Index

Unit – 4 ⇝ Graph Theory - I

Method 1 ⇝ Basic Terminologies

Introduction

→ Graph theory in discrete mathematics a branch of mathematics that deals with thestudy of graphs, which are mathematical structures used to represent relationships between objects.

→ The applications of the linear graph are used not only in Mathematics but also in other fields such as Computer Science, Social Sciences, Biology, Transportation, Cyber Security, Finance, etc.

→ In real – life, the best example of graph structure is GPS, where you can track the path or know the direction of the road.

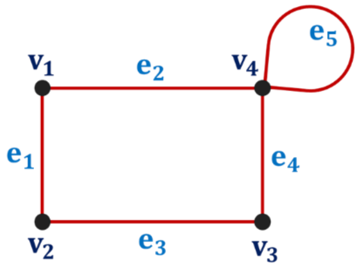
→ Graph theory continues to play a crucial role in advancing our understanding of complex systems and optimizing processes in our interconnected world.

Basic Terminologies

(1) Graph

→ A graph $G = (V,\ E)$ consists of a non – empty set $V = \{\, v_1,\ v_2, v_3, \dots \}$ of vertices and a set $E = \{\, e_1,\ e_2, e_3, \dots \}$ of edges such that each edge is incident to an ordered/unordered pair $(v_i,\ v_j)$ of vertices.

→ Example:



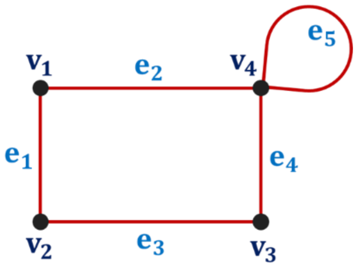
(2) Vertex

→ A vertex (plural vertices) is represented by circle or dot.

→ The set of vertices of a graph G is denoted as V or $V(G)$.

→ Vertex is known as Node, Point, Dot or Junction.

→ Example:



- Vertex Set $V = \{v_1, v_2, v_3, v_4\}$

(3) Edge

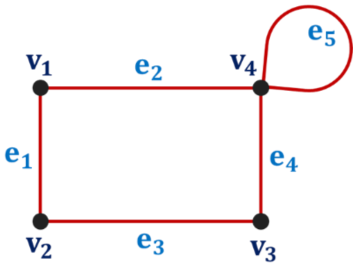
→ An edge is represented by line or arc.

→ The set of edges of a graph G is denoted as E or E(G).

→ $E(G) = \{ e = (u, v) \mid u, v \in V(G) \}$

→ Edge is known as Branch, Line or Arc.

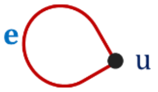
→ Example:



- Edge Set $V = \{e_1, e_2, e_3, e_4, e_5\}$

→ Any edge e can be made by one or two vertices.

- Example:



$e = (u, u)$

$$\left.\begin{array}{c} e = (u, v) \\ \text{or} \\ e = (v, u) \end{array}\right\} \text{unordered pair}$$

(4) Self – Loop

→ An edge e of a graph G that joins a vertex u to itself is called a Self - Loop.

→ Self – Loop is also known as Loop or Sling.

→ Example:

i.e., a self – loop is an edge $e = (u, u)$.

(5) Adjacent Vertices

→ If two vertices u and v are joined by an edge e, then u and v are said to be adjacent vertices.

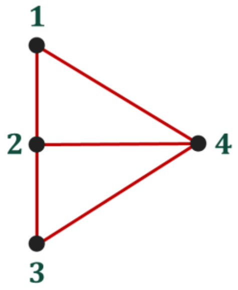
(6) Incident Edge

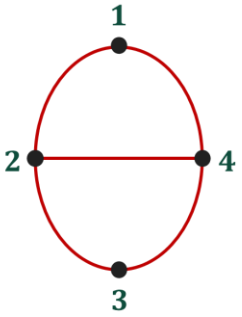
→ If an edge e from the edge set joins the vertices u and v, then the edge e is said to be incident to the vertices u and v.

→ The vertices u and v are the end points of an edge e.

Remark

→ In drawing a graph, it is immaterial whether the edges or lines are drawn straight or curved, long or short, what is important is the incidence between edges and vertices are the same in both cases.

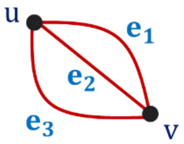




(7) Parallel Edges

→ If two vertices of a graph are joined by more than one edge then these edges are called parallel edges or multiple edges.

→ Example:

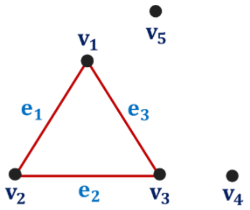


- Here e_1, e_2 and e_3 are parallel edges.

(8) Isolated Vertex

→ In a graph, a vertex which is not adjacent/associated to any other vertex is known as isolated vertex.

→ Example:



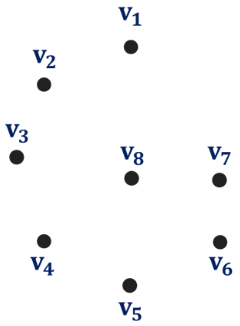
- Here, vertex v_4 and v_5 are isolated vertices.

(9) Null Graph

→ A graph without edges is known as null graph.

→ Also, a graph which contains only isolated vertices is known as null graph.

→ Example:

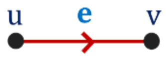


(10) Directed Edge

→ In a graph G an edge e which is associated with an ordered pair of vertices u to v is called directed edge of a graph G.

→ Directed edge from vertex u to v is denoted as $\langle u, v\rangle$.

→ Example:



- Edge **e** $= \langle u, v\rangle$ is a directed edge.

→ Suppose $e = \langle u, v\rangle$ is a directed edge, then

- vertex u is known as the initial vertex and vertex v is known as the terminal vertex of e.

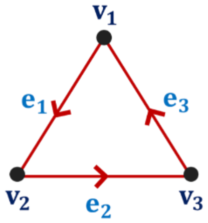
- **e** is said to be incident from u to v.

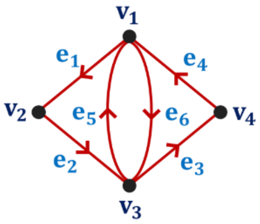
(11) Directed Graph

→ A graph in which every edge is directed edge is known as Directed Graph or Digraph.

→ A digraph is denoted as $G = \langle V, E \rangle$; where V = Vertex Set and E = Edge Set.

→ Example:





(12) Undirected Edge

→ In a graph G, an edge e which is associated with an unordered pair of vertices u to v is called an undirected edge of a graph G.

→ Undirected edge from vertex u to v is denoted as (u, v).

→ Example:

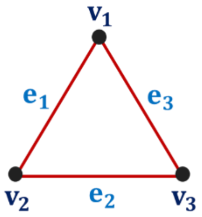
- Edge $e = (u, v)$ is an undirected edge.

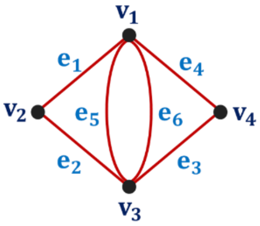
(13) Undirected Graph

→ A graph in which every edge is undirected edge is known as undirected graph.

→ An undirected graph is denoted as $G = (V, E)$; where V = Vertex Set and E = Edge Set

→ Example:

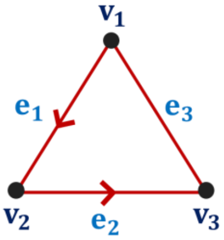


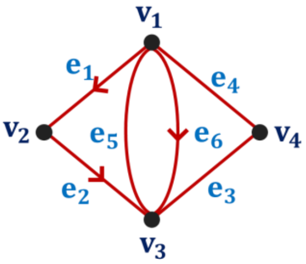


(14) Mixed Graph

→ If some edges of a graph G are directed and some edges are undirected, then G is known as a mixed graph.

→ Example:

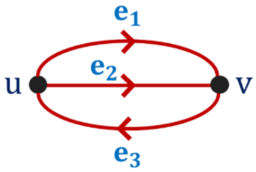




(15) Distinct Edge

→ In a directed graph, the two possible edges between a pair of vertices which are opposite in direction are known as distinct edges.

→ Example:

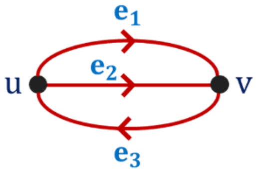


- Here, edge $\mathbf{e_1}$ and $\mathbf{e_3}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ are distinct edges.
- Edge $\mathbf{e_1}$ and $\mathbf{e_2}$, are not distinct edges as they are in same direction.

