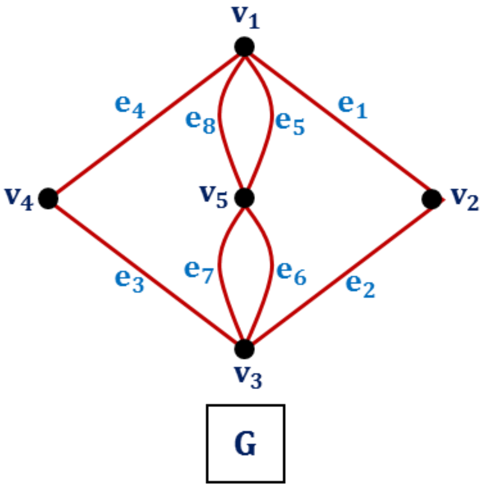
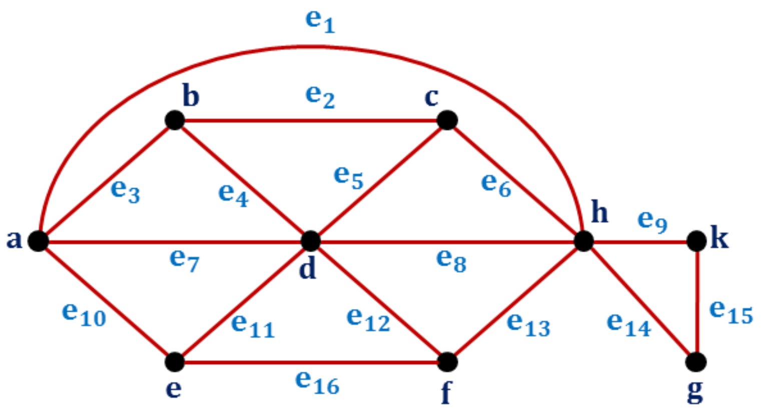
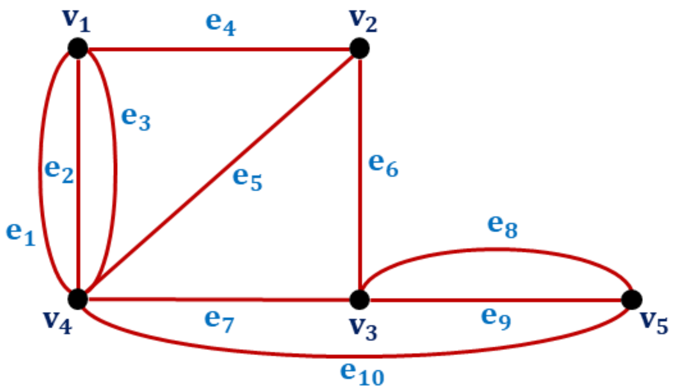
Index

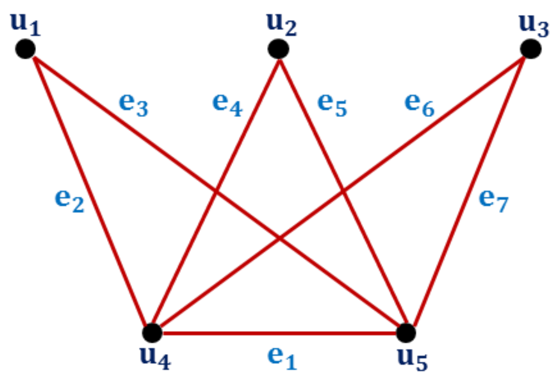
Unit – 5.1 ⇝ Graph Theory – II

Method 1 ⇝ Euler Paths and Circuits

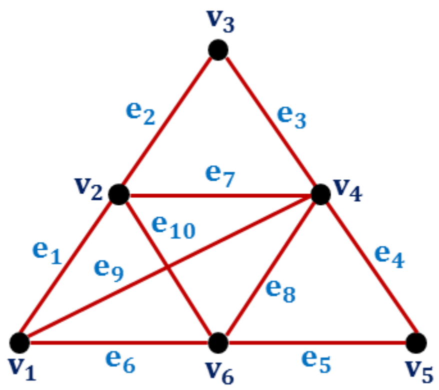
A	1	Define the following terms: Euler Circuit, Euler Path, Eulerian Graph **Answer: Refer Theory**
A	2	Determine whether the given graph has an Euler circuit. Give such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and give such a path if one exists.  **Hint: Euler circuit of G:** $v_4e_4v_1e_1v_2e_2v_3e_7v_5e_8v_1e_5v_5e_6v_3e_3v_4$
A	3	For which values of n do the graphs K_n and C_n have an Euler path but no Euler circuit?

A	4	Determine whether the given graph has an Euler circuit. Give such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and give such a path if one exists.  **H** **Hint: H has neither Euler circuit nor Euler path**
A	5	Determine whether the given graph has an Euler circuit. Give such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and give such a path if one exists.  **G** **Hint: Euler path of G:** $v_5e_{10}v_4e_1v_1e_2v_4e_3v_1e_4v_2e_6v_3e_8v_5e_9v_3e_7v_4e_5v_2$

<table>
<tr>
<td>B</td>
<td>6</td>
<td>
Determine whether the following graphs has an Euler circuit. Give such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and give such a path if one exists.



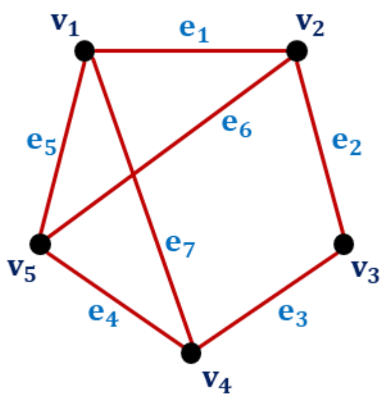
H



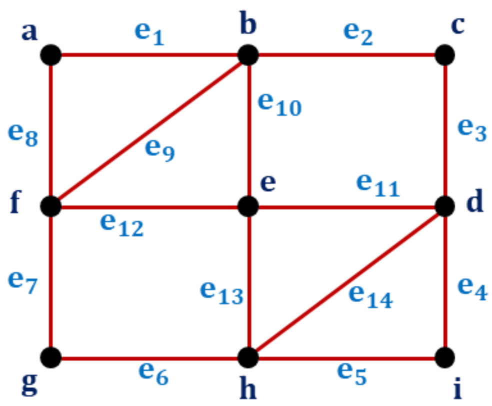
G

Hint: Euler circuit of H: $u_4e_2u_1e_3u_5e_7u_3e_6u_4e_4u_2e_5u_5e_1u_4$,

Euler path of G: $v_4e_4v_5e_5v_6e_6\ v_1e_1v_2e_2v_3e_3v_4e_7v_2e_{10}v_6e_8v_4e_9v_1$
</td>
</tr>
<tr>
<td>B</td>
<td>7</td>
<td>
Determine whether the given graph has an Euler circuit. Give such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and give such a path if one exists.



