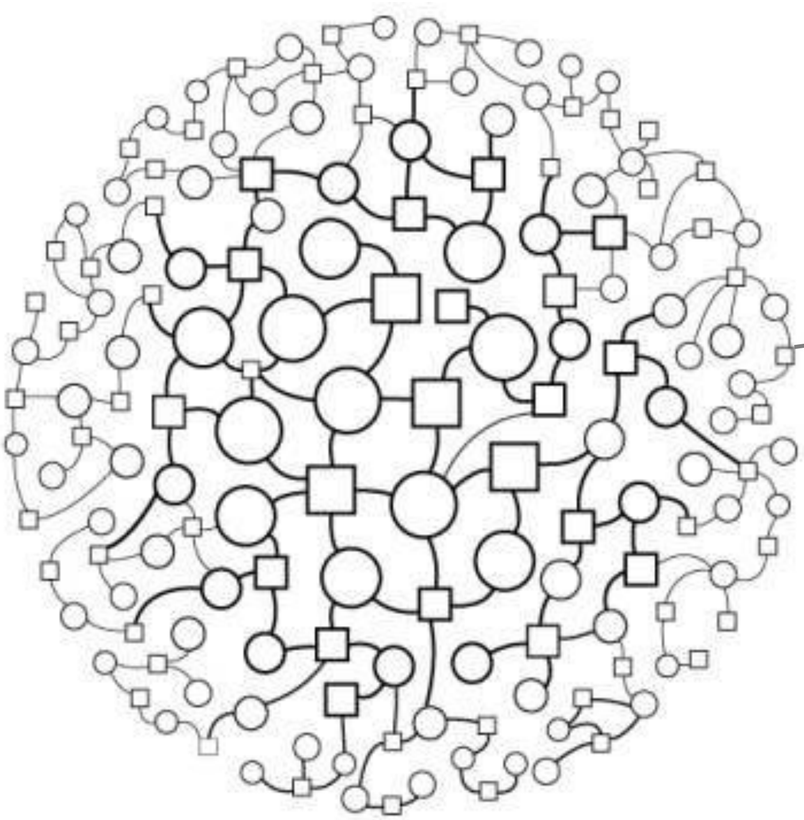
GRAPH THEORY FOR COMPUTING

RESEARCH CYCLE-1 REPORT



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➢ <u>Assigned Problem:</u>

1) Solve the following problems using graph theoretic concepts and justify your answers.

a) Draw the graphs whose vertices and edges correspond to the vertices and edges of the Platonic solids.

b) Which Platonic solids are represented by complete graphs?

c) Find the chromatic number, the edge chromatic number, and the minimum number of colours needed to colour the map represented by each of the complete graphs you found in part b).

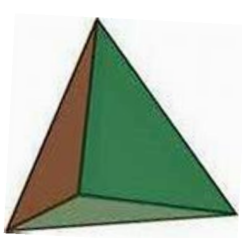
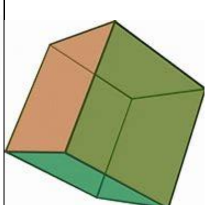
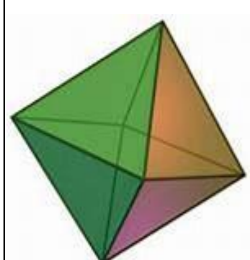
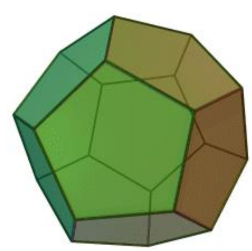
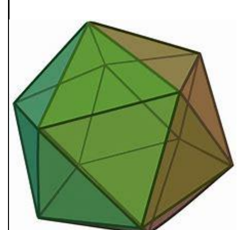
d) Which of the graphs in part a) are i) Eulerian? ii) Hamiltonian?

