Deep Learning Basics
Lecture 8: Autoencoder & DBM

Princeton University COS 495