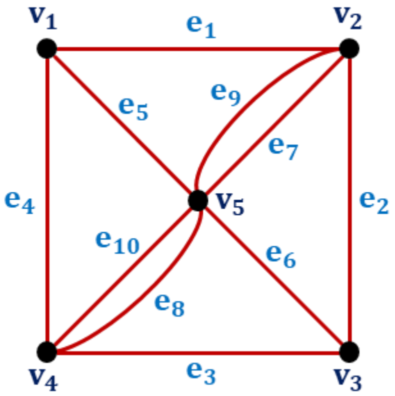
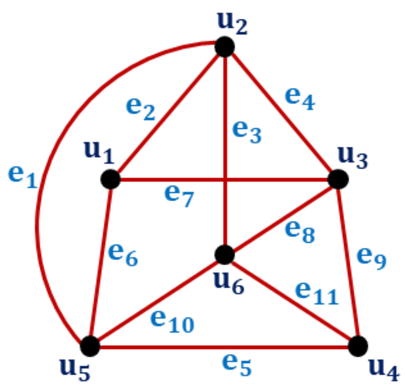
G

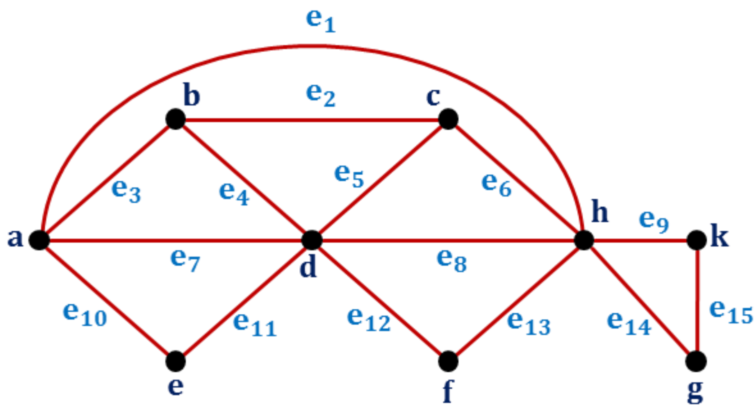


H

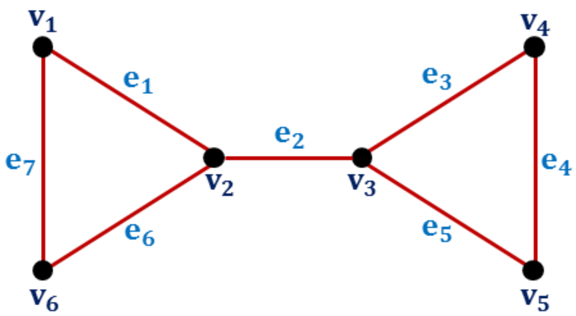
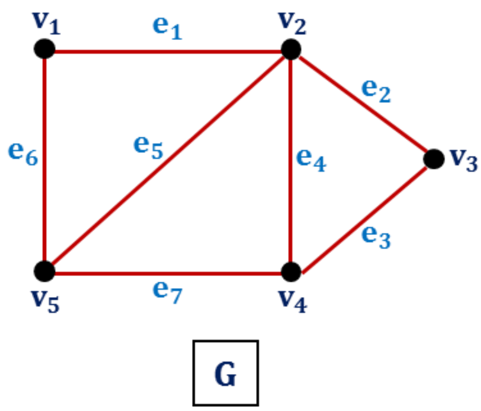
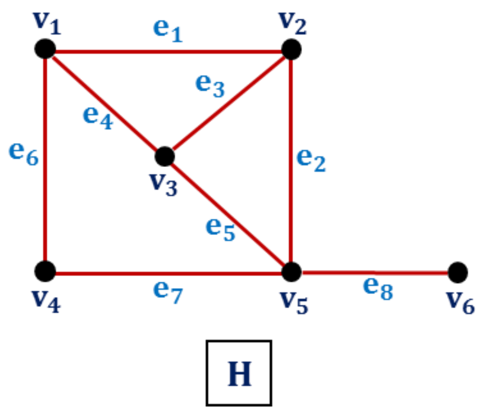
Hint: Euler circuit of H:

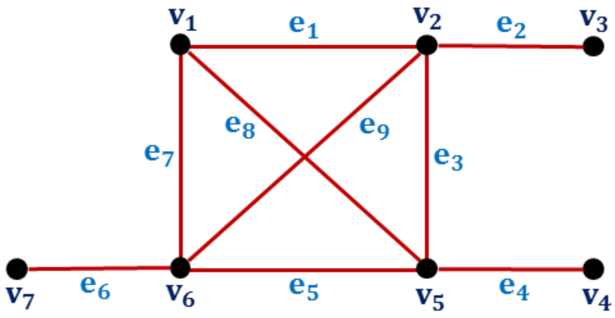
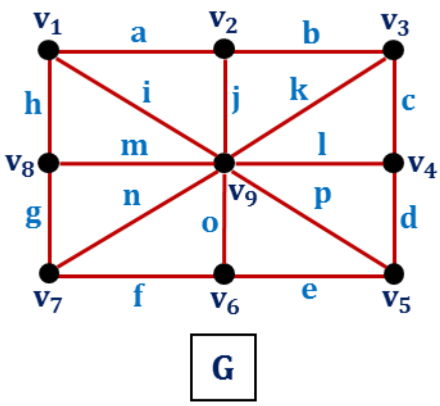
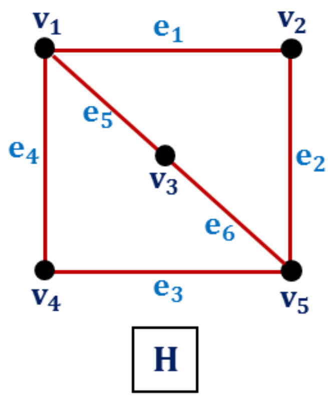
$ae_1be_2ce_3de_4ie_5he_{14}de_{11}ee_{10}be_9fe_{12}ee_{13}he_6ge_7fe_8a$
</td>
</tr>
</table>

