Index

Unit – 3 ⇝ Relation

Cartesian Product

→ Cartesian product of sets is set of ordered pair.

→ Cartesian product of sets A and B is denoted by $\mathbf{A \times B}$ which is read as "A cross B" and defined as follow:

$A \times B = \{ (\mathbf{a}, \mathbf{b}) : \mathbf{a} \in \mathbf{A} \text{ and } \mathbf{b} \in \mathbf{B} \}$

→ Cartesian product of sets B and A is denoted by $\mathbf{B \times A}$ which is read as "B cross A" and defined as follow:

$B \times A = \{ (\mathbf{b}, \mathbf{a}) : \mathbf{b} \in \mathbf{B} \text{ and } \mathbf{a} \in \mathbf{A} \}$

→ For example:

Let $A = \{ a, \ b \}$ and $B = \{ 1, \ 2 \}$.

- Cartesian product of set A and B is

$$A \times B = \{ a, \ b \} \times \{ 1, \ 2 \}$$

$$= \{ (a, 1), \ (a, 2), \ (b, 1), \ (b, 2) \}$$

- Similarly, cartesian product of set B and A is

$$B \times A = \{ 1, \ 2 \} \times \{ a, \ b \}$$

$$= \{ (1, a), \ (1, b), \ (2, a), \ (2, b) \}$$

→ Properties of Cartesian product

(1) If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.

(2) If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$.

(3) Generally, $A \times B \neq B \times A$.

(4) $A \times B = B \times A$ if and only if $A = B$.

Relation or Binary Relation

→ Let A and B be two non-empty sets.

→ **Subset** of Cartesian product $\mathbf{A \times B}$ is known as relation from A to B.

→ It is denoted by **R** and read as "relation R".

i.e.

$$\mathbf{R = \{ (a, b) \mid a \in A \text{ and } b \in B \} \subseteq A \times B}$$

→ Note that, if $| A | = p$ and $| B | = q$,

then total number of relations from A to B or B to A is $\mathbf{2^{pq}}$.

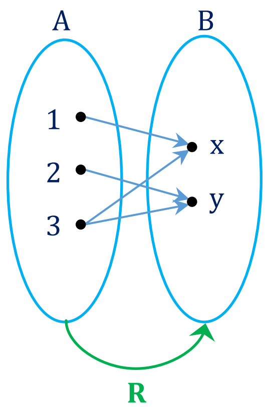
→ In any relation ordered pair of the form (a, a) is known as **diagonal pair**.

→ For example:

- Let $A = \{ 1, 2, 3 \}$ and $B = \{ x, y \}$.

  $A \times B = \{ (1, x), (1, y), (2, x), (2, y), (3, x), (3, y) \}$.

  Let, $R = \{ (1, x), (2, y), (3, x), (3, y) \}$



This diagram is known as **Arrow Diagram** of relation R.

→ For any relation R,

- If $(a, b) \in R$, then it is denoted by **aRb** and read as "a is related to b".
- If $(a, b) \notin R$, then it is denoted by $\mathbf{a \not R b}$ and read as "a is **not** related to b".

→ For example:

- $R = \{ (1, x), (2, y), (3, x), (3, y) \}$

  $(1, x) \in R$ so, it is denoted as **1Rx**.

  $(1, y) \notin R$ so, it is denoted as $\mathbf{1 \not R y}$.

Relation on a Set:

→ Relation on a set A is a relation from A to A.

i.e., $R \subseteq A \times A$

→ For example:

- Let $A = \{ 1, 2, 3 \}$

  So, relation R on a set A can be

  $R = \{ (1, 1), (1, 3), (2, 3), (3, 2), (3, 3) \}$

  Here, diagonal pairs are $(1, 1)$ and $(3, 3)$.

Method – 1 ⇝ Properties of Relation

Properties of Relation

→ There are several properties that are used to classify relations on a set.

(1) Reflexive Relation

(2) Irreflexive Relation

(3) Symmetric Relation

(4) Asymmetric Relation

(5) Anti-symmetric Relation

(6) Transitive Relation

Reflexive Relation

→ A relation R on a set A is **reflexive** if $(a, a) \in R$, $\forall a \in A$. "∀" means "for every"

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 1), (1, 2), (1, 4), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)\}$

    Here, $(1, 1), (2, 2), (3, 3), (4, 4) \in R_1$

    Hence, R_1 is reflexive.
  - $R_2 = \{(1, 1), (1, 4), (2, 2), (2, 4), (3, 3), (3, 1)\}$

    Here, $(4, 4) \notin R_2$

    Hence, R_2 is **not** reflexive.

Irreflexive Relation

→ A relation R on a set A is **irreflexive** if $(a, a) \notin R$, $\forall a \in A$.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 2), (1, 4), (2, 3), (3, 4)\}$

    Here, $(1, 1), (2, 2), (3, 3), (4, 4) \notin R_1$.

