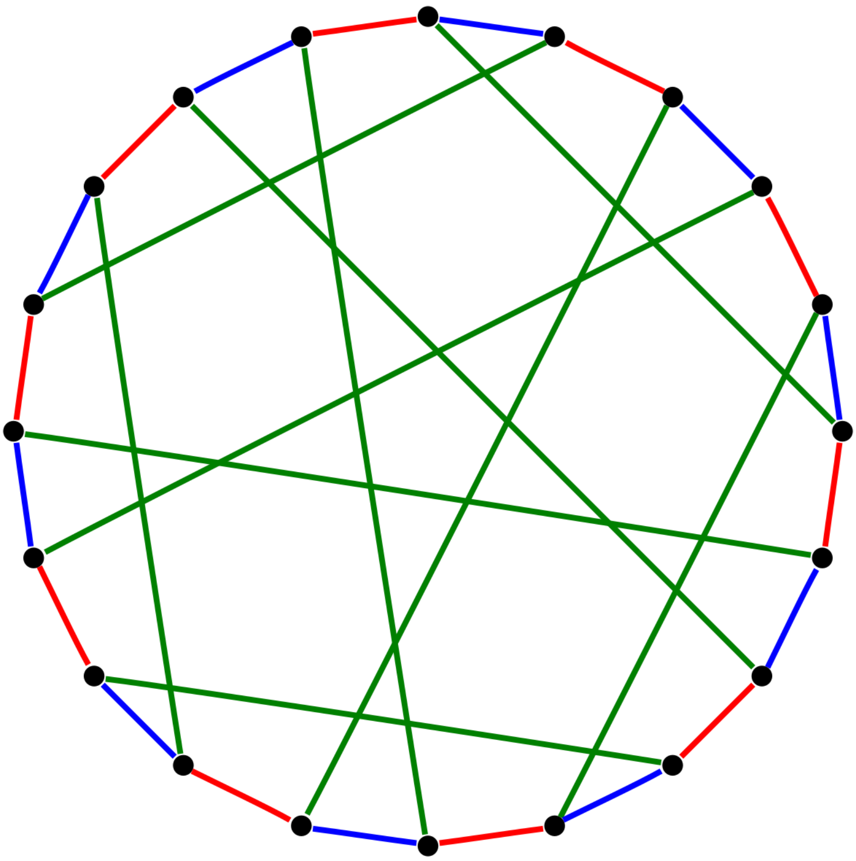
CS 2150 GRAPH THEORY

RESEARCH CYCLE-1

PROJECT REPORT

GROUP C6



Group Members

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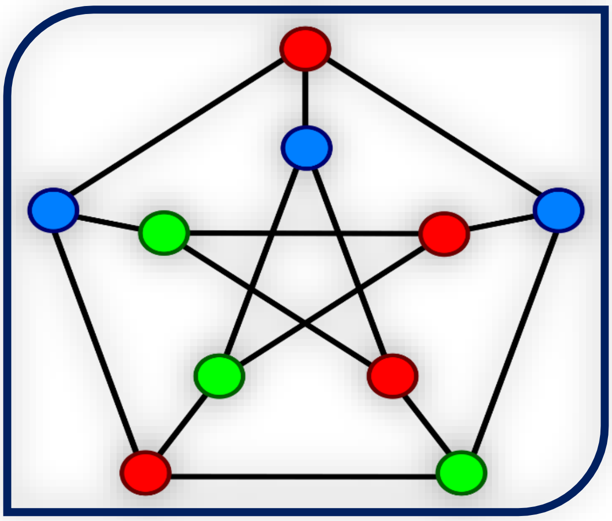
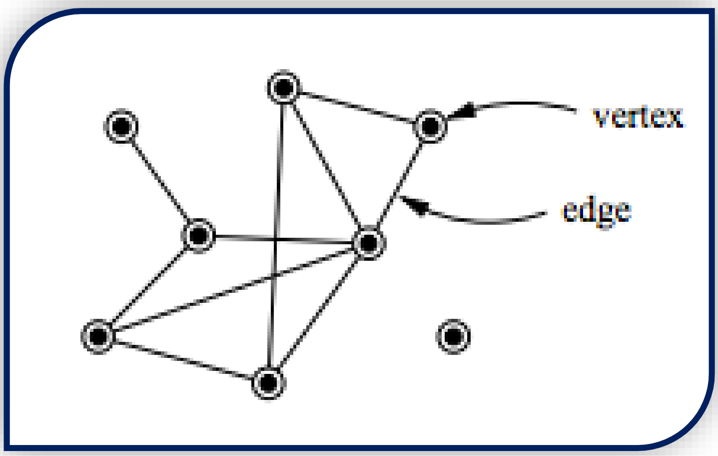
Definitions

