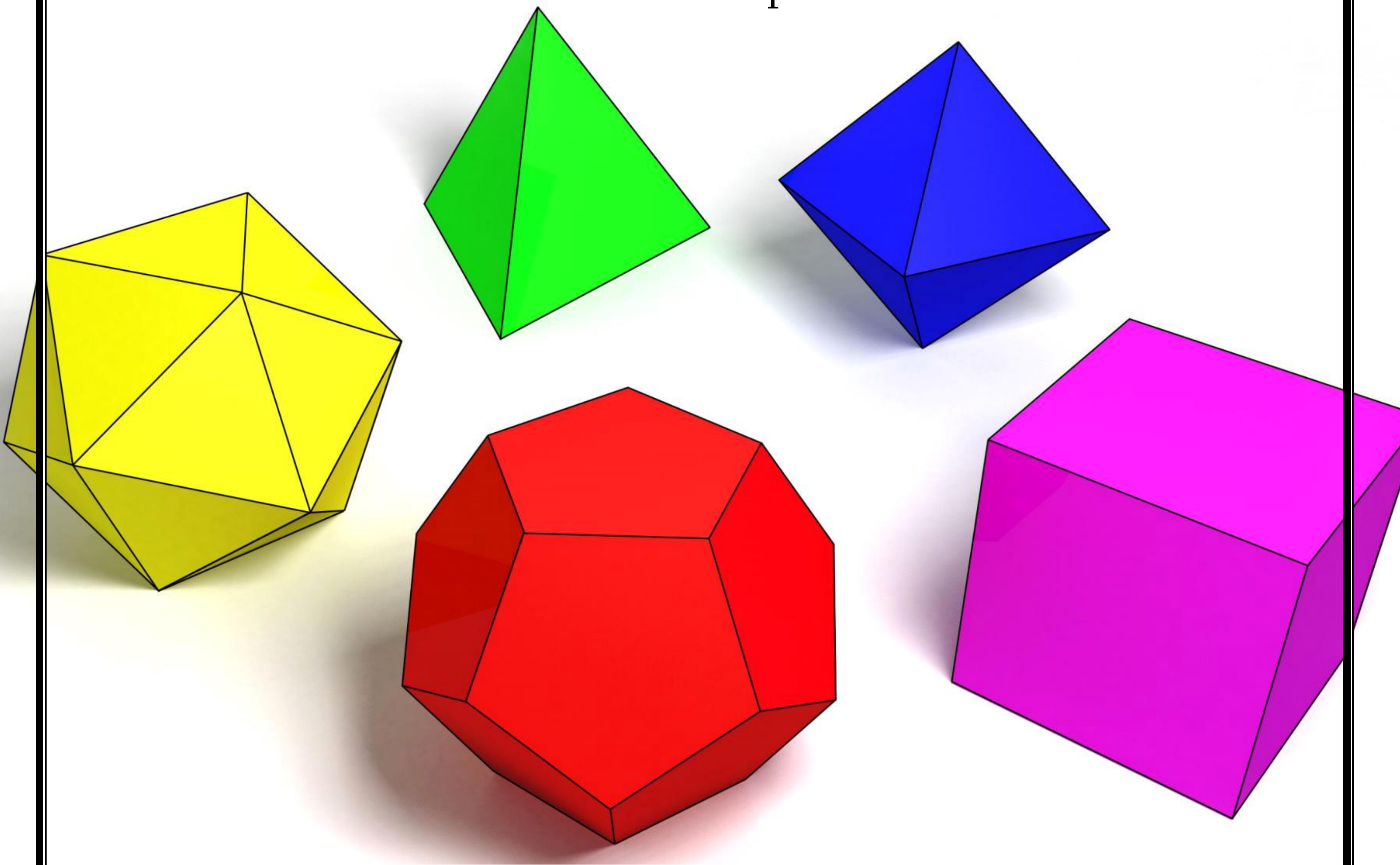
CS 2150: GRAPH THEORY FOR COMPUTING
RESEARCH CYCLE - 1

Research Report



Graph Theory

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Definitions and Theorems

1. Graph

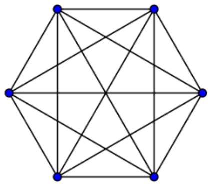
A graph is a collection of points and lines connecting some (possibly empty) subset of them. The points of a graph are known as graph vertices (nodes) The lines connecting the vertices of a graph are known as graph edges (arcs) [1]