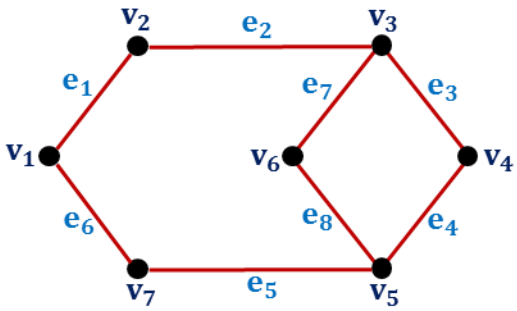
C	8	Determine whether the given graph has an Euler circuit. Give such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and give such a path if one exists.  G_1  G_2 **Hint**: **Euler path of G_1**: $v_1e_1v_2e_2v_3e_6v_5e_7v_2e_9v_5e_8v_4e_{10}v_5e_5v_1e_4v_4e_3v_3$ **Euler path of G_2**: $u_4e_5u_5e_{10}u_6e_{11}u_4e_9u_3e_8u_6e_3u_2e_4u_3e_7u_1e_2u_2e_1u_5e_6u_1$

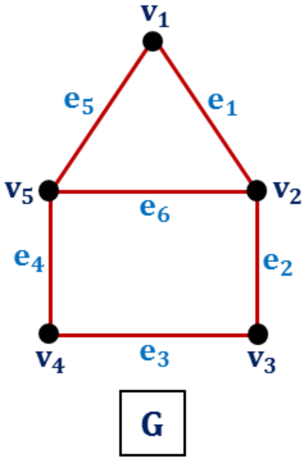
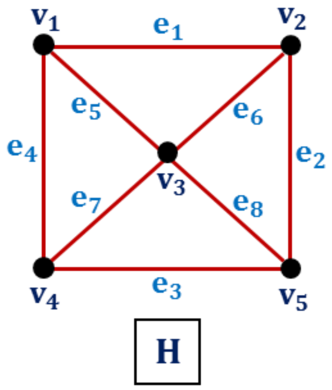
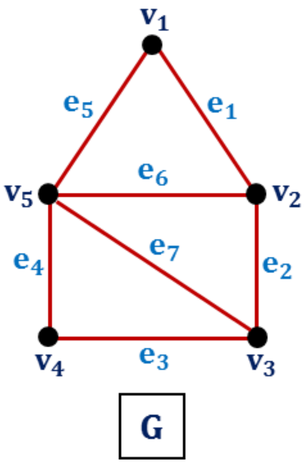
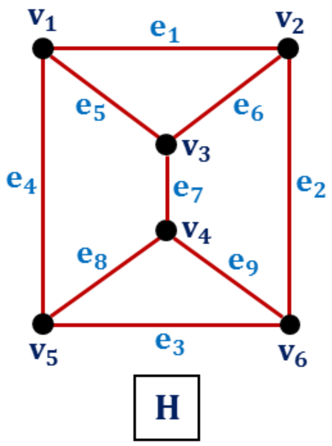
C	9	Determine whether the given graph has an Euler circuit. Give such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and give such a path if one exists.  H **Hint**: **Euler path**: $be_3ae_1he_{13}fe_{12}de_{11}ee_{10}ae_7de_8he_9ke_{15}ge_{14}he_6ce_5de_4be_2c$

Method 2 ⇝ Hamiltonian Paths and Circuits

A	1	Define the following terms: Hamiltonian Path, Hamiltonian Circuit, Hamiltonian Graph **Answer: Refer Theory**
A	2	Determine whether the given graph has a Hamilton circuit. If it does, find such a circuit.  **Answer: The given graph does not have a Hamilton circuit.**
A	3	Determine whether the following graphs has a Hamilton circuit. If it does, find such a circuit.  **Answer: Graph G has a Hamiltonian circuit, $v_1e_1v_2e_2v_3e_3v_4e_7v_5e_6v_1$** **Graph H doesn′t have a Hamiltonian circuit.**

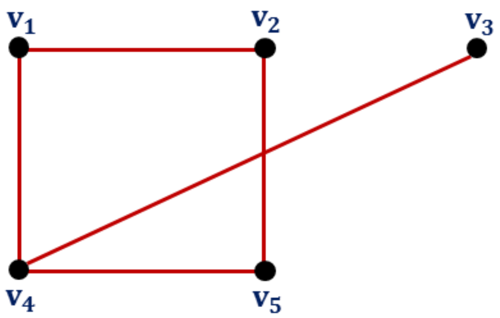
A	4	Determine whether the following graph has a Hamilton circuit. If it does, find such a circuit.  **Answer: The given graph does not have a Hamiltonian circuit.**
A	5	For what values of n does the complete graph K_n with n vertices contains a Hamiltonian circuit?
B	6	Determine whether the following graphs has a Hamilton circuit. If it does, find such a circuit.   **Answer: The graph G has a Hamiltonian circuit,** $v_7 g v_8 h v_1 a v_2 j v_9 k v_3 c v_4 d v_5 e v_6 f v_7$, **The graph H oes not have a Hamiltonian circuit.**

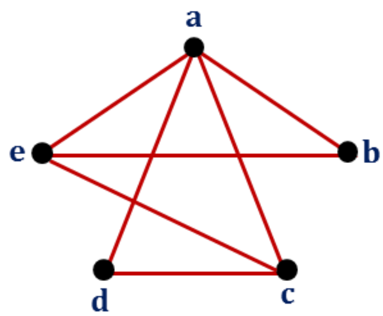
B	7	Determine whether the following graphs has a Hamilton circuit. If it does, find such a circuit. Also, Does the graph have a Hamiltonian path? If so, find such a path.  **Answer: The given does not have a Hamilton circuit but it has a Hamiltonian path, $v_6e_7v_3e_3v_4e_4v_5e_5v_7e_6v_1e_1v_2$**
B	8	Give two examples of a graph that has an Eulerian Circuit which is also a Hamiltonian Circuit.
B	9	Give two examples of a graph that has an Eulerian Circuit and a Hamiltonian Circuit, which are distinct.
B	10	Give two examples of a graph which has an Eulerian Circuit but not a Hamiltonian Circuit.
B	11	Give two examples of a graph which has a Hamiltonian Circuit but not an Eulerian Circuit.
B	12	Give two examples of a graph that has neither a Hamiltonian Circuit nor an Eulerian Circuit.
B	13	Give two examples of a graph that has Hamiltonian path but not a Hamiltonian circuit.
B	14	For what values of m and n does the graph $K_{m,n}$ contains a Hamiltonian circuit?

