Index

Unit – 2 ⇝ Recurrence Relation and Propositional Logic

Unit – 2.1 ⇝ Recurrence Relation

Method – 1 ⇝ Recurrence Relation

Fibonacci Sequence

→ The Fibonacci sequence, also known as Fibonacci numbers, is defined as the sequence of numbers in which each number in the sequence is equal to the sum of two numbers before it.

→ The Fibonacci Sequence is given as:

$$\underbrace{0, \quad 1,}_{+} \quad \underbrace{1,}_{+} \quad \underbrace{2,}_{+} \quad \underbrace{3,}_{+} \quad \underbrace{5,}_{+} \quad 8, \quad 13, \quad 21, \quad \dots$$

→ So, if we take $F_0 = 0$ and $F_1 = 1$.

→ $F_2 = F_0 + F_1 = 1, \qquad F_3 = F_1 + F_2 = 2, \qquad F_4 = F_2 + F_3 = 3$

→ In general, Fibonacci sequence can be defined as $\mathbf{F_n = F_{n-1} + F_{n-2}, \ n \geq 2}$

provided $F_0 = 0$ & $F_1 = 1$.

→ Here, $F_n = F_{n-1} + F_{n-2}$ is known as recurrence relation and

$F_0 = 0$ & $F_1 = 1$ are known as initial conditions of recurrence relation.

Recurrence Relation

→ A recurrence relation for the sequence is a formula that expresses $\{a_n\}$ in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \dots, a_{n-1}$.

Method – 2 ⇝ Solution of Linear Recurrence Relation using Undetermined Coefficient Method

Linear Recurrence Relation

→ A recurrence relation of the form

$$\mathbf{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = F(n), \quad n \geq k} \qquad \dots(\mathbf{1})$$

is known as linear recurrence relation of k^{th} order with constant coefficients, where $\mathbf{c_0}, \mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_k}$ are constants with $c_k \neq 0$ and $c_0 \neq 0$, and $F(n)$ is a function of n.

→ If $\mathbf{F(n) = 0}$ then the relation is known as **homogeneous**, otherwise it is **non-homogeneous**.

→ For example:

- $2a_n + 3a_{n-1} = 0$ **Homogeneous**
- $3a_n - 5a_{n-1} + 2a_{n-2} = n^2 + 5$ **Non-homogeneous**

→ Note that, the recurrence relation

$$\mathbf{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0}$$

is known as **associated homogeneous recurrence relation** of given recurrence relation $(\mathbf{1})$.

→ It plays an important role in the solution of the non-homogeneous recurrence relation.

Solution of Linear Recurrence Relation using Undetermined Coefficient Method

→ If $\mathbf{a_n^{(p)}}$ is a particular solution of non-homogeneous linear recurrence relation

$$\mathbf{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = F(n)}$$

then total solution is of the from

$$\mathbf{a_n = a_n^{(h)} + a_n^{(p)}},$$

where $\mathbf{a_n^{(h)}}$ is a solution of the associated homogeneous recurrence relation

$$\mathbf{c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0}.$$

→ In case of **homogeneous** linear recurrence relation, the total solution is

$$\mathbf{a_n = a_n^{(h)}}.$$

→ **Steps to compute $a_n^{(h)}$:**

- **Step 1**: Write characteristic equation.
  - If associated homogeneous recurrence relation is

    $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = 0,$

    then characteristic equation will be

    $c_0 \lambda^k + c_1 \lambda^{k-1} + c_2 \lambda^{k-2} + \cdots + c_k = 0$
- **Step 2**: Solve characteristic equation for k values of λ.
- **Step 3**: Write $a_n^{(h)}$ according to λ.

$a_n^{(h)}$ according to λ	
λ	$a_n^{(h)}$
$\lambda_1 \neq \lambda_2$	$C_1(\lambda_1)^n + C_2(\lambda_2)^n$
$\lambda_1 = \lambda_2 = \lambda_3$	$(C_1 + C_2 n + C_3 n^2)(\lambda_1)^n$
$\lambda_1 = \lambda_2 \neq \lambda_3$	$(C_1 + C_2 n)(\lambda_1)^n + C_3(\lambda_3)^n$

→ **Steps to compute $a_n^{(p)}$:**

- **Step 1**: Write $a_n^{(p)}$ according to $F(n)$.
- **Step 2**: Write $a_{n-1}^{(p)}, a_{n-2}^{(p)}, \ldots, a_{n-k}^{(p)}$ according to $a_n^{(p)}$.