    Hence, R_1 is irreflexive.

- $R_2 = \{ (2, 3), (2, 4), (4, 4) \}$

  Here, $(4, 4) \in R_1$.

  Hence, R_2 is **not** irreflexive.

Symmetric Relation

→ A relation R on a set A is **symmetric** if whenever **(a, b) ∈ R**, then **(b, a) ∈ R**, $\forall$ a, b ∈ R.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{ 1, 2, 3, 4 \}$.
  - $R_1 = \{ (1, 2), (2, 1), (2, 3), (3, 2), (3, 3)\}$

    Here, $(1, 2), (2, 1) \in R_1$, $(2, 3), (3, 2) \in R_1$ and $(3, 3) \in R_1$.

    Hence, R_1 is symmetric.
  - $R_2 = \{ (2, 1), (2, 3) \}$

    Here, $(2, 1) \in R_2$ but $(1, 2) \notin R_2$.

    Hence, R_2 is **not** symmetric.

Asymmetric Relation

→ A relation R on a set A is **asymmetric** if whenever **(a, b) ∈ R**, then **(b, a) ∉ R**, $\forall$ a, b ∈ R.

→ Asymmetric relation does not contain diagonal pairs.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{ 1, 2, 3, 4 \}$.
  - $R_1 = \{ (1, 2), (2, 3), (3, 2) \}$

    Here, $(2, 3), (3, 2) \in R_1$.

    Hence, R_1 is **not** asymmetric.
  - $R_2 = \{ (2, 1), (2, 3) \}$

    Here, $(2, 1) \in R_2$ but $(1, 2) \notin R_2$ and $(2, 3) \in R_2$ but $(3, 2) \notin R_2$.

    Hence, R_2 is asymmetric.

Antisymmetric Relation

→ A relation R on a set A is **antisymmetric** if whenever $(a, b) \in R$ and $(b, a) \in R$, then $a = b$, $\forall a, b \in R$.

→ If $(a, b) \in R$ and $(b, a) \notin R$, then there is no need to discuss $a = b$ or $a \neq b$.

OR

→ A relation R on a set A is antisymmetric if whenever $(a, b) \in R$ and $a \neq b$, then $(b, a) \notin R$, $\forall a, b \in R$.

→ Antisymmetric relation may contain diagonal pairs.

→ For example:

- Let R_1 and R_2 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 2), (2, 3), (2, 2)\}$

    Here, $(1, 2) \in R_1$ but $(2, 1) \notin R_1$, $(2, 3) \in R_1$ but $(3, 2) \notin R_1$.

    So, no need to discuss $a = b$.

    Hence, R_1 is antisymmetric.
  - $R_2 = \{(2, 1), (1, 2)\}$

    Here, $(2, 1), (1, 2) \in R_2$ but $1 \neq 2$.

    Hence, R_2 is **not** antisymmetric.

Transitive Relation

→ A relation R on a set A is **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, $\forall a, b, c \in R$.

→ If $(a, b) \in R$ and $(b, c) \notin R$, then there is no need to discuss $(a, c) \in R$ or $(a, c) \notin R$.

→ For example:

- Let R_1, R_2 and R_3 be relations on a set $A = \{1, 2, 3, 4\}$.
  - $R_1 = \{(1, 2), (2, 3), (2, 2)\}$

    Here, $(1, 2), (2, 3) \in R_1$ but $(1, 3) \notin R_1$.

    Hence, R_1 is **not** transitive.
  - $R_2 = \{(2, 1), (3, 2), (3, 1)\}$

    Here, for $(3, 2), (2, 1) \in R_2$, $(3, 1) \in R_2$.

    Hence, R_2 is transitive.

- $R_3 = \{ (2, 3), (2, 1), (4, 1)\}$

  Here, for $(a, b) \in R_2$, $(b, c) \notin R_2$.

  So, there is no need to discuss about (a, c).

  Hence, R_3 is transitive.

Examples of Method-1: Properties of Relation

C	1	For each of these relations on the set $A = \{ 1, 2, 3, 4 \}$, determine whether it is reflexive, irreflexive, symmetric, asymmetric, antisymmetric or transitive. **(1)** $R_1 = \{ (1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (4, 4) \}$ **(2)** $R_2 = \{ (1, 1), (2, 2), (3, 3), (4, 4) \}$ **(3)** $R_3 = \{ (1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4) \}$ **Answer: R_1 is reflexive, symmetric and transitive** **R_2 is reflexive, symmetric, antisymmetric and transitive** **R_3 is irreflexive**
C	2	Let $A = \{ 1, 2, 3, 4, 5, 6 \}$ and define a relation R on A as $R = \{ (x, y) \mid y \text{ is divisible by } x \}$ Check whether R is reflexive, symmetric or transitive. **Answer: R is reflexive and transitive.**

Method – 2 ⇝ Matrix and Graph Representation of a Relation

Matrix Representation of a Relation

→ A relation between finite sets can be represented using a zero–one matrix.