C	15	For each of these graphs, determine (i) whether Dirac's theorem can be used to show that the graph has a Hamiltonian circuit, (ii) whether Ore's theorem can be used to show that the graph has a Hamiltonian circuit, and (iii) whether the graph has a Hamiltonian circuit.   **Answer: For G ⇝ No, No, Yes, For H ⇝ Yes, Yes, Yes**
C	16	For each of these graphs, determine (i) whether Dirac's theorem can be used to show that the graph has a Hamiltonian circuit, (ii) whether Ore's theorem can be used to show that the graph has a Hamiltonian circuit, and (iii) whether the graph has a Hamiltonian circuit.   **Answer: For G ⇝ No, No, Yes, For H ⇝ Yes, No, Yes**

Method 3 ⇝ Introduction to Planar Graph

<table>
<tr><td>A</td><td>1</td><td>Define the following terms :
Planar Graph, Region of a Graph, Finite Region, Infinite Region

Answer: Refer Theory</td></tr>
<tr><td>A</td><td>2</td><td>Show that each is planar graph by redrawing it so that no edges cross. Also, verify the Euler's formula.


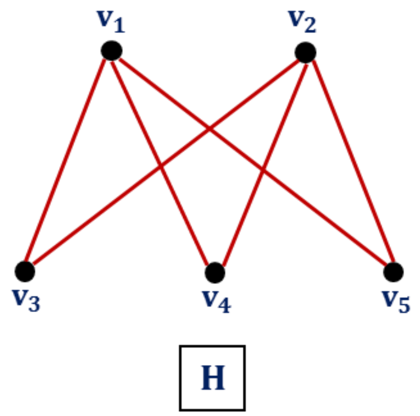
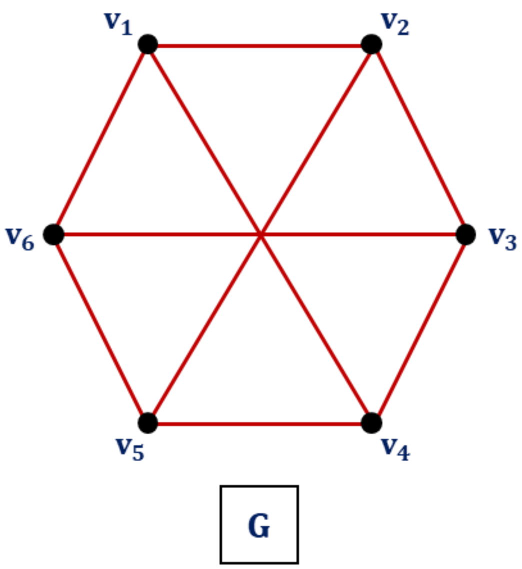
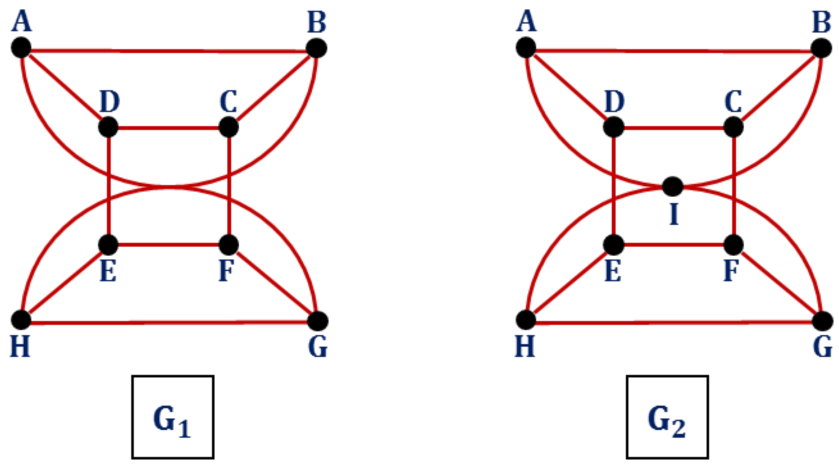
G


H</td></tr>
<tr><td>A</td><td>3</td><td>Suppose that a connected planar graph has 30 edges. If a planar representation of this graph divides the plane into 20 regions, how many vertices does this graph have?

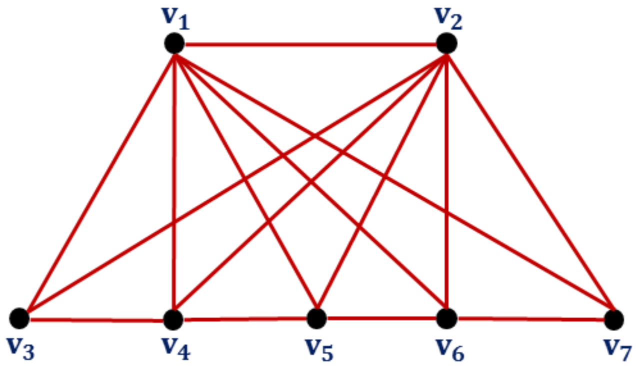
Answer: 12</td></tr>
<tr><td>B</td><td>4</td><td>Suppose that a connected planar graph has six vertices, each of degree four. Into how many regions is the plane divided by a planar representation of this graph?

Answer: 8</td></tr>
<tr><td>B</td><td>5</td><td>Suppose that a connected planar graph has eight vertices, each of degree three. Into how many regions is the plane divided by a planar representation of this graph?