$a_n^{(P)}$ according to $F(n)$	
$F(n)$	$a_n^{(P)}$
Constant	P_0
$a + bn + cn^2 + dn^3 + \cdots$	$P_0 + P_1 n + P_2 n^2 + P_3 n^3 + \cdots$
$ab^n \ (b \neq \lambda)$	$P_0 b^n$
$ab^n \ (b = \lambda \text{ with multiplicity } m)$	$P_0 n^m b^n$

- For example: If $a_n^{(p)} = P_0 + P_1 n$, then
  $a_{n-1}^{(p)} = P_0 + P_1(n-1)$, $a_{n-2}^{(p)} = P_0 + P_1(n-2)$ and so on.
  Put these values in given non homogeneous relation and find values P_i's.
- Substitute values P_i's in $a_n^{(p)}$.
- Write total solution $a_n = a_n^{(h)} + a_n^{(p)}$

→ **Find constants using initial condition(s) if given.**

- **Step 1**: Find C_i's using initial condition(s) if given.
- **Step 2**: Substitute values of C_i's in a_n.

Examples of Method-2: Solution of Linear Recurrence Relation using Undetermined Coefficient Method

C	1	Solve the recurrence relations $a_n = 3a_{n-1} - 2a_{n-2},\ n \geq 2;\ a_1 = -2,\ a_2 = 4$ using the method of undetermined coefficients. **Answer**: $a_n = (-8)(1)^n + 3(2)^n$
C	2	Solve the recurrence relations $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2$ using the method of undetermined coefficients. **Answer**: $a_n = C_1(-2)^n + C_2(-3)^n + \frac{115}{288} + \frac{17}{24}n + \frac{1}{4}n^2$
C	3	Solve the recurrence relations using the method of undetermined coefficients: $a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 2^n,\ n \geq 3;\ a_0 = a_1 = 0,\ a_2 = 2$ **Answer**: $a_n = \left(-\frac{1}{8} + \frac{1}{4}n\right)(-2)^n + (2)^{(n-3)}$
C	4	Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = n + 2^n; n \geq 2$, $a_0 = 1,\ a_1 = 1$ using the method of undetermined coefficients. **Answer**: $a_n = -5(2)^n + \frac{17}{4}(3)^n + \frac{7}{4} + \frac{1}{2}n - 2n2^n$

Unit – 2.2 ⇝ Propositional Logic

Logic

→ Logic is concerned with the study of the principles and techniques of reasoning.

→ Logic was developed by **ARISTOTLE** (384 B.C.-322 B.C.).

→ The rules of logic give precise meaning to mathematical statements.

→ Logic is used whether mathematical arguments is valid or invalid.

→ Logic have provided the theoretical basis for many areas for Computer Science such as Digital Logic Design, Automata Theory, and Computability, Artificial intelligence etc.

Types of Logic

→ There are two types of logic:

(1) Proposition Logic

(2) Predicate Logic

Proposition Logic

→ It deals with statement with values true and false and is concerned with analysis of propositions.

Predicate Logic

→ It is a generalization of propositional calculus.

→ It contains all the components of propositional calculus, including propositional variables and constants.

Method 1 ⇝ Statements

Statements or Propositions

→ A sentence which is either **true** or **false** but **not both** is known as Proposition.

→ Proposition is also known as **Statement**.

→ For Example:

(1) " India got freedom in 1947 " is a statement.

(2) " $3 * 4 = 15$ " is a statement.

(3) " Open the door " is **not** a statement.

(4) " Where are you going? " is **not** a statement.

(5) " $X + 2 = 5$ " is **not** a statement.

Propositional Variables

→ Propositional variables are used to represent proposition.

→ Propositional variables are denoted by p, q, r, etc.

→ For Example:

(1) $p : (a+b)^2 = a^2 + 2ab + b^2$

(2) q : The sum of the interior angles of a triangle is 180°.

(3) r : There exists an even prime number.

Here p, q, r are propositional variable.

Types of Propositions

→ There are two types of propositions:

(1) Simple proposition or Atomic proposition

(2) Compound proposition

(1) Simple Proposition or Atomic Proposition

→ A proposition which **cannot be divided** further is known as simple proposition.

→ For Example:

- p : Delhi is capital of India.
- $q : 2 \in N$.

(2) **Compound Proposition**

→ A proposition which is the **combination** of **two or more** propositions is known as compound proposition.

→ For Example:

- p : The milk is white or $6 + 5 = 11$ ".
- q : Roses are red and violets are blue.

Truth Value of Proposition

→ There are two types of truth value of proposition:

(1) **T**

(2) **F**

→ If proposition is **true**, then truth value of proposition is **T**.

→ If proposition is **false**, then truth value of proposition is **F**.

→ For Example:

Sr. No.	Proposition	Truth value
1.	p : $9 > 13$	F
2.	q : Drink it quickly.	No truth value

Examples of Method-1: Statements

C	1	Which of the following are proposition? p : Calcutta is capital of India. q : $x + 2 = 5$ if $x = 1$ r : $x + 3 = 8$ **Answer: p and r are propositions**

C	2	Assign the truth value to the following sentences if it is proposition. $p : 4 + 2 = 6$ q : 5 is a prime number. r : Paris is in Bangladesh. $s : x + 4 = 5$ **Answer: p : T, q : T, r : F,** **s : No truth value as it is not a proposition**

Method 2 ⇝ Logical Connectives

Logical Connectives

→ Logical connectives are known as logical operators.