Remark

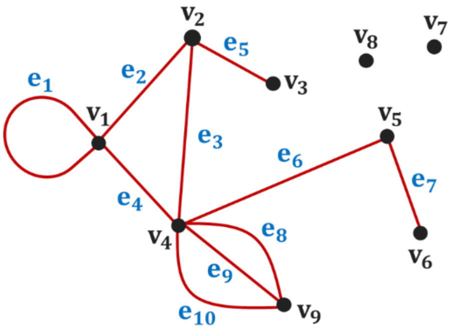
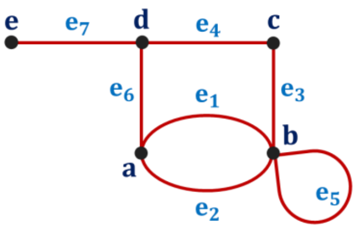
→ In a directed graph, more than one directed edge in a particular direction is considered as parallel edge.

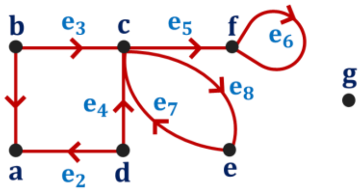
→ Example:



- Edge $\mathbf{e_1}$ and $\mathbf{e_2}$ are parallel edges as they are in same direction.

Examples of Method-1: Basic Terminologies

C	1	Determine the vertex set, edge set and incidence relation of the following graph G:  G **Answer**: **Vertex Set** $\mathbf{V = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9 \}}$, **Edge Set** $\mathbf{E = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9 \}}$, **Incidence Relation** $\rightsquigarrow \mathbf{e_1 = (v_1, v_1)}$, $\mathbf{e_2 = (v_1, v_2)}$, $\mathbf{e_3 = (v_2, v_4)}$, $\mathbf{e_4 = (v_1, v_4), e_5 = (v_2, v_3), e_6 = (v_4, v_5), e_7 = (v_5, v_6), e_8 = (v_4, v_9)}$, $\mathbf{e_9 = (v_1, v_4), e_{10} = (v_1, v_4)}$
C	2	Draw the undirected graph $G = (V, E)$, where $V = \{a, b, c, d, e\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and its incidence relations given as: $e_1 = (a, b)$, $e_2 = (a, b)$, $e_3 = (b, c)$, $e_4 = (c, d)$, $e_5 = (b, b)$, $e_6 = (a, d)$ & $e_7 = (e, d)$. Determine self – loop, parallel edges, one pair of adjacent and non – adjacent vertices, isolated vertex. **Answer**:  **(1) edge $\mathbf{e_5}$ is self – loop,** **(2) $\mathbf{e_1}$ and $\mathbf{e_2}$ are parallel edges,** **(3) Vertices e and d are adjacent, while e and c are non – adjacent,** **(4) There are no isolated vertices.**

C	3	Draw the directed graph $G = (V, E)$, where $V = \{a, b, c, d, e, f, g\}$ and $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ and its incidence relation given as: $e_1 = < b, a >, e_2 = < d, a >,\ e_3 = < b, c >,\ e_4 = < d, c >, e_5 = < c, f >,$ $e_6 = < f, f >, e_7 = < e, c > \ \&\ e_8 = < c, e >.$ Determine self – loop, parallel edges, one pair of adjacent and non – adjacent vertices, isolated vertex. **Answer**:  **(1) edge e_6 is self – loop, (2) There are no parallel edges,** **(3) Vertices b and c are adjacent, while b and f are non – adjacent,** **(4) Vertex g is isolated.**

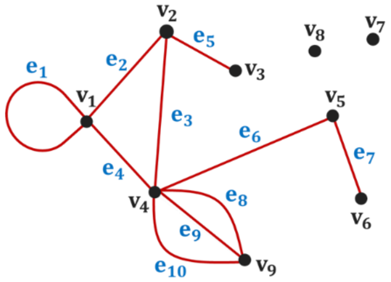
Method 2 ⇝ First and Second Theorem of Graph Theory

(1) Order of a Graph

→ The number of vertices in a graph G is called order of a graph G.

→ The order of a graph G with the vertex set V is denoted as |V(G)|.

→ Example:



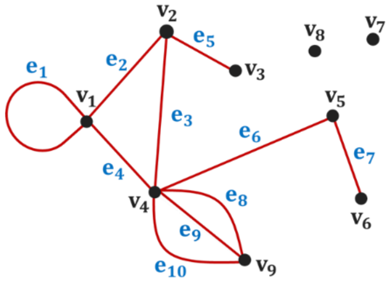
- Here, $|V(G)| = 8$

(2) Size of a Graph

→ The number of edges in a graph G is called size of a graph G.

→ The size of a graph G with the edge set E is denoted as |E(G)|.

→ Example:



- Here, $|E(G)| = 10$

(3) Degree of a Vertex

→ Let G be an undirected graph then, the degree of a vertex $v \in G$ is defined as the number of edges incident on v.

→ The degree of a vertex $v \in G$ is denoted by $d(v)$ OR $d_G(v)$ OR $\deg(v)$.

→ Note that, self - loop will be counted twice in the degree of a corresponding vertex.

(4) Odd Vertex

→ A vertex with odd degree is known as odd vertex.

(5) Even Vertex

→ A vertex with even degree is known as even vertex.

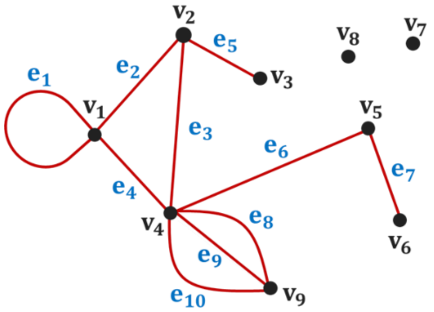
(6) Isolated Vertex

→ A vertex with degree zero is known as isolated vertex.

(7) Pendent Vertex

→ A vertex with degree one is known as pendent vertex.

→ Example:



- $d(v_1) = 4, d(v_2) = 3, d(v_3) = 1, d(v_4) = 6, d(v_5) = 2, d(v_6) = 1, d(v_7) = 0,$ $d(v_8) = 0, d(v_9) = 3$
- In above graph, odd vertices $\rightsquigarrow v_2, v_3, v_6, v_9,$ even vertices $\rightsquigarrow v_1, v_4, v_5, v_7, v_8$ isolated vertices $\rightsquigarrow v_7, v_8,$ pendent vertices $\rightsquigarrow v_3, v_6$

First Theorem of Graph Theory

→ **Statement:**

- Any graph G with ‘n’ vertices $v_1, v_2, \dots, v_n$ and ‘e’ edges,

$$\sum_{i=1}^{n} d(v_i) = d(v_1) + d(v_2) + \cdots + d(v_n) = 2e$$

→ i.e., the sum of degree of all the vertices is twice the number of edges.

→ i.e., the sum of degree of all the vertices is even.

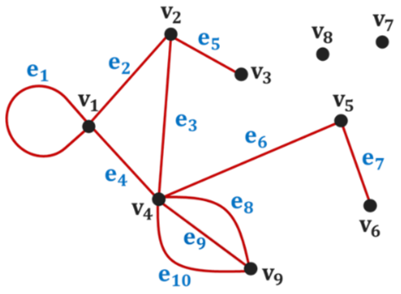
→ This theorem is also known as Handshaking Theorem or Degree Sum Formula.

Second Theorem of Graph theory

→ **Statement:**

- In any undirected graph G, the number of odd vertices is even.

→ Example:



G

- Graph G has 4(even) odd vertices.

(8) Out – Degree of a Vertex

→ In a directed graph, the number of edges directed outwards vertex is called out – degree of the vertex.

→ Let G be a directed graph, then for any vertex $u \in G$, the number of edges which have u as their initial vertex is called the out – degree of the vertex u.

→ The out – degree of the vertex u is denoted by $d^+(u)$.

(9) In – Degree of a vertex

→ In a directed graph, the number of edges directed towards vertex is called in – degree of the vertex.

→ Let G be a directed graph, then for any vertex $u \in G$, the number of edges which have u as their terminal vertex is called the in – degree of the vertex u.

→ The in – degree of the vertex u is denoted by $d^-(u)$.

(10) Total Degree of a vertex

→ Let G be a directed graph, then the sum of indegree and outdegree of vertex is called the total degree of the vertex.

→ The total degree of the vertex u is denoted by $d(u)$.

