Research Cycle Report 1

Graph Theory for Computing

Hand shake problem and three jug question

Group B5

Group Members

160046C	Udesh Athukorala
160452R	Dulaj Sanjeewa
160001K	Aathman Tharmasanthiran
160214P	Madhushani Hewagama
160670H	Pinsara Weerasingha

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Definitions and Theorems

Basic Definitions

Graph G:

G = (V, E) is an ordered pair of finite sets with $E \subseteq V^2$. Elements of V are vertices/nodes, and elements of E are called edges/arcs. V is the vertex set of G, and E is the edge set. [1]

Edges/Arcs:

An edge (a set of two elements) is drawn as a line connecting two vertices. The edge set of G is usually denoted by E (G). [2]

Vertices/Nodes:

Simply drawn as a node or a dot. The vertex set of G is usually denoted by V (G). [3]

Walk

If u and v are two vertices in a graph G, a u-v walk is an alternating sequence of vertices and edges starting with u and ending at v. Consecutive vertices and edges are incident. [4]

Trail

A trail is a walk with no repeated edges. [5]

Degree of a graph Vertex

The degree of a graph is the number of graph edges which touch the graph vertex, also called the local degree. [6]

Breadth First Search

A search algorithm of a graph which explores all nodes adjacent to the current node before moving on. [7]

Weighted Graphs

A weighted graph is a graph in which each edge is given a numerical weight. [8]

Directed Graphs

A graph in which each edge is specified as going in a particular direction. [9]

Problem 1 (Handshake Problem)

Professor M and her husband Mr. M invited 4 married couples for a party. Some pairs of people shake hands when they meet, but naturally no couple shakes hands with each other. At the end of the party Professor M asked everyone else how many people they have shaken hands with, and she received 9 different answers.
How many people shook hands with Mr. M?

Solution

There are 10 persons were at the party. (4 couples and Professor M and his wife)

There are two conditions associated with this problem. First one is a obvious condition & the second condition is given in problem statement.
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1. A person cannot shake hands with himself
2. A person cannot shake hands with his or her partner

When considering the first condition, the maximum number of handshakes one person can get is the 9 (10 - 1)
When considering the second condition, the maximum number of handshakes one person can get is; 8 (9 – 1)
However according to the question Professor M got 9 different values for the number of handshakes.

Since the possible answers for number of handshakes would be:

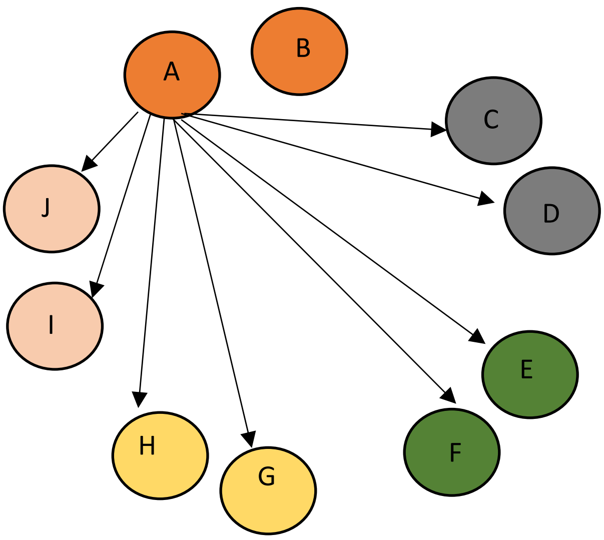
0,1,2,3,4,5,6,7,8.----------------(Result 1)

In order to come up with a solution using a graph we have come up with a graph notation for the above question as shown below,

There are 10 people in this scenario.
So we can represent them as, A, B, C, D, E, F, G, H, I and J

Let's represent them as vertices and if they have shaken hands with someone, connect them with a line (an edge).

Let's Consider the person with 8 handshake case. Assume A is that person.



