UNIVERSITY OF MORATUWA

FACULTY OF ENGINEERING

Department of Computer Science and Engineering

CS2150 – Graph Theory

RESERCH CYCLE - 01

Project Report

A3

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Definitions & Theorems

Definitions

Degree of a Vertex:

The number of edges incident to a vertex is called the degree of a vertex.

Directed Graph (Digraph)

A directed graph as $G=(V(G),E(G))$ consisting of the set $V(G)$ of vertices and the set $E(G)$ of edges, which are ordered pairs of elements of $V(G)$.

In-degree of a vertex

For a directed graph $G=(V(G),E(G))$ and a vertex $v_1 \in V(G)$, The In-Degree of v_1 refers to the number of arcs incident to v_1. That is, the number of arcs directed towards the vertex v_1.

Out-degree of a vertex

For a directed graph $G=(V(G),E(G))$ and a vertex $v_1 \in V(G)$, the Out-Degree of v_1 refers to the number of arcs incident from v_1. That is, the number of arcs directed away from the vertex v_1.

Undirected Graph

An undirected graph as $G=(V(G),E(G))$ consisting of the set $V(G)$ of vertices and the set $E(G)$ of edges, which are unordered pairs of elements of $V(G)$.

Weighted Graph

With each edge $E(G)$ of graph G let there be associated a real number $w(E)$, called its weight. Then graph G, together with these weights on its edges, is called a weighted graph.

Theorems

Handshaking Lemma

If $G = (V,E)$ is an undirected graph with n edges,

$$\sum_{v \in V(G)} deg(v) = 2|E|$$

Corollary (A)

A finite undirected graph $G=(V,E)$ Contains an Even number of vertices with odd degrees.

Corollary (B)

If G is a graph, then the sum of its vertex degrees is nonnegative and even.

Proofs

Proof of Handshaking Lemma

Let's use ***induction method*** to prove this,

Base case:

Consider when $|V| = 0$,

$$|V| = 0 \rightarrow |E| = 0, \sum_{v \in V(G)} deg(v) = 0$$

Therefore, the result is true for the base case.

Inductive step:

Assume $2|E(G')| = \sum_{v \in V(G)} deg(v)$ for $|V| = n \geq 0$

Now consider insertion of vertex V_1 to G' graph.

This gives $G = (V, E)$ graph with $k+1$ number of vertices.

Insertion of V_1 will increase total degree sum in $2e$. (Because each edge will be connected to two vertices.)

Therefore,

$$\sum_{v \in V(G)} deg(v) = 2|E(G')| + 2e = 2(|E(G')| + e) = 2|E(G') + e| = 2|E(G)|$$

Therefore, the result is true for $k+1$ vertices.

So, we have proved the result is true for the base case. And assuming result is true for n vertices, we have proved that the result stands for $n+1$. Therefore, by mathematical induction, the result is true for any number of n. $(n \in \mathbb{Z}^+)$

Proof of Corollary (A)

Partition V into two disjoint subsets: V_e is the subset of V that contains only vertices with even degrees; and V_o is the subset of V with only vertices of odd degrees.

That is, $V = V_e \cup V_o$ and $V_e \cap V_o = \emptyset$. From above Theorem, we have

$$\sum_{v \in V} deg(v) = \sum_{v \in V_e} deg(v) + \sum_{v \in V_O} deg(v) = 2|E|$$