→ Suppose that R is a relation from $A = \{ a_1, a_2, \dots, a_m \}$ to $B = \{ b_1, b_2, \dots, b_n \}$.

→ The relation R can be represented by the matrix $M_R = [\, m_{ij} \,]_{m \times n}$, where

$$m_{ij} = \begin{cases} 1, & \text{if } (a_i, b_j) \in R, \\ 0, & \text{if } (a_i, b_j) \notin R. \end{cases}$$

→ A matrix of relation R on a set A is always square matrix.

→ For example:

- Let $A = \{ 1, 2, 3 \}$, $B = \{ x, y \}$ and R be the relation from A to B defined as $R = \{ (1, x), (2, y), (3, x), (3, y) \}$.

  The matrix of R is

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \!\!\! \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}$$

Graph (Digraph) Representation of a Relation

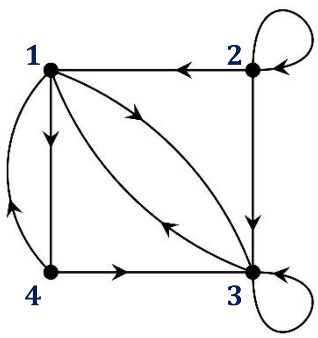
→ The relation R on a set A is represented by the **directed graph** that has the **elements of A** as its **vertices** and the **ordered pairs** $(a, b) \in R$, as **edges**.

→ For example:

- Let $A = \{ 1, 2, 3, 4 \}$ and R be the relation on A defined as

  $R = \{ (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3), (4, 1), (4, 3) \}$

  The directed graph of the elation R is as follow:



Examples of Method-2: Matrix and Graph Representation of a Relation

<table>
<tr><td>C</td><td>1</td><td>

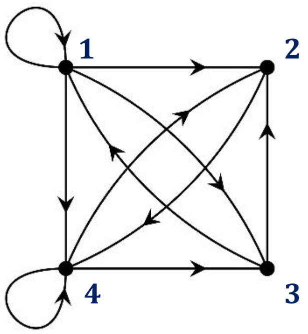
Represent given relation R on $\{1, 2, 3, 4\}$ with a matrix also draw a directed graph of it.

$$R = \begin{Bmatrix} (1,1), & (1,2), & (1,3), & (1,4), & (2,4), \\ (3,1), & (3,2), & (4,2), & (4,3), & (4,4) \end{Bmatrix}$$

Answer: Matrix Representation of a Relation:

$$M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Digraph of a Relation:



</td></tr>
</table>

Method – 3 ⇝ Partition and Covering of a Set

Covering and Partition of a Set

→ Let S be a nonempty finite set. Let $A_1,\ A_2,\ \dots,\ A_k$ be nonempty subsets of S.

→ If $\bigcup_{i=1}^{k} \mathbf{A_i} = \mathbf{S}$, then $\{\ A_1,\ A_2,\ \dots,\ A_k\ \}$ is known as **covering** of a set S.

→ Furthermore, if $\mathbf{A_i} \cap \mathbf{A_j} = \boldsymbol{\phi}, \forall\ \mathbf{A_i} \neq \mathbf{A_j}$, then $\{\ A_1,\ A_2,\ \dots,\ A_k\ \}$ is known as **partition** of a set S.

→ The sets $A_1,\ A_2,\ \dots,\ A_k$ are known as **block** of partition.

→ Note that, every partition is covering but covering may not be partition.

→ For example:

- Let B_1, B_2 and B_3 be collections of subsets of $S = \{\ 1,\ 2,\ \dots,\ 8,\ 9\ \}$
  - $B_1 = \{\ \{\ 1,\ 3,\ 5\ \},\ \ \{\ 2,\ 6\ \},\ \ \{\ 4,\ 8,\ 9\ \}\ \}$

    Here, **7** does not belong to any of the subset.

    Hence, B_1 is neither covering nor partition of S.
  - $B_2 = \{\ \{\ 1,\ 3,\ 5\ \},\ \ \{\ 2,\ 4,\ 6,\ 8\ \},\ \ \{\ 5,\ 7,\ 9\ \}\ \}$

    Here, $\{\ 1,\ 3,\ 5\ \} \cup \{\ 2,\ 4,\ 6,\ 8\ \} \cup \{\ 5,\ 7,\ 9\ \} = S$ but

    $$\{\ 1,\ 3,\ 5\ \} \cap \{\ 5,\ 7,\ 9\ \} \neq \boldsymbol{\phi}$$

    Hence, B_2 is covering but not partition.
  - $B_3 = \{\ \{\ 1,\ 3,\ 5\ \},\ \ \{\ 2,\ 4,\ 6,\ 8\ \},\ \ \{\ 7,\ 9\ \}\ \}$

    Here, $\{\ 1,\ 3,\ 5\ \} \cup \{\ 2,\ 4,\ 6,\ 8\ \} \cup \{\ 5,\ 7,\ 9\ \} = S$ and

    all subsets are mutually disjoint.

