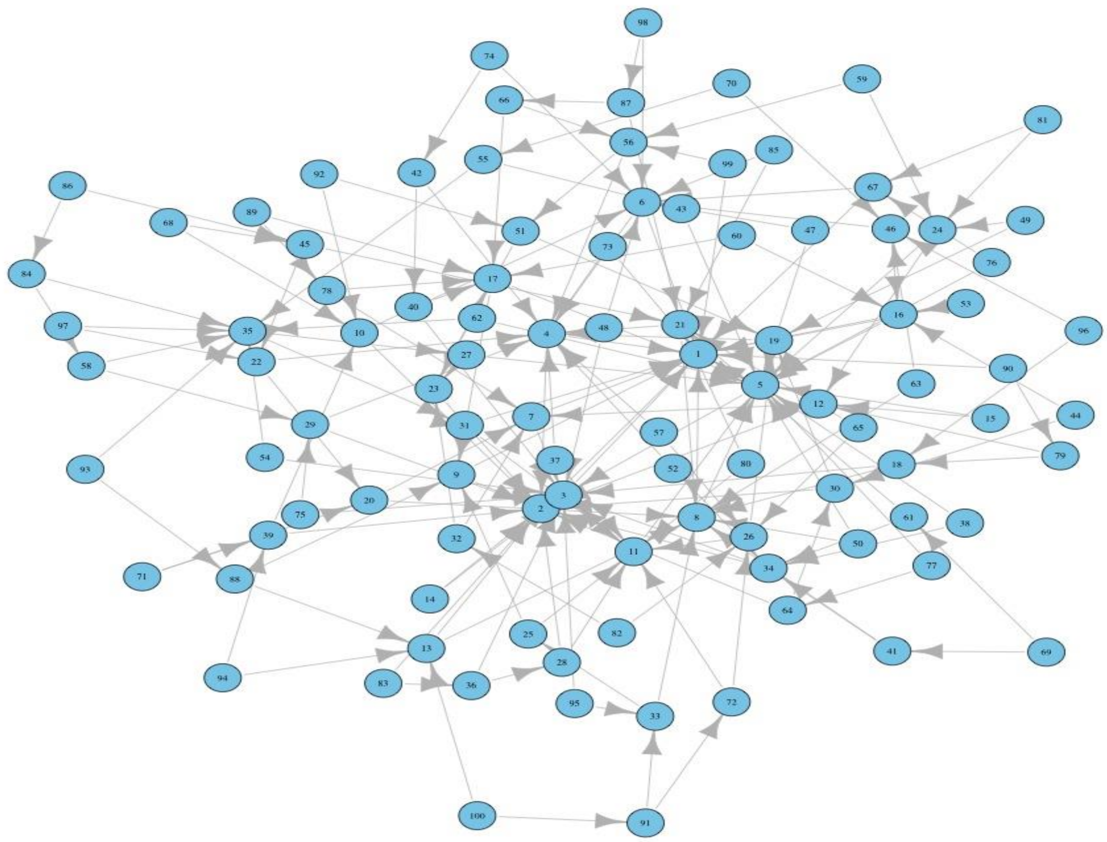

RESEARCH REPORT 01

GROUP B3

GROUP MEMBERS:

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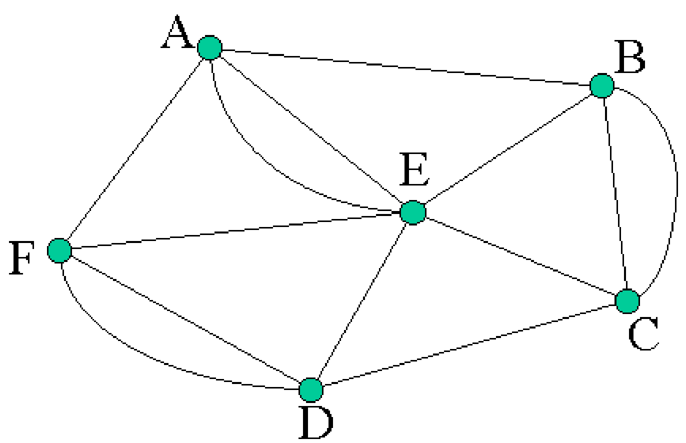
DEFINITIONS