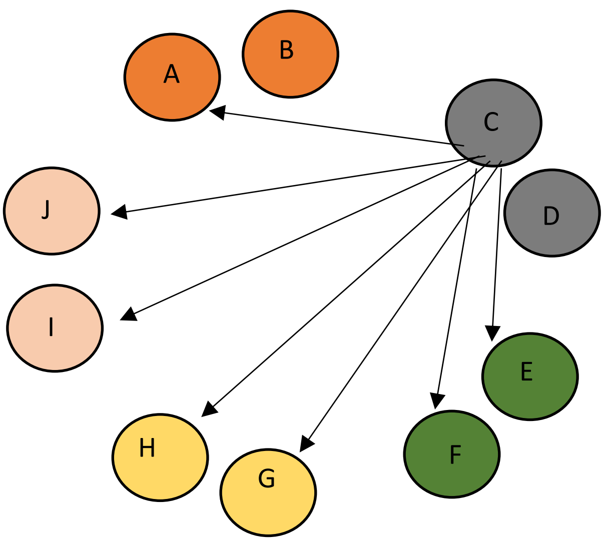
Person A can shake hands with all others except himself and his partner.

So, he shook hands with 8 others.

Since all others except A's partner shook hands with A, all the others had at least 1 handshake.

From Result 1 there should be a person who shook hands with 0 people and that is B. (A's partner)

Consider the person with 7 handshake case. Assume C is that person,



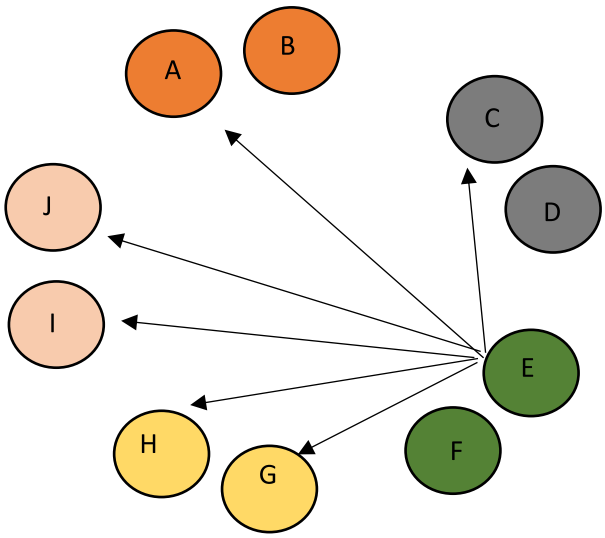
Person C can shake hands with all others except himself, his partner and B.

So, he shook hands with 7 others.

Since all the others except his partner shook hands with A & C, all the others had at least 2 handshakes

From Result 1 there should be a person with 1 handshake that is D (C's partner).

Consider the person with 6 handshake case. Assume that person E,



Person E can shake hands with 6 persons.

From Result 1 there should be a person with 2 handshakes, and since it is similar to the second case we can say that the person having 2 handshakes is F (E’s partner).