→ They are used to connect two or more propositions.

→ There are five types of logical connectives which is

(1) Conjunction

(2) Disjunction

(3) Negation

(4) Conditional

(5) Biconditional

2.1 Conjunction

→ If two or more propositions are joined by the word **"and"** to form a compound proposition, then compound proposition is known as conjunction.

→ It is denoted by "$\mathbf{p \wedge q}$" and read as "p and q" or "p conjunction q" or "p product q".

→ For Example:

p : 3 is an integer.

q : 10 is an even number.

$\mathbf{p \wedge q}$: 3 is an integer **and** 10 is an even number.

→ For $p \wedge q$, propositions p and q are known as sub-propositions.

→ "But" and "while" are used instead of "and".

→ Truth value of $\mathbf{p \wedge q}$ and $\mathbf{q \wedge p}$ are same.

→ Truth table for $p \wedge q$ as follows:

p	**q**	**$p \wedge q$**
T	**T**	**T**
T	F	F
F	T	F
F	F	F

Examples of Method-2.1: Conjunction

C	1	Write the following propositions in symbolic form: **(1)** He is tall and he is handsome. **(2)** All integers are rational and all natural numbers are integers. **Answer**: (1) $p \wedge q$, (2) $p \wedge q$
C	2	Define the truth value for the following conjunction: **(1)** $4 + 2 = 6$ and $2 + 2 = 4$ **(2)** Delhi is in India and $4 \times 2 = 10$ **Answer**: (1) T, (2) F
C	3	Construct the truth table for $(p \wedge q) \wedge r$. **Answer**: (see table below)

p	q	r	$(p \wedge q) \wedge r$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

2.2 Disjunction

→ If two or more propositions are joined by the word **“or”** to form a compound proposition, then compound proposition is known as disjunction.

→ It is denoted by "**$p \vee q$**" and read as “**p or q**” or “**p disjunction q**” or “**p sum q**”.

→ For Example:

p : It is snowing.

q : I am cold.

$p \vee q$: It is snowing **or** I am cold.

→ For $p \vee q$, propositions p and q are known as sub-propositions.

→ Truth value of **$p \vee q$** and **$q \vee p$** are same.

→ Truth table for $p \vee q$ as follows:

p	**q**	**$p \vee q$**
T	T	T
T	F	T
F	T	T
F	**F**	**F**

Examples of Method-2.2: Disjunction

C	1	Write the following propositions in symbolic form: **(1)** Mumbai is in Maharashtra or Mumbai is in Gujarat. **(2)** $x^2 - 8 = 0$ if $x = 2$ or River ganga passes throug tamilnadu. **Answer**: **(1) $p \vee q$, (2) $p \vee q$**
C	2	Determine the truth value of following disjunction. 1. 5 + 4 = 21 or 2 × 3 = 5 2. Charminar is in Hyderabad or $7 + 1 = 6$ **Answer**: **(1) T, (2) T**

C	3	Construct the truth table for $(p \vee q) \vee r$. **Answer**:

p	q	r	$(p \vee q) \vee r$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

2.3 Negation

→ If p is a proposition, then "**not p**" is known as negation of proposition p.

→ It is denoted by "**$\neg$p or ~p**" and read as "**negation of p**".

→ Negation of proposition p can be formed by writing following lines before p.

(1) " It is **not** the case that … "

(2) " It is **false** that… "

→ For Example:

q : Michael's PC has 16 GB of RAM.

$\neg$ q : Michael's PC **does not** have 16 GB of RAM.

or

$\neg$ q : It is **not** the case that Michael's PC has 16 GB of RAM.

→ Truth table for $\neg$p as follows:

p	**$\neg$ p**
T	F
F	T

Examples of Method-2.3: Negation

C	1	Write the negation for each of the following propositions: **(1)** p : Violet are blue. **(2)** q : Delhi is in America. **(3)** r : $3 + 3 = 7$ **Answer**: **(1) $\neg$ p : Violets are not blue** **(2) $\neg$ q : Delhi is not in America** **(3) $\neg$ r : $3 + 3 \neq 7$**

<table>
<tr><td>C</td><td>2</td><td>Write each of following propositions in symbolic form using p and q:
Where, p : He is tall.
q : He is handsome.
(1) He is not tall but he is handsome.
(2) He is tall but not handsome.
(3) He is neither tall nor handsome.

Answer: (1) $\neg p \wedge q$, (2) $p \wedge \neg q$, (3) $\neg p \wedge \neg q$</td></tr>
<tr><td>C</td><td>3</td><td>Construct the truth table for compound proposition $p \wedge \neg q$.

Answer:
<table><tr><th>p</th><th>q</th><th>$p \wedge \neg q$</th></tr><tr><td>T</td><td>T</td><td>F</td></tr><tr><td>T</td><td>F</td><td>T</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>F</td><td>F</td><td>F</td></tr></table></td></tr>
<tr><td>C</td><td>4</td><td>Construct the truth table for compound proposition $(p \wedge \neg p) \wedge \neg q$.