1. **<u>Graph</u>**



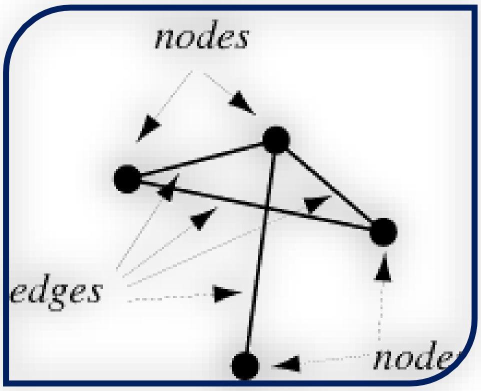
A *graph* G = (V, E) is an ordered pair of finite sets. Elements of V are called vertices or nodes, and elements of E are called edges or arcs. We refer to V as the vertex set of G, with E being the edge set.

2. **<u>Vertex</u>**



A *vertex* is a point where multiple lines meet. It is also called a node. Similar to points, a vertex is also denoted by an alphabet.

3. **Edge**



An *edge* is the mathematical term for a line that connects two vertices. Many edges can be formed from a single vertex. Without a vertex, an edge cannot be formed. There must be a starting vertex and an ending vertex for an edge.

4. **Degree of a vertex**

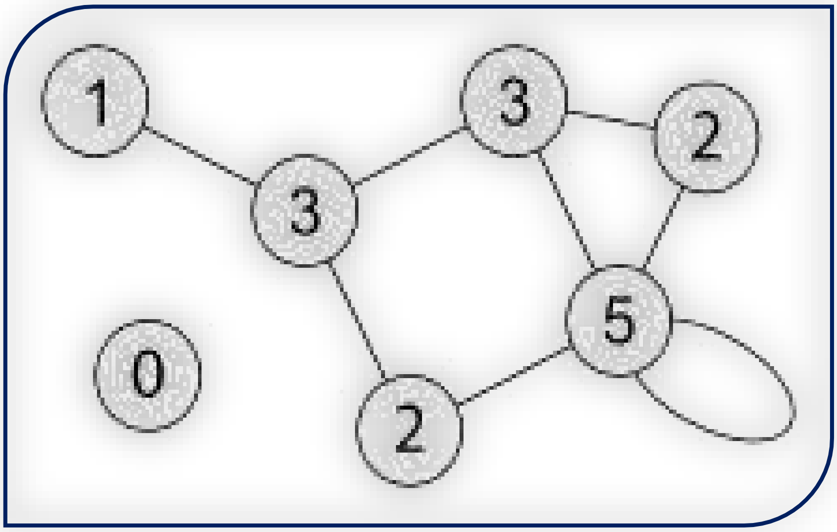
The *degree of vertex* in a graph is number of edges that enter or exit from the vertex. A loop contributes 2 to the degree of its vertex.

Directed graphs have two types of degrees, known as the indegree and the outdegree.