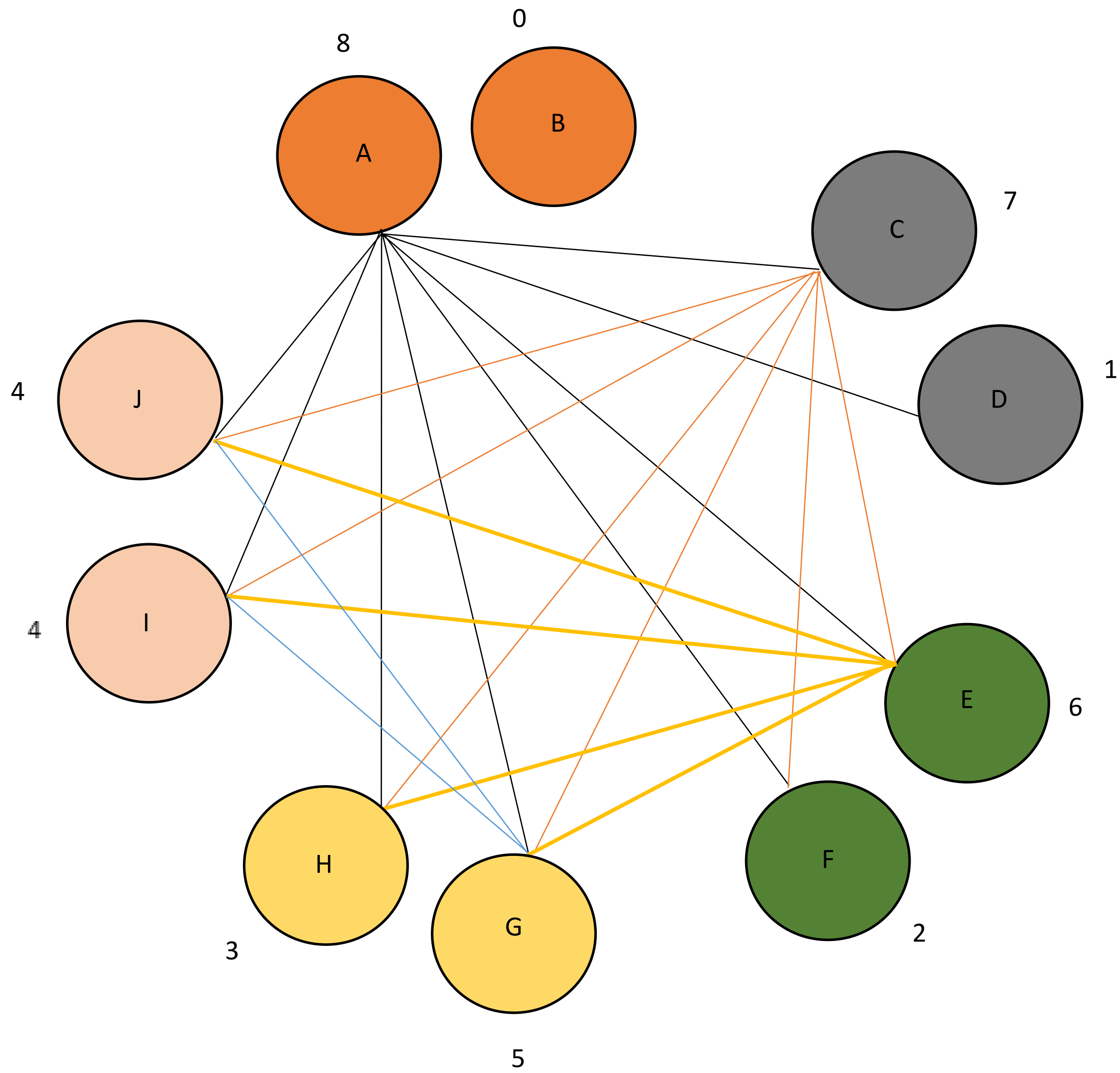
In those scenarios we can see a pattern of the number of handshakes,

- Degree of the Vertex A+ Degree of his partner(B) = 8+0 = 8
- Degree of the Vertex C+ Degree of his partner(D) = 7+1 = 8
- Degree of the Vertex E+ Degree of his partner(F) = 6+2 = 8

So here, a pattern was identified that the sum of degree of couple equals to 8

Hence similarly we can find the number of handshakes each couple had.

(8, 0), (7, 1), (6, 2), (5, 3) (4, 4) are the handshakes done by each couple.



Since we should have different answers for each person the couples with different answers should be the guests while the couple with value 4 should be Mr. M and Mrs. M.

So, the number of handshakes Mr. M had should be 4.

Problem 2 (3 Jug Problem)

Suppose we have 3 jugs labelled A, B, C with capacities of 8, 5, and 3 gallons respectively. Suppose jug A is full and the others are empty. We wish to divide the liquid into 2 equal parts by pouring from one jug into another. *When pouring, we must pour until one jug is empty or the other jug is full.*
Can this be done? *Hint:* Try to represent the problem as a graph where the vertices and arcs represent the amounts in the jugs and the amounts being poured.

Solution

Let a, b and c be the amounts of liquids in the jugs A, B and C respectively at a given stage.

Let's represent the problem by a graph where a vertex is a 3-tuple (a, b, c) representing the amount of gallons in each jug, at each stage of pouring and arcs the amounts being poured.

Those tuples should satisfy following constraints.

1, A, B, C with capacities of 8, 5, and 3

$0 \le a \le 8 \;\&\; 0 \le b \le 5 \;\&\; 0 \le c \le 3$

2. Total of the amount of gallons, in each jug, at each stage equals to 8

$a + b + c = 8$

We can find the above coordinates by using simple python code,

```
for a in range(9):
    for b in range(6):
        for c in range(4):
            if(a+b+c==8):
                print (a, b, c)
```

We got 24 tuples that satisfy the above conditions.