which can be re-arranged as

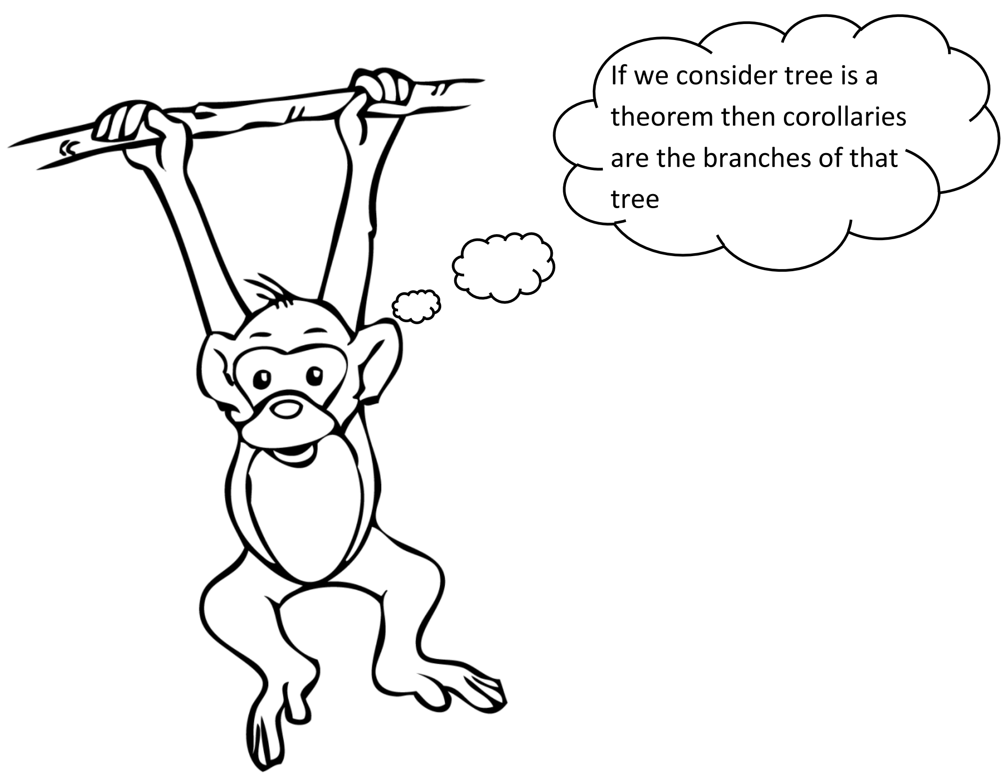
$$\sum_{v \in V_O} deg(v) = \sum_{v \in V} deg(v) - \sum_{v \in V_e} deg(v)$$

As $\sum_{v \in V} deg(v)$ and $\sum_{v \in V_e} deg(v)$ are both even, their difference is also even.

So that $\sum_{v \in V_o} deg(v)$ is even.

Proof of Corollary (B)

As $E \subseteq V_2$, then E can be the empty set, in which case the total degree of $G = (V, E)$ is zero where $E \neq \emptyset$, then the total degree of G is greater than zero. Therefore the sum of all the vertex degrees in a graph is non negative and even.



Problems and Solutions

Problem 1

Suppose 11 teams (A, B, C, … , K) qualify to play in the next ICC World Cup. One official suggests that in the first round, each team should play five games against different opponents. Give a method for conducting the first round or explain why it is not possible.

Solution

Let's represent this problem with a graph where a vertex represents a team and an edge represents a match.

