{bmatrix} 0 & -\omega_{ib,z}^{b} & \omega_{ib,y}^{b} \\ \omega_{ib,z}^{b} & 0 & -\omega_{ib,x}^{b} \\ -\omega_{ib,y}^{b} & \omega_{ib,x}^{b} & 0 \end{bmatrix} \tag{4}$$

$$\boldsymbol{\Omega}_{ie}^{n} = \omega_{ie}\begin{bmatrix} 0 & \sin L_b & 0 \\ -\sin L_b & 0 & -\cos L_b \\ 0 & \cos L_b & 0 \end{bmatrix} \tag{5}$$

$$\boldsymbol{f}_{ib}^{n} = \frac{1}{2}\left(\boldsymbol{C}_{b}^{n}(t_0) + \boldsymbol{C}_{b}^{n}(t_i)\right)\boldsymbol{f}_{ib}^{b} \tag{10}$$

$$\boldsymbol{v}_{eb}^{n}(t_i) = \boldsymbol{v}_{eb}^{n}(t_0) + \left[\boldsymbol{f}_{ib}^{n} + \boldsymbol{g}_{b}^{n}\left(L_b(t_0),\, h_b(t_0)\right) - 2\,\boldsymbol{\Omega}_{ie}^{n}(t_0)\boldsymbol{v}_{eb}^{n}(t_0)\right]\tau_i \tag{11*}$$

$$g_{b,D}^{n} = g_0(L_b)\left\{1 - \frac{2}{a}\left[1 + f\left(1 - 2\sin^2 L_b\right)\right] h_b + \frac{3}{a^2}\, {h_b}^2\right\}, \tag{14*}$$

$$g_0 = 9.7803253359\,\frac{\left(1 + 0.001931853\,\sin^2 L_b\right)}{\sqrt{1 - e^2 \sin^2 L_b}} \tag{15}$$

$$h_b(t_i) = h_b(t_0) - \frac{\tau_i}{2}\left(v_{eb,D}^{n}(t_0) + v_{eb,D}^{n}(t_i)\right) \tag{17}$$

$$L_b(t_i) = L_b(t_0) + \frac{\tau_i}{2}\left(\frac{v_{eb,N}^{n}(t_0)}{R_N(L_b(t_0)) + h_b(t_0)} + \frac{v_{eb,N}^{n}(t_i)}{R_N(L_b(t_0)) + h_b(t_i)}\right) \tag{18}$$

$$\lambda_b(t_i) = \lambda_b(t_0) + \frac{\tau_i}{2}\left(\frac{v_{eb,E}^{n}(t_0)}{\left(R_E(L_b(t_0)) + h_b(t_0)\right)\cos L_b(t_0)} + \frac{v_{eb,E}^{n}(t_i)}{\left(R_E(L_b(t_i)) + h_b(t_i)\right)\cos L_b(t_i)}\right) \tag{19}$$

** - simplified functions*

7. Important considerations and notes!

- For all input values in the algorithm the equations always use meters, seconds, radians and their derivatives. Transform them later into the desired format.
- Estimating the position using the Strapdown navigational equations is an iterative process and therefore after each update your newly estimated values will become initial values for the next iteration. Every update cycle ends with:

  $\boldsymbol{C}_{b}^{n}(t_0) = \boldsymbol{C}_{b}^{n}(t_i)$, $\boldsymbol{v}_{eb}^{n}(t_0) = \boldsymbol{v}_{eb}^{n}(t_i)$, $L_b(t_0) = L_b(t_i)$, $\lambda_b(t_0) = \lambda_b(t_i)$, & $h_b(t_0) = h_b(t_i)$.
- The navigation solution acquired using the equations described in Chapter 6 of this supplementary document should be compared with the solution using precision navigation equations described in reference literature (Groves 2015, Ch. 4.c). The precision navigation solution will be provided as a separate ".txt" file and the Python script will be provided for the comparison.
- In addition, the estimated trajectory should be graphically compared with the reference trajectory defined by the GNSS measurements of our kinematic multi sensor system.

- An explanation of how to extract the measurement data, initial values and the GNSS reference solution from the “raw” data structure is provided in a separate document. This will also be explained during the preliminary discussion of the exercise.
- To obtain the highest possible precision the presumption of linear change of the MSS states should be rejected, and more complex integral functions should be used (Groves 2013.,Ch. 5.4).
- It is worth noting that after the simplifications made in Chapter 5 of this supplementary document the navigational equations in the local navigation frame represent the most simple case of the navigation functions in the inertial navigation frame (Groves 2013, Ch. 5.1). The only mayor difference is the representation of the body frame position.
- Example Matlab code for calculating the acceleration due to gravity based on accelerometer observations (Equation 16):

```
g_emp = median(sqrt(allmsm_f_ib_b(1:obs_nr,1).^2 +
allmsm_f_ib_b(1:obs_nr,2).^2 + allmsm_f_ib_b(1:obs_nr,3).^2));
```

8. *Constants*

- $a = 6378137.0\ m$ (Earth’s ellipsoid radius at equator)
- $b = 6\,356\,752.3142\ \mathrm{m}$ (Earth’s ellipsoid radius at poles)
- $e = 0.0818191908426$ (Earth’s ellipsoid eccentricity)
- $\omega_{ie} = \ 7.2921150\ \times 10^{-5}\ \mathrm{rad}\ s^{-1}$ (Earth’s rotational speed)
- $\mu = 3.986004418 \times 10^{14}\ m^3 s^{-2}$ (Geocentric gravitational constant)
- $f = 1/298.257223563$ (Earth’s ellipsoid flattening)