→ $d(u) = d^+(u) + d^-(u)$

Remark

→ The in – degree and out – degree of an isolated vertex is 0.

Thus, the total degree of an isolated vertex is always 0.

→ A self – loop at a vertex contributes 1 to both the in – degree and out – degree of that vertex.

Degree Sum Formula for a Directed Graph

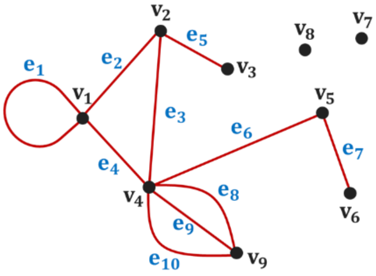
→ **Statement:**

→ In a directed graph $G = \langle V, E \rangle$ with ‘n’ vertices $v_1, v_2, \dots, v_n$ and ‘e’ edges,

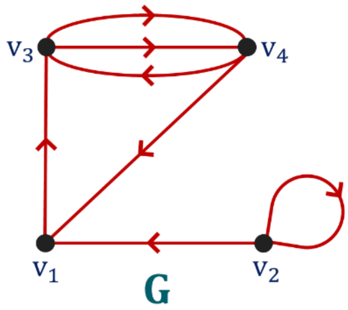
$$\sum_{i=1}^{n} d^{+}(v_i) = \sum_{i=1}^{n} d^{-}(v_i) = e \text{ and } \sum_{i=1}^{n} d(v_i) = 2e.$$

→ i.e., the sum of in – degrees of all the vertices of a digraph is equal to sum of out – degrees of all its vertices which is equal to the number of edges of the graph.

Examples of Method-2: First and Second Theorem of Graph Theory

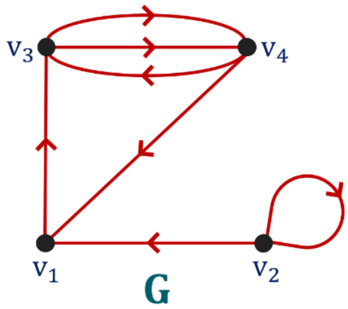
C	1	Verify Handshaking theorem for the following graph G:  G

<table>
<tr>
<td>C</td>
<td>2</td>
<td>
Find out – degree, in – degree and total degree of all the vertices of following graph G:



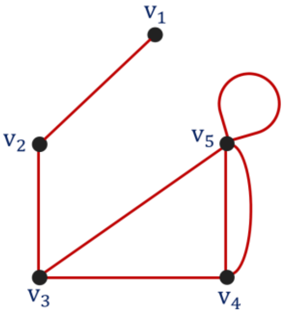
Answer:

<table>
<tr><th>In – degree</th><th>Out – degree</th><th>Total degree</th></tr>
<tr><td>$\mathbf{d^-(v_1) = 2}$</td><td>$\mathbf{d^+(v_1) = 1}$</td><td>$\mathbf{d(v_1) = 3}$</td></tr>
<tr><td>$\mathbf{d^-(v_2) = 1}$</td><td>$\mathbf{d^+(v_2) = 2}$</td><td>$\mathbf{d(v_2) = 3}$</td></tr>
<tr><td>$\mathbf{d^-(v_3) = 2}$</td><td>$\mathbf{d^+(v_3) = 2}$</td><td>$\mathbf{d(v_3) = 4}$</td></tr>
<tr><td>$\mathbf{d^-(v_4) = 2}$</td><td>$\mathbf{d^+(v_4) = 2}$</td><td>$\mathbf{d(v_4) = 4}$</td></tr>
</table>
</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>
Verify degree sum formula for the following directed graph G:



</td>
</tr>
<tr>
<td>C</td>
<td>4</td>
<td>
Find the number of vertices in an undirected graph $G = (V, E)$ with 27 edges in which 6 vertices of degree 2, 3 vertices of degree 4 and remaining vertices of degree 3.

Answer: **19**
</td>
</tr>
</table>

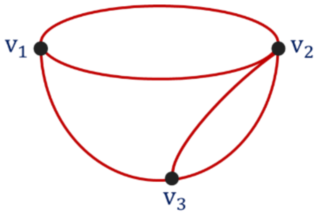
C	5	Either draw a graph with specified properties or explain why no such graph exists. a) graph with four vertices of degrees 1, 2, 3 and 3 b) graph with five vertices of degrees 1, 2, 3, 3 and 5 **Answer**: **a) It is not possible to draw such graph as in given data as number of odd vertices are 3 and number of odd vertices must be even.** **b)** 

Method 3 ⇝ Special Types of a Graph

(1) Undirected Multigraph

→ An undirected graph which contains some parallel edges and does not contain self – loop is known as undirected multigraph.

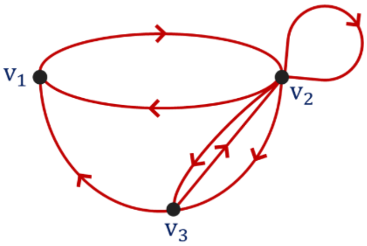
→ Example:



(2) Directed Multigraph

→ A directed graph which contains parallel edges and self – loop is known as directed multigraph.

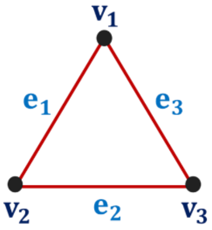
→ Example:



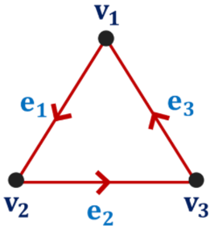
(3) Simple Graph

→ A graph without parallel edges and self – loop is known as Simple Graph.

→ Example:



Undirected Simple Graph



Directed Simple Graph

Remark

Type	Edges	Multiple Edges	Self - Loop
Simple Graph	Directed	Not Allowed	Not Allowed
	Undirected		
Multigraph	Directed	Allowed	Allowed
	Undirected	Allowed	Not Allowed
Mixed Graph	Directed	Allowed	Allowed
	Undirected		

(4) Regular Graph

→ A graph is a regular graph, if the degree of each vertex is same.

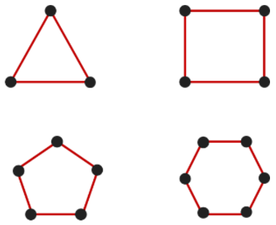
→ If the degree of each vertex of a graph G is k ($k \in \mathbb{N} \cup \{0\}$), then the graph G is known as k – regular graph.

→ Note that, if graph G has n vertices and is regular graph of degree k, then G has $\frac{kn}{2}$ edges.

→ Example:

0 – regular

1 – regular



2 – regular

(5) Complete Graph

→ A simple graph G in which there is an edge between every pair of vertices is known as complete graph.

→ A complete graph with n vertices is denoted by K_n.

→ The complete graph with n vertices is $(n-1)$ - regular.

→ Thus, the degree of each vertex of complete graph with n vertices is $(n-1)$.

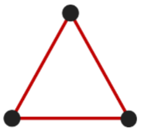
→ Example:

K_1

$0-$ regular

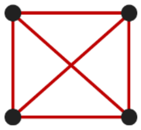
K_2

$1-$ regular



K_3

$2-$ regular



K_4

$3-$ regular

K_5

$4-$ regular

K_6

$5-$ regular

→ The complete graph with n vertices (K_n) has

$$\frac{n(n-1)}{2} \text{ edges.}$$

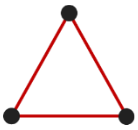
→ Example:

The complete graph with $n = 5$ vertices (K_5) has $\frac{5(5-1)}{2} = 10$ edges.

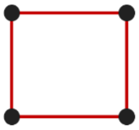
(6) Cycle Graph

→ The cycle graph C_n $(n \geq 3)$ of length n is a graph which contains n vertices and n edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), \dots, (v_{n-1}, v_n), (v_n, v_1)$.

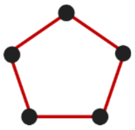
→ Example:



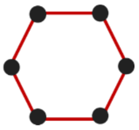
C_3



C_4



C_5



C_6

→ Notes

- C_n $(n \geq 3)$ is a regular graph of degree 2.
- Every self – loop is a cycle graph.

(7) Bipartite Graph

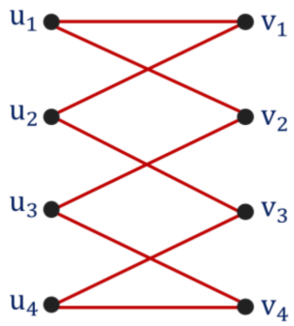
→ The simple graph G is known as bipartite if,

- the vertex set V can be partitioned into two non – empty subsets V_1 and V_2 such that each edge of G has one end vertex in V_1 and another in V_2
- $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$

→ Here, V_1 and V_2 are called bipartition of the vertex set V.