    Hence, B_3 is **partition** of S.

    Since, it is partition it is **covering**.

    $\{\ 1,\ 3,\ 5\ \},\ \ \{\ 2,\ 4,\ 6,\ 8\ \}$ and $\{\ 7,\ 9\ \}$ are blocks of partition.

Examples of Method-3: Partition and Covering of a Set

C	1	Let $S = \{a, b, c, d, e, f, g\}$ be a set and $A_1 = \{a, c, e\}$, $A_2 = \{b\}$, $A_3 = \{d, g\}$, $A_4 = \{d, f\}$, $A_5 = \{f\}$ be subsets of S. Determine which of the collection is covering or partition: $P_1 = \{A_1, A_2, A_3\}$, $P_2 = \{A_1, A_2, A_3, A_4, A_5\}$ $P_3 = \{A_1, A_2, A_3, A_5\}$ **Answer: P_1 is neither covering nor partition,** **P_2 is covering but not partition,** **P_3 is covering and partition.**

Method – 4 ⇝ Equivalence Relation

Equivalence Relation

→ A relation on a set A is called an **equivalence relation** if it is **reflexive**, **symmetric**, and **transitive**.

→ For example:

- Let $A = \{ 1, 2, 3, 4 \}$ and

  $R = \{ (1, 1), (1, 4), (4, 1), (4, 4), (2, 2), (2, 3), (3, 2), (3, 3) \}$

  Here R is reflexive, symmetric and transitive.

  So, R is an equivalence relation.

→ Let R be an equivalence relation on a set A.

→ The set of all elements that are **related to an element "a"** of A is known as the equivalence class of "a".

→ The equivalence class of "a" with respect to R is denoted by $[\,a\,]_R$ or simply $[\,a\,]$.

→ In other words, if R is an equivalence relation on a set A, the equivalence class of the element "a" is

$[\,\mathbf{a}\,]_{\mathbf{R}} = \{ \mathbf{x} \in \mathbf{A} \mid (\mathbf{x}, \mathbf{a}) \in \mathbf{R} \}$ or $[\,\mathbf{a}\,]_{\mathbf{R}} = \{ \mathbf{x} \in \mathbf{A} \mid \mathbf{xRa} \}$.

→ If $b \in [\,a\,]$, then b is called a **representative** of this equivalence class.

→ Any element of a class can be used as a representative of this class.

Properties of Equivalence Class

→ Let R be an equivalence relation on a set A and let $a, b \in A$.

(1) For all $a \in A$ we have $a \in [\,a\,]$.

(2) $[\,a\,] = [\,b\,] \Leftrightarrow aRb$.

(3) If $[\,a\,] \neq [\,b\,]$, then $[\,a\,] \cap [\,b\,] = \phi$.

Examples of Method-4: Equivalence Relation

C	1	Let L is a set of straight lines. Let R be a relation on set L defined as $R = \{ (l_1, l_2) \mid l_1 \parallel l_2 \text{ and } l_1, l_2 \in L \}$. Prove that R is an equivalence relation.

C	2	Let R be a relation on a set $\mathbb{Z}$ defined as follow: (**1**) $aRb \Leftrightarrow a - b$ is divisible by 3. (**2**) $(a, b) \in R \Leftrightarrow a + b$ is odd. Determine whether R is an equivalence relation or not. **Answer**: (**1**) **R is an equivalance relation** (**2**) **R is not an equivalance relation**
C	3	Let R be an equivalence relation on a set $A = \{0, 1, 2, 3, 4\}$ defined as $R = \left\{ \begin{matrix} (0, 0), & (0, 4), & (1, 1), & (1, 3), & (2, 2), \\ (3, 1), & (3, 3), & (4, 0), & (4, 4) & \end{matrix} \right\}$ Find the distinct equivalence classes of R. **Answer**: **Distinct equivalence classes**: $\{\mathbf{0}, \mathbf{4}\}$, $\{\mathbf{1}, \mathbf{3}\}$, $\{\mathbf{2}\}$

Method – 5 ⇝ Partially Ordered Relation

→ A relation R on a set P is known as a **partial ordering** or **partial order** if it is **reflexive**, **antisymmetric** and **transitive**.

→ A set P together with a partial ordering R is known as a **partially ordered set**, or **poset**.

→ Poset is denoted by $(P, \preccurlyeq)$.