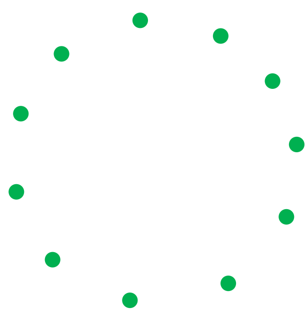


Figure 1.1

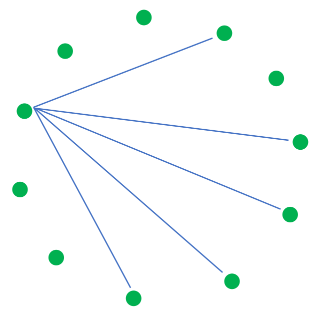
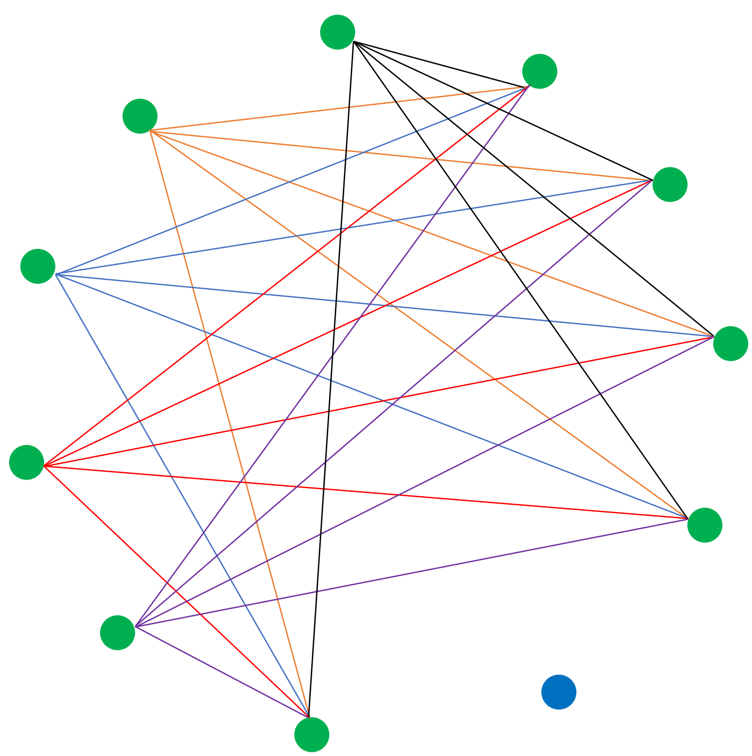


Figure 1.2



A possible graph representation of the solution

Figure 1.1 shows the 11 teams. Each team is to play with 5 different teams. The degree of each team would be 5. (Figure 1.2)

The sum of the degrees of all vertices = $\sum_{i=1}^{11} \text{degree}(i) = 11 \times 5 = 55$ .It is odd.

Let’s consider a graph with n number of vertices without edges. As we connect two vertices with an edge following happens.

Figure 1.3

Figure 1.4

- The degree of each vertex is increased by 1.
- The sum of degrees of all the vertices is increased by 2.

∴ The sum of degrees = 2 * Number of edges

The sum of degrees has to be even.

∴ By contradiction each team playing with five different opponents is not possible.

Problem 2

Suppose the round robin stage of the next Asian Cricket Cup between Bangladesh, India, Pakistan, and Sri Lanka has the following results: SL defeats B; SL defeats P; SL defeats I; B defeats I; I defeats P; P defeats B. Using graph theory concepts explain how to rank the 4 teams from best to worst, or why it is not possible to do so.

Solution - I

According to the definition of graph theory we represented a team as a vertex. We used an edge to represent a match between two teams. More over with the definition of the directed graph, we used an outward edge of a vertex to represent a win of the team of that particular vertex.

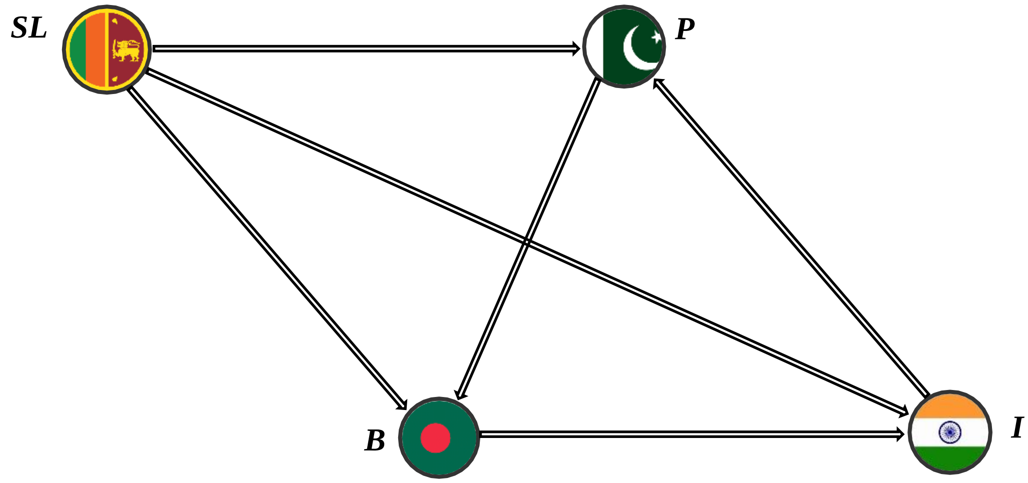


Figure 2.1

According to the graph the wins of each teams are as follows:

- Sri Lanka -3
- Bangladesh -1
- India-1
- Pakistan -1

We can notice that 3 teams have same number of wins. So it is not possible to rank the teams.