➢ Definitions and theorems :

1 <u>Platonic solids:</u>

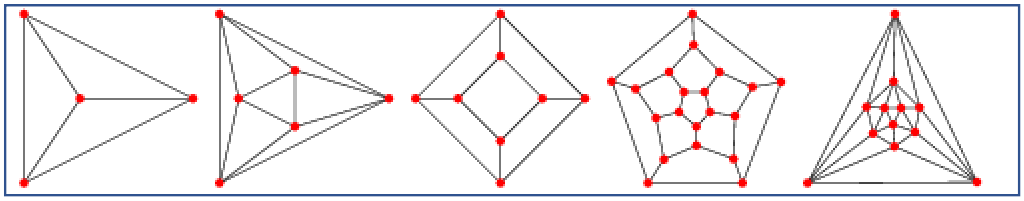
In three-dimensional space, a **Platonic solid** is a regular, convex polyhedron. It is constructed by congruent (identical in shape and size) regular (all angles equal and all sides equal) polygonal faces with the same number of faces meeting at each vertex.

Only five of solids have met the criteria:

Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Four faces	Six faces	Eight faces	Ten faces	Twenty faces
				

2 Platonic graphs:

A polyhedral graph corresponding to the skeleton of a Platonic solid. The five platonic graphs : the tetrahedral graph, cubical graph, octahedral graph, dodecahedral graph, and icosahedral graph, are illustrated below.



3 Complete Graphs:

A complete graph is a graph in which each pair of graph vertices is connected by an edge.

4 Eulerin Graph:

An Eulerian cycle, also called an Eulerian circuit, Euler circuit, Eulerian tour, or Euler tour, is a trail which starts and ends at the same graph vertex. In other words, it is a graph cycle which uses each graph edge exactly once. For technical reasons, Eulerian cycles are mathematically easier to study than are Hamiltonian cycles.

5 Hamiltonian Graph:

Hamilton cycle, or Hamilton circuit, is a graph cycle (i.e., closed loop) through a graph that visits each node exactly once A graph possessing a Hamiltonian cycle is said to be a Hamiltonian graph.

6 Chromatic number:

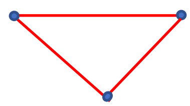
The chromatic number of a graph $G$is the smallest number of colors needed to color the vertices of Gso that no two adjacent vertices share the same color i.e., the smallest value of $k$possible to obtain a *k*-coloring.

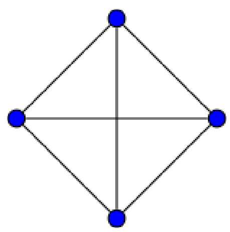
7 Chromatic index (Edge chromatic number):

The edge chromatic number, sometimes also called the chromatic index, of a graph $G$is fewest number of colors necessary to color each edge of Gsuch that no two edges incident on the same vertex have the same color. In other words, it is the number of distinct colors in a minimum edge coloring.

8 Planar graphs:

Graph is planar if it can be drawn in a plane without graph edges crossing (i.e., it has graph crossing number 0).

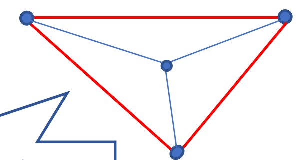
Eg: 



Is this a planar graph??

Answer: No, because the edges cross one another.

But a similar graph can be converted as following using isomorphism (Same number of edges and vertices and connections)



Isomorphic planar graph of the graph given

9 Mapping:

It is colouring the spaces in a way such that no two adjacent spaces of a planar graph sharing a common edge have same colour. For this external common space is also considered.

- ## Eluers Therorem:

Euler Theorem 01 :

For a connected multi-graph G, G contains an Euler circuit if and only if every vertex has even degree. And now it’s a grapgh to be cloured.Follow edge or vertex colouring.

Lemma 1: If G is a graph with $\delta(G) \geq 2$, then the graph G must contain a cycle.

Lemma 2: A Graph G where each vertex has an even degree can be split into cycles by which no cycle has a common edge.

Proof of Euler's theorem:

- Lemma 1: If G is a graph with $\delta(G) \geq 2$, then the graph G must contain a cycle.