→ Because of the special role played by the $\leq$ relation in the study of partial order relations, the symbol "$\preccurlyeq$" is often used to refer to a general partial order relation.

→ For example:

- Let P be a set of real numbers and define the "less than or equal to" relation, $\leq$ on P as follows:

  For all real numbers x and y in P, $\mathbf{x \leq y \Leftrightarrow x < y \text{ or } x = y}$.

  - **Reflexive:**

    Let $x \in P$.

    We know that

    $x = x \quad \Rightarrow \quad x \leq x$

    So, $\leq$ is reflexive.

  - **Antisymmetric:**

    Let $x, y \in \mathbb{Z}$ such that $x \leq y$ and $y \leq x$.

    $\Rightarrow x < y$ or $x = y$ and $y < x$ or $y = x$

    $\Rightarrow x = y$ or $y = x$ So, $\leq$ is antisymmetric.

  - **Transitive:**

    Let $x, y, z \in \mathbb{Z}$ such that $x \leq y$ and $y \leq z$.

    $\Rightarrow x < y$ or $x = y$ and $y < z$ or $y = z$

    $\Rightarrow x < z$ or $x = z$

    So, $\leq$ is transitive.

    So, a relation $\leq$ is reflexive, antisymmetric and transitive.

    Hence, a relation $\leq$ is a partial order relation.

Examples of Method-5: Partially Ordered Relation

C	1	Determine whether a relation R defined as “divisibility” on $\mathbb{Z}^+$ is partially ordered relation or not. **Answer: R is partially ordered relation.**
C	2	Prove that $(\mathbb{Z}, \preccurlyeq)$ is a poset where, $\preccurlyeq$ be a relation defined on $\mathbb{Z}$ as follow: $a \preccurlyeq b \Leftrightarrow b = a^r$ for some $r \in \mathbb{N}, \forall\, a, b \in \mathbb{Z}$

Method 6 ⇝ Hasse–diagram

Cover

→ Let $(A, \leq) = (A, R)$ be a poset and $a, b \in A$ with **aRb** then element **b** is known cover of an element **a** if there is no $\mathbf{c} \in \mathbf{A}$ such that **aRc** and **cRb**, where $c \neq a, c \neq b$.

→ For Example:

$(\{2,\ 4,\ 6,\ 8,\ 10,\ 12\},\ D)$ is a poset.

Cover of $2 = 4,\ 6,\ 10$

But **12 cannot** be a cover of 2, because **2R6** and **6R12** exist in relation R.

Cover of $4 = 8,\ 12$

Cover of $6 = 12$

Cover of 8 is **not** possible

Cover of 10 is **not** possible

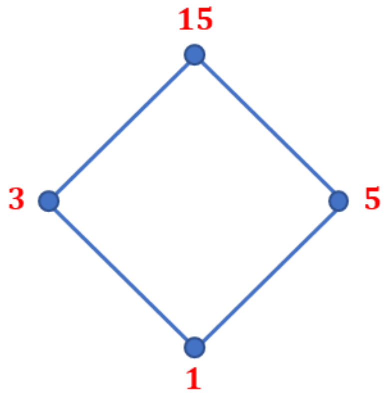
Cover of 12 is **not** possible.

Hasse Diagram

→ Let $(A, \leq) = (A, R)$ be a poset. Representation of poset as diagram in plane is known as Hasse diagram.

→ For Example:

$(\{ 1,\ 3,\ 5,\ 15 \},\ D)$, Where "**aDb**" means a divides b

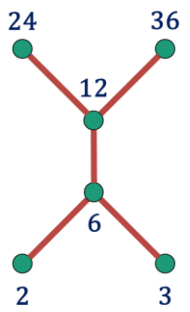
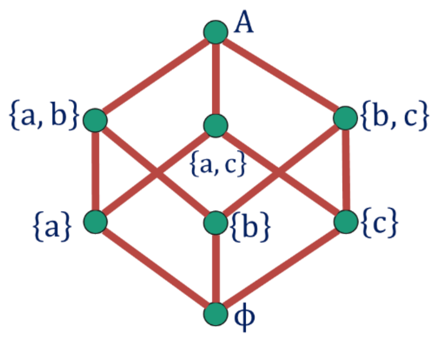


→ **How to draw Hasse–diagram?**

- Each element of set is represented by circle or dot.
- If **"a"** is covered by **"b"** then draw the circle for **"a"** below the circle of **"b"** and draw a single line segment between **"a"** and **"b"**.
- If **"a"** is not covered by **"b"** then do not connect **"a"** and **"b"** by a line segment directly.
- In this manner obtained diagram is known as **Hasse–diagram**.