Solution - II

In here we represented a team as a vertex. We used an edge to represent a match between two teams. arrow goes from the winning team to the losing team. Before we are going to this solution we should know about Hamiltonian path and Hamiltonian cycle.

- **Hamiltonian path :**

  Hamilton path, is a graph path between two vertices of a graph that visits each vertex exactly once.

- **Hamiltonian cycle :**

  If a Hamiltonian path exists whose endpoints are adjacent, then the resulting graph cycle is called a Hamiltonian cycle (or Hamiltonian cycle).

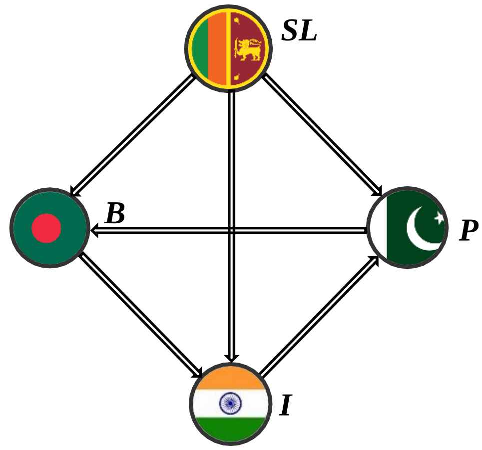


Figure 2.2

We are going to find all the available Hamiltonian paths.

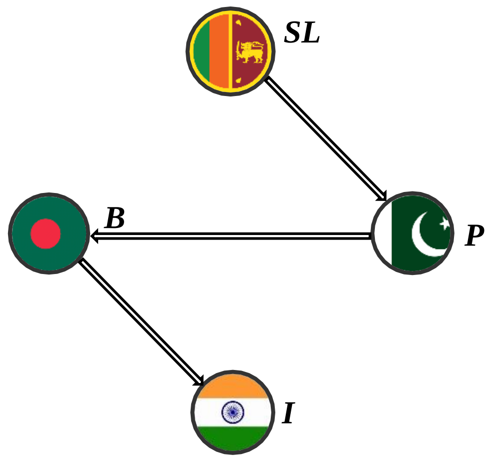


Figure 2.2.1

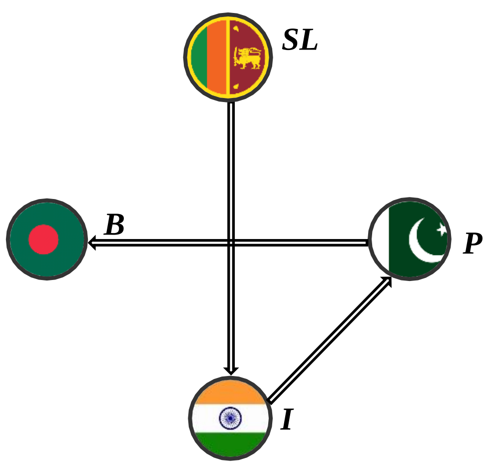


Figure 2.2.2

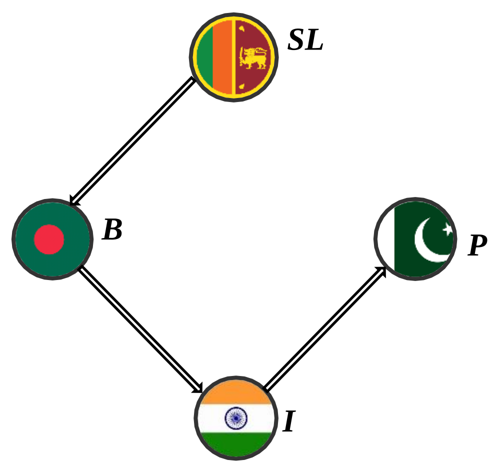


Figure 2.2.3

In here we could draw 3 paths (see Figure 2.2.1, Figure 2.2.2, Figure 2.2.3).