  [$\delta(G)$ refers to the minimum degree of a graph]

- **proof for Lemma 1:**

  Suppose we have a graph G=(V(G), E(G)) such that $\delta(G)\geq 2$. Let's let P be any maximal path in the graph G (a maximal path P is a path that cannot be further extended). The vertices of graph P are set ,V(P)= {x1, x2,...,xn}.But P is a maximal graph, so all neighbours of the vertex x1 are in the set {x2, x3,…,xn}. Since $\delta(G)\geq 2$, then the degree of all other vertices is at least 2. So,the vertex x1 has at least two neighbours on the path P. A cycle can thus be formed using x1 and xj where xj is any neighbour of x1.

➢ Lemma 2: A Graph G where each vertex has an even degree can be split into cycles by which no cycle has a common edge.

- **proof for Lemma 2:**

  Suppose we have a graph G=(V(G), E(G)) where each vertex in V(G) has an even degree. We can first form a cycle by starting at a vertex and traversing around the graph. Note that since all vertices have even degree, then when you enter a vertex from an edge, you are always able to exit from another edge. Since the graph G has a finite number of vertices, we will eventually return to the vertex we started at. We will call this cycle C1.

  We can now remove this cycle from the graph G to obtain a subgraph of G. This subgraph may or may not be connected, but that doesn't matter. Every vertex of this subgraph will have an even degree because when we passed by a vertex, we subtracted 2 from the degree as we entered using one edge, and exited using another edge. Hence, the next cycle C2 can be obtained in the same manner, and the process can be repeated until there are no edges left to traverse. All of the cycles C1, C2, …, Cn do not share any edges.

- **Theorem 1 (Euler's Theorem):**

  A connected graph G=(V(G), E(G)) is Eulerian if and only if all vertices in V(G) have an even degree.

  We now have the necessities to prove Euler's theorem on Eulerian graphs. We will prove this theorem using mathematical induction.

  For a connected graph G=(V(G),E(G)), for each $m\geq 0$, let S(m) be the statement that if G has m edges and all of the degrees of vertices in V(G) are even, then the graph G is Eulerian.

  Base Step (m=0): S(0) has no edges. Since the graph is connected, the only possibly way a connected graph can have no edges is that the graph is a single vertex, let's call it x1. deg(x1) =0, which is even, and is trivially Eulerian.

Induction Step $S(0) \wedge S(1) \wedge \ldots \wedge S(k-1) \Longrightarrow S(k)$): Let $k \geq 1$ and assume that S(1),S(2),...,S(k-1) is true. We want to prove S(k) is value. Let G be a graph with k-edges, is connected, and all vertices of G have even degrees.

Since G is a connected graph, there are no isolated vertices, so it follows that the smallest degree $\delta(G) \geq 1$. But all degrees are even, so $\delta(G) \geq 2$. From above, this graph G must contain a cycle, let's call it C.

Now let's create a new graph H by removing all of the edges that are in graph C from graph G. Note that the graph H may be disconnected. We can say the graph H is the union of the connected components H1, H2,...,Ht. The degree is each Hi must be even since the degrees drop only by 0 or 2.

Applying the induction hypothesis to each Hi that is S(| E(H1)|),...,S(| E(Ht)|), each Hi will have an Eulerian circuit, let's say Ci.

We can now create a Eulerian circuit for G by splicing together the graph C with the Ci's. First start on any vertex of Ci and traverse until you hit some Hi. Then traverse Ci and continue back on C until you hit the next Hi.

Conclusion: Thus it follows that G must be Eulerian. This completes the inductive step as $S(0) \wedge S(1) \wedge \ldots \wedge S(k-1) \Longrightarrow S(k)$). By the principle of strong mathematical induction, for $m \geq 0$, S(m) is true.

- **Proof 2:**

➔ Each time the path passes through a vertex it contributes 2 to the vertex degree, except the starting and ending vertices. If the path terminates where it started it will contribute 2 to the degree as well.

🡐 Let G be the connected graph with each vertices of even degree.