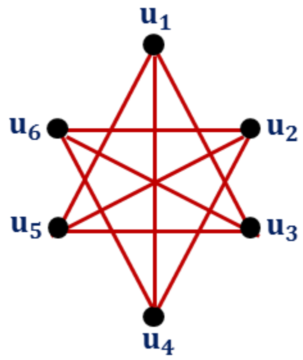
Answer: 6</td></tr>
</table>

<table>
<tr><td>B</td><td>6</td><td>Show that the following graph is planar by redrawing it so that no edges cross. Also, verify the Euler's formula.
</td></tr>
<tr><td>B</td><td>7</td><td>Show that the following graph G is non – planar.
</td></tr>
<tr><td>C</td><td>8</td><td>Draw the given planar graphs without any crossings.
</td></tr>
</table>

C	9	Show that the following graph is planar by redrawing it so that no edges cross. Also, verify the Euler's formula.

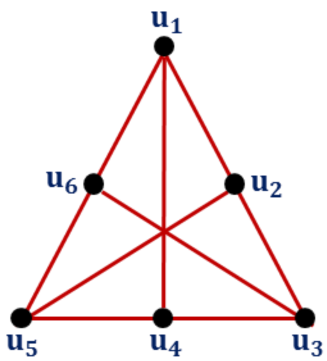


G

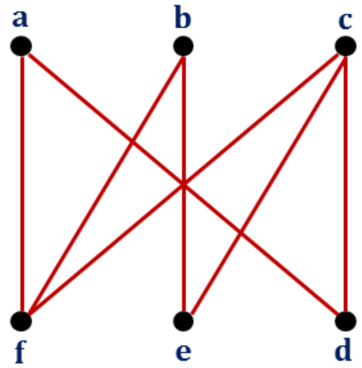


H

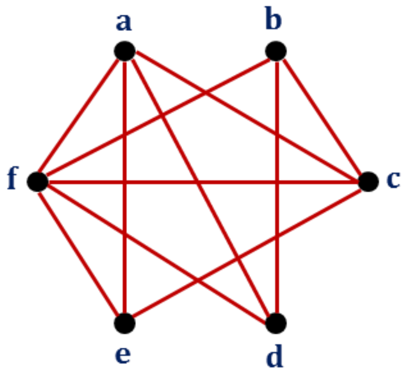
C	10	Determine whether the given graph is planar. If so, draw it so that no edges cross. Also, verify the Euler's formula.



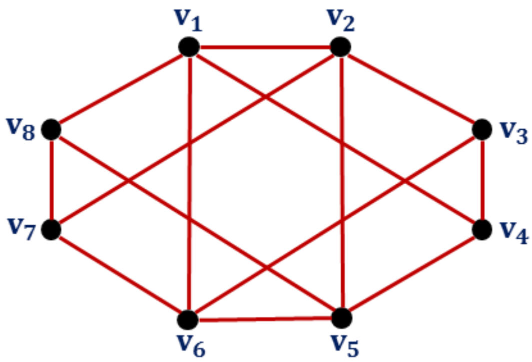
G_1



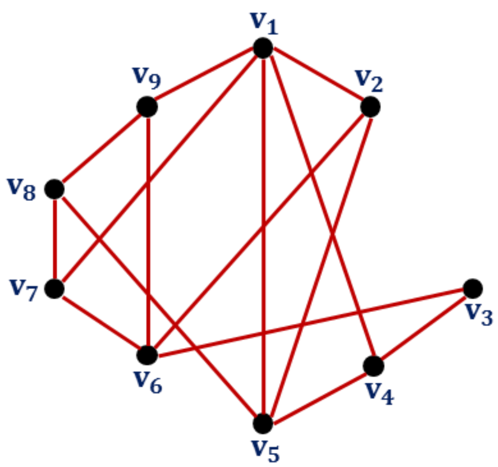
G_2



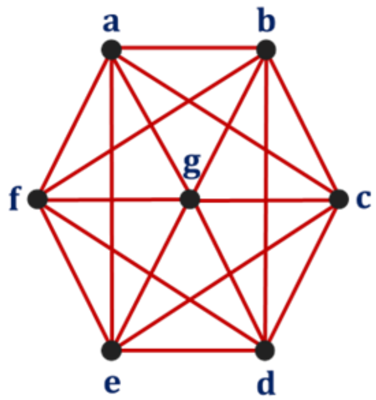
G_3



G_4

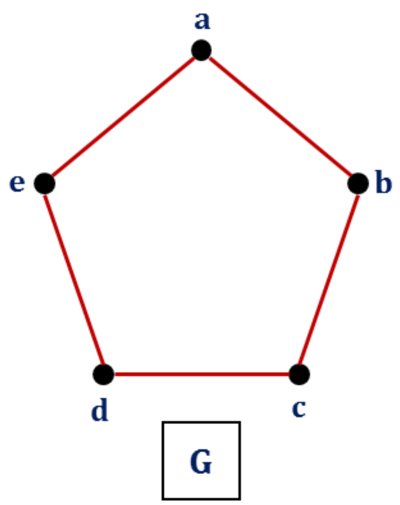
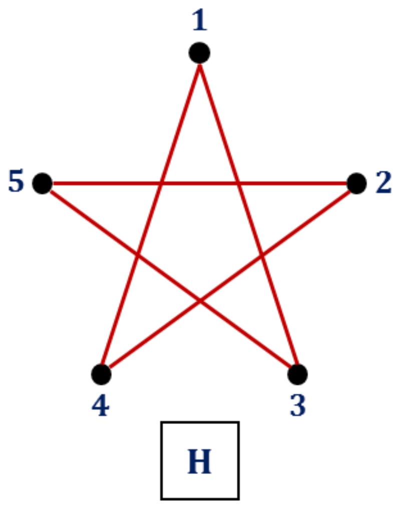
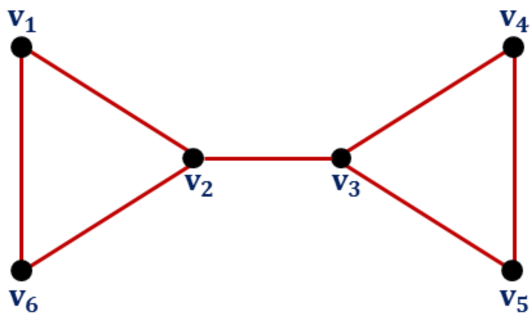