Indegree - The number of inward directed graph edges from a given graph vertex

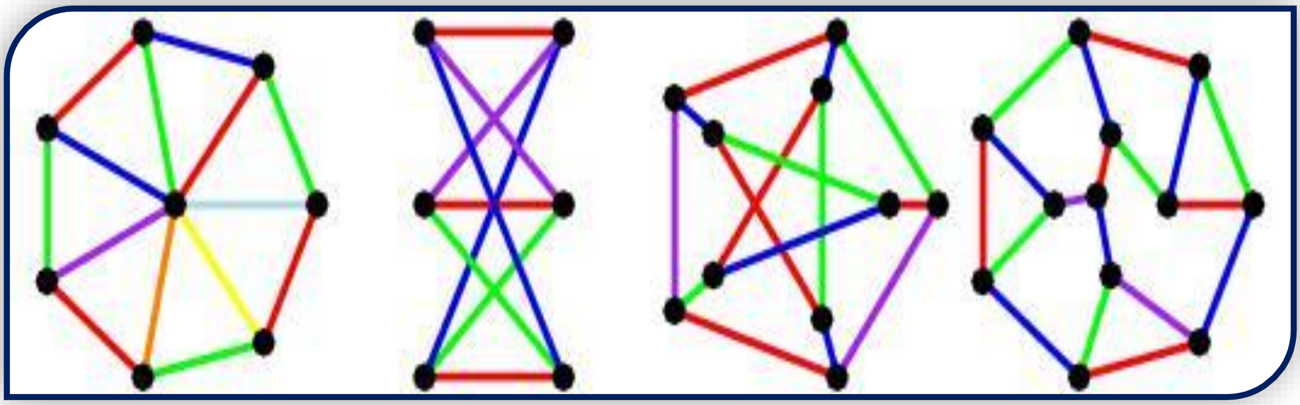
Outdegree - The number of outward directed graph edges from a given graph.

In the following figure, degree of vertex is shown inside vertex.



5. <u>Proper Edge Colouring</u>

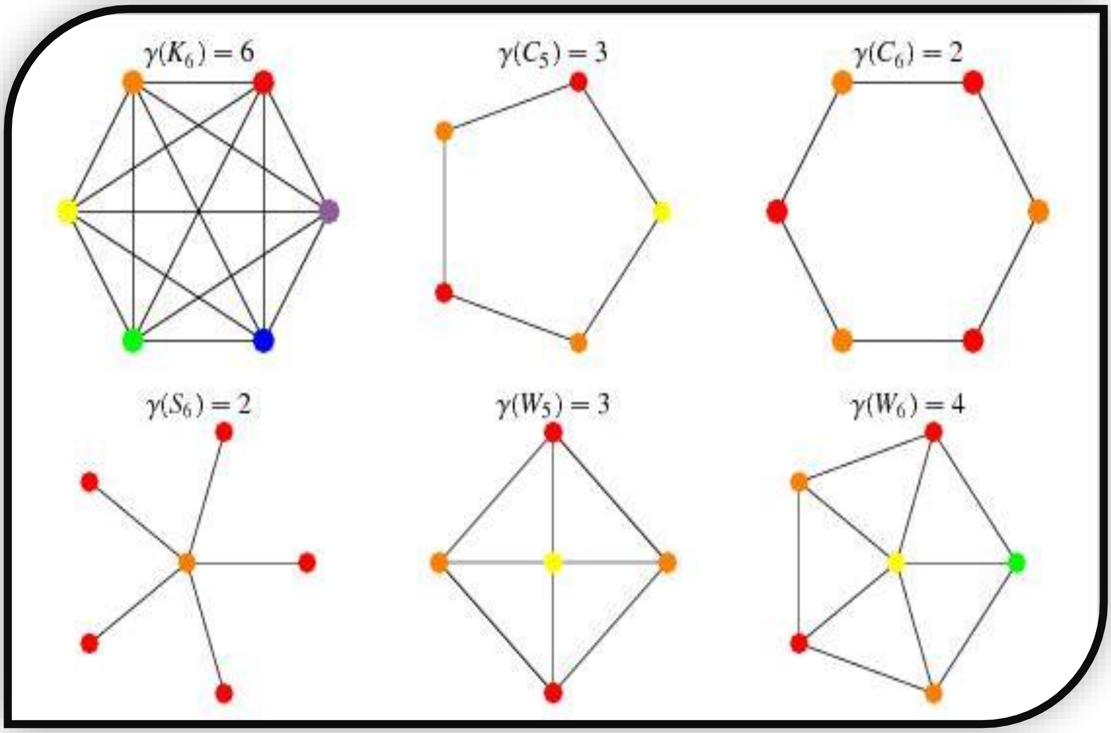
An *edge colouring* of a graph is an assignment of colours to edges of a graph so that no vertex is incident to two edges of the same colour. An edge colouring with k colours is called a **k-edge-colouring**.