➢ GRAPH :

A graph $G=(V,E)$ is an ordered pair of finite sets where, elements of V are called vertices or nodes and elements of $E \subseteq V(2)$ are called edges or arcs.[1]

➢ DEGREE OF VERTEX:

The degree (d(v)) of a vertex v is the number |E(v)| of edges at v or the number of neighbors of v.[2]



d(A) – 4

d(B) - 4

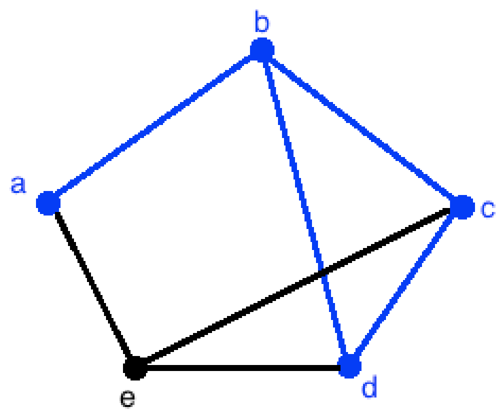
d(c) - 4

d(D) – 4

d(E) - 6

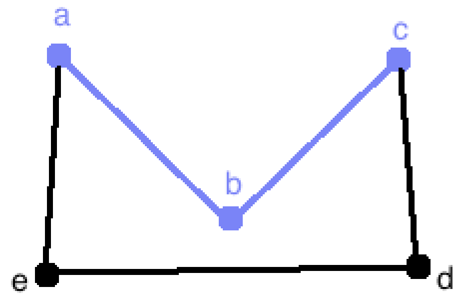
➢ WALK :

A walk is a sequence v_0, e_1, v_1, ..., v_k of graph vertices v_i and graph edges e_i such that for $1 \le i \le k$, the edge e_i has endpoints v_{i-1} and v_i .The length of a walk is its number of edges.[3]



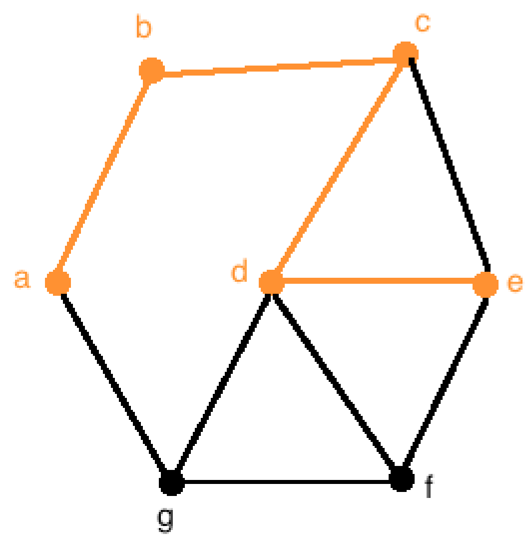
➢ TRAIL :

A trail is a walk $v_0, e_1, v_1, ..., v_k$ with no repeated edge. The length of a trail is its number of edges.[4]



➢ PATH :

A path is a walk with no repeating vertices, except possibly the start and end vertices.