(0, 5, 3)	(1, 4, 3)	(1, 5, 2)	(2, 3, 3)	(2, 4, 2)	(2, 5, 1)
(3, 2, 3)	(3, 3, 2)	(3, 4, 1)	(3, 5, 0)	(4, 1, 3)	(4, 2, 2)
(4, 3, 1)	(4, 4, 0)	(5, 0, 3)	(5, 1, 2)	(5, 2, 1)	(5, 3, 0)
(6, 0, 2)	(6, 1, 1)	(6, 2, 0)	(7, 0, 1)	(7, 1, 0)	(8, 0, 0)

The constraint given in the problem statement when pouring into jugs is as follows;

3.We must pour until one jug is empty or the other jug is full

a = 0 or a = 8 or b = 0 or b = 5 or c = 0 or c = 3

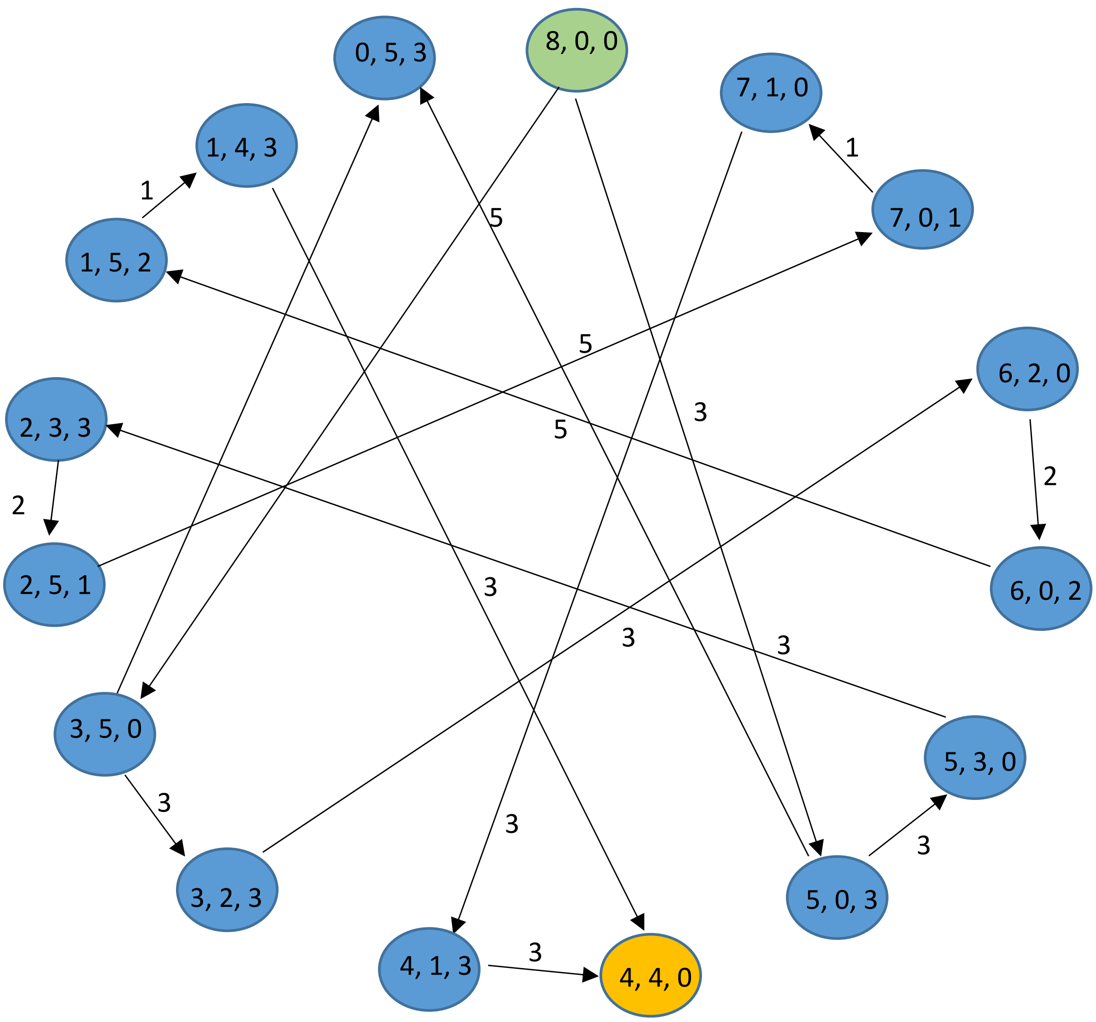
If we consider these constraints we will have only 16 tuples remaining.