6. <u>Chromatic Number</u>

The *chromatic number* of a graph G is the smallest number of colours needed to colour the vertices of G so that no two adjacent vertices share the same colour.

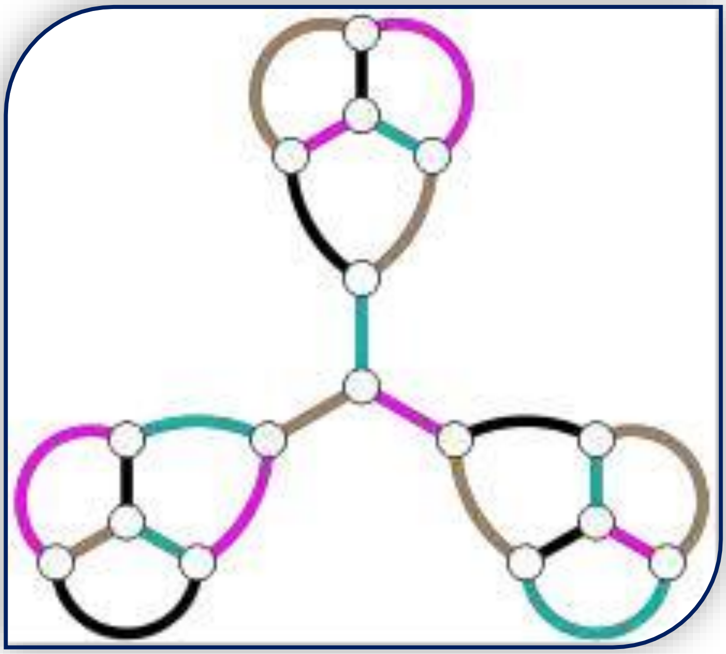
The chromatic of a graph G is most commonly denoted χ(G) .



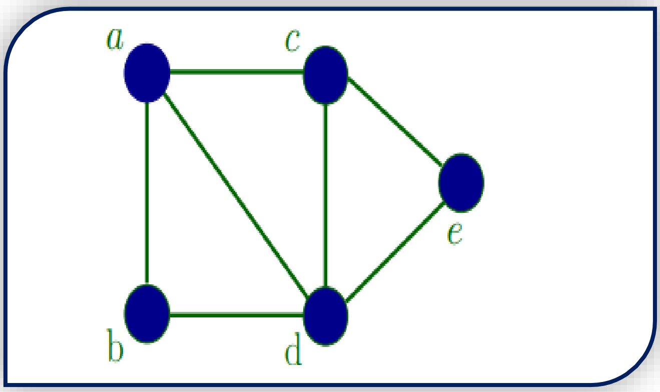
7. Edge Chromatic Number

The *edge chromatic number*, sometimes also called the chromatic index, of a graph G is fewest number of colours necessary to colour each edge of G such that no two edges incident on the same vertex have the same colour. In other words, it is the number of distinct colours in a minimum edge colouring.

The edge chromatic number of following figure is 4.