G_5



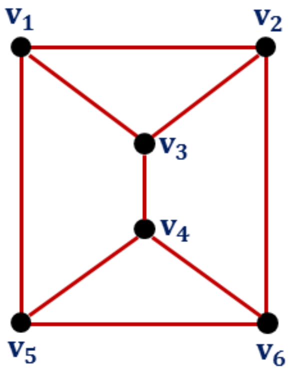
G_6

Method 4 ⇝ Introduction to Graph Coloring

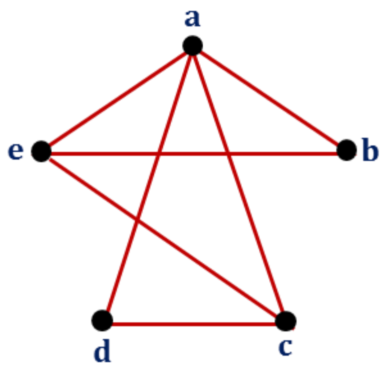
A	1	Define the following terms with examples: Vertex Coloring, Chromatic Number of a Graph **Answer: Refer Theory**
A	2	Determine the chromatic number of the following graphs.  **Answer:** $\chi(G) = 3, \qquad \chi(H) = 3$
A	3	Determine the chromatic number of the following graph.  **Answer: 3**

<table>
<tr>
<td>A</td>
<td>4</td>
<td>

Determine the chromatic number of the following graphs.



G

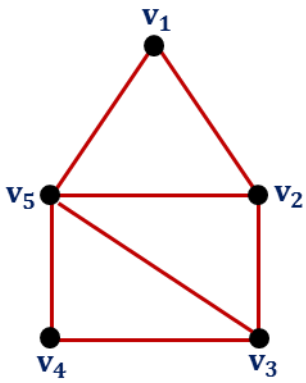


H

Answer: $\chi(G) = 3$, $\chi(H) = 3$

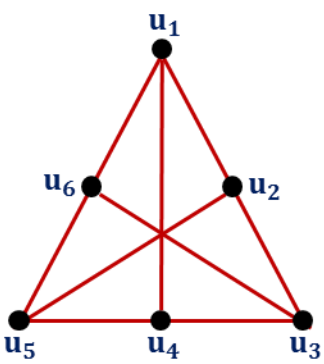
</td>
</tr>
<tr>
<td>A</td>
<td>5</td>
<td>

Show that the following graph is 3 – chromatic.

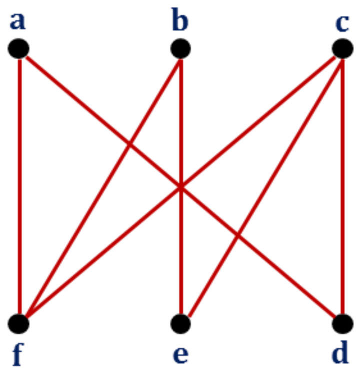


</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>

Find the chromatic number of the following graphs.



