Boolean Function and Quantum gates

Introduction to Quantum Computing

Jothishwaran C.A.

Department of Electronics and Communication Engineering
Indian Institute of Technology Roorkee

May 20, 2023

Outline

Boolean Variables and Operations

- A Boolean variable is an element belongs to the set $\{0, 1\}$
- Some Boolean operations are defined as follows:

- The AND operation or conjunction '$\cdot$'

$$\begin{aligned} 0 \cdot 0 &= 0 \\ 0 \cdot 1 &= 0 \\ 1 \cdot 0 &= 0 \\ 1 \cdot 1 &= 1 \end{aligned}$$

- The OR operation or disjunction '$\vee$'

$$\begin{aligned} 0 \vee 0 &= 0 \\ 0 \vee 1 &= 1 \\ 1 \vee 0 &= 1 \\ 1 \vee 1 &= 1 \end{aligned}$$

- These operations have their corresponding physical versions known as logic gates.

Variables and Operations [contd.]

- The XOR operation or bit addition: '$\oplus$' or '$+$'

$$
\begin{aligned}
0+0 &= 0\\
0+1 &= 1\\
1+0 &= 1\\
1+1 &= 0
\end{aligned}
$$

- The NOT operation or negation: '$\neg$' or '$^{-}$'

$$
\begin{aligned}
\neg 0 &= \bar{0} = 1\\
\neg 1 &= \bar{1} = 0
\end{aligned}
$$

- In general we can write, $\bar{x} = 1 + x$.

- The XOR gate can also be seen as integer addition *modulo* 2, i.e. adding the numbers and taking the remainder after dividing by 2.
- Similarly, The conjunction (AND) operation can be seen as multiplication *modulo* 2.
- It is also to be noted that some of these logic gates form sets known as universal gates. This implies any Boolean operator can be expressed in terms of the universal operators.

Classical bits and registers

- The single Boolean variable is the same as a classical bit, $x \in \{0, 1\}$ and is the quantity stored in a flip-flop.
- A register containing n-bits can be represented by the Boolean string $x \equiv x_0 x_1 \dots x_{n-1}$; where , $x_i \in \{0, 1\}$: $i \in [0, n-1]$.
- The i^{th} element of the string will be denoted by $x^{(i)} = x_i$.
- There are 2^n possible values x, if treated as integers to the base 2 these strings will have values from 0 to $2^n - 1$.
- For instance, $n = 2$ the strings x are $\{00, 01, 10, 11\}$ corresponding to the integers 0 to $2^2 - 1$.
- These strings are also referred to as Boolean vectors and defined as, $x \in \{0, 1\}^n$.

Boolean functions and their forms

- A n variable Boolean function, F takes a Boolean string $x \in \{0,1\}^n$ as an input and gives a single Boolean variable $y \in \{0,1\}$:

$$F \; : \; \{0,1\}^n \rightarrow \{0,1\} \; : \; F(x) = y$$

- It is possible to represent any Boolean function using the universal gate sets. These representations are known as forms

- If the set $\{\neg, \cdot, \vee\}$ is used, there are two forms, the *Conjunctive Normal Form* (CNF) and the *Disjunctive Normal Form* (DNF).

- If the set $\{+, \cdot\}$ is used, the function is referred to be in the *Algebraic Normal Form* (ANF).

Single variable functions

- A single variable function can be represented as, $F(x) = y$. There are only four possible single variable Boolean functions.

$$F_0(x) = 0 \;\; ; \;\; F_1(x) = 1$$
$$\text{and} \;\; F_2(x) = x \;\; ; \;\; F_3(x) = \bar{x} = 1 + x$$

- These functions can also be represented as truth tables.
- The above functions are all represented in their Algebraic Normal Form.

Two variable functions

- A two variable function can be represented as, $F(x) = y \ : x \in \{0,1\}^2$.

- There are many two variable Boolean functions. Some examples are given below:

$$F(x) = 0 \ ; \ F(x) = x_1$$
$$\text{and} \ \ F(x) = x_0 \cdot x_1 \ ; \ F(x) = \overline{x_0 \cdot x_1} = \overline{x_0} \vee \overline{x_1}$$

- There are 16 different two variable Boolean functions, 256 three variable Boolean functions and 65536 four variable functions.

The CNOT gate

- Consider a system of two qubits where each qubit is in either the $|0\rangle$ or the $|1\rangle$ state.
- The possible states of the two-qubit system can now be labelled as $|x_0\rangle\,|x_1\rangle$, where $x_0, x_1 \in \{0, 1\}$.
- Applying the $CNOT_1^0$ gate two this system, the general result is as shown:

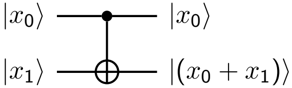

Figure 1: The *CNOT* gate

- Just like the the X-gate was the quantum analogue of the the classical NOT operation, The *CNOT* gate is the analogue of the classical XOR gate.

Two controllers

- Consider a system of three qubits where each qubit is in either the $|0\rangle$ or the $|1\rangle$ state.
- The possible states of the two-qubit system can now be labelled as $|x_0\rangle\,|x_1\rangle\,|x_2\rangle$, where $x_0, x_1, x_2 \in \{0, 1\}$.
- Consider a new gate where $|x_0\rangle$ and $|x_1\rangle$ both control the application of the X-gate on $|x_2\rangle$

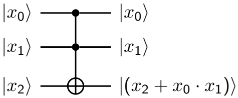

Figure 2: A new gate

- This new gate is called the *CCX* gate or the Toffoli gate and is crucial for realising Boolean functions on quantum computers.
- If the state $|x_2\rangle$ is set to $|0\rangle$ the this gate is a quantum implementation of the classical AND gate.