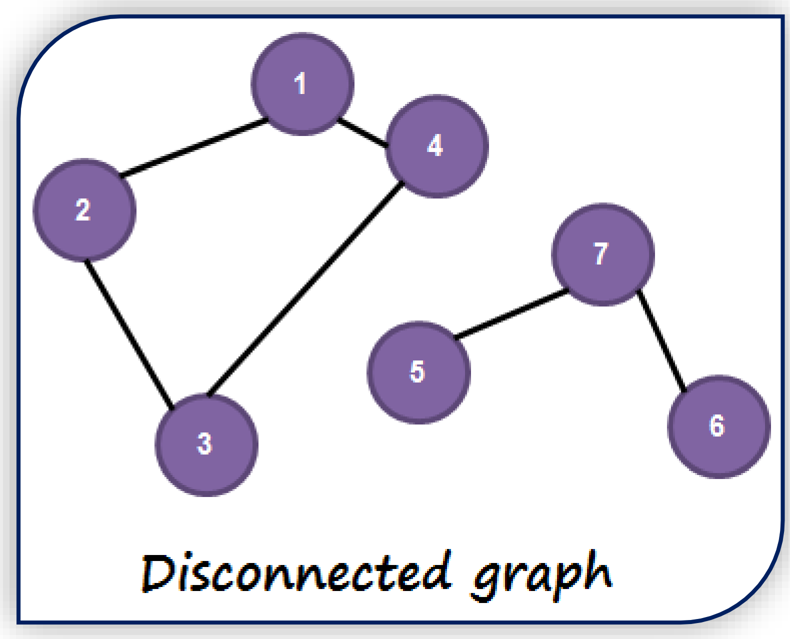


8. Connected Graph



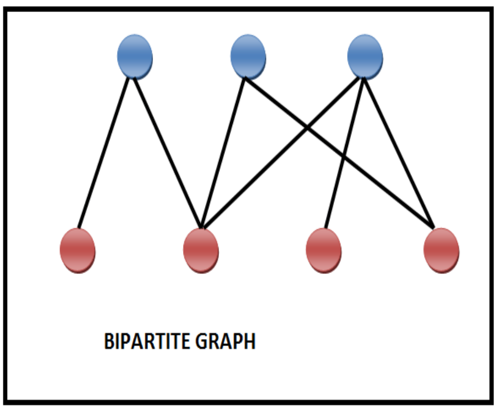
A graph G is said to be *connected* **if there exists a path between every pair of vertices**. There should be at least one edge for every vertex in the graph. So that we can say that it is connected to some other vertex at the other side of the edge.

9. Disconnected Graph



A graph G is *disconnected*, if it does not contain at least two connected vertices.

10. Bipartite Graph

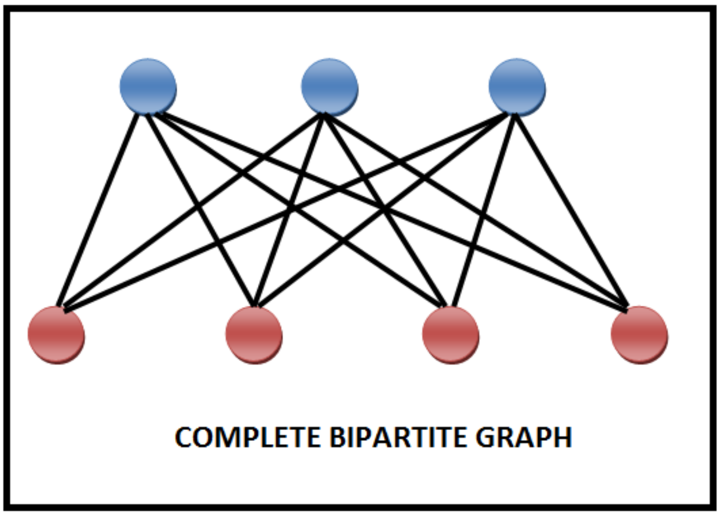


A simple graph G = (V, E) with vertex partition V = {V_1, V_2} is called a *bipartite graph* **if every edge of E joins a vertex in V_1 to a vertex in V_2.**

In general, a Bipartite graph has two sets of vertices, let us say, V_1 and V_2, and if an edge is drawn, it should connect any vertex in set V_1 to any vertex in set V_2.