G_1



G_2

Answer: $\chi(G_1) = 2$, $\chi(G_2) = 2$

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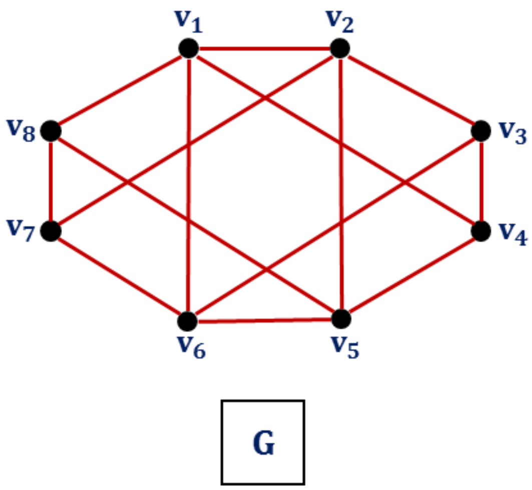
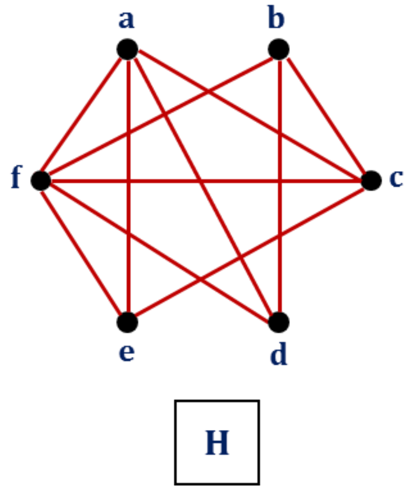
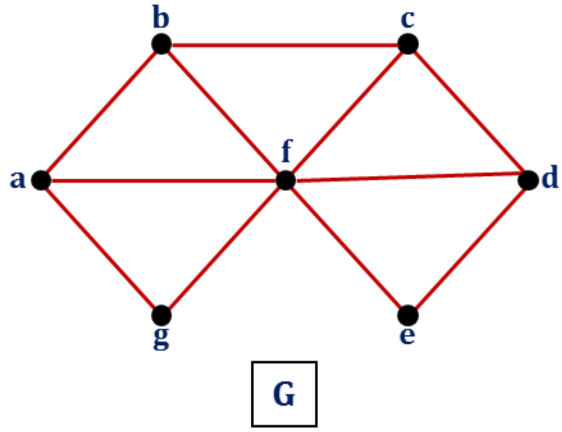
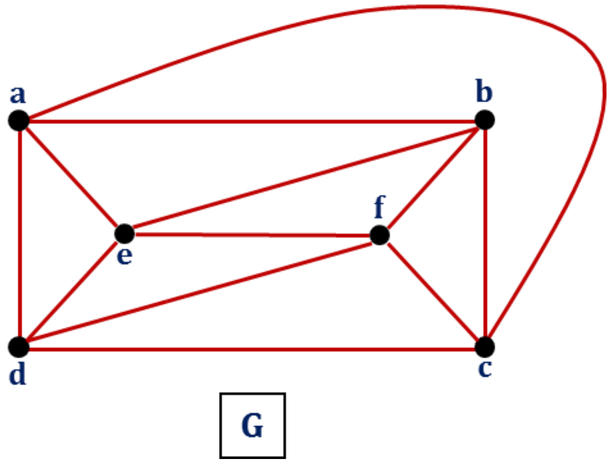
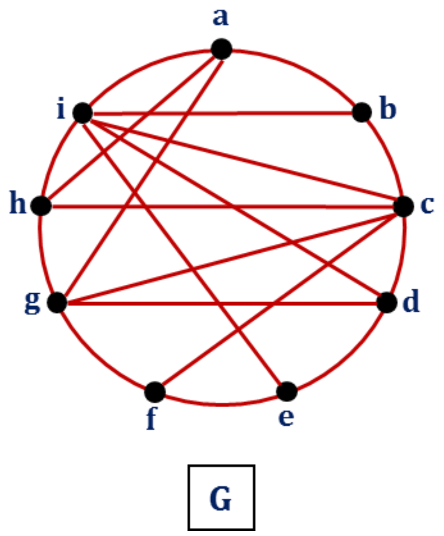
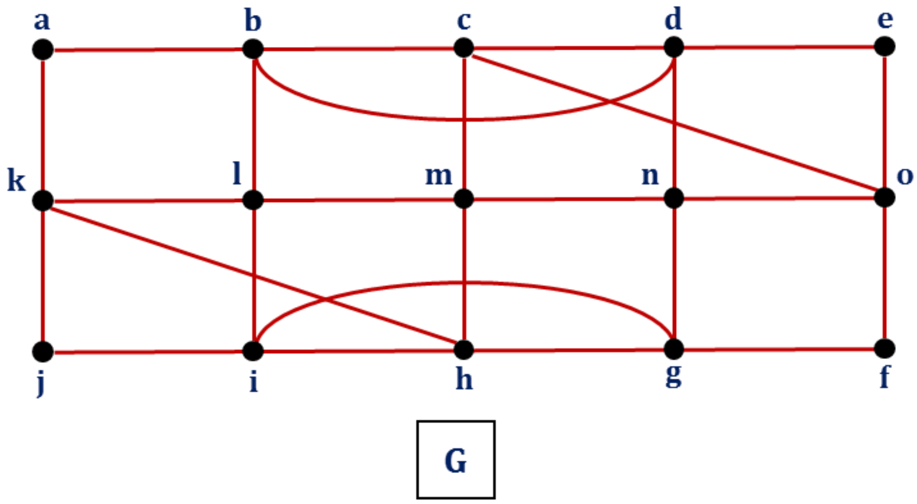
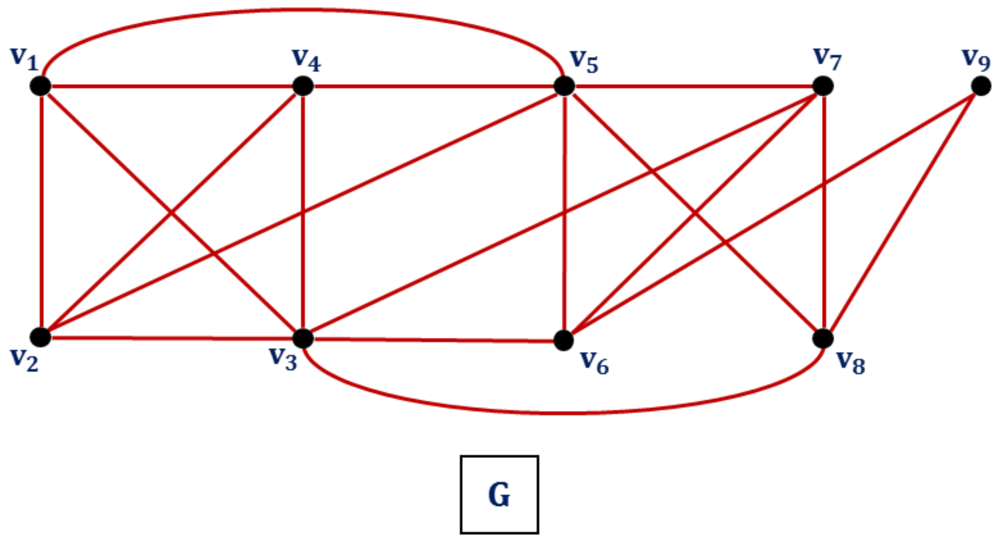
</table>

B	7	Find the chromatic number of the following graphs.   **Answer**: $\chi(G) = 2, \quad \chi(H) = 4$
B	8	Find the chromatic number of the following graph.  **Answer**: $\chi(G) = 3$

C	9	Find the chromatic number of the following graph.  **Answer**: $\chi(G) = 3$
C	10	Find the chromatic number of the following graph.  **Answer**: $\chi(G) = 3$

C	11	Find the chromatic number of the following graph.  **Answer**: $\boldsymbol{\chi(G) = 3}$
C	12	Find the chromatic number of the following graph.  **Answer**: $\boldsymbol{\chi(G) = 4}$

Unit 5.2 Group Theory

Method 1 ⇝ Binary Operation

Example of Method-1: Binary Operation

A	1	On the set $\mathbb{Z}^+$, check whether $*$ is binary operation or not. (1) $m * n = m - n$ (2) $m * n = \frac{m}{n}$ **Answer:** (1) **No,** (2) **No**
A	2	On the set $\mathbb{Q}$, check whether the binary operation $*$ is associative or not. (1) $m * n = mn + 1$ (2) $m * n = \frac{m}{n}$ (3) $m * n = 10$ **Answer:** (1) **No,** (2) **No,** (3) **Yes**
A	3	On the set $\mathbb{Q}$, check whether the binary operation $*$ is commutative or not. (1) $m * n = m^n$ (2) $m * n = \frac{mn}{3}$ (3) $m * n = 10$ **Answer:** (1) **No,** (2) **Yes,** (3) **Yes**
B	4	Construct composition table for set $S = \{1, -1, i, -i\}$ where $i^2 = -1$ with binary operation multiplication $\times$. **Answer:**

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

B	5	Write down the composition table for $(\mathbb{Z}_7,\ +_7)$.

Answer:

$+_7$	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

B	6	Write down the composition table for $(\mathbb{Z}_6,\ +_6)$.

Answer:

$+_6$	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

B	7	Write down the composition table for $(\mathbb{Z}_7^*,\ \times_7)$.

Answer:

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

B	8	Write down the composition table for $(\mathbb{Z}_6^*, \times_6)$.

