Entangled Quantum States

Introduction to Quantum Computing

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Outline

Two-Qubit basis and the state criteria

- The inner product[1] [2] relations for the two-qubit basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ are given as follows.

  $$\langle 00|00\rangle = \langle 01|01\rangle = \langle 10|10\rangle = \langle 11|11\rangle = 1$$

  all inner products between different basis vectors are zero.
- The given basis is therefore an orthonormal basis and is referred to as the two-qubit computational state
- Any vector $|\Psi\rangle$[3] that is a linear combination of the basis and obey the criteria $\langle\Psi|\Psi\rangle = 1$.
- Following the same approach as before, It should be possible to parameterize the state $|\Psi\rangle$ in a way similar to the Bloch sphere.

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[1] The transposed (Hermitian) conjugate of $|ab\rangle$ is given by, $(|a\rangle |b\rangle)^\dagger = \langle a| \langle b| = \langle ab|$

[2] The inner product $\langle ab|cd\rangle = \langle a|c\rangle \langle b|d\rangle$.

[3] Upper case letters (Ψ, Φ) are used to denote multi-qubit states.

Visualization of two-qubit states

- Consider the two-qubit system $|\Psi\rangle = |\psi\rangle \otimes |\chi\rangle$. The single qubit states $|\psi\rangle$ and $|\chi\rangle$ can be represented as vectors on their respective Bloch spheres.

$$\begin{aligned} |\psi\rangle &= \cos\frac{\theta_1}{2}|0\rangle + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}|1\rangle \\ |\chi\rangle &= \cos\frac{\theta_2}{2}|0\rangle + e^{\mathrm{i}\phi_2}\sin\frac{\theta_2}{2}|1\rangle \end{aligned}$$

- The two-qubit state can be represented in terms of the parameters θ_1, θ_2, ϕ_1 and ϕ_2.

$$\begin{aligned} |\psi\rangle \otimes |\chi\rangle &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle + e^{\mathrm{i}\phi_2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\ &\quad + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|10\rangle + e^{\mathrm{i}(\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle \end{aligned}$$

- It can be easily verified that $\langle\Psi|\Psi\rangle = 1$, these states are also referred to as unit vectors.

More unit vectors

- Consider the following vectors

$$\begin{aligned}
|\Psi'\rangle &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|01\rangle \\
&\quad + e^{\mathrm{i}\phi_2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|10\rangle + e^{\mathrm{i}(\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle \\
|\Psi''\rangle &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle + e^{\mathrm{i}\phi_2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\
&\quad + e^{\mathrm{i}(\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|10\rangle + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|11\rangle \\
|\Psi'''\rangle &= \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle + e^{\mathrm{i}\phi_2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\
&\quad + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|10\rangle + e^{\mathrm{i}\phi_3}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|11\rangle
\end{aligned}$$

- All of the above vectors are also unit vectors.

Separable states

- The state $|\Psi\rangle$ can be"factored" and expressed as:

$$|\Psi\rangle \;=\; (\cos\frac{\theta_1}{2}\,|0\rangle + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}\,|1\rangle)\otimes(\cos\frac{\theta_2}{2}\,|0\rangle + e^{\mathrm{i}\phi_2}\sin\frac{\theta_2}{2}\,|1\rangle)$$

- Similarly the state $|\Psi'\rangle$ an be factored as:

$$|\Psi'\rangle \;=\; (\cos\frac{\theta_2}{2}\,|0\rangle + e^{\mathrm{i}\phi_2}\sin\frac{\theta_2}{2}\,|1\rangle)\otimes(\cos\frac{\theta_1}{2}\,|0\rangle + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}\,|1\rangle)$$

- While the factored states are valid single qubit states in case of $|\Psi\rangle$ and $|\Psi'\rangle$. This is not possible in the case of $|\Psi''\rangle$ and $|\Psi'''\rangle$.
- Two-qubit states that can be factored into single qubit states are called separable states. The two-qubit computational basis vectors are all separable.

Non-separable unit vectors = Two-qubit states?

- The state $|\Psi\rangle$ is considered to be a two-qubit state because it is simply a combination of the two single qubit states.
- The operations required to transform $|0\rangle$ to any state are known and so any separable two-qubit state can be physically prepared.
- Therefore, if there exist transformations that can be used to prepare the non-separable unit vectors, then those vectors are also valid two-qubit states.
- For the following discussion, the state $|\Psi''\rangle$ shall be considered.

$$
\begin{aligned}
|\Psi''\rangle = {} & \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|00\rangle + e^{\mathrm{i}\phi_2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|01\rangle \\
& + e^{\mathrm{i}(\phi_1+\phi_2)}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}|10\rangle + e^{\mathrm{i}\phi_1}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}|11\rangle
\end{aligned}
$$

Two-qubit actions

- It can be observed that $|\Psi''\rangle$ can be obtained from $|\Psi\rangle$ by interchanging $|10\rangle$ and $|11\rangle$ in its expansion.
- If there exists a transformation M that can convert $|\Psi\rangle$ into $|\Psi''\rangle$, the action of M on the two qubit computational basis are as follows:

$$\begin{aligned} M|00\rangle &= |00\rangle \\ M|01\rangle &= |01\rangle \\ M|10\rangle &= |11\rangle \\ M|11\rangle &= |10\rangle \end{aligned}$$

- The operator M seems to invert the second qubit state when the first qubit state is in $|1\rangle$
- M acts as if the first qubit is controlling the application of an X gate on the second qubit.