- ✓ Let W be the longest trail in G.
- ✓ W: $\mathbf{V_0E_1V_2E_2 \ldots\ldots\ldots\ldots E_{n-1}V_{n-1}E_nV_n}$

Since all the edges incident with $\mathbf{V_n}$ are used in W and $\mathbf{V_n}$ has even degree;

- ✓ $\mathbf{V_n=V_0}$ => **W** is closed path
- ✓ Suppose W is not Euler tour.
- ✓ Since G is connected, there is an edge **f – V_iU € G $\cancel{€}$ E(w)**

Then $\mathbf{fe_{i+1} \ldots\ldots e_n e \ldots\ldots e_i}$ is longer walk **(Contradiction)**

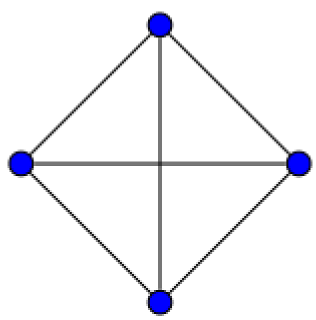
So, W is an **Eulers Graph.**

Solutions to the Questions:

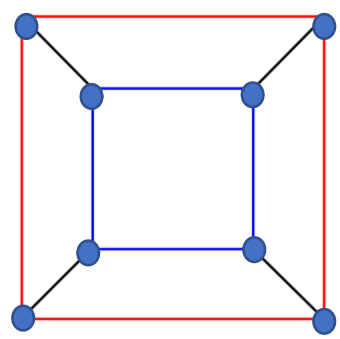
a) Draw the graphs whose vertices and edges correspond to the vertices and edges of the Platonic solids.

Solution:

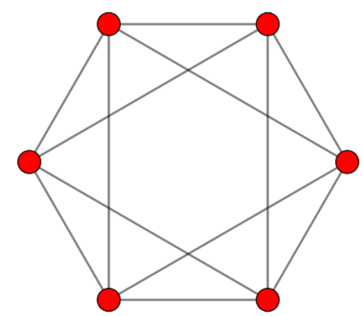
Considering the platonic solids here, we pretend to draw the planar platonic graphs.



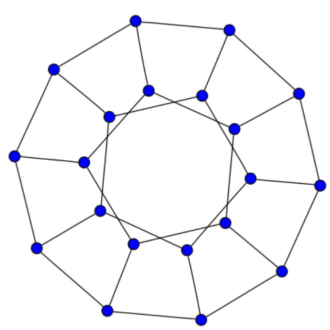
Tetrahedron



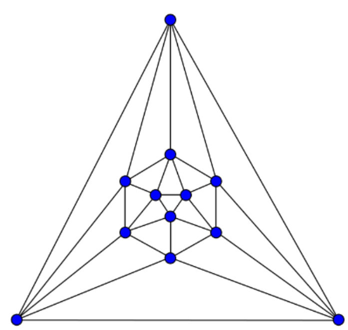
Cube



Octahedron



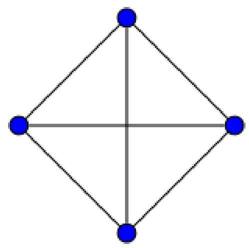
Dodecahedron



Icosahedron

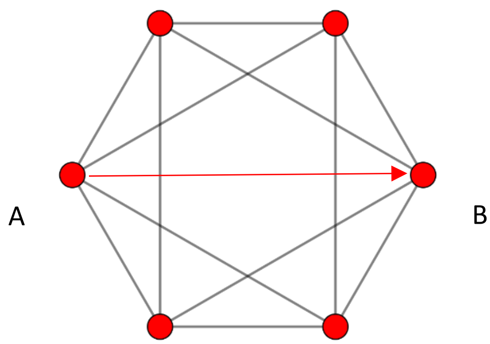
b) Only platonic graph corresponding to four faces have a complete graph:

Solution: The complete graph can be drawn only for platonic solid with four faces.



Here all vertices are connected.

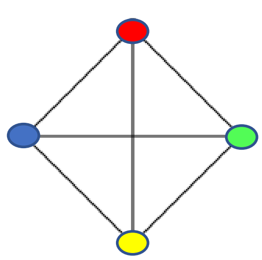
- Do you think it is a complete graph?