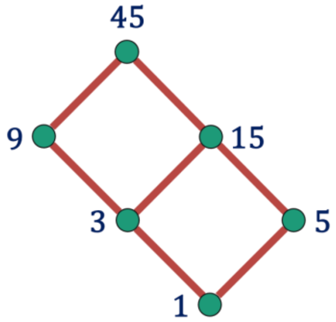
→ **Remark:** In Hasse–diagram elements at the same level are not comparable.

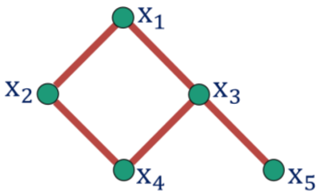
Examples of Method-6: Hasse–diagram

C	1	Draw the Hasse–diagram for poset $(A, \le), A = \{2, 3, 6, 12, 24, 36\}$ And "$a \le b$" if "a divides b". Find cover of each element of set A if possible. **Answer :** **Cover of 2 = 6 Cover of 3 = 6** **Cover of 6 = 12 Cover of 12 = 24, 36** 
C	2	Draw the Hasse–diagram of poset $(P(X), \subseteq)$, Where $X = \{a, b, c\}$. Find cover of each element of set P(X) if possible. **Answer :** 

<table>
<tr><td>C</td><td>3</td><td>Draw the Hasse diagram of $(S_{45}, |)$, where S_n is set of factors (divisors) of positive integer n.

Answer :


</td></tr>
<tr><td>C</td><td>4</td><td>Hasse diagram of a poset (P, R), where $P = \{x_1, x_2, x_3, x_4, x_5\}$, is given below. Find out which of the followings are true?



<table>
<tr><td>(a) $x_1 \, R \, x_2$</td><td>(b) $x_3 \, R \, x_5$</td><td>(c) $x_1 \, R \, x_1$</td><td>(d) $x_4 \, R \, x_5$</td></tr>
<tr><td>(e) $x_4 \, R \, x_1$</td><td>(f) $x_2 \, R \, x_5$</td><td>(g) $x_2 \, R \, x_3$</td><td></td></tr>
</table>
Answer:
(e) and (c)</td></tr>
</table>

Method 7 ⇝ Totally Ordered Set

Chains

→ Let $(A, \leq)$ be a poset and B be a subset of A. Subset B is known as chain if every elements are comparable (related).

For Example:

$(A, \leq)$ be a poset and $A = \{ 1, 2, 3, 5, 6, 10, 15, 30 \}$.

aRb if a divides b, then

Chains are:

$\{ 1, 2, 6, 30 \}$, $\{ 1, 3, 6, 30 \}$, $\{ 1, 2, 10, 30 \}$, $\{ 1, 3, 15, 30 \}$etc.

AntiChains

→ Let $(A, \leq)$ be a poset and B be a subset of A. Subset B is known as antichain if no two distinct elements are comparable.

→ For Example:

From above example,

AntiChains are:

$\{ 2, 3 \}$, $\{ 2, 5 \}$, $\{ 3, 10 \}$, $\{ 5, 6 \}$ etc.

Totally Ordered Set

→ Let $(A, \leq)$ be any poset. If any two elements of set A are comparable, then poset $(A, \leq)$ is known as totally ordered set or **Toset**.

→ For Example:

(4) Let$A = \{ 1, 2, 3 \}$and

$R = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$

Then $(A, \leq)$ is totally ordered set,

Because, any two elements of A are comparable in relation R.

(5) Let$A = \{ 1, 2, 3 \}$and

$R = \{(1,1), (1,3), (2,2), (2,3), (3,3)\}$

Then $(A, \leq)$ is **not** totally ordered set,

Because, elements 1 and 2 of A are **not comparable** in relation R.

→ Totally ordered set is known as **Linearly Ordered Set (LOSET)** or **simply ordered set** or **chain** also.

→ **Remark:**

(6) Every toset is poset.

(7) Every poset may not be toset.

- For Example:

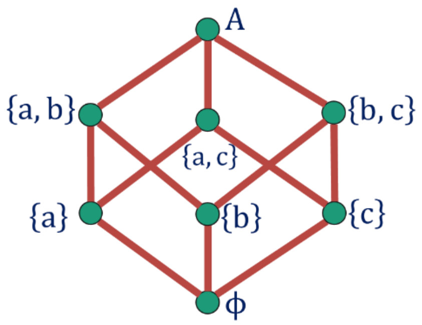
  $(\mathbb{N}, D)$ is a poset but not toset, Where D is relation of divisibility.

  Because, $(2,3) \notin D$.

  Hence 2 and 3 are not comparable.

  $\therefore$ $(\mathbb{N}, D)$ is not toset.