2. Platonic Solids

The Platonic solids are convex polyhedra with equivalent faces composed of congruent convex regular polygons. [2]

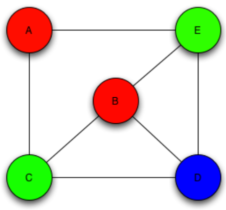
3. Complete Graph

A complete graph is a graph in which each pair of graph vertices is connected by an edge. [3]



4. Chromatic Number

The chromatic number of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. [4]

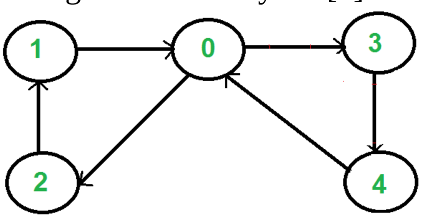


5. Edge Chromatic Number

The edge chromatic number, sometimes also called the chromatic index, of a graph G is fewest number of colors necessary to color each edge of G such that no two edges incident on the same vertex have the same color. [5]

6. Eulerian Graph

An Eulerian graph is a graph containing an Eulerian cycle. [6]



7. Eulerian Cycle

An Eulerian cycle, also called an Eulerian circuit, Euler circuit, Eulerian tour, or Euler tour, is a trail which starts and ends at the same graph vertex. [7]

8. Trail

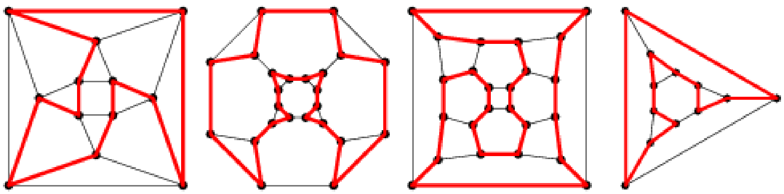
A trail is a walk with no repeated edge [8]

9. Walk

A walk is a sequence of graph vertices and graph edges. [9]

10. Hamiltonian Graph

a Hamilton graph, is a graph possessing a Hamiltonian cycle. [10]



11. Hamiltonian Cycle

A Hamiltonian cycle, also called a Hamiltonian circuit, Hamilton cycle, or Hamilton circuit, is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once. [11]

12. Graph Cycle

A cycle of a graph G, also called a circuit if the first vertex is not specified, is a subset of the edge set of G that forms a path such that the first node of the path corresponds to the last. [12]

13. Path

A graph path is a sequence $\{x_1,x_2,...,x_n\}$ such that (x_1,x_2), (x_2,x_3), ..., $(x_{(n-1)},x_n)$ are graph edges of the graph and the x_i are distinct. [13]

- Euler's Theorem for identifying Eulerian graphs : A connected graph has an Eulerian cycle iff it has no graph vertices of odd degree. [7]

Proof

To prove the above theorem we must first prove the lemma that "Every connected graph of two or more vertices has a vertex that can be removed (along with its incident edges) without disconnecting the remaining graph."

Proof for lemma: P(n) = "A connected graph with n vertices has two distinct vertices, each of which can be removed individually (along with incident edges) without disconnecting the remaining graph"

Base Case: P(2): Either one of the vertices can be removed. The remaining graph is a single vertex in both cases, which is a connected graph.

Inductive Case:
1. Consider arbitrary connected graph G with n + 1 vertices.

2. Remove an arbitrary vertex v to get G"

3. Case 1: G" is connected

(a) Since P(n) is true (Inductive Hypothesis), there are two distinct vertices w and u that (individually) can be removed from G" without disconnecting it.
(b) If w were removed from G instead of v, the resulting graph would
still be connected.
(c) Therefore, two distinct vertices can be safely removed from G, and we are done.

4. Case 2: G" is disconnected

5. Then G" is made up of k connected component subgraphs $C_1, \ldots, C_k$.
6. $k \geq 2$ because with at least n+1 vertices in G where $n \geq 2$, the remaining n vertices (at least 2) must be in separate components if G" is disconnected.