Introducing the $CNOT$ gate

- The transformation described thus far is formally known as the $CNOT_2^1$ gate or the CX gate, the names stand for controlled-NOT and controlled-X respectively.
- In its matrix form the $CNOT_2^1$ is given as:

$$CNOT_2^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Introducing the $CNOT$ gate

- This matrix is real and symmetric, $CX^\dagger = CX$ and $CX \circ CX = I$ [4], therefore this is a unitary transformation.
- This matrix cannot be represented a direct product of two matrices $A \otimes B$, this is because the equations shown below have no solution.

$$A_{00}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \; A_{11}\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

.

[4] The symbol $\circ$ denotes matrix multiplcation and I in this case defines the 4×4 identity matrix

Controllers and Targets

- The numbers on $CNOT_2^1$ indicates which qubit controls the transformation, in this case it indicates that the first qubit controls the transformation of the second qubit (referred to as the target qubit).

- In case the second qubit controls the transformation of the first qubit, the operation is given the symbol $CNOT_1^2$

$$CNOT_1^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- When no index is given, it is assumed that the first qubit is the control qubit.

$$CNOT \equiv CNOT_2^1$$

The controlled-U gate

- It is possible to perform the controlled version of any unitary transformation $U = \begin{pmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{pmatrix}$
- The matrix form of the controlled-U gate is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix}$$

- When the first qubit is in state $|1\rangle$ the operation U is applied to the second qubit. If the first qubit is in state $|0\rangle$, no transformation is performed.
- The state $|\Psi'''\rangle$ can be prepared from $|\Psi\rangle$ using a controlled-P_ϕ gate. The state $|\Psi'\rangle$ is prepared by applying the $SWAP$ gate to $|\Psi\rangle$.

A preparation scheme

- Considering the following transformation applied to a system of two qubits; The Hadamard transformation is applied only to the first qubit and then the $CNOT$ operation is applied to the system.
- The transformation is given as $CNOT \circ (H \otimes I)$ where I is the 2×2 identity matrix.
- The action of this operation on the $|00\rangle$ state is given as

$$
\begin{aligned}
CNOT \circ (H \otimes I) |0\rangle |0\rangle &= CNOT \left(\frac{|00\rangle + |10\rangle}{\sqrt{2}} \right) \\
&= \frac{1}{\sqrt{2}} (CNOT |00\rangle + CNOT |10\rangle) \\
&= \frac{|00\rangle + |11\rangle}{\sqrt{2}}
\end{aligned}
$$

Arriving at Entanglement

- Similarly, the action of the operation $CNOT \circ (H \otimes I)$ on the remaining elements of the computational basis are as shown

$$CNOT \circ (H \otimes I)\,|01\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$
$$CNOT \circ (H \otimes I)\,|10\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$
$$CNOT \circ (H \otimes I)\,|11\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

- Each of these states are "non-separable" and can be prepared from an initial separable state. Multi-qubit states that are not separable are known as **Entangled states**.

The Bell Basis

- The four entangled states prepared by the action of $CNOT \circ (H \otimes I)$ on the computational basis also form a basis known as the Bell basis.
- These states may be labelled based on the initial state from which they are prepared.

$$
\begin{aligned}
|B_{00}\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
|B_{01}\rangle &= \frac{|01\rangle + |10\rangle}{\sqrt{2}} \\
|B_{10}\rangle &= \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
|B_{11}\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}
\end{aligned}
$$

- These states have several special properties associate with measurement. These properties are however not being considered for this discussion

The Physical Perspective

- The core idea of entanglement is that a system of two or more distinct elements gains an identity of its own.
- This idea is very familiar in physics, it is in fact older than quantum mechanics itself. In spectrometry, the many electrons present in an atom have to be treated as a single entity.
- The behaviour of entangled systems, especially their measurement can only be explained through quantum physics, this is the basic idea behind Bell's Inequality.
- In many cases the presence of entanglement behaviour is used to confirm that a system can show quantum behaviour. Entanglement plays a central role in quantum information.
- The complete explanation for the behaviour of entangled systems is an open problem.

Visualization of two-qubit states [contd.]

- Let the state $|\Psi\rangle$ be represented in the computational basis. Using polar coordinates for the coefficients gives a state defined by 8 real numbers.

$$|\Psi\rangle \;=\; r_0 e^{\mathrm{i}\theta_0}\,|00\rangle + r_1 e^{\mathrm{i}\theta_1}\,|01\rangle + r_2 e^{\mathrm{i}\theta_2}\,|10\rangle + r_3 e^{1\theta_3}\,|11\rangle$$

- The global phase can be eliminated yields 3 independent phase variables.

$$|\Psi\rangle \;=\; r_0\,|00\rangle + r_1 e^{\mathrm{i}\phi_1}\,|01\rangle + r_2 e^{\mathrm{i}\phi_2}\,|10\rangle + r_3 e^{1\phi_3}\,|11\rangle$$

- The normalization criteria $\langle\Psi|\Psi\rangle = 1$ gives

$$|r_0|^2 + |r_1|^2 + |r_2|^2 + |r_3|^2 = 1$$

- This implies the three out of the four r values are independent. Therefore, a two-qubit state is represented by 6 independent real variables and therefore it is impossible to visually represent a general two-qubit state.