Examples of Method-7: Totally Ordered Set

C	1	Draw the Hasse diagram of the following sets under the partial order relation "divides" and indicate those which are totally ordered sets. (a) $\{2,6,24\}$, (b) $\{3,5,15\}$, (c) $\{1,2,3,6,12\}$ and (d) $\{3,9,27,54\}$ **Answer : (a) and (d)**
C	2	From following hasse diagram find any two Chains and AntiChains.  **Answer : Chains** $= \{\{a\}, \{a, b\}\}$, $\{\{b, c\}, \{a, b, c\}\}$ **AntiChains** $= \{\{a\}, \{b\}\}$, $\{\{b\}, \{c\}\}$

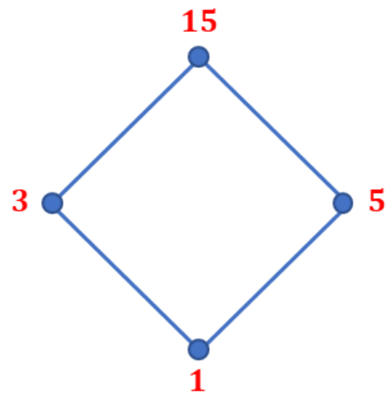
Method 8(a) ⇝Least Member and Greatest Member

Least Member

Let (A, R) be any poset. An element x ∈ A is known as Least Member of A if xRy, ∀y ∈ A.

→ For Example:

Hasse–diagram of poset ({ 1, 3, 5, 15 }, D), Where **D** indicates divisible relation.



From Hasse–diagram Least Member is **1**.

Greatest Member

→ Let (A, R) be any poset. An element x ∈ A is known as Greatest Member of A

if yRx, ∀y ∈ A

→ For Example:

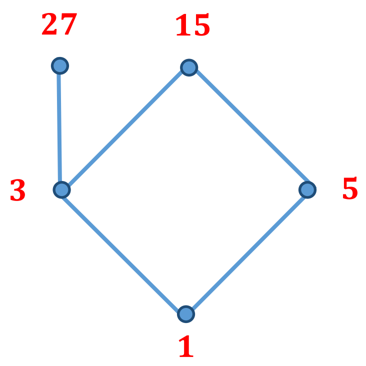
From above Hasse–diagram Greatest Member is **15**.

→ **Note:**

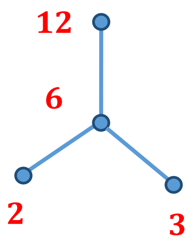
- Least Member and Greatest Member can be found from Hasse–diagram very easily.
- If Least Member and Greatest Member exist, then they are unique.
- It is possible that either Least Member or Greatest Member or both does not exist.
- In chain, Least Member and Greatest Member always exist.

→ For Example:

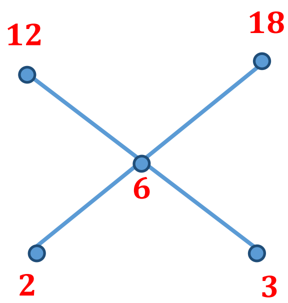
- For following Hasse–diagram: Only Least Member **"1"** exists but Greatest Member does not exist.



- For following Hasse–diagram: Only Greatest Member **"12"** exists but Least Member does not exist.



- For following Hasse–diagram: Neither Least Member nor Greatest Member exist.



Method 8(b) ⇝ Minimal Elements and Maximal Elements

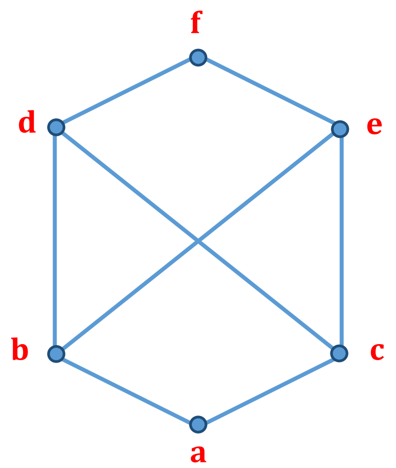
Minimal Element

→ Let (A, R) be any poset. An element x ∈ A is known as Minimal Element of A,

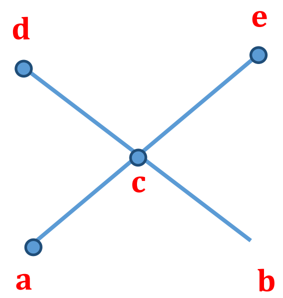
if there is no y ∈ A such that yRx or (y, x) ∈ R and y ≠ x.

→ For Example:

(8) From below Hasse–diagram Minimal Element is **"a"**, because no element is related to a.



(9) From below Hasse–diagram Minimal Elements are **"a"** and **"b"**, because no element is related to **"a"** and **"b"**.