7. Each component C_i has a vertex v_i that was a neighbor of v in G.

8. Case 2.1: For each C_i , vertex v_i is the only vertex in the component.

(a) Add v back to G" get G again, and then remove v_i instead.
(b) Alternately, since $k \geq 2$, we could have removed v_k instead.
(c) Now we have removed a vertex from G in two different ways, and each results in a connected graph.

9. Case 2.2: At least one of C_i , call it C_q, has 2 or more vertices.

(a) The Strong Inductive Hypothesis applies to C_q, so there are two distinct vertices that can be removed from C_q without disconnecting it.
(b) Since there are two, and they are distinct, at least one of them is not v_q. Call the other one x.
(c) Add v back to G" to get G again.
(d) We have already identified one vertex, x, that can be safely removed from G without disconnecting it.
(e) We also know that there is at least one other Cp where p 6= q because $k \geq 2$.
(f) If C_p is just one vertex, we can remove it (v_p) as in Case 2.1.
(g) If C_p is more than one vertex, then the I.H. allows us to find and remove one vertex (not v_p) as in Case 2.2.
(h) The vertex removed from C_p accounts for the second vertex that we could remove from G without disconnecting it.
10. In all cases, P(n+1) is true because we found two distinct vertices, either of which could be removed without disconnecting G.

Now we can use this lemma to prove the **sufficient condition** for a Eulerian cycle to exist.

P(n) = "A connected graph with n vertices, each of even degree, has an Euler circuit"

Base Case: P(2):

1. Because vertex degrees are even, there must be an even number of edges between these two vertices.

2. Call the vertices a and b, and assume there are 2k edges.

3. Then going from a to b and then back again to a k times results in an Euler circuit.

Inductive Case: $P(n) \rightarrow P(n + 1)$

1. Take arbitrary connect graph G with n+1 vertices, each of even degree.

2. By the lemma, we can remove a vertex v (and incident edges) that does not disconnect the graph. Call the result G0.

3. v had an even degree in G, so we can arbitrarily pair up all of v's incident edges.

4. For every such pair of edges (x, v), (v, y) that existed in G, add one edge (x, y) to G0.

5. The degree of each remaining vertex in G0 stays the same as in G, so all are still even.

6. The Inductive Hypothesis then indicates that G0 has an Euler circuit. Call it C.

7. Add v back to get G again, and restore the edges to their original state.

8. Create a path in G from C: whenever an edge (x, y) is traversed that existed in G0, but does not exist in G, traverse (x, v) then (v, y) instead.

9. The resulting path is an Euler circuit in G.

10. Therefore G is an Eulerian Graph.

$\therefore$ P(n) is true for all $n \geq 2$

Proof for necessity

Suppose G contains an Eulerian circuit C. Then, for any choice of vertex v, C contains all the edges that are adjacent to v. Furthermore, as we traverse along C, we must enter and leave v the same number of times, and it follows that degree of v must be even.

So we can see that every vertex of a graph having an even degree is a necessary and sufficient condition for an Eulerian cycle to exist and make it an Eulerian graph.

Proof from [14] and [15]

Assigned Problems

Solve the following problems using graph theoretic concepts and justify your answers.

a) Draw the graphs whose vertices and edges correspond to the vertices and edges of the Platonic solids.

b) Which Platonic solids are represented by complete graphs?