Answer:
<table><tr><th>p</th><th>q</th><th>$(p \wedge \neg p) \wedge \neg q$</th></tr><tr><td>T</td><td>T</td><td>F</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>F</td><td>F</td><td>F</td></tr></table></td></tr>
</table>

2.4 Conditional

→ A proposition of the form **"If p, then q"** is known as conditional connective or implication.

→ It is denoted by "**$p \to q$**" and read as "**if p, then q**"

→ For Example:

p : Jay learns discrete mathematics.

q : He will get job.

$p \to q$: If Jay learns discrete mathematics, then he will get a job.

→ If $p \to q$ is a conditional proposition, then

(1) **$q \to p$** is known as **converse**

(2) **$\neg p \to \neg q$** is known as **inverse**

(3) **$\neg q \to \neg p$** is known as **contrapositive**.

→ Truth table for $p \to q$ as follows:

p	**q**	**$p \to q$**
T	T	T
T	**F**	**F**
F	T	T
F	F	T

Examples of Method-2.4: Conditional

C	1	Express the given statement in symbolic form. "If Mohan takes calculus or Gopal takes English, then Radha takes Sanskrit". **Answer**: $(p \vee q) \to r$

C	2	Express the following compound propositions as an English sentence by using p, q, r: Where, p : You have the flue. q : You miss the final examination. r : You pass the course. **(1)** $p \rightarrow q$ **(2)** $q \rightarrow \neg r$ **(3)** $(p \rightarrow \neg r) \vee (q \rightarrow \neg r)$ **Answer**: (1) **If you have the flue, then you miss the final exam.** (2) **If you miss the final exam, then you don't pass the course.** (3) **If you have the flue, then you will not pass the course or** **If you miss the final exam, then you don't pass the course.**
C	3	Determine whether each of the following conditional statements is true or false: **(1)** If $1 + 1 = 2$, then $2 + 2 = 5$. **(2)** If $1 + 1 = 3$, then $2 + 2 = 4$. **(3)** If $1 + 1 = 3$, then $2 + 2 = 5$. **Answer**: (1) **F**, (2) **T**, (3) **T**
C	4	Construct the truth table for $(p \vee q) \rightarrow (p \wedge q)$. **Answer**:

p	q	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	T

C	5	Construct the truth table for $(q \rightarrow \neg p) \vee (\neg p \rightarrow \neg q)$. **Answer:** <table><tr><th>p</th><th>q</th><th>$(q \rightarrow \neg p) \vee (\neg p \rightarrow \neg q)$</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>T</td></tr><tr><td>F</td><td>T</td><td>T</td></tr><tr><td>F</td><td>F</td><td>T</td></tr></table>
C	6	Express the converse, inverse and contrapositive statements of conditional statement given “ If $x < 2$, then $x + 4 < 6$.” **Answer:** (1) **Converse** : **If $x + 4 < 6$, then $x < 2$.** (2) **Inverse** : **If $x \nless 2$, then $x + 4 \nless 6$.** (3) **Inverse** : **If $x + 4 \nless 6$, then $x \nless 2$.**

2.5 Biconditional

→ A proposition of the form "**p if and only if q**" is known as biconditional connective.

→ It is denoted by "**$p \rightleftarrows q$**" and read as "**p if and only if q**".

→ For Example:

p : I am breathing.

q : I am alive.

$p \rightleftarrows q$: I am breathing if and only if i am alive.

→ Truth table for $p \rightleftarrows q$ as follows:

p	**q**	**$p \rightleftarrows q$**
T	**T**	**T**
T	F	F
F	T	F
F	**F**	**T**

Examples of Method-2.5: Biconditional

C	1	Determine whether each of the following biconditional statements is true or false: **(1)** $1+1 = 2$ if and only if $2+2 = 4$. **(2)** $1+1 = 3$ if and only if $2+2 = 4$. **(3)** $1+1 = 3$ if and only if $2+2 = 5$. **Answer**: (1) **T**, (2) **F**, (3) **T**
C	2	Construct the truth table for $(p \vee q) \rightleftarrows (p \wedge q)$. **Answer**:

p	**q**	**$(p \vee q) \rightleftarrows (p \wedge q)$**
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

C	3	Construct the truth table for $(\neg p \rightleftarrows \neg q) \rightleftarrows (q \rightleftarrows r)$. **Answer**:

p	q	r	$(\neg p \rightleftarrows \neg q) \rightleftarrows (q \rightleftarrows r)$
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	T

Method 3 ⇝ Precedence Rule

Precedence Rule

→ Precedence rule is order of preference.

→ Precedence rule helps to decide which operator will get evaluated first in complicated compound propositions.