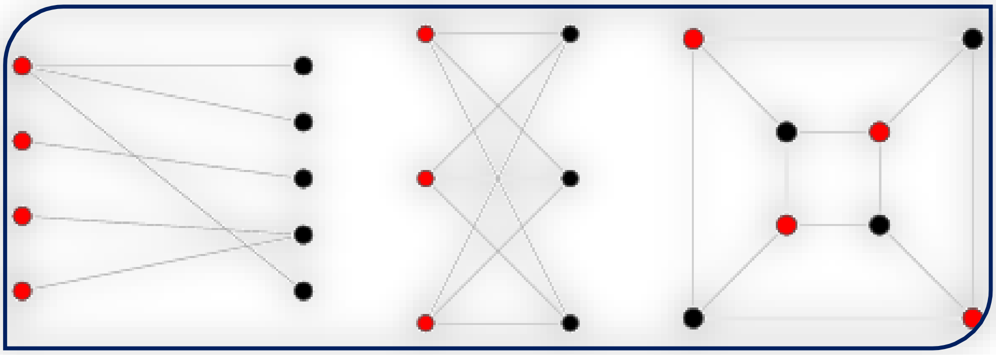
11. <u>Complete Bipartite Graph</u>

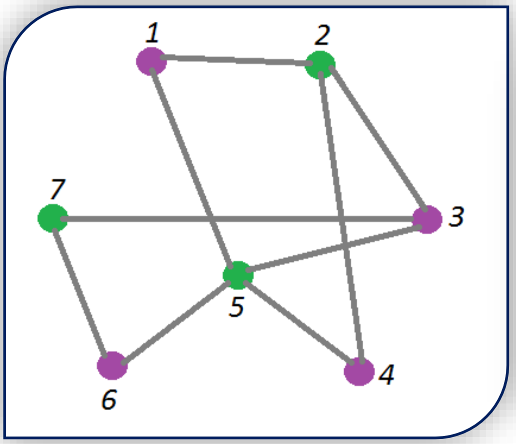
A bipartite graph 'G', G = (V, E) with partition V = {V_1, V_2} is said to be a *complete bipartite graph* if every vertex in V_1 is connected to every vertex of V_2.

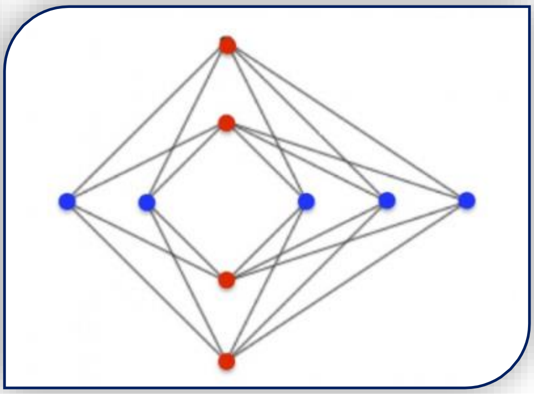


In general, a complete bipartite graph connects each vertex from set V_1 to each vertex from set V_2.

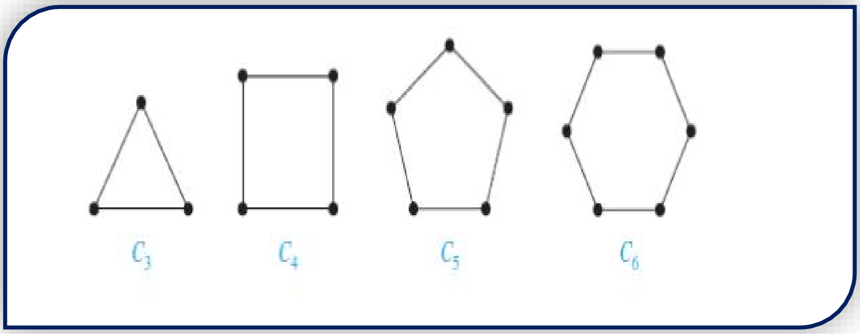
Some bipartite graphs examples:







12. <u>Cycle Graph</u>



A simple graph with ‘n’ vertices (n >= 3) and ‘n’ edges is called a cycle graph if all its edges form a cycle of length ‘n’.