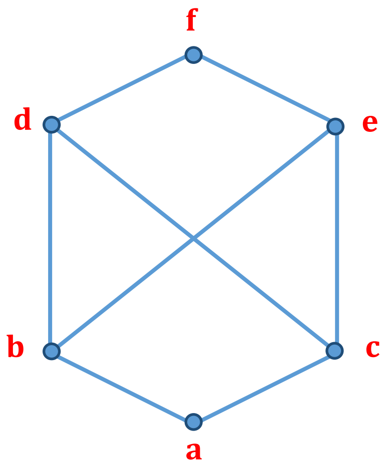
Maximal Elements

→ Let (A, R) be any poset. An element x ∈ A is known as Maximal Element of A,

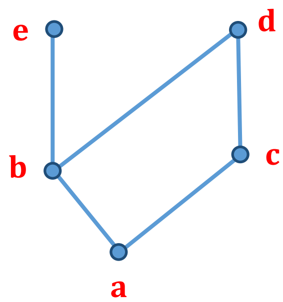
if there is no y ∈ A such that xRy or (x, y) ∈ R and y ≠ x.

→ For Example:

(10) From below Hasse–diagram Maximal Element is **"f "** because **" f "** is not related to any element.



(11) From below Hasse–diagram Maximal Elements are **"e"** and **"d"** because **"e"** and **"d"** are not related to any element.



→ **Note:**

- Minimal and Maximal elements can be found from Hasse–diagram very easily.
- Minimal and Maximal elements can be more than one.
- You are **maximal/minimal** when there is nobody **above/below** you.
- You are **greatest/least** when you are **above/below** everyone else.

Method 8(c) ⇝ Least Upper Bound and Greatest Lower Bound

Least Upper Bound

→ Let (A, R) be any poset and $B \subseteq A$. An element $x \in A$ is known as Least Upper Bound for B if x is an upper bound for B and xRy, Where y is upper bound of B.

→ It is denoted by **"LUB"**.

→ For Example:

Let ({ 2, 3, 6, 12, 24, 36 }, D), Where **D** indicates divisible relation.

For subset B = { 6, 12 }, upper bounds are 12, 24 and 36.

But among them 12 is least upper bound.

∴**12** is LUB of subset B.

→ Least Upper Bound is known as **Supremum (sup)** also.

Greatest Lower Bound

→ Let ⟨ A, R⟩ be any poset and $B \subseteq A$. An element $x \in A$ is known as Greatest Lower Bound for B if x is an lower bound for B and yRx, Where y is lower bound for B.

→ It is denoted by **"GLB"**.

→ For Example:

Let ({ 2, 3, 6, 12, 24, 36 }, D), Where **D** indicates divisible relation.

For subset B = { 6, 12 }, lower bounds are 2, 3 and 6.

But among them 6 is greatest lower bound.

→ ∴ **6** is GLB of subsetB.

→ Greatest Lower Bound is known as **Infimum (inf)** also.

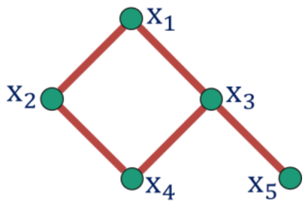
→ **Note:**

- If LUB and GLB exist, then they are unique.
- For a chain, every subset has LUB and GLB.
- For a poset, it is not necessary that every subset has LUB and GLB.

Examples of Method-8: Least and Greatest Members, Minimal and Maximal Elements, & Least Upper Bound and Greatest Lower Bound

C 1

For the given poset,
(1) Find least and greatest member in **P** if exists.
(2) Find minimal and maximal elements.



Answer :
(1) Greatest member is x_1 and least member does not exist.
(2) Minimal elements are x_4 and x_5 and maximal element is x_1.

C 2

Let $\mathbf{P} = \{\mathbf{2},\ \mathbf{3},\ \mathbf{6},\ \mathbf{12},\ \mathbf{24},\ \mathbf{36}\}$ and the relation $\leq$ be such that $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}$ divides $\mathbf{y}$. Find upper bounds, lower bounds, LUB and GLB if exists for sets:

$(\mathbf{a})\ \{\mathbf{2},\ \mathbf{3},\ \mathbf{6}\}, (\mathbf{b})\ \{\mathbf{2},\ \mathbf{3}\}, (\mathbf{c})\ \{\mathbf{12},\ \mathbf{6}\}, (\mathbf{d})\ \{\mathbf{24},\ \mathbf{36}\}, (\mathbf{e})\ \{\mathbf{3},\ \mathbf{12},\ \mathbf{24}\}.$

Answer :

	(a)	(b)	(c)	(d)	(e)
Lower Bound	**DNE**	**DNE**	**2, 3, 6**	**2, 3, 6, 12**	**3**
GLB	**DNE**	**DNE**	**6**	**12**	**3**
Upper Bound	**6,12,24,36**	**6,12,24,36**	**12,24,36**	**DNE**	**24**
LUB	**6**	**6**	**12**	**DNE**	**24**

*** * * * * End of the Unit * * * * ***