Notes on the behavior of the Cellular Automata parity rule

The parity rule can be written as

$$s_{i,j}(t+1) = s_{i-1,j}(t) \oplus s_{i+1,j}(t) \oplus s_{i,j-1}(t) \oplus s_{i,j+1}(t) \tag{1}$$

where $s_{i,j}(t) \in \{0,1\}$ is the state of cell (i,j) at time t and $\oplus$ is the sum modulo 2, or the `XOR` logical operation.

For periodic boundary conditons and a lattice size $\times n$ where n is a power of two ($n = 2^k$, with k an integer number), the rule exhibits the unexpected property that

$$s_{i,j}(t^* = 2^{k-1}) = 0 \qquad \forall i,j \tag{2}$$

and then, obviously, for all $t > t^*$, $s_{i,j}(t) = 0$, too.

This property is demonstrated in [1]. Here we give a proof for the 1D version of this rule (actually rule 90 according to Wolfram's numbering scheme)

$$s_i(t+1) = s_{i-1}(t) \oplus s_{i+1}(t) \tag{3}$$

From this expression we obtain

$$\begin{aligned} s_i(t+2) &= s_{i-1}(t+1) \oplus s_{i+1}(t+1) \\ &= s_{i-2}(t) \oplus s_i(t) \oplus s_i(t) \oplus s_{i+2}(t) \\ &= s_{i-2}(t) \oplus s_{i+2}(t) \end{aligned} \tag{4}$$

because $s_i(t) \oplus s_i(t) = 0$, always.

Let us assume for now that

$$s_i(t+2^\ell) = s_{i-2^\ell}(t) \oplus s_{i+2^\ell}(t) \tag{5}$$

holds for all t. Let us then show that eq. (5) is also true for $t + 2^{\ell+1}$ if it is true for ℓ:

$$\begin{aligned} s_i(t+2^{\ell+1}) &= s_i(t+2^\ell+2^\ell) = s_i(t'+2^\ell) \qquad \text{where } t' = t+2^\ell \\ &= s_{i-2^\ell}(t') \oplus s_{i+2^\ell}(t') \\ &= s_{i-2^\ell}(t+2^\ell) \oplus s_{i+2^\ell}(t+2^\ell) \\ &= s_{i-2^\ell-2^\ell}(t) \oplus s_i(t) \oplus s_i(t) \oplus s_{i+2^\ell+2^\ell}(t) \\ &= s_{i-2^{\ell+1}}(t) \oplus s_{i+2^{\ell+1}}(t) \end{aligned} \tag{6}$$

As eq. (5) is true for $\ell = 0$ (by the very definition of the rule) and for $\ell = 1$, see eq. (4), it is true for all ℓ by the principle of mathematical induction.

Therefore we have for any k

$$s_i(2^{k-1}) = s_{i-2^{k-1}}(0) \oplus s_{i+2^{k-1}}(0) \tag{7}$$

If the system size is $n = 2^k$, $s_{i-2^{k-1}} = s_{i+2^{k-1}}$ due to periodic conditions. Indeed

$$(i-2^{k-1}) \bmod n = (i-2^{k-1}+n) \bmod n = (i-2^{k-1}+2^k) \bmod n = (i+2^{k-1}) \bmod n \tag{8}$$

As a result,

$$s_i(2^{k-1}) = s_{i+2^{k-1}} \oplus s_{i+2^{k-1}} = 0 \tag{9}$$

This means that for a periodic system of size $n = 2^k$, rule 90 (and similarly the parity rule) reaches a whole-zero state after $t = 2^{k-1}$ iterations, for any initial condition.

References

[1] B. Chopard and M. Droz. *Cellular Automata Modeling of Physical Systems.* Cambridge University Press, 1998.