Answer:

$\times_6$	1	2	3	4	5
1	**1**	**2**	**3**	**4**	**5**
2	**2**	**4**	**0**	**2**	**4**
3	**3**	**0**	**3**	**0**	**3**
4	**4**	**2**	**0**	**4**	**2**
5	**5**	**4**	**3**	**2**	**1**

Method 2 ⇝ Group

Example of Method-2: Group

A	1	Show that $(\mathbb{Z}, +)$ is group.
A	2	Show that $(\mathbb{R} - \{\,0\,\},\ \times)$ is group.
A	3	Show that $(\mathbb{Q} - \{\,0\,\},\ \times)$ is group.
B	4	Show that the set of square root of unity forms a group under multiplication.
C	5	Show that the set of fourth root of unity forms a group under multiplication.
B	6	Prove that the set $\mathrm{G} = \{\,3^{\mathrm{m}} \cdot 3^{\mathrm{n}}\ ;\quad \mathrm{m}, \mathrm{n} \in \mathbb{Z}\,\}$ is a group under multiplication.
A	7	Check whether $(\{\,15,\ \ 25,\ \ 35\,\},\ \times_{40})$ is a group or not? **Answer: Not group, as closure property not satisfied**

Method 3 ⇝ Subgroup

Example of Method-3: Subgroup

A	1	Show that $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$.
B	2	Show that $\mathrm{H} = n\mathbb{Z} = \{ nx;\ x \in \mathbb{Z} \}$ is a subgroup of $(\mathbb{Z}, +)$.
B	3	Find all subgroups of $(\mathbb{Z}_{18}, +_{18})$. **Answer**: $\mathbb{Z}_{18} = \{ \mathbf{0},\ \mathbf{1},\ \mathbf{2},\ \mathbf{3} \dots\dots \mathbf{17} \}$ $\langle \mathbf{1} \rangle = \{ \mathbf{0},\ \mathbf{1},\ \mathbf{2},\ \mathbf{3}, \dots\dots\ \mathbf{17} \}$ $\langle \mathbf{2} \rangle = \{ \mathbf{0},\ \mathbf{2},\ \mathbf{4},\ \mathbf{6},\ \mathbf{8},\ \mathbf{10},\ \mathbf{12},\ \mathbf{14},\ \mathbf{16} \}$ $\langle \mathbf{3} \rangle = \{ \mathbf{0},\ \mathbf{3},\ \mathbf{6},\ \mathbf{9},\ \mathbf{12},\ \mathbf{15} \}$ $\langle \mathbf{6} \rangle = \{ \mathbf{0},\ \mathbf{6},\ \mathbf{12} \}$ $\langle \mathbf{9} \rangle = \{ \mathbf{0},\ \mathbf{9} \}$ $\langle \mathbf{18} \rangle = \{ \mathbf{0} \}$
C	4	Find all subgroups of $(\mathbb{Z}_7^*,\ \times_7)$. **Answer**: $\mathbb{Z}_7^* = \{ \mathbf{1},\ \mathbf{2},\ \mathbf{3} \dots\dots \mathbf{6} \}$ $\langle \mathbf{1} \rangle = \{ \mathbf{1} \}$ $\langle \mathbf{2} \rangle = \{ \mathbf{2},\ \mathbf{4},\ \mathbf{8} \}$ $\langle \mathbf{3} \rangle = \{ \mathbf{1},\ \mathbf{2},\ \mathbf{3},\ \mathbf{4},\ \mathbf{5},\ \mathbf{6} \}$ $\langle \mathbf{6} \rangle = \{ \mathbf{1},\ \mathbf{6} \}$

Method 4⇝ Abelian group

Example of Method-4: Abelian group

B	1	Show that $G = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} / a, b \in \mathbb{R} \right\}$ is a commutative group under matrix addition.
B	2	Show that $(\mathbb{Z}_4,\ +_4)$ is an abelian group.
B	3	Let $*$ be a binary operation defined on $\mathbb{Q}$ defined by $a * b = a + b - ab$. $(\mathbf{1})$ Find $2 * 3,\ \ 3 * (-5),\ \ 7 * \left(\frac{1}{2}\right)$ $(\mathbf{2})$ Is $(\mathbb{Q},\ *)$ group? $(\mathbf{3})$ Is it commutative? **Answer**: $(\mathbf{1})\ -\mathbf{1},\quad \mathbf{13},\quad \mathbf{4}$ $(\mathbf{2})$ **Yes** (3) **Yes**
B	4	Let $*$ be a binary operation on $\mathbb{Q} \times \mathbb{Q}$ defined by $(a,\ b) * (x,\ y) = (ax,\ ay + b)$ then $(\mathbf{1})$ Find $(3,\ 4) * (1,\ 2),\ \ (-1,\ 3) * (5,\ 2)$. $(\mathbf{2})$ Is it group? $(\mathbf{3})$ Is it commutative? **Answer**: $(\mathbf{1})\ (\mathbf{3},\ \mathbf{10}),\ (-\mathbf{5},\ \mathbf{1})$ $(\mathbf{2})$ **Yes** $(\mathbf{3})$ **No**

Method 5 ⇝ Order of an Element of a Group

Example of Method-5: Order of an Element of a Group

A	1	Find the order of each element of $(\mathbb{Z}_6, +_6)$.
A	2	Find the order of each element of $(\mathbb{Z}_7^*, \times_7)$.

Method 6 ⇝ Cyclic Group

Example of Method-6: Cyclic Group

A	1	Prove that $(\mathbb{Z}, +)$ is cyclic group.
A	2	Prove that $(\mathbb{Z}_3, +_3)$ is cyclic group.
A	3	Prove that $G = \{1, \ -1\}$ is cyclic group under multiplication.
B	4	Consider the group $G = \{1, \ 2, \ 3, \ 4, \ 5, \ 6\}$ under multiplication modulo 7. (1) Find multiplication table of G. (2) Find 2^{-1}, 3^{-1} and 6^{-1}. (3) Is group G cyclic? **Answer: (1)**

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

(2) $2^{-1} = 4, \quad 3^{-1} = 5, \quad 6^{-1} = 6$

(3) Yes, since it is generated by 3

B	5	Consider the group $G = \{1, 2, 4, 7, 8, 11, 13, 14\}$ under multiplication modulo 15. (1) Find multiplication table of G. (2) Find 2^{-1}, 7^{-1} and 11^{-1}. (3) Is group G cyclic?

Answer: (1)

$\times_{15}$	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

(2) $2^{-1} = 8$, $7^{-1} = 13$, $11^{-1} = 11$

(3) **No, since no element generates G**