→ Preference table

Connectives	Names	Preference
$\neg$	Negation	1
$\wedge$	Conjunction	2
$\vee$	Disjunction	3
$\rightarrow$	Implication	4
$\rightleftarrows$	Biconditional	5

→ For Example:

(1) " $p \rightarrow q \wedge \neg p$ " is equivalent to the expression " $\mathbf{p} \rightarrow (\mathbf{q} \wedge (\neg \mathbf{p}))$"

(2) " $\neg p \vee q \wedge r$ " is equivalent to the expression " $(\neg \mathbf{p}) \vee (\mathbf{q} \wedge \mathbf{r})$ "

(3) " $p \wedge q \vee r$ " is **not** equivalent to the expression " $\mathbf{p} \wedge (\mathbf{q} \vee \mathbf{r})$ "

is equivalent to the expression " $(\mathbf{p} \wedge \mathbf{q}) \vee \mathbf{r}$ "

Examples of Method-3: Precedence Rule

C	1	By using preference rule put parentheses at appropriate place for following compound propositions: **(1)** $p \wedge q \vee \neg p \wedge q$ **(2)** $\neg p \vee q \wedge r \rightleftarrows p \wedge q \vee p \wedge r$ **Answer**: **(1)** $(\mathbf{p} \wedge \mathbf{q}) \vee (\neg \mathbf{p} \wedge \mathbf{q})$ **(2)** $((\neg \mathbf{p}) \vee (\mathbf{q} \wedge \mathbf{r})) \rightleftarrows ((\mathbf{p} \wedge \mathbf{q}) \vee (\mathbf{p} \wedge \mathbf{r}))$

Method 4 ⇝ Well Formed Formula

Well Formed Formula

→ Well-Formed Formula (WFF) is an expression of **variables, parentheses** and **connective** symbols.

→ Rules for Well-Formed Formulas:

(1) A propositional variable standing alone is a Well-Formed Formula.

(2) If " **p** " is WFF, then "**¬p**" is also a WFF.

(3) If **p** and **q** are WFFs, then $(p \wedge q)$, $(p \vee q)$, $(p \rightleftarrows q)$, $(p \rightarrow q)$ etc. are also WFFs.

→ Examples which are WFF:

(1) $\neg\neg p$ **Reason**: Since, $\neg p$ is a WFF.
So, $\neg\neg p$ is also a WFF.

(2) $(\neg p \wedge q)$ **Reason:** $(\neg p \wedge q)$ is a compound statement and it is placed inside parenthesis. So, It is a WFF.

→ Examples which are not WFF:

(1) (p)

Reason: placing 'p' inside parenthesis is not considered as WFF by any rule.

(2) $((p \rightarrow q))$

Reason: Since, $(p \rightarrow q)$ is WFF.

Suppose, $(p \rightarrow q) = A$.

So, $((p \rightarrow q)) = (A)$, which is not a valid WFF.

Examples of Method-4: Well Formed Formula

C	1	Which of the following are well-formed formula? **(1)** $\neg(p \wedge q)$ **(2)** $(p \rightarrow q) \rightarrow (\wedge q)$ **(3)** $(p \rightarrow (q \rightarrow r))$ **(4)** $\neg p \wedge q \rightarrow r$ **(5)** $((p \rightarrow q) \wedge (q \rightarrow r))$ **Answer**: (1), (3), (5)

Method 5 ⇝ Tautologies

Tautology

→ If a compound proposition is always **true** for all possible truth vales of its propositional variables, then compound proposition is known as **Tautology**.

→ The truth table of such proposition contains only truth value **T** in the last column.

→ For Example:

$(p \vee \neg p)$ is a tautology.

$\mathbf{p}$	$\neg \mathbf{p}$	$\mathbf{p} \vee \neg \mathbf{p}$
T	F	T
F	T	T

Contradiction

→ If a compound proposition is always **false** for all possible truth vales of its propositional variable, then compound proposition is known as **Contradiction**.

→ The truth table of such proposition contains only truth value **F** in the last column.

→ For Example:

$\neg(p \vee \neg p)$ is a contradiction.

$\mathbf{p}$	$\neg \mathbf{p}$	$\mathbf{p} \vee \neg \mathbf{p}$	$\neg(\mathbf{p} \vee \neg \mathbf{p})$
T	F	T	F
F	T	T	F

Contingency

→ A compound proposition which is **neither** tautology **nor** contradiction is known as **Contingency**.

→ The truth table of such proposition contains **T and F** in the last column.

→ For Example:

$(p \vee q)$ is a contingency.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Examples of Method-5: Tautologies

<table>
<tr><td>C</td><td>1</td><td>Show that the following proposition is tautology:
$(p \rightarrow q) \rightleftarrows (\neg q \rightarrow \neg p)$

Answer:
<table>
<tr><th>p</th><th>q</th><th>$(p \rightarrow q) \rightleftarrows (\neg q \rightarrow \neg p)$</th></tr>
<tr><td>T</td><td>T</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>T</td></tr>
<tr><td>F</td><td>F</td><td>T</td></tr>
</table></td></tr>
<tr><td>C</td><td>2</td><td>Show that the following proposition is contradiction:
$(p \vee q) \wedge (\neg p \wedge \neg q)$