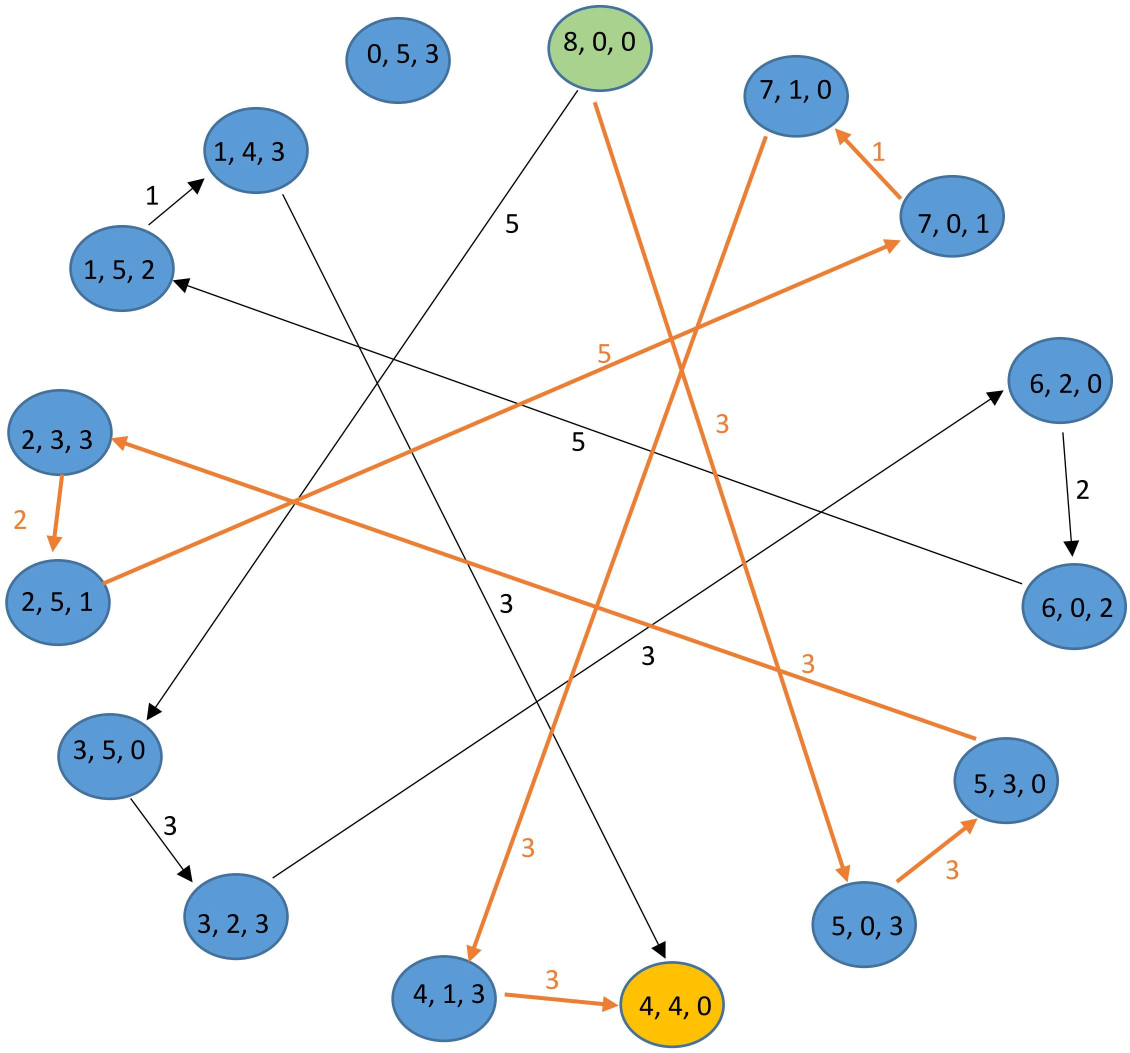
(8, 0, 0),	(7, 1, 0),	(7, 0, 1),	(6, 2, 0),	(6, 0, 2),	(5, 3, 0),
(5, 0, 3),	(4, 4, 0),	(4, 1, 3),	(3, 5, 0),	(3, 2, 3),	(2, 5, 1),
(2, 3, 3),	(1, 5, 2),	(1, 4, 3),	(0, 5, 3)		

We can represent these states in a directed graph. A vertex (a, b, c) is joined to (a', b', c') by an edge directed towards (a', b', c') if the state (a', b', c') can be obtained from state (a, b, c) using a single pouring.