➢ **A walk/trail/path of length n $\geq$ 3 whose start and end vertices are the same is called a closed walk/trail/path.**

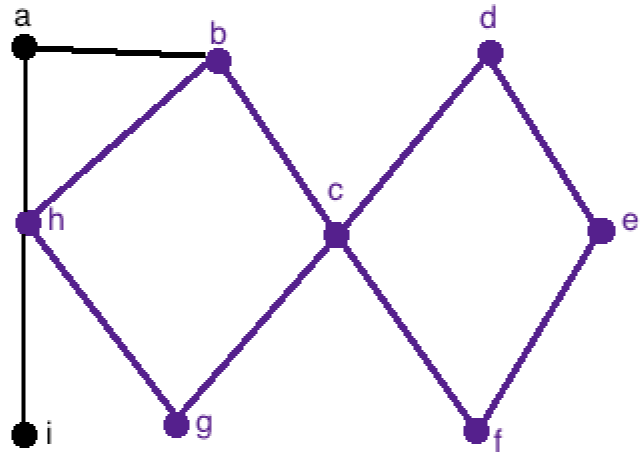
Walk : a > b > c > e > d > b > a

Path : a > b > c > e > d

Trail : a > b > c > e > d > b

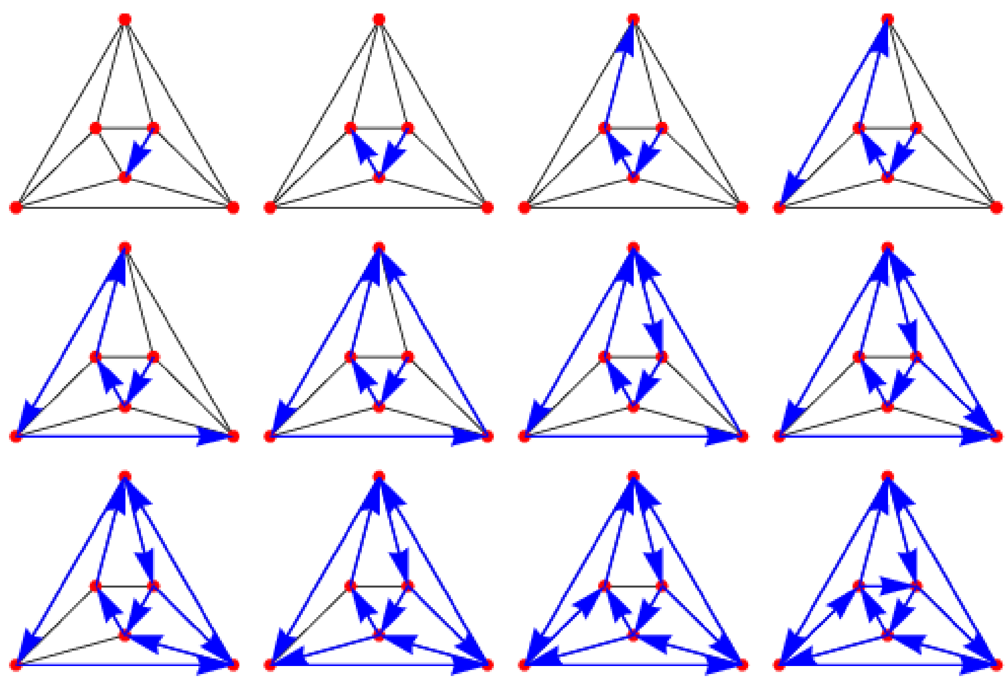
➢ CIRCUIT:

A **Circuit** is a closed trail. That is, a circuit has no repeated edges but may have repeated vertices.[5]



