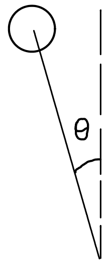
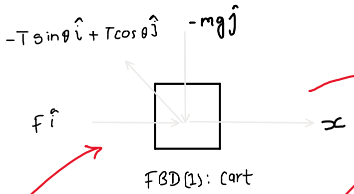
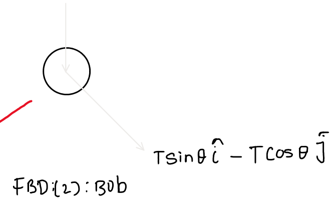
Note on Inverted Pendulum

Thursday, July 10, 2025 6:58 PM







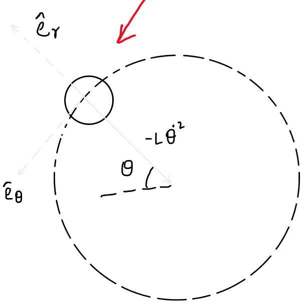


fig: Kinematics of pendulum

- m = mass of pendulum
- M = mass of cart
- θ = angle made by pendulum with vertical axis
- f = applied force
- T = Tension on pendulum rod
- L = length of pendulum

#1 For cart:

FBD(1):

$\hat{i}$ direction: $F - T\sin\theta = M\ddot{x}$ — (1)

#2 For pendulum:

FBD 2:

$\hat{i}: T\sin\theta = m a_{px}$ — (2)

$\hat{j}: -T\cos\theta - mg = m a_{py}$ — (3)

#3 Kinematics:

$$a_p = a_c + a_{p/c}$$

$$= \ddot{x} + [L\ddot{\theta}\hat{e}_\theta - L\dot{\theta}^2\hat{e}_r]$$

- a_p = accⁿ of bob
- a_c = accⁿ of cart
- $a_{p/c}$ = accⁿ of pendulum w.r.to cart

$$= \ddot{x}\hat{i} + L\ddot{\theta}[-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}] - L\dot{\theta}^2[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}] \quad \text{— (4)}$$

Use equation (4) in (2) and (3)

$$T\sin\theta = m(\ddot{x} - L\cos\theta\cdot\ddot{\theta} + L\dot{\theta}^2\sin\theta) \quad \text{— (5)}$$

$$-T\cos\theta - mg = m(-L\ddot{\theta}\sin\theta - L\dot{\theta}^2\cos\theta) \quad \text{— (6)}$$

Solving (5) × cos θ & (6) × sin θ and add,

$$-mg\sin\theta = m\ddot{x}\cos\theta - mL\ddot{\theta} \quad \text{— (7)}$$

use eqn (5) in (1)

$$F + mL\ddot{\theta}\cos\theta - mL\dot{\theta}^2\sin\theta = (M+m)\ddot{x} \quad \text{— (8)}$$

From (7) and (8)

$$\ddot{\theta} = \frac{F\cos\theta - mL\dot{\theta}^2\cos\theta\sin\theta + mg\cos^2\theta\sin\theta}{L[M + m(1-\cos^2\theta)]} \quad \text{— (9)}$$

From (9) and (7)

$$\ddot{x} = \frac{F - mL\dot{\theta}^2\sin\theta + mg\cos\theta\cdot\sin\theta}{M + m(1-\cos^2\theta)} \quad \text{— (10)}$$

Linearizing about fixed point ($\theta = 0 \rightarrow$ upright, $\theta = \pi \rightarrow$ down)

$\sin\theta \simeq \theta$

$\cos\theta \simeq 1$

$\dot{\theta}^2\theta \simeq 0$

#4

$$\left.\begin{aligned} \ddot{x} &= \frac{1}{M}[F + mg\theta] \\ \ddot{\theta} &= \frac{1}{ML}[F + g(m+M)\theta] \end{aligned}\right\} \text{Linearized equation}$$

State $\underline{x} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \longrightarrow \dot{\underline{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix}$

State space equation

#5

$$\frac{d}{dt}[\overline{X}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{m+M}{ML}g & 0 \end{bmatrix}[\overline{X}] + \begin{bmatrix} 0 \\ 1/ML \\ 0 \\ 1/ML \end{bmatrix}[u] \quad \text{— (*)}$$

#6 Now we need MATLAB where we can

1) Define eigen vector by ourselves, like we can chose any -ve poles for stable output (K_f)

2) Use LQR method to find optimal value of K_f for given A, B

LQR : Linear Quadratic Regulator

$\dot{\underline{x}} = \underline{A}\,\underline{x} + \underline{B}\,\underline{u}$

$\underline{u} = -K\underline{x}$ (input)

$\dot{\underline{x}} = (\underline{A} - \underline{B}K)\,\underline{x}$

Cost function: $J = \int_0^\infty (x^T Q x + u^T R u)\,dt$

state weighing matrix — penalizes deviation in state $\underline{x}$

control effort weighing matrix → penalizes use of control u

For MATLAB we chose

$$Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}$$

Penalty for position

Penalty for velocity

Penalty for angular posn.

Penalty for angular velocity

Here, among four states, we have prioritized θ and $\dot{\theta}$ over linear position and velocity of cart.

$R = 0.001$ → we have as much energy available for use.

Note *

1.) For example we can reach to final state either quickly or slow. If we want system be agile, we compromise power and energy consumption of motor. Similarly, if we want high efficiency but ok with slow response we can do that as well.

2) The required response can be obtained by changing Q and R according to our system requirement.