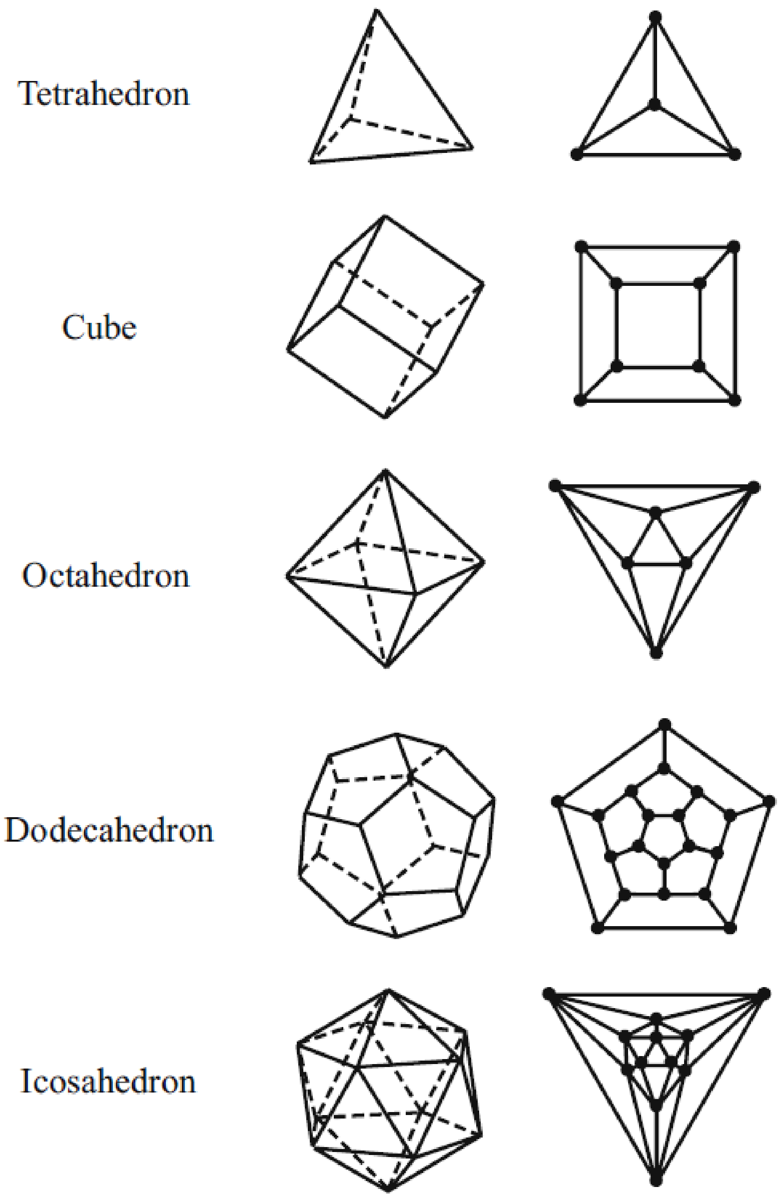
c) Find the chromatic number, the edge chromatic number, and the minimum number of colours needed to colour the map represented by each of the complete graphs you found in part b).

d) Which of the graphs in part a) are

i) Eulerian?
ii) Hamiltonian?

Solutions

a)

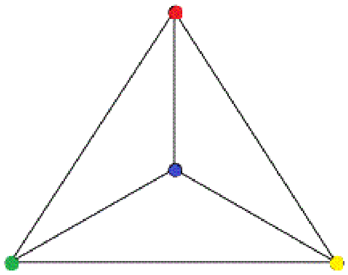


b) Out of all the Platonic graphs **tetrahedral graph** is the only complete graph since only it fulfills the requirements in the definition for a complete graph.

c) Since tetrahedral graph is the only complete graph out of the Platonic graphs, we can find the required properties for a tetrahedral graph.

c.1) Chromatic number

Because a tetrahedral graph is a complete graph and each vertex is adjacent to all other vertices, each vertex will need to be coloured in a different colour. So we need **4 colours** and this cannot be accomplished with 3 or less colours. Therefore the **chromatic number for the graph is 4**.



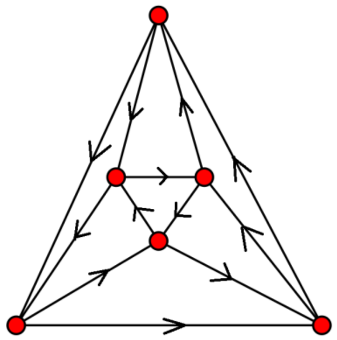
c.2) Edge chromatic number

Since this is a complete graph with 4 vertices, every vertex has 3 vertices connected with it and there are 3 edges incident on each vertex. Therefore 3 colours are required to colour the 3 edges in each vertex.