Answer:
<table>
<tr><th>p</th><th>q</th><th>$(p \vee q) \wedge (\neg p \wedge \neg q)$</th></tr>
<tr><td>T</td><td>T</td><td>F</td></tr>
<tr><td>T</td><td>F</td><td>F</td></tr>
<tr><td>F</td><td>T</td><td>F</td></tr>
<tr><td>F</td><td>F</td><td>F</td></tr>
</table></td></tr>
</table>

C	3	Show that the following proposition is contingency: $(p \rightarrow q) \wedge ((q \rightarrow r) \rightarrow (p \rightarrow r))$

Answer:

p	**q**	**r**	$(p \rightarrow q) \wedge ((q \rightarrow r) \rightarrow (p \rightarrow r))$
T	**T**	**T**	**T**
T	**T**	**F**	**T**
T	**F**	**T**	**F**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**T**
F	**F**	**T**	**T**
F	**F**	**F**	**T**

Method 6 ⇝ Equivalence of Formulas

Equivalence or Logical Equivalence

→ Let P (p , q , ...) and Q (p , q , ...) are two compound propositions.

If P and Q have **same truth values**, then P and Q are known as **Logically Equivalent** or simply **Equivalent**.

→ It is denoted by **"P ≡ Q"** and read as "P is equivalent to Q".

Laws of Logic

→ **Idempotent Laws:**

(1) $p \vee p \equiv p$

(2) $p \wedge p \equiv p$

→ **Commutative Laws:**

(1) $p \vee q \equiv q \vee p$

(2) $p \wedge q \equiv q \wedge p$

→ **Associative Laws:**

(1) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(2) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

→ **Distributive Laws:**

(1) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(2) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

→ **Identity Laws:**

(1) $p \wedge T \equiv p$

(2) $p \vee F \equiv p$

(3) $p \wedge F \equiv F$

(4) $p \vee T \equiv T$

→ **Complement Laws:**

(1) $p \vee \neg p \equiv T$

(2) $p \wedge \neg p \equiv F$

(3) $\neg T \equiv F$

(4) $\neg F \equiv T$

→ **De Morgan's Laws:**

(1) $\neg (p \vee q) \equiv (\neg p \wedge \neg q)$

(2) $\neg (p \wedge q) \equiv (\neg p \vee \neg q)$

→ **Involution Law:**

(1) $\neg\neg p \equiv p$

Examples of Method- 6: Equivalence of Formulas

<table>
<tr>
<td>C</td>
<td>1</td>
<td>Show that, $(p \rightleftarrows q) \equiv ((\neg p \wedge \neg q) \vee (p \wedge q))$.

Hint:
<table>
<tr><th>p</th><th>q</th><th>$p \rightleftarrows q$</th></tr>
<tr><td>T</td><td>T</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>F</td></tr>
<tr><td>F</td><td>T</td><td>F</td></tr>
<tr><td>F</td><td>F</td><td>T</td></tr>
</table>
</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>Check whether the compound propositions $\neg(p \rightleftarrows q)$ and $(p \rightleftarrows \neg q)$ are equivalent or not?

Hint: Yes
<table>
<tr><th>p</th><th>q</th><th>$(p \rightleftarrows \neg q)$</th></tr>
<tr><td>T</td><td>T</td><td>F</td></tr>
<tr><td>T</td><td>F</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>T</td></tr>
<tr><td>F</td><td>F</td><td>F</td></tr>
</table>
</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>Check whether the compound propositions $(p \rightarrow (q \vee r))$ and $((r \rightarrow q) \vee (p \rightarrow r))$ are equivalent or not?

Hint: Not equivalent
<table>
<tr><th>p</th><th>q</th><th>r</th><th>$(p \rightarrow (q \vee r))$</th><th>$((r \rightarrow q) \vee (p \rightarrow r))$</th></tr>
<tr><td>T</td><td>T</td><td>T</td><td>T</td><td>T</td></tr>
<tr><td>T</td><td>T</td><td>F</td><td>T</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>T</td><td>T</td><td>T</td></tr>
<tr><td>T</td><td>F</td><td>F</td><td>F</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>T</td><td>T</td><td>T</td></tr>
<tr><td>F</td><td>T</td><td>F</td><td>T</td><td>T</td></tr>
<tr><td>F</td><td>F</td><td>T</td><td>T</td><td>T</td></tr>
<tr><td>F</td><td>F</td><td>F</td><td>T</td><td>T</td></tr>
</table>
</td>
</tr>
</table>

Method 7 ⇝ Normal Forms

Introduction

→ In propositional logic if there are many variables, then construction of truth table becomes complicated. So, it is required to convert it in easier form which is known as **Normal Forms**.

→ It is also known as Canonical Forms.

→ There are four types of normal forms:

(1) Disjunctive Normal Form (DNF)

(2) Conjunctive Normal Form (CNF)

(3) Principle Disjunctive Normal Form (PDNF)

(4) Principle Conjunctive Normal Form (PCNF)

Elementary Sum

→ A sum of the variables and their negations is known as **Elementary Sum**.