So we have to check whether there is a trail from (8, 0, 0) to (4, 4, 0)

We shall use breadth first algorithm here. Since, breadth first search explores all the neighbor vertices before going to the next level we will get the shortest path from (8, 0, 0) to (4, 4, 0) .





There are 2 correct paths for this problem. They are

Path 1

(8, 0, 0)=> (5, 0, 3)=> (5, 3, 0)=> (2, 3, 3)=> (2, 5, 1)=> (7, 0, 1)=> (7, 1, 0)=> (4, 1, 3)=> (4, 4, 0)

Path 2

(8, 0, 0)=> (3, 5, 0)=> (3, 2, 3)=> (6, 2, 0)=> (6, 0, 2)=> (1, 5, 2)=> (1, 4, 3)=> (4, 4, 0)

So, the liquid can be divided into 2 equal parts.

3 Questions

Q1 Computational problem.

The Wolf, the Sheep and the Turnip is a classic problem! Dr Julia is standing on the side of a river with a wolf, a sheep and a turnip. She has a boat that is so small that it can only hold herself and one of the three objects. However, if she leaves the wolf alone with the sheep on the side of the river then the wolf will eat the sheep, and if she leaves the sheep with the turnip then the sheep will eat the turnip. Thankfully, wolves do not like to eat turnips. How can Dr Julia get all the items safely to the other side of the river? And how can this be formulated in terms of graphs?

Q2 True/False question.

The graph with the degree sequence $6; 6; 6; 4; 4; 2; 2$ exist?

Q3 True/False question.

Every graph on at least two vertices contains two vertices of equal degree.

Solutions

Solution for Q1

- First we need to design some notation to tell us what state the animals are in. We are going to use the symbol (W,S,T) where each of W, S and T are 0 if the object is on the left bank and is 1 if the object is on the right bank. So we have: (0,0,0) Our initial state, with all three on the left side of the river. (1,0,0) The wolf has crossed the river, but not the sheep or the turnip. (0,1,0) The sheep has crossed the river, but not the wolf or the turnip. (0,0,1) The turnip has crossed the river, but not the wolf or the sheep. (1,1,0) The wolf and the sheep have crossed the river, but not the turnip. (0,1,1) The sheep and the turnip have crossed the river, but not the wolf. (1,0,1) The wolf and turnip have crossed the river, but not the sheep. (1,1,1) Our desired state, where all three have crossed the river. We can think of these symbols as 3-dimensional coordinates and draw ourselves a nice picture: Now we need to delete the forbidden edges, so we that we don't leave behind any of the objects that might eat each other. This means the edge joining (0,0,0) to (1,0,0) is forbidden, for example, because the sheep and the turnip are left behind and the sheep would eat the turnip.

- Now all we need to do is find a path from (0,0,0) to (1,1,1)! It is clear that there are two possible solutions: 1. Move sheep to other side. 2. Move turnip to other side. 3. Move sheep back. 4. Move wolf to other side. 5. Move sheep to other side. 1. Move sheep to other side. 2. Move wolf to other side. 3. Move sheep back. 4. Move turnip to other side. 5. Move sheep to other side.

Solution for Q2

False, since otherwise we have 3 vertices of degree 6 which are adjacent to all other vertices of the graph; so each vertex in the graph must be of degree at least 3.

Solution for Q3

True. Suppose that the n vertices all have different degrees, and look at the set of degrees. Since the degree of a vertex is at most n - 1, the set of degrees must be
$\{0; 1; 2; : : : ; n - 2; n - 1\}$
But that's not possible, because the vertex with degree n - 1 would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex

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Q2, Q3 https://dcg.epfl.ch/files/content/sites/dcg/files/...%20Graph%20Theory/solutions1.pdf