→ Note that, a bipartite graph can have no self – loop as self – loop connects the same vertex which is not permitted in bipartite graph.

→ Example:



- Here, $V = \{ v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4 \}$
- Partitions: $V_1 = \{ u_1, u_2, u_3, u_4 \}$, $V_2 = \{ v_1, v_2, v_3, v_4 \}$
- Also, $V_1 \cup V_2 = V$ and $V_1 \cap V_2 = \phi$

(8) Complete Bipartite Graph

→ A bipartite graph is known as complete bipartite if, each vertex of set V_1 is joined with each vertex of set V_2.

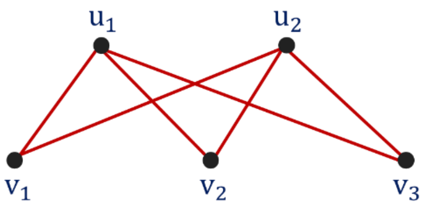
→ A complete bipartite graph is denoted by $K_{m,\,n}$

where, $m =$ number of vertices of set V_1 and $n =$ number of vertices of set V_2

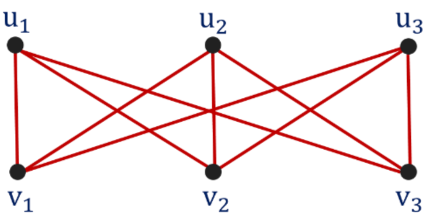
→ **Note**

- $K_{m,\,n}$ has $m + n$ vertices and mn edges; $m, n \in \mathbb{N}$.
- A complete bipartite graph $K_{m,\,n}$ is regular if $m = n$.

→ Example:



$K_{2,3}$



$K_{3,3}$

Subgraph

→ Consider a graph $G = (V,\ E)$. A graph $H = (V',\ E')$ is called subgraph of G, if $V'(H) \subseteq V(G),\ E'(H) \subseteq E(G)$.

→ Here G is known as Super graph of H.

→ **Properties**

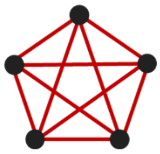
- Every graph is a subgraph of itself.
- Every single vertex in a graph G is a subgraph of G.
- Every single edge in a graph G is a subgraph of G.

→ Example:

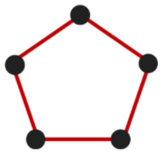
Graph **Subgraphs**

G

$\mathbf{H_1}$



$\mathbf{H_2}$



$\mathbf{H_3}$

$\mathbf{H_4}$

→ For a graph $G = (V,\ E)$, let $|\,V\,| = m$ and $|\,E\,| = n$, then the total number of subgraphs is equal to $(2^{m-1}) \times 2^n$.

- Example:

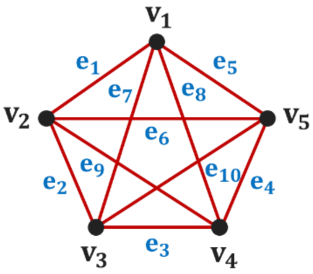
  - Here, $|\,V\,| = m = 5$ and $|\,E\,| = n = 10$

- The total number of subgraphs is equal to $(2^{m-1}) \times 2^n$

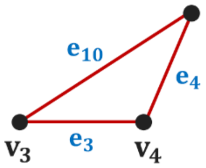
$$= (2^{5-1}) \times 2^{10} = 2^{14}$$

→ Vertex Deleted Subgraph

- The graph obtain by deleting a vertex v and all the edges incident to it from a given graph $G = (V, E)$ is called a vertex deleted subgraph of G.
- The resulting subgraph is denoted by $G - v$.
- If we remove more than one vertex from a graph $G = (V, E)$, then vertex deleted subgraph will be denoted as $H = (V \setminus V', E \setminus E')$; where V' = the set of removed vertices and E' = the set of edges which are incident with removed vertices
- Example:



$G = (V, E)$



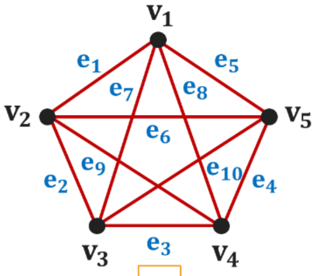
$H = (V \setminus V', E \setminus E')$

$V' = \{v_1, v_2\}$

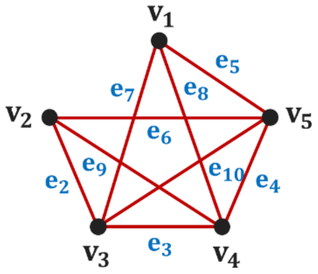
$E' = \{e_1, e_5, e_7, e_8, e_6, e_9, e_2\}$

→ Edge Deleted Subgraph

- The graph obtain by deleting an edge e from a given graph $G = (V, E)$ is called an edge deleted subgraph of G.
- The resulting subgraph is denoted by $G - e$.
- If we remove more than one edge from a graph $G = (V, E)$, then edge deleted subgraph will be denoted as $H = (V, E \setminus E')$; where E' = the set of removed edges.
- Example:

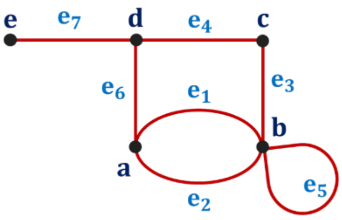


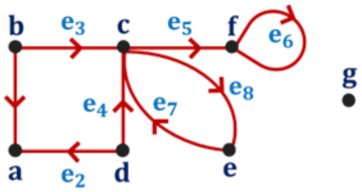
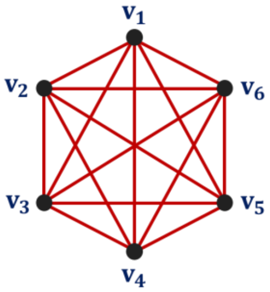
G

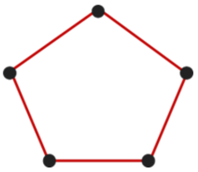
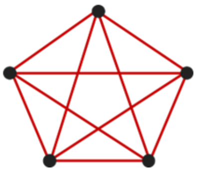
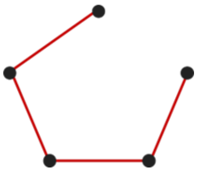
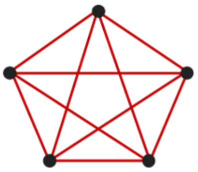
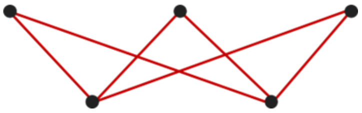


$G - e_1$

Examples of Method-3: Special Types of a Graph

C	1	Answer the following questions for the undirected graph G:  G (1). Check whether the graph is simple or not. Justify it. (2). Check whether the graph is multigraph or not. Justify it. (3). Check whether the graph is mixed or not. Justify it. **Answer**: **(1) The graph is not simple, as it contains self – loop e_5 and parallel edges e_1 and e_2.** **(2) The graph is not a multigraph as in undirected multigraph self – loops are not allowed.** **(3) The graph is not a mixed graph as all the edges are undirected.**

C	2	Answer the following questions for the directed graph G:  (1). Check whether the graph is simple or not. Justify it. (2). Check whether the graph is multigraph or not. Justify it. (3). Check whether the graph is mixed or not. Justify it. **Answer**: **(1) The graph is not simple, as it contains self – loop e_6.** **(2) The graph is a multigraph as in directed multigraph self – loops are allowed.** **(3) The graph is not a mixed graph as all the edges are directed.**
C	3	Draw a complete graph K_6 and answer the following questions: (1). Check whether K_6 is regular graph or not. Justify it. (2). Find the total number of edges of K_6. (3). Find the total number of subgraphs of K_6. **Answer**:  **(1) K_6 is a regular graph, as degree of each vertex is 5.** **(2) K_6 has 16 edges.** **(3) K_6 has 2^{20} subgraphs.**

C	4	Draw a graph which is (1). regular but not complete (2). regular and complete (3). neither regular nor complete (4). regular but not cycle (5). cycle but not complete (6). bipartite but not regular **Answer**: (1)  (2)  (3)  (4)  (5)  (6) 

Method 4 ⇝ Graph Isomorphism

Graph Isomorphism

→ Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a function $f: V_1 \rightarrow V_2$ such that

- f is one – one and onto
- (a, b) is an edge in E_1 if an only if $\big(f(a), f(b)\big)$ is an edge in E_2 ; for any $a,\ b \in V_1$

→ Such a function f is known as graph isomorphism.