→ For Example:

(1) $p \vee q$

(2) $p \vee \neg p$

(3) $\neg p \vee p \vee \neg q$

Elementary Product

→ A product of the variables and their negations is known as **Elementary Product**.

→ For Example:

(1) $p \wedge q$

(2) $p \wedge \neg p$

(3) $\neg p \wedge q \wedge \neg q$

Minterm

→ Let p and q be two statement variables, then $\mathbf{p \wedge q}$, $\mathbf{p \wedge \neg q}$, $\mathbf{\neg p \wedge q}$ and $\mathbf{\neg p \wedge \neg q}$ are known as **Minterms** of p and q.

→ It may be noted that none of the minterms should contain both a variable and its negation.

i.e. $p \wedge \neg p$ is **not minterm.**

→ The number of minterms in n variables is 2^n.

→ Let p, q and r be three statement variables. Minterms for three variables are as below:

$p \wedge q \wedge r$	$p \wedge \neg q \wedge r$	$p \wedge q \wedge \neg r$	$p \wedge \neg q \wedge \neg r$
$\neg p \wedge q \wedge r$	$\neg p \wedge \neg q \wedge r$	$\neg p \wedge q \wedge \neg r$	$\neg p \wedge \neg q \wedge \neg r$

Maxterm

→ Let p and q be two statement variables, then $p \vee q$, $p \vee \neg q$, $\neg p \vee q$ and $\neg p \vee \neg q$ are known as **Maxterms** of p and q.

→ It may be noted that none of the maxterms should contain both a variable and its negation.

i.e. $p \vee \neg p$ is **not maxterm.**

→ Negation of minterm is maxterm and vice versa.

→ The number of maxterms in n variables is 2^n.

→ Let p, q and r be three statement variables. Maxterms for three variables are as below:

$p \vee q \vee r$	$p \vee \neg q \vee r$	$p \vee q \vee \neg r$	$p \vee \neg q \vee \neg r$
$\neg p \vee q \vee r$	$\neg p \vee \neg q \vee r$	$\neg p \vee q \vee \neg r$	$\neg p \vee \neg q \vee \neg r$

7.1 Disjunctive Normal Form

→ A logical expression, if it is the sum of elementary product is known as **Disjunctive Normal Form**.

→ For Example:

(1) $(p \wedge q) \vee (p \wedge \neg q)$ is a **DNF**.

(2) $(\neg p \wedge q) \vee (\neg p \vee \neg q)$ is not a **DNF**.

→ DNF of a given logical expression is **not unique**, but it is **equivalent** to given logical expression.

→ For Example:

$p \vee (q \wedge r)$ is already a DNF, but we can write it as below

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$\equiv (p \wedge p) \vee (p \wedge r) \vee (q \wedge p) \vee (q \wedge r)$, which is also a **DNF**.

→ **Procedure to construct DNF:**

(1) Replace " $\rightarrow$, $\rightleftarrows$ " by using logical connectives "$\neg, \vee, \wedge$".

(2) Use De Morgan's laws to eliminate " $\neg$ " before sum or products.

(3) Apply distributive law until a sum of elementary product is obtained.

→ Useful equivalence relation:

(1) $p \rightarrow q \equiv \neg p \vee q$

(2) $p \rightleftarrows q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

(3) $p \rightleftarrows q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

Factor

→ For any DNF, each part is known as **Factor**.

→ Each factor is fundamental conjunction.

→ For Example:

$p \vee (q \wedge r)$ is DNF.

Factors: p, $q \wedge r$

→ In DNF, if every factor is F, then DNF is False.

i.e. Given logical expression is Contradiction.

Examples of Method-7.1: Disjunctive Normal Form (DNF)

C	1	Find disjunctive normal form of $p \wedge (p \rightarrow q)$. **Answer:** $(\mathbf{p} \wedge \neg \mathbf{p}) \vee (\mathbf{p} \wedge \mathbf{q})$
C	2	Find disjunctive normal form of $p \vee \left(\neg p \rightarrow (q \vee (q \rightarrow \neg r))\right)$. **Answer:** $\mathbf{p} \vee \mathbf{q} \vee \neg \mathbf{q} \vee \neg \mathbf{r}$
C	3	Find disjunctive normal form of $p \rightarrow \left((p \rightarrow q) \wedge \neg(\neg q \vee \neg p)\right)$. **Answer:** $\neg \mathbf{p} \vee (\mathbf{p} \wedge \mathbf{q})$

7.2 Conjunctive Normal Form

→ A logical expression, if it is the product of elementary sum is known as **Conjunctive Normal Form**.

→ For Example:

(1) $(p \vee q) \wedge (\neg p \vee \neg q)$ is a **CNF**.

(2) $(p \vee q \wedge r) \wedge (\neg p \vee r)$ is **not a CNF**.

(3) $(p \wedge \neg (q \vee r) \vee r)$ is neither a **CNF** nor **DNF**.