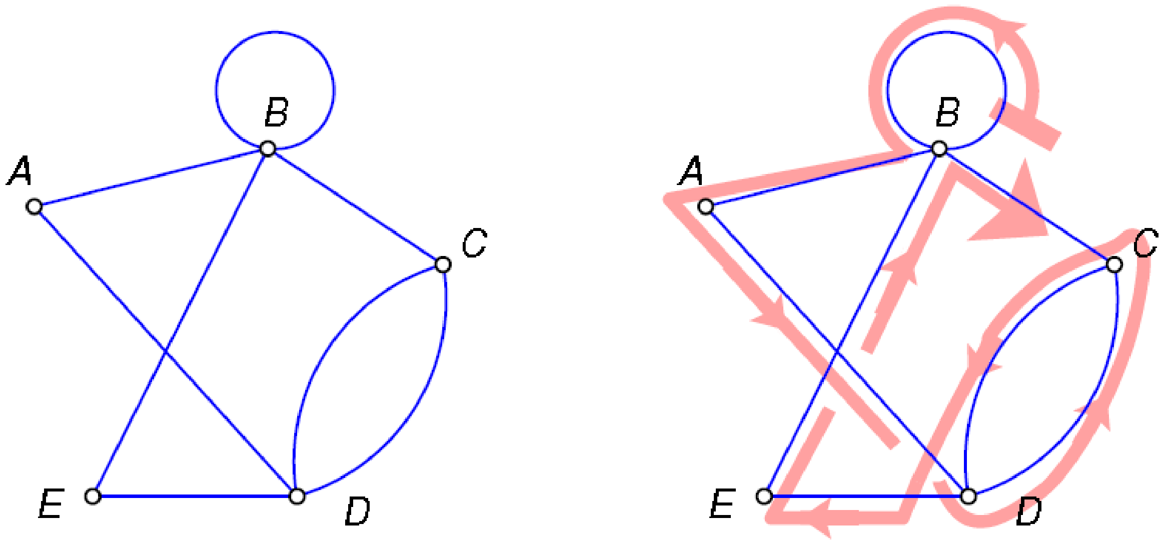
➢ EULERIAN CYCLE :

An Eulerian cycle, also called an Eulerian circuit, Euler circuit, Eulerian tour, or Euler tour, is a trail which starts and ends at the same graph vertex. In other words, it is a graph cycle which uses each graph edge exactly once.[6]



➢ EULERIAN PATH :

An Eulerian path, also called an Euler chain, Euler trail, Euler walk, or "Eulerian" version of any of these variants, is a walk on the graph edge of a graph which uses each graph edge in the original graph exactly once.[7]



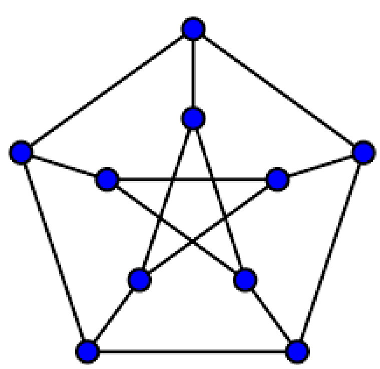
An Euler path: BBADCDEBC

➢ EULARIAN GRAPH :

An Eulerian graph is a graph containing an Eulerian cycle.

➢ CONNECTED GRAPH :

A graph is connected when there is a path between every pair of vertices. In a connected graph, there are no unreachable vertices.[8]



ASSIGNED PROBLEMS

1. Show that any connected graph where the vertex degrees are even has an Eulerian cycle.

2. Show that in any connected graph where there are exactly two vertices *a* and *b* of odd degree, there is an Eulerian path from *a* to *b,* but if there are more than two vertices of odd degree, then it is impossible to construct an Eulerian path.

SOLUTIONS

1. By Induction

- ✓ For a connected graph $G=(V(G),E(G))$, for each $m\geq 0$, let $S(m)$ be the statement that if G has m edges and all of the degrees of vertices in $V(G)$ are even, then the graph G is Eulerian.

- ✓ **Base Step (m=0):** $S(0)$ has no edges. Since the graph is connected, the only possibly way a connected graph can have no edges is that the graph is a single vertex, let's call it $x1$. $\deg(x1)=0$, which is even, and is trivially Eulerian.