→ If G_1 and G_2 are isomorphic, then it is denoted as $\mathbf{G_1 \cong G_2}$

→ If G_1 and G_2 are not isomorphic, then it is denoted as $\mathbf{G_1 \ncong G_2}$

Another Definition of Graph Isomorphism

→ Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if

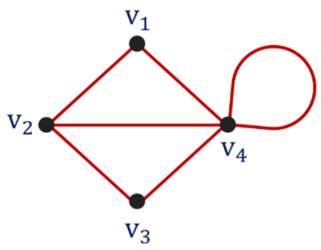
- $|V_1| = |V_2|$ i.e., number of vertices are same
- $|E_1| = |E_2|$ i.e., number of edges are same
- G_1 and G_2 have same degree sequence
    - i.e., if G_1 has n vertices of degree k, then G_2 must have exactly n vertices of degree k.
- adjacency is preserved.
    - i.e., if edge e is incident on vertices v_1 and v_2 in G_1, then the corresponding edge e' in in G_2 must be incident on the vertices v_1' and v_2' that correspond to v_1 and v_2 respectively.

Degree Sequence of a Graph

→ For a graph $G = (V_1, E_1)$, if $v_1, v_2, v_3, \dots, v_n$ are n vertices of G and let $d_1, d_2, d_3, \dots, d_n$ be their degrees respectively.

→ If the sequence $(d_1, d_2, d_3, \dots, d_n)$ is monotonically increasing $(\text{i.e.}, \ d_1 \le d_2 \le d_3 \le \dots \le d_n)$, then it called degree sequence of graph G.

→ Example:



→ The degree sequence of the graph shown in figure is $(2,\ 2,\ 3,\ 5)$ as $d(v_1) = 2, d(v_2) = 3, d(v_3) = 2$ and $d(v_4) = 5$.

→ **Procedure to check whether the undirected graphs are isomorphic or not:**

- Let $G_1 = (V_1,\ E_1)$ and $G_2 = (V_2,\ E_2)$ be two undirected graphs.
    - **Step:1** $|V_1| = |V_2|$
    - **Step:2** $|E_1| = |E_2|$
    - **Step:3** G_1 and G_2 must have same degree sequence.

      i.e., if G_1 has n vertices of degree k, then G_2 must have exactly n vertices of degree k.
    - **Step:4** Adjacency must be preserved.

      i.e., if edge e is incident on vertices v_1 and v_2 in G_1, then the corresponding edge e' in in G_2 must be incident on the vertices v_1' and v_2' that correspond to v_1 and v_2 respectively.
- If all the four steps are satisfied, then we can say G_1 and G_2 are isomorphic.
- If any of the above step does not satisfy, then G_1 and G_2 are not isomorphic.

→ **Procedure to check whether the directed graphs are isomorphic or not:**

- Let $G_1 = \langle V_1,\ E_1 \rangle$ and $G_2 = \langle V_2,\ E_2 \rangle$ be two directed graphs.
    - **Step:1** $|V_1| = |V_2|$
    - **Step:2** $|E_1| = |E_2|$
    - **Step:3** If G_1 has n vertices of in – degree k_1 and out – degree k_2, then G_2 must have exactly n vertices in – degree k_1 and out – degree k_2.
    - **Step:4** Adjacency must be preserved.

      i.e., if edge e is incident on vertices v_1 and v_2 in G_1, then the corresponding edge e' in in G_2 must be incident on the vertices v_1' and v_2' that correspond to v_1 and v_2 respectively.
- If all the four steps are satisfied, then we can say G_1 and G_2 are isomorphic.
- If any of the above step does not satisfy, then G_1 and G_2 are not isomorphic.

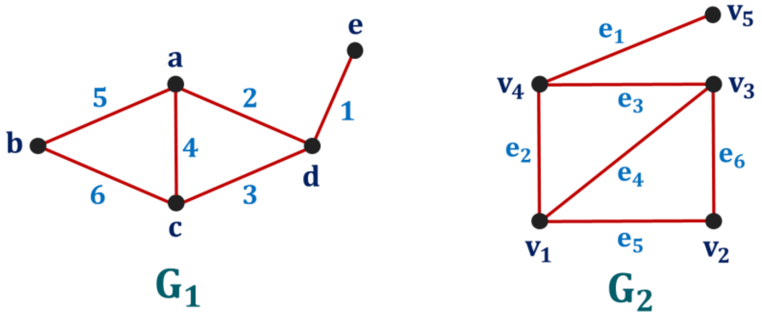
Remark

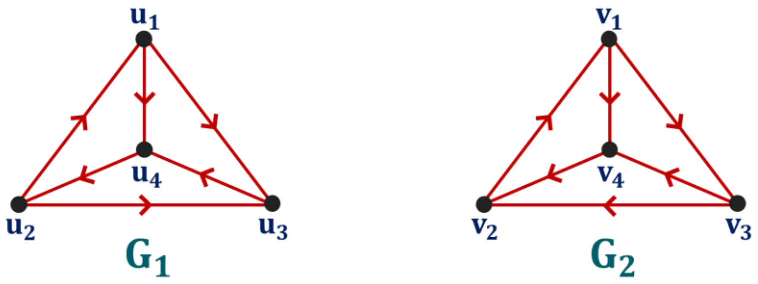
→ Every graph G is isomorphic to itself. **(Reflexive)**

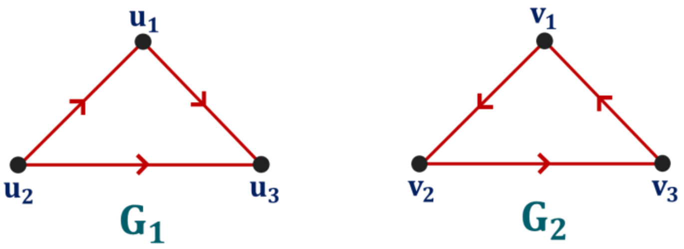
→ If $G_1 \cong G_2 \Rightarrow G_2 \cong G_1$ **(Symmetric)**

→ If $G_1 \cong G_2$ and $G_2 \cong G_3 \Rightarrow G_1 \cong G_3$ **(Transitive)**

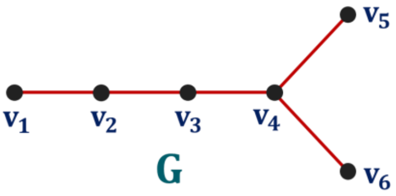
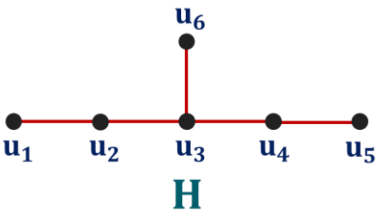
Examples of Method-4: Graph Isomorphism

<table>
<tr><td>C</td><td>1</td><td>Check whether the given pair of graphs are isomorphic or not.


Answer: $\mathbf{G_1 \cong G_2}$</td></tr>
<tr><td>C</td><td>2</td><td>Check whether the given pair of graphs are isomorphic or not.


Answer: $\mathbf{G_1 \cong G_2}$</td></tr>
<tr><td>C</td><td>3</td><td>Check whether the given pair of graphs are isomorphic or not.


Answer: $\mathbf{G_1 \ncong G_2}$</td></tr>
</table>

C	4	Check whether the given pair of graphs are isomorphic or not.   **Answer: $G \ncong H$**

Method 5 ⇝ Connectivity

Path of Graph

→ A path of a graph G is an alternating sequence of vertices and edges of the form $\mathbf{v_1e_1v_2e_2v_3 \dots v_{n-1}e_nv_n}$; where the vertices v_i and v_{i-1} are the end points of the edge e_i, $i = 1, 2, 3, \dots, n-1$

- Here, v_1 is a starting or initial vertex, v_n is end vertex and $v_2, v_3, \dots, v_{n-1}$ are called internal vertices.

→ If $\mathbf{v_1e_1v_2e_2v_3 \dots v_{n-1}e_nv_n}$, we can say the path is from v_1 to v_n or the path between v_1 and v_n or connects v_1 to v_n.

→ If $\mathbf{v_1 = v_n}$ (i.e., starting and end vertices are same), then the path is known as closed path.

→ The total number of edges which are present in a path is called length of a path.

- A path of length n is denoted by P_n.
- A path of length zero consists of a single vertex.