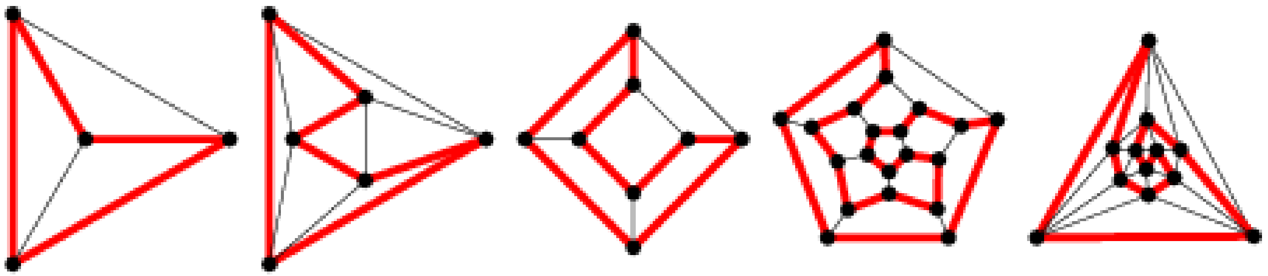
c.3) Minimum number of colours needed to colour the map

In colouring a map each adjacent area of the map should be assigned different colours. Since the tetrahedral graph is a representation of a tetrahedron the 4 faces of the tetrahedron are represented in the graph as areas. Because each of these faces are adjacent to each other we can see that we need a **minimum of 4 colours to colour the map**.

d) i) The graph representation of a octahedron is the only graph out of all the Platonic graphs that doesn't have a vertex of odd degree. Therefore by Euler's theorem (proven above) Octahedral graph is the only Eulerian graph from the Platonic graphs.

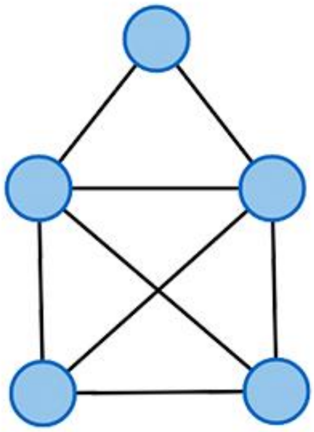


d.ii) All of the Platonic solids are Hamiltonian graphs.



Questions for the reader

1. You can draw the following line drawing starting from a blue dot and finish at the same blue dot, without lifting your pen or drawing over previous lines except at blue dots. [True/False]



2. The edge chromatic number of a graph is equal to the degree of the vertex with the maximum degree. [True/False]

3. Mentioned below is a group of people and their favourite TV shows.

Kamal	The Cartoon, Flies, News1, SportsPlus
Sunil	SportsPlus, Morning Show, The Cartoon, Random Drama
Nimali	Song Show, The Cartoon, Flies, Random Drama
Amali	Random Drama, SportsPlus, Flies, Just Ask

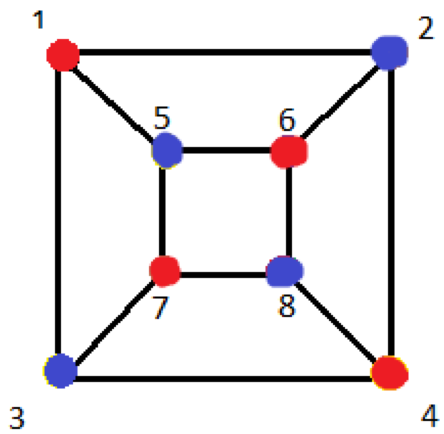
Assuming every show is from a different channel and each has a TV slot of same run time, what is the minimum number of TV slots required for everyone to watch their favourite shows.

Answers

1. If we take the blue dots as vertices and the lines as edges it is evident that the resulting graph needs to be Eulerian, to be drawn according to given instructions. But by the Eulerian theorem, the graph is not an Eulerian graph since it has vertices of odd degree. Therefore the statement is **False**.

2. If the vertex with the maximum degree (v) is of n degree we would need n colours to colour these n edges since they should be different. Vertices from other edges would require colours less than n. So we need minimum n colours so it is the edge chromatic number. Therefore the statement is **True**.

3. If we take every TV show as graph vertices and, connect each show with shows that they share a viewer with, by an edge, we get a cubic graph. By coloring it's vertices we can assign TV slots to them as no adjacent TV show (vertex) can share a slot. Therefore the number of minimum TV slots will be equal to the chromatic number of a cubic graph, which is **2**.
[1. News1 2. The Cartoon 3. Flies 4. Song Show 5. SportsPlus 6. Morning Show 7. Just Ask 8.Random Drama]



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