→ CNF of a given logical expression is **not unique**, but it is **equivalent** to given logical expression.

→ For Example:

$p \wedge (q \vee r)$ is already a CNF, but we can write it as below.

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$\equiv (p \vee p) \wedge (p \vee r) \wedge (q \vee p) \wedge (q \vee r)$, which is also a **CNF**.

→ **Procedure to construct CNF:**

(1) Replace " $\rightarrow$, $\rightleftarrows$ " by using logical connectives "$\neg , \vee , \wedge$".

(2) Use De Morgan's laws to eliminate " $\neg$ " before sum or products.

(3) Apply distributive law until a product of elementary sum is obtained.

→ Useful equivalence relations:

(1) $p \rightarrow q \equiv \neg p \vee q$

(2) $p \rightleftarrows q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

(3) $p \rightleftarrows q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$

Factor

→ For any CNF, each part is known as **Factor**.

→ Each factor is fundamental disjunction.

→ For Example:

$p \wedge (q \vee r)$ is CNF.

Factors: p , $q \vee r$

→ In CNF, if every factor is T, then CNF is True.

i.e. Given logical expression is Tautology.

Examples of Method-7.2: Conjunctive Normal Form

C	1	Find conjunctive normal form of $p \wedge (p \to q)$. **Answer**: $\mathbf{p} \wedge (\neg \mathbf{p} \vee \mathbf{q})$
C	2	Find conjunctive normal form of $\neg (p \vee q) \rightleftarrows (p \wedge q)$. **Answer**: $(\mathbf{p} \vee \mathbf{q}) \wedge (\mathbf{p} \vee \mathbf{q}) \wedge (\neg \mathbf{p} \vee \neg \mathbf{q}) \wedge (\neg \mathbf{p} \vee \neg \mathbf{q})$

7.3 Principal Disjunctive Normal Form

→ Principal disjunctive normal form of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms only.

→ This is also known as **Sum of Products Canonical Form**.

→ For Example:

(1) $(\neg p \wedge q) \vee (q \wedge \neg p)$ is a **PDNF**.

(2) $(\neg p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r)$ is a **PDNF**.

(3) $(\neg p \vee q \vee \neg r) \vee (p \vee \neg q)$ is a **DNF but not PDNF**.

→ **Procedure to construct PDNF:**

(1) Construct a truth table of the given compound propositions.

(2) For every truth value T of the given proposition, select the minterm, which also has the truth value T for the same combination of the truth value of the statement variables.

(3) Join the every selected minterms in step (2) by disjunction.

→ **Advantages of Principal Disjunctive Normal Forms:**

(1) The principal disjunctive normal form of a given formula is unique.

(2) Two formulas are equivalent if and only if their principal disjunctive normal forms are same.

(3) If the given compound proposition is a tautology, then its principal disjunctive normal form will contain all possible minterms of its components.

Examples of Method-7.3: Principal Disjunctive Normal Form (PDNF)

C	1	Find principal disjunctive normal form of $(p \to q) \wedge (\neg p \wedge q)$. **Answer:** $\neg \mathbf{p} \wedge \mathbf{q}$
C	2	Find principal disjunctive normal form of $\big(q \vee (p \wedge r)\big) \wedge \neg\big((p \vee r) \wedge q\big)$. **Answer:** $(\neg \mathbf{p} \wedge \mathbf{q} \wedge \neg \mathbf{r}) \vee (\mathbf{p} \wedge \neg \mathbf{q} \wedge \mathbf{r})$

7.4 Principal Conjunctive Normal Form

→ Principal conjunctive normal form of a given formula can be defined as an equivalent formula consisting of conjunctives of maxterms only.

→ This is also known as **Product of Sums Canonical Form**.

→ For Example:

(1) $(\neg p \vee q) \wedge (q \vee \neg p)$ is a **PCNF**.

(2) $(\neg p \vee q \vee \neg r) \wedge (p \vee \neg q \vee r)$ is a **PCNF**.

(3) $(\neg p \vee q \vee \neg r) \wedge (p \vee r)$ is a **CNF but not PCNF**.

→ **Procedure to construct PCNF:**

(1) Construct a truth table of the given compound propositions.

(2) For every truth value F if the given proposition, select the maxterm, which also has the truth value F for the same combination of the truth value of the statement variables.

(3) Join the every selected maxterms in step (2) by conjunction.

→ **Advantages of Principal Conjunctive Normal Forms:**

(1) The principal conjunctive normal form of a given formula is unique.

(2) Every compound proposition, which is not a tautology, has an equivalent principal conjunctive normal form.

(3) If the given compound proposition is a contradiction, then its principal conjunctive normal form will contain all possible maxterms of its components.

Examples of Method-7.4: Principal Conjunctive Normal Form (PCNF)

C	1	Find principal conjunctive normal form of $(p \rightarrow q) \wedge (\neg p \wedge q)$. **Answer**: $(\neg p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee q)$

*** * * * * End of the Unit * * * * ***