→ **Simple Path**

- A path in a graph is known as simple if the edges do not repeat in path.

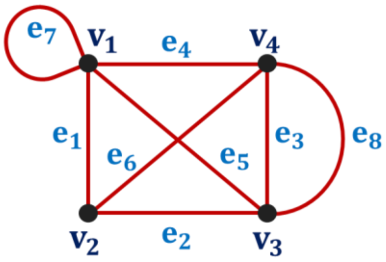
→ **Elementary Path**

- A path in a graph is known as elementary if the vertices do not repeat in path.

→ **Note**

- Every elementary path is always a simple path.
- Closed path can never be elementary path.

→ Example:



Path	Length	Closed Path	Elementary Path	Simple Path
$v_3e_5v_1$	1	No	Yes	Yes
$v_3e_3v_4e_4v_1$	2	No	Yes	Yes
$v_3e_8v_4e_3v_3e_5v_1$	3	No	No	Yes
$v_1e_7v_1$	1	Yes	No	Yes
$v_1e_5v_3e_3v_4e_4v_1$	3	Yes	No	Yes
$v_2e_6v_4e_8v_3e_3v_4e_6v_2$	4	Yes	No	No

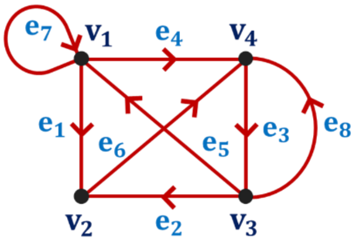
Cycle or Circuit

→ A closed path in a graph G is known as a cycle or circuit.

i.e., a path of length greater than zero that begins and ends at the same vertex is called a circuit or cycle.

→ A cycle of length n is known as n – cycle and is denoted as C_n.

→ Example:



- Cycle of length 3 $\rightsquigarrow$ $v_3e_2v_2e_6v_4e_3v_3$
- Cycle of length 4 $\rightsquigarrow$ $v_1e_4v_4e_3v_3e_2v_2e_1v_1$
- Cycle of length 5 $\rightsquigarrow$ $v_3e_8v_4e_3v_3e_2v_2e_6v_4e_3v_3$

→ **Elementary Cycle**

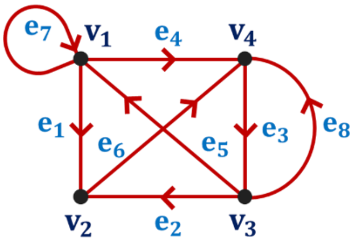
- A cycle is known as elementary if all the vertices in a cycle are distinct except the end vertices.

→ **Simple Cycle or Trail**

- A cycle is known as simple if all the edges in cycle are distinct.

  i.e., a simple closed path is known as trail or simple cycle.

→ Example:



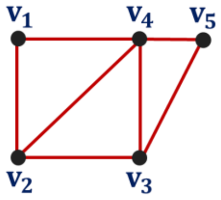
- $C_3 = v_3e_2v_2e_6v_4e_3v_3$ is an elementary as well as simple cycle.
- $C_5 = v_3e_8v_4e_3v_3e_2v_2e_6v_4e_3v_3$ is neither elementary nor simple cycle.

Connected Graph

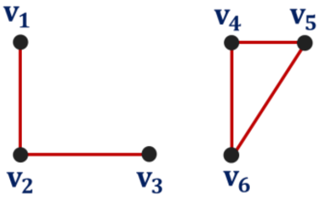
→ An undirected graph is called connected if there is at least one path between every pair of distinct vertices of the graph.

→ An undirected graph that is not connected is called disconnected.

→ We can produce a disconnected subgraph by removing vertices or edges, or both.



Connected Graph

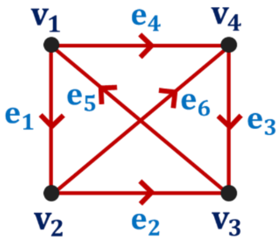


Disconnected Graph

Strongly Connected Digraph

→ A directed graph G is said to be strongly connected if for each pair of vertices u, v in G, there must be a path from u to v and from v to u.

→ Example:



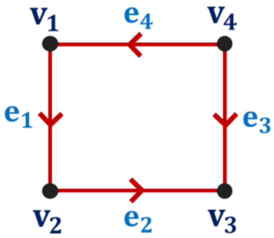
- Here, there is a path between every pair of vertices.

- Hence, the above graph is strongly connected.

Unilaterally Connected Digraph

→ A directed graph G is said to be unilaterally connected if for each pair of vertices u, v in G, either there is a path from u to v or there is a path from v to u.

→ Example:



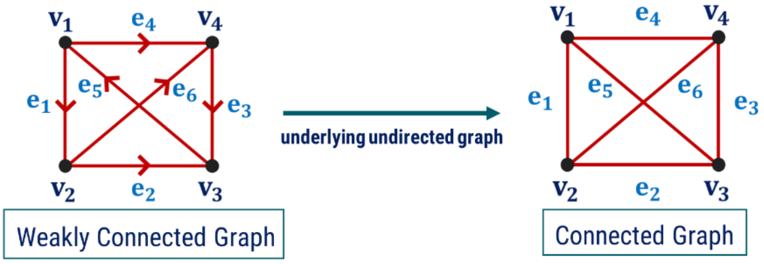
- Here, there is a path from v_1 to v_2 but there is no path from v_2 to v_1.
- Hence, the above graph is unilaterally connected.

Weakly Connected Graph

→ A directed graph G is said to be weakly connected if its underlying undirected graph is connected.

→ i.e., a directed graph G is said to be weakly connected if there is a path between every two vertices when the directions of edges are removed.

→ Example:



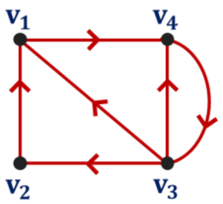
Remarks

→ Any strongly connected directed graph is unilaterally connected as well as weakly connected.

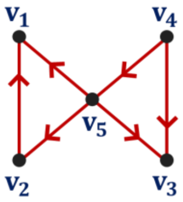
→ Any unilaterally connected directed graph is weakly connected.

→ But every weakly connected directed graph is not necessarily unilaterally connected directed graph.

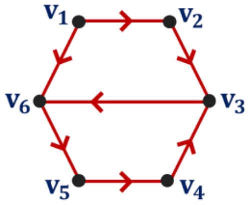
→ Example:



G_1



G_2



G_3

- Here, digraph G_1 is strongly connected.
- Digraph G_2 is weakly connected but not strongly and unilaterally connected as there is no path from v_1 to v_3 and/or v_3 to v_1.
- Digraph G_3 is unilaterally connected but not strongly connected as there is no path from v_6 to v_1.

Strongly Connected Components (SCC)

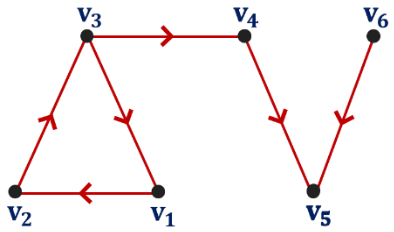
→ The maximal strongly connected subgraphs, are called the strongly connected components or strong components of a digraph G.

→ Maximal strongly connected subgraphs means a subgraph S of G which is strongly connected but no super graph of S which is strongly connected.

→ **Algorithm to find Strongly Connected Components**

- Find largest possible cycle in a given digraph. That cycle is an SCC.
- The remaining vertices independently makes SCC.
- If there is no cycle, then all the vertices make SCC independently.

→ Example:



G

- Here, vertex v_1, v_2 and v_3 forms a cycle, so they form a SCC S_1.
- Also, there is a path from v_4 to v_5 and v_6 to v_5 but there is no path from v_5 to v_4 and v_5 to v_6. So, the vertices v_4, v_5 and v_6 forms a single SCC.

- Strongly Connected Components: $\{ v_1, v_2, v_3 \}, \{ v_4 \}, \{ v_5 \}, \{ v_6 \}$

Unilaterally Connected Components (UCC)

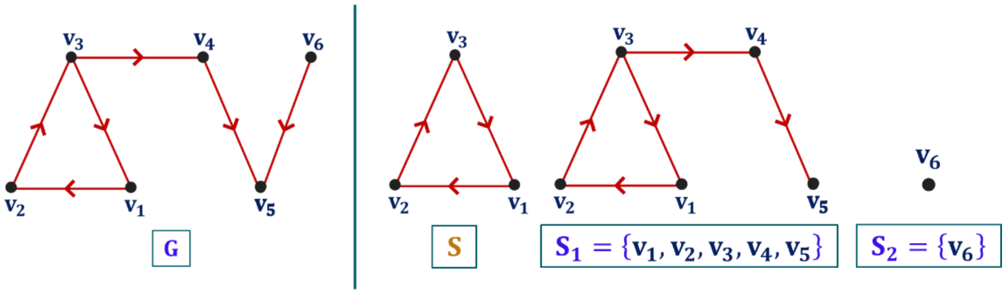
→ The maximal unilaterally connected subgraphs, are called the unilaterally connected components or unilateral components of a digraph G.