If the **degree of each vertex in the graph is two,** then it is called a *Cycle Graph*.

Notation − C_n

13. <u>Odd Cycle</u>

An *odd cycle* is a cycle with odd length, that is, with an odd number of edges.

14. <u>Even Cycle</u>

An *even cycle* is a cycle with even length, that is, with an even number of edges.

Problem 1– Group C6

Prove that the number of colours required for a proper edge colouring of a graph G is ≥ the maximum degree of any vertex of G.

Let's prove the statement by Contra positive method.

Statement:

The number of colours required for a proper edge colouring of a graph G is ≥ The maximum degree of any vertex of G

Let,

- The number of colours required for proper edge colouring of G (Edge chromatic number) be 'n'
- The maximum degree of any vertex of G be 'x'.

Prepositions in the Statement:

P : Graph G is properly coloured

! P : Graph G is not properly coloured

Q : $n \geq x$ (number of colours required for proper edge colouring is greater than or equal to the Maximum degree of any vertex of G)

! Q : $n < x$

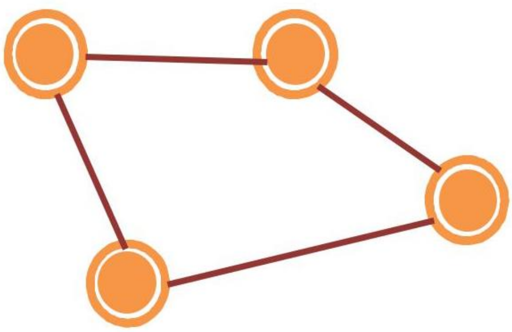
To prove:

P => Q

By contra positive method, we have to prove that, ! Q =>! P That is, if $n < x$, then graph G is not properly coloured.

Proof:

Let's take the following graph



Here maximum degree of any vertex = 2

So $x = 2$

For $n<x$ to be true, we must be able to colour the edges using at most 1 colour

This violates the definition of proper edge colouring because two edges incident on the same vertex have the same colour.

That is the graph G is not properly coloured. (! P)

Taking !Q as true, we have proved that !P. (!Q=>!P).

By the contrapositive method, P =>Q

If Graph G is properly coloured, then the number of colours required is ≥ maximum degree of any vertex of G.

Problem 2– Group C6

Prove that a graph G is bipartite if and only if it does not have a cycle of odd length.

⇒ **<u>G is bipartite => G does not have a cycle of odd length</u>**

Let G is bipartite then

$V(G) = A \cup B$ $A \cap B = \emptyset$

For every $e \in E(G)$ $e= ab$ where $a \in A$, $b \in B$

(This is definition of bipartite graph)

Suppose G has (at least) one odd length cycle.

Let the length of C be n

Let $C = (v_1, v_2, \ldots\ldots, v_n)$

Let V1 ϵ A then V2 ϵ B then V3 ϵ A and so on…

Hence we see that for all k ={1,2,………….,n}

$V_k \in A$: k odd

$V_k \in B$: k even

But as n is odd, Vn ϵ A

$V_1, V_n \in A$

So $V_1V_n \in E(G)$ which contradicts the assumption that G is bipartite

Therefore, G is bipartite => G does not have a cycle of odd length

⇐ **<u>G does not have a cycle of odd length => G is bipartite</u>**

If G is disconnected then any cycle in G is contained in one of the connected components. It suffices to show the claim for connected graphs. Assume G is connected graph.

Suppose G has no odd length cycle.