- ✓ **Induction Step $S(0)\wedge S(1)\wedge\ldots\wedge S(k-1)\Longrightarrow S(k)$):** Let $k\geq 1$ and assume that $S(1),S(2),\ldots,S(k+1)$ is true. We want to prove $S(k)$ is value. Let G be a graph with k-edges, is connected, and all vertices of G have even degrees.
- ✓ Since G is a connected graph, there are no isolated vertices, so it follows that the smallest degree $\delta(G)\geq 1$. But all degrees are even, so $\delta(G)\geq 2$. From above, this graph G must contain a cycle, let's call it C.
- ✓ Now let's create a new graph H by removing all of the edges that are in graph C from graph G. Note that the graph H may be disconnected. We can say the graph H is the union of the connected components $H1,H2,\ldots,Ht$. The degree is each Hi must be even since the degrees drop only by 0 or 2.
- ✓ Applying the induction hypothesis to each Hi that is $S(|E(H1)|),\ldots,S(|E(Ht)|)$, each Hi will have an Eulerian circuit, let's say Ci.
- ✓ We can now create a Eulerian circuit for G by splicing together the graph C with the Ci's. First start on any vertex of Ci and traverse until you hit some Hi. Then traverse Ci and continue back on C until you hit the next Hi.

- ✓ **Conclusion:** Thus, it follows that G must be Eulerian. This completes the inductive step as $S(0)\wedge S(1)\wedge\ldots\wedge S(k-1)\Longrightarrow S(k)$). By the principle of strong mathematical induction, for $m\geq 0$, $S(m)$ is true

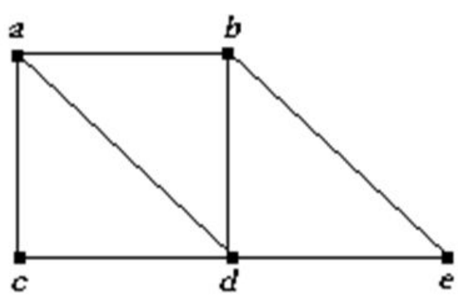
2. A graph G that is connected has a Eulerian path if two vertices in the graph have odd degree. let G=(V(G),E(G)) be a connected graph with exactly two vertices of odd degree, let's call them $x1$ and $x2$. By adding an edge to $x1$ and $x2$ then each vertex in G has an even degree. Thus, an Eulerian trail exists. The removal of this edge between $x1$ and $x2$ results in an open trail which contains all edges of G. Thus, G must have a Eulerian path and when there exist more than two odd vertices (assume n) there won't be any retrieving from n-1 vertices therefore impossible to construct a Eulerian path.

SAMPLE QUESTIONS

1. Prove that a complete graph with n vertices contains n(n - 1)=2 edges.
2. Both following statements are TRUE or FALSE ?

- **A graph with one odd vertex will have an Euler Path but not an Euler Circuit.**
- **In a Euler's Circuit or Path, you cannot use a vertex twice.**

3. Following statement is TRUE or FALSE?



This graph does not contain an Euler circuit but does contain two Euler path.

ANSWERS

1. This is easy to prove by induction. If n = 1, zero edges are required, and 1(1- 0)=2 = 0. Assume that a complete graph with k vertices has k(k - 1)=2. When we add the (k + 1)st vertex, we need to connect it to the k original vertices, requiring k additional edges. We will then have k(k- 1)=2 + k = (k + 1)((k + 1)- 1)=2 vertices, and we are done.

2. By the previous theorem Euler circuit shout have even degree vertices. By the definition of Euler path, one odd vertex will have an Euler path. Therefore, this statement is true.

   By the definition of Euler path, it cannot use an edge twice but not the vertex. So that this statement is false.

   The answer is **FALSE.**

3. Since this graph has the vertices which have odd degree of vertex , by the Euler's Theorem, this graph does not contain any Euler circuits.

   This graph has two Euler paths.(b<c<d<c<a<b<d<a) and (a<c<d<e<b<d<a<b)
   So this statement is TRUE.

REFERENCES

[1] - http://mathworld.wolfram.com/Graph.html

[2] - http://mathworld.wolfram.com/Graph.html

[3] – http://mathworld.wolfram.com/Walk.html

[4] –http://mathworld.wolfram.com/Trail.html

[5] - http://mathonline.wikidot.com/walks-trails-paths-cycles-and-circuits

[6] – http://mathworld.wolfram.com/EulerianCycle.html

[7] – http://mathworld.wolfram.com/EulerianPath.html

[8] - http://mathworld.wolfram.com/ConnectedGraph.html