According to Figure 2.2.1 : *SL>B>I>P*

According to Figure 2.2.2 : *SL>I>P>B*

According to Figure 2.2.3 : *SL>P>B>I*

- It is obvious that the best team is SL but there is no worst team. 3 paths give us 3 different worst teams. Therefore, it is not possible to rank the 4 teams, best to worst.

Problem 3

An odd fellow wishes to have an odd party that is attended by an odd number of odd people each of whom is acquainted with an odd number of other odd people at the party. Can this situation occur? Justify your answer.

Solution

The question has many unnecessary “odd”s. Let’s first simplify the question by removing them.

A fellow wishes to have a party that is attended by an odd number of people each of whom is acquainted with an odd number of other people at the party.

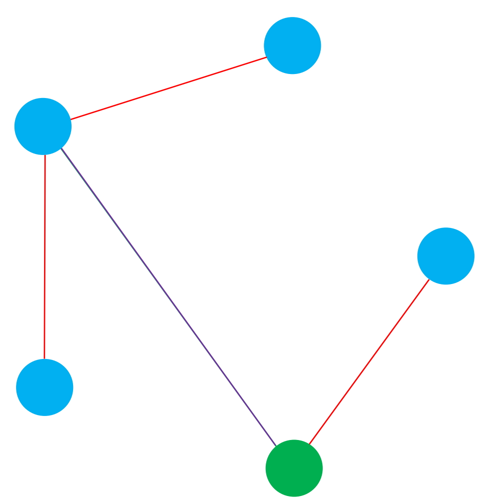
When representing the above scenario in a graph let’s take a person as a vertex and a connection between 2 people (acquaintance) as an edge.

The party is attended by an odd number of people → Odd number of vertices in a graph

Each is person is acquainted with an odd number of people → Each vertex has an odd degree

But per the proven corollary above, there should be even number of odd degree vertices in a graph.

Therefore, this is not possible. (Corollary 1(A))



A possible graph representation of the solution

Related Problems

Problem 1

Khansaheb Civil Engineering L.L.C is a famous construction company in Dubai which want to build a small island in the Persian Gulf. They want to build 17 cities on that island but the king Mohammed bin Rashid Al Maktoum said that each city should connect with eleven different cities on this island. Imagine you are one of the engineer in Khansaheb Civil Engineering L.L.C. Give a method to connect the cities in the way the king wants or explain why it is not possible to the king.

Problem 2

Suppose the round robin stage of the 2018 World Chess Championship between Viswanathan Anand, Wesley So, Magnus Carlsen, Levon Aronian and Vladimir Kramnik has the following results: Anand defeats So; Anand defeats Carlsen; Anand defeats Aronian; Anand defeats Kramnik; Carlsen defeats Aronian; Aronian defeats Kramnik; Kramnik defeats So; So defeats Carlsen; Kramnik defeats Carlsen; So defeats Aronian. Using graph theory concepts explain how to rank these five players from best to worst, or why is not possible to do so

Problem 3

In a group of 15 people, is it possible for each person to have exactly 7 friends (assuming that friendship is a symmetric relationship; i.e., if X is a friend of Y, then Y is a friend of X) Is it true or false? Explain your answer.

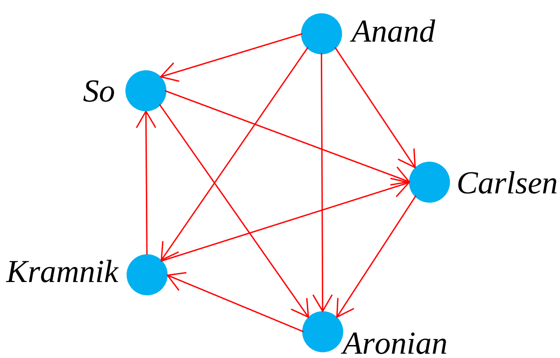
Solutions for related problems

Solution for Problem 1

No, we can't

Let's represent the problem by a graph. The 17 cities are represented by vertices and the roads are represented by an edge. Assume that the scenario can happen. Then the degree of each vertex is 11 which make the sum of the degrees 11*17 = 187. But according to the handshaking lemma the sum of the degrees should be twice the number of edges. That is even. Therefore, by contradiction this scenario cannot happen.