→ Maximal unilaterally connected subgraphs means a subgraph S of G which is unilaterally connected but no super graph of S which is unilaterally connected.

→ **Algorithm to find Unilaterally Connected Components**

- Find largest possible path in a given digraph. That path is a UCC.
- The remaining vertices independently makes UCC.

→ Example:



- Here, S is unilaterally connected but it is not maximal unilaterally connected subgraph as S_1 is a super graph of S and S_1 is maximal unilaterally connected subgraph.
- Unilaterally Connected Components: $\{ v_1, v_2, v_3, v_4, v_5 \}, \{ v_6 \}$

Weakly Connected Components (WCC)

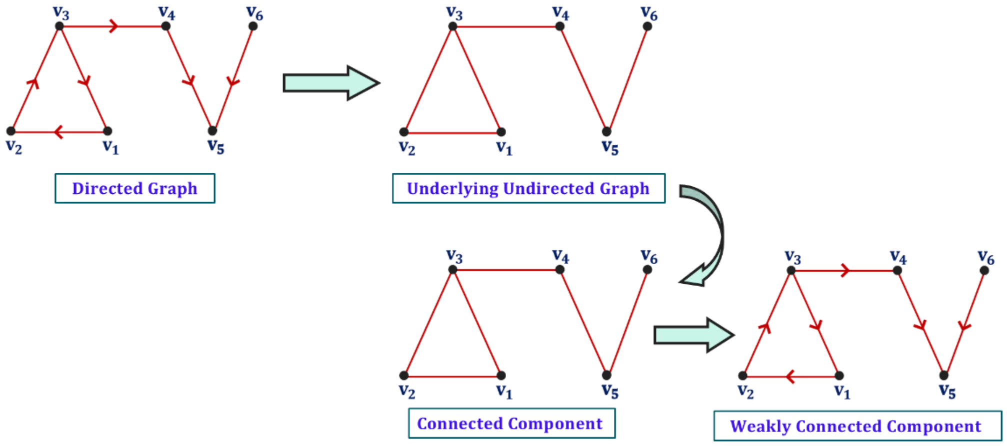
→ The maximal weakly connected subgraphs, are called the weakly connected components or weak components of a digraph G.

→ Maximal weakly connected subgraphs means a subgraph S of G which is weakly connected but no super graph of S which is weakly connected.

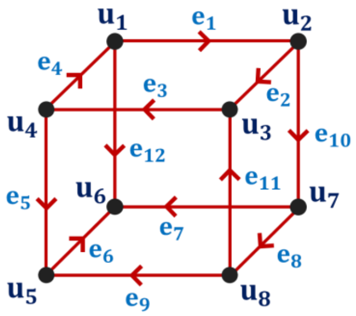
→ **Algorithm to find Weakly Connected Components**

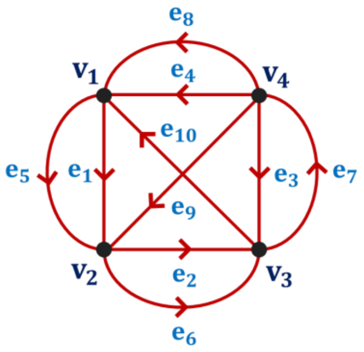
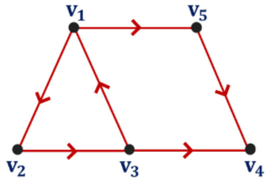
- Construct the underlying undirected graph of the given undirected graph.
- Find all the connected components of the undirected graph.
- The connected components of the undirected graph with the directions will be the weakly connected components of the given directed graph.

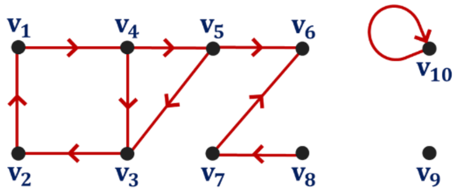
→ Example



Examples of Method-5: Connectivity

C	1	From the given directed graph determine (1) path of length 3, 4, 5 and 6 (2) closed path of length 4 (3) an elementary path length 7 (4) a simple but not elementary path  **Answer**: **(1)** $\mathbf{P_3 = u_8e_{11}u_3e_3u_4e_5u_5,}$ $\mathbf{P_4 = u_1e_1u_2e_2u_3e_3u_4e_4u_1,}$ $\mathbf{P_5 = u_1e_1u_2e_2u_3e_3u_4e_5u_5e_6u_6,}$ $\mathbf{P_6 = u_1e_1u_2e_{10}u_7e_8u_8e_{11}u_3e_3u_4e_4u_1}$ **(2)** $\mathbf{P_4 = u_1e_1u_2e_2u_3e_3u_4e_4u_1}$ **(3)** $\mathbf{P_7 = u_3e_3u_4e_4u_1e_1u_2e_{10}u_7e_8u_8e_9u_5e_6u_6}$ **(4)** $\mathbf{P_4 = u_1e_1u_2e_2u_3e_3u_4e_4u_1,}$ $\mathbf{P_5 = u_4e_4u_1e_1u_2e_2u_3e_3u_4e_5u_5}$

C	2	From the given directed graph determine (1) circuit of length 3, 4, 5 (2) elementary cycle length 4 (3) trail of length 7 (4) an elementary as well as simple cycle (5) simple but not elementary cycle (6) neither elementary nor simple cycle  **Answer**: (1) $C_3 = v_1e_5v_2e_2v_3e_{10}v_1$, $C_4 = v_3e_7v_4e_8v_1e_5v_2e_2v_3$ $C_5 = v_1e_5v_2e_2v_3e_7v_4e_3v_3e_{10}v_1$ (2) $C_4 = v_2e_2v_3e_7v_4e_8v_1e_5v_2$ (3) $C_7 = v_1e_5v_2e_6v_3e_7v_4e_8v_1e_1v_2e_2v_3e_{10}v_1$ (4) $C_4 = v_1e_5v_2e_6v_3e_7v_4e_4v_1$ (5) $C_7 = v_2e_6v_3e_7v_4e_4v_1e_5v_2e_2v_3e_{10}v_1e_1v_2$ (6) $C_8 = v_2e_6v_3e_7v_4e_4v_1e_5v_2e_2v_3e_7v_4e_4v_1e_1v_2$
C	3	Check whether the graph G is strongly connected, unilaterally connected or weakly connected. Also, find its component.  G **Answer**: **G is unilaterally as well as weakly connected graph.** **SCC**: $\{v_1, v_2, v_3\}, \{v_4\}, \{v_5\}$, **UCC & WCC**: $\{v_1, v_2, v_3, v_4, v_5\}$

C	4	Check whether the graph G is strongly connected, unilaterally connected or weakly connected. Also, find its component.  **Answer**: **The given graph is disconnected.** **SCC**: $\{ v_1, v_4, v_5, v_3, v_2 \}, \{ v_6 \}, \{ v_7 \}, \{ v_8 \}, \{ v_9 \}, \{ v_{10} \}$ **UCC**: $\{ v_1, v_4, v_3, v_2, v_5, v_6 \}, \ \{ v_7, v_8 \}, \ \{ v_9 \}, \ \{ v_{10} \}$ **WCC**: $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}, \ \{ v_9 \}, \ \{ v_{10} \}$

Method 6 ⇝ Matrix Representation of a Graph

Adjacency Matrix for an Undirected Graph

→ Let $G = (V, E)$ be an undirected graph with n vertices and without parallel edges.

→ The adjacency matrix of graph $G = (V, E)$ is denoted as A or A_G and is defined as an $n \times n$ matrix $A = [a_{ij}]$ whose elements a_{ij} are given as follows:

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between the vertex } v_i \text{ and } v_j \\ 0, & \text{otherwise} \end{cases}$$

Adjacency Matrix for a Directed Graph

→ Let $G = \langle V, E \rangle$ be a directed graph with n vertices and without multi edges.