A→ B is not connected. So it is not a connected graph.

c) Find the chromatic number, the edge chromatic number, and the minimum number of colours needed to colour the map represented by each of the complete graphs you found in part b).

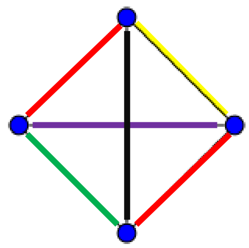
Solution :



Since it is a complete graph, as its all nodes are connected to **colour the nodes** it needs at least the number of nodes.

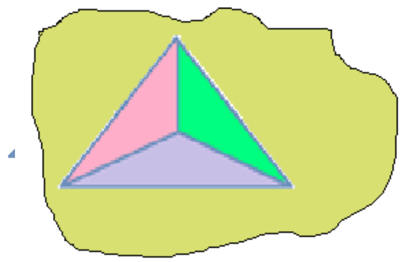
So,

Chromatic number = No.of nodes = 4



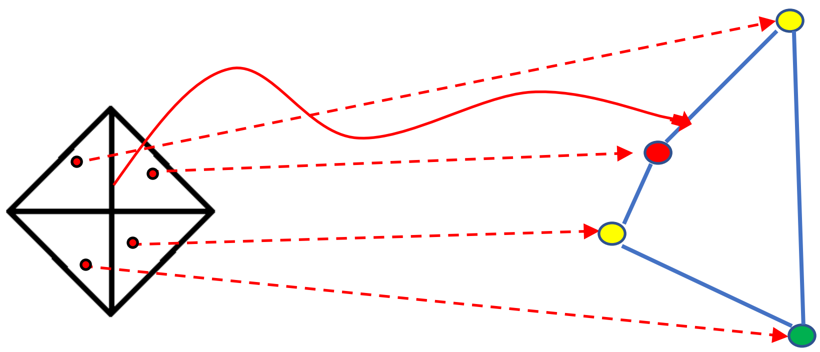
In **edge colouring**, maximum degree would be the Edge chromatic number (Chromatic index) as we have to colour edges such that edges from same vertex have different colours.

Chromatic Index=Max degree of graph=3



➔ This is how we can **colour the spaces**. Since it is a complete graph we need four colours(including the external space).

And if we want we can try drawing a new graph by keeping the spaces as vertex and edges diving the adjacent spaces as connecting edge of the vertices.



Can simply say by edge chromatic number:

Maximum degree of graph is 3. So, chromatic number is 3.(here we consider the vertex as spaces and coloured them)

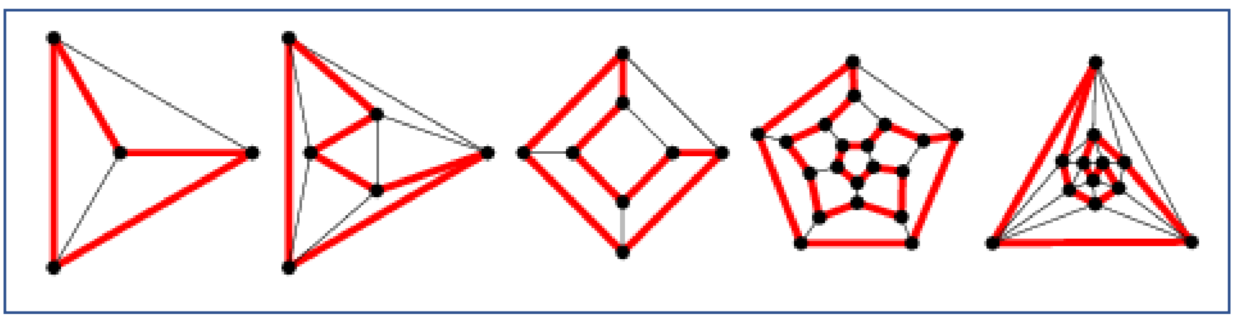
And we need one more colour for external space. So, we need altogether four colours.

d) Which of the graphs in part a) are i) Eulerian? ii) Hamiltonian?