Solution for Problem 2



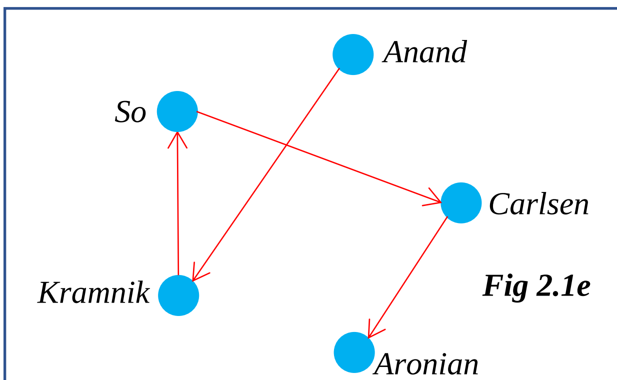


Fig 2.1e

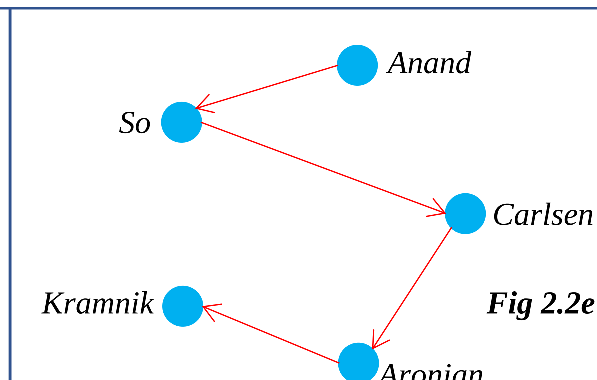


Fig 2.2e

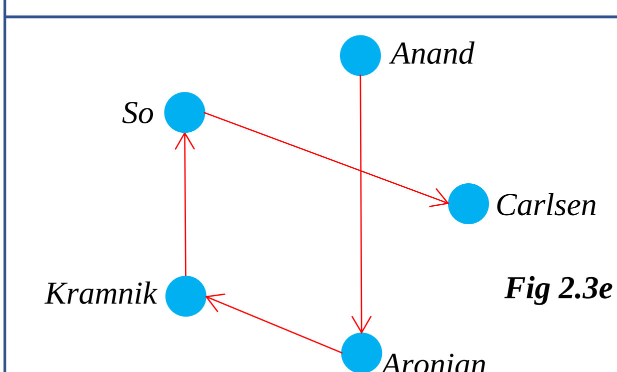


Fig 2.3e

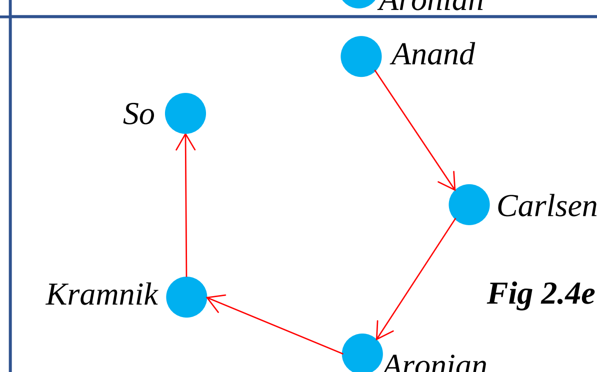


Fig 2.4e

According to Figure 2.1e : Anand>Kramnik>So>Carlsen>Aronian

According to Figure 2.2e : Anand> So>Carlsen>Aronian>Kramnik

According to Figure 2.3e : Anand>Aronian >Kramnik>So>Carlsen

According to Figure 2.4e : Anand> Carlsen>Aronian> Kramnik >So

- It is obvious that the best player is Anand but there is no worst players. 4 paths give us 4 different worst players. Therefore, it is not possible to rank the 5 players, best to worst.

Solution for Problem 3

We denotes each person by a vertex and their friendship by an edge. Then the problem becomes whether it is possible to draw a 15-vertex graph with degree of each vertex equals 7. Consider that total degree of all vertices = 15 x 7 = 105, which is odd. This contradicts with handshaking lemma, which states the total degree of vertices should be even. Therefore, it is not possible for each person to have exactly 7 friends in a group of 15 people.

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