Let v ϵ V(G)
A= {a/ d ϵ (v, a) is even }
B= {b/ d ϵ (v, b) is odd}
d- shortest path

Since G is connected,
A U B = V(G)   A∩B = ø

We need to show that any e ϵ E(G) is at the form e = ab a ϵ A, b ϵ B
Need to show that a_1,a_2 ϵ E(G) a_1,a_2 ϵ A OR b_1,b_2 ϵ E(G) b_1,b_2 ϵ E(G) it is impossible.

Suppose a_1, a_2 ϵ A are adjacent, a_1a_2 ϵ E(G)

Then there would be a closed walk at odd length (v, ……a_1, a_2, ………v)
But from graph containing close walk of odd length also contains odd cycle it follows that G would then contain an odd cycle.

This contradicts our assumption G has no odd length cycles,
Therefore $a_1a_2 \notin$ E(G)

By same argument if b_1, b_2 ϵ B => $b_1b_2 \notin$ E(G)

Therefore, any e ϵ E(G) is the form e=ab a ϵ A, b ϵ B

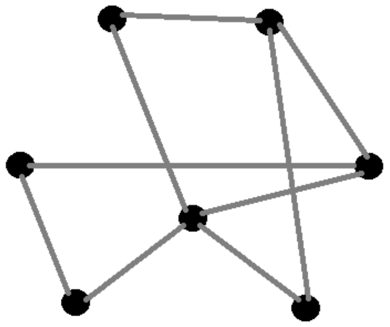
Therefore, G is bipartite.

References

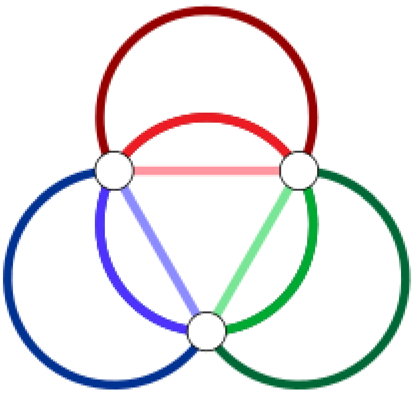
- https://www.tutorialspoint.com/index.htm
- http://mathonline.wikidot.com/
- http://mathworld.wolfram.com/
- https://proofwiki.org/wiki/Graph_is_Bipartite_iff_No_Odd_Cycles
- https://www.youtube.com/watch?v=YiGFhWxtHjQ

Questions

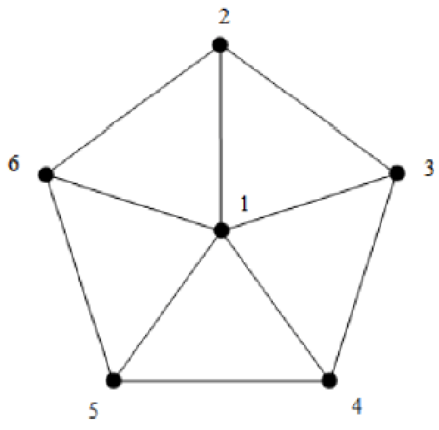
a) The following graph is a bipartite graph. (True/False)



b) The following graph edge chromatic number is 9. (True/False)

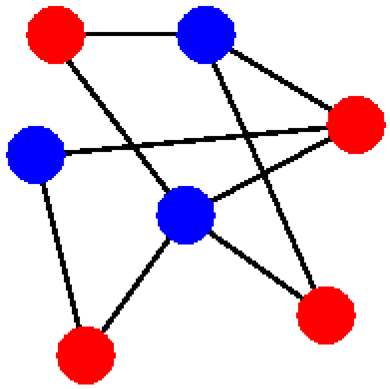


c) What is the chromatic number of following wheel graph?



Solutions

a) True



b) False

Edge chromatic number is 3.

c) Chromatic number is 4.

W_n ($n >= 3$) the wheel graph with $n+1$ vertices then

$\chi(W_n) = 3$; if n is even

$\chi(W_n) = 4$; if n is odd