Solution :1) Since each vertex has equal number of faces in Platonic solids, each vertex of a platonic graph has same degree and it is easy for us to find out which are Eulerian through Euler's theorem that we have come across earlier before.

Platonic graphs (resembling platonic solids)	Number of edges per vertex(n)	Is it Eulerian??
Tetrahedron	3	n is odd. By Euler's Theorem it is not Eulerian graph
Cube	3	n is odd. By Euler's Theorem it is not Eulerian graph
Octahedron	4	n is even. By Euler's Theorem it is **Eulerian** graph
Dodecahedron	3	n is odd. By Euler's Theorem it is not Eulerian graph
Icosahedron	5	n is odd. By Euler's Theorem it is not Eulerian graph

2) Since we could find at least one cycle where the path flows through all vertices exactly once, all platonic graphs are Hamiltonian.



➢ Summary:

Platonic graphs	Vertices	Edges	Regularity Faces	Edges per vertex(n)	Eulerian Graph	Hamiltonian Graph	Complete Graph
Tetrahedron	4	6	Triangle	3	No	Yes	Yes
Cube	8	12	Square	3	No	Yes	No
Octahedron	6	12	Triangle	4	Yes	Yes	No
Dodecahedron	20	30	Pentagon	3	No	Yes	No
Icosahedron	12	30	Triangle	5	No	Yes	No

➢ Questions :

1. Does there exist a polyhedron with exactly thirteen faces, all of which are triangles? (Y/N)

   Answer: Can't

   Proof: The first thing you should do is try to draw such a polyhedron as a planar graph, keeping in mind that the outside face must be a triangle. You can try doing this all day, but what you'll find is that the task is impossible. And hopefully, you'll also find that you can have such polyhedra with exactly ten faces or twelve faces or fourteen faces. What this tells you is that there is something to do with oddness and evenness going on in this problem and that suggests that we are going to use the handshaking lemma So now let's suppose that there does exist a polyhedron with exactly thirteen faces, all of which are triangles. The trick here is to use the handshaking lemma on the dual graph. This asserts that the sum of the numbers of edges around each face is equal to twice the number of edges. In our case, this means that 13×3 is equal to twice the number of edges or, in other words, that the number of edges must be 1912 . Of course, this is a contradiction, so we can deduce that there does not exist a polyhedron with exactly thirteen faces, all of which are triangles.

2. If G is a simple connected graph, and if the largest vertex-degree of G is n (> 3), then G is n colourable. (Y/N)

   Answer: No, consider a complete graph which required n+1 colours.

3. Prove that connected graph G is Eulerian if and only if its edge set can be decomposed into cycles.

Proof: Let G (V, E) be a connected graph and let G be decomposed into cycles. If k of these cycles is incident at a particular vertex v, then d(v) = 2k. Therefore, the degree of every vertex of G is even and hence G is Eulerian. Conversely, let G be Eulerian. We show G can be decomposed into cycles. To prove this, we use induction on the number of edges. Since $d(v) \geq 2$ for each $v \in V$, G has a cycle C. Then G−E(C) is possibly a disconnected graph, each of whose components C1, C2, ..., Ck is an even degree graph and hence Eulerian. By the induction hypothesis, each Ci is a disjoint union of cycles. These together with C provide a partition of E(G) into cycles.

❖ References:

1.Platonic Solid -- from Wolfram MathWorld
http://mathworld.wolfram.com/PlatonicSolid.html

2.Platonic Graphs - from Wolfram MathWorld
http://mathworld.wolfram.com/search/?query=platonic+graphs&x=0&y=0

3. Planer Graph - Wolfram MathWorld
http://mathworld.wolfram.com/search/?query=planer+graph&x=1&y=3

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http://mathworld.wolfram.com/EulerGraph.html

5. Euler's Theorem - from Wolfram MathWorld
http://mathworld.wolfram.com/EulersTheorem.html

6. Hamiltonian Graph -- from Wolfram MathWorld
http://mathworld.wolfram.com/HamiltonianGraph.html

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