→ The adjacency matrix of graph $G = \langle V, E \rangle$ is denoted as A or A_G and is defined as an $n \times n$ matrix $A = [a_{ij}]$ whose elements a_{ij} are given as follows:

$$a_{ij} = \begin{cases} 1, & \text{if there is a directed edge between the vertex } v_i \text{ and } v_j \\ 0, & \text{otherwise} \end{cases}$$

→ **Observations**

- For a directed graph,
  - The sum of all 1's in a row indicates the out – degree of the corresponding vertex.
  - The sum of all 1's in a column indicates the in – degree of the corresponding vertex.
- For a null graph which consists of only n vertices but no edges, the adjacency matrix is a null matrix.
- If a graph has no self – loops, then the diagonal entries of the adjacency matrix are zero.
- If there are loops at each vertex but no other edges in the graph, then the adjacency matrix is the identity matrix.

Incidence Matrix for an Undirected Graph

→ Let G be an undirected graph with n vertices, m edges and without self – loops.

→ The incidence matrix of graph G is denoted as M or M_G and is defined as an $n \times m$ matrix $M = [a_{ij}]$ whose rows corresponds to vertices and columns corresponds to edges such that

$$a_{ij} = \begin{cases} 1, & \text{if edge } e_j \text{ is incident on the vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$

Incidence Matrix for a Digraph

→ Let G be a directed graph with n vertices, m edges and without self – loops.

→ The incidence matrix of graph G is denoted as M or M_G and is defined as an $n \times m$ matrix $M = [a_{ij}]$ whose rows corresponds to vertices and columns corresponds to edges such that

$$a_{ij} = \begin{cases} 1, & \text{if edge } e_j \text{ is incident out of the vertex } v_i \\ -1, & \text{if edge } e_j \text{ is incident in of the vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$

→ **Observations**

- For the Undirected Graph
  - Since every edge is incident on exactly two vertices, each column of incidence matrix has exactly two $1'$s.
  - The number of $1'$s in each row equals the degree of the corresponding vertex.
  - A row with all the entries zero in an incidence matrix represents isolated vertex.
  - The columns corresponding to parallel edges in an incidence matrix are always same.

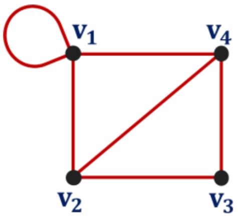
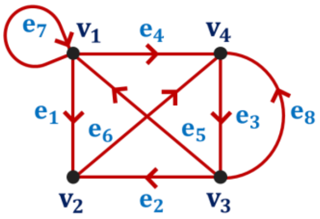
Path Matrix for a Directed Graph

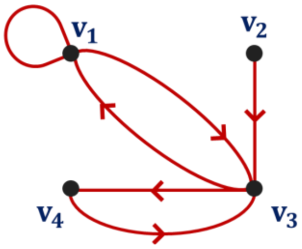
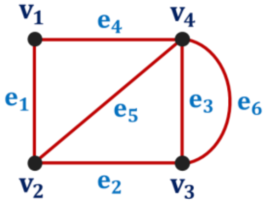
→ Let G be a simple digraph with n vertices $v_1, v_2, \dots, v_n$.

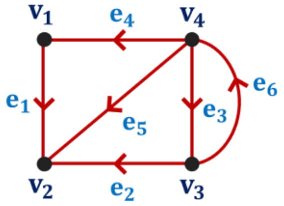
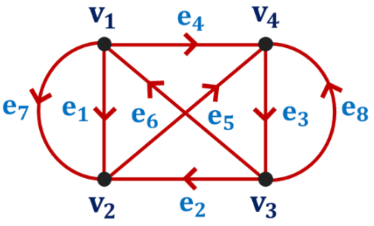
- The path matrix of a digraph graph G is denoted as P and is defined as an $n \times n$ matrix $P = [a_{ij}]$ whose elements a_{ij} are given as follows:

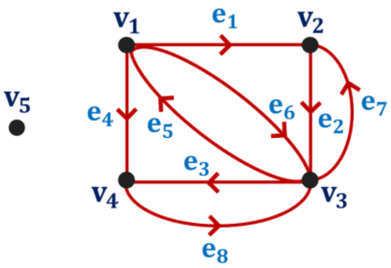
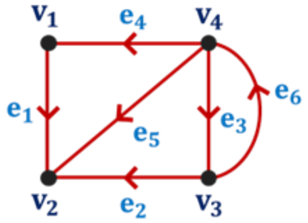
$$a_{ij} = \begin{cases} 1, & \text{if there is a path from the vertex } v_i \text{ to } v_j \\ 0, & \text{otherwise} \end{cases}$$

Examples of Method-6: Matrix Representation of a Graph

C	1	Find an adjacency matrix for the following graph:  **Answer**: $A = \left[\begin{array}{ccccc} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 1 & 1 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 1 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 1 & 1 & 0 & 0 \end{array}\right]$
C	2	Find an adjacency matrix for the following directed graph:  **Answer**: $A = \left[\begin{array}{ccccc} & v_1 & v_2 & v_3 & v_4 \\ v_1 & 1 & 1 & 0 & 1 \\ v_2 & 0 & 0 & 0 & 1 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 \end{array}\right]$

C	3	Draw the directed graph having adjacency matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. **Answer**: 
C	4	Find an incidence matrix for the following graph:  **Answer**: $M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$

C	5	Find an incidence matrix for the following directed graph:  **Answer**: $$M = \begin{array}{c\|cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\ \hline v_1 & 1 & 0 & 0 & -1 & 0 & 0 \\ v_2 & -1 & -1 & 0 & 0 & -1 & 0 \\ v_3 & 0 & 1 & -1 & 0 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 1 & 1 & -1 \end{array}$$
C	6	Determine the incidence matrix for the following directed graph:  **Answer**: $$M = \begin{array}{c\|cccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \\ \hline v_1 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ v_2 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 \\ v_3 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & -1 \end{array}$$

C	7	Draw the directed graph having incidence matrix $A = \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **Answer**: 
C	8	Find the path matrix for the following directed graph:  **Answer**: $\mathbf{P} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

Method 7 ⇝ Warshall's Algorithm

Warshall's Algorithm to Produce a Path Matrix

→ Let G be a directed graph with n vertices $v_1, v_2, \dots, v_n$ and suppose we want to find the path matrix P of the graph G.

→ Define $n + 1$ square Boolean matrices $P_0, P_1, P_2, \dots, P_n$ as follows:

- Let $P_k[i, j]$ denotes the ij^{th} entry of the matrix P_k. Then

$$P_k[i,j] = \begin{cases} 1, & \text{if there is a simple path from } v_i \text{ and } v_j \text{ which does not} \\ & \text{use any other vertices except possible } v_1, v_2, \dots, v_k \\ 0, & \text{otherwise} \end{cases}$$

- That means,
  - $P_0[i,\ j] = 1$, if there is an edge from v_i to v_j.

    Note that, $P_0 = A$ is the adjacency matrix of G.
  - $P_1[i,\ j] = 1$, if there is a simple path from v_i to v_j which does not use any vertex except possibly v_1.
  - $P_2[i,\ j] = 1$, if there is a simple path from v_i to v_j which does not use any vertex except possibly v_1 and v_2.
  - Continue this process until we obtain $P_n[i,\ j]$ or P_n.
- Since G has only n vertices, the last matrix $P_n = P$ is the path matrix of G.

Computation of P_k from P_{k-1}

→ Keep all 1's of P_{k-1} as it is in P_k.

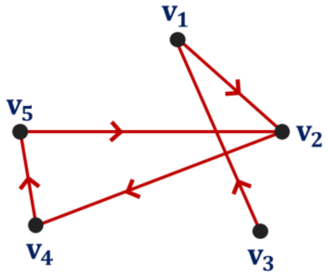
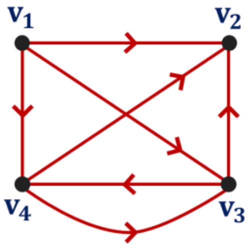
- i.e., if an element of P_{k-1} is 1, then the corresponding entry of P_k is also 1.

→ Consider the k^{th} column of P_{k-1} and list out locations/positions $u_1, u_2, \dots, u_r$; $1 \le r \le n$

→ Consider the k^{th} row of P_{k-1} and list out locations/positions $v_1, v_2, \dots, v_t$; $1 \le t \le n$

→ Place 1 at the location (u_i, v_j) in P_k if 1 is not already there.

Examples of Method-7: Warshall’s Algorithm

C	1	Apply Warshall’s algorithm to produce a path matrix for the given graph.  **Answer**: $P = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$
C	2	Apply Warshall’s algorithm to produce a path matrix for the given graph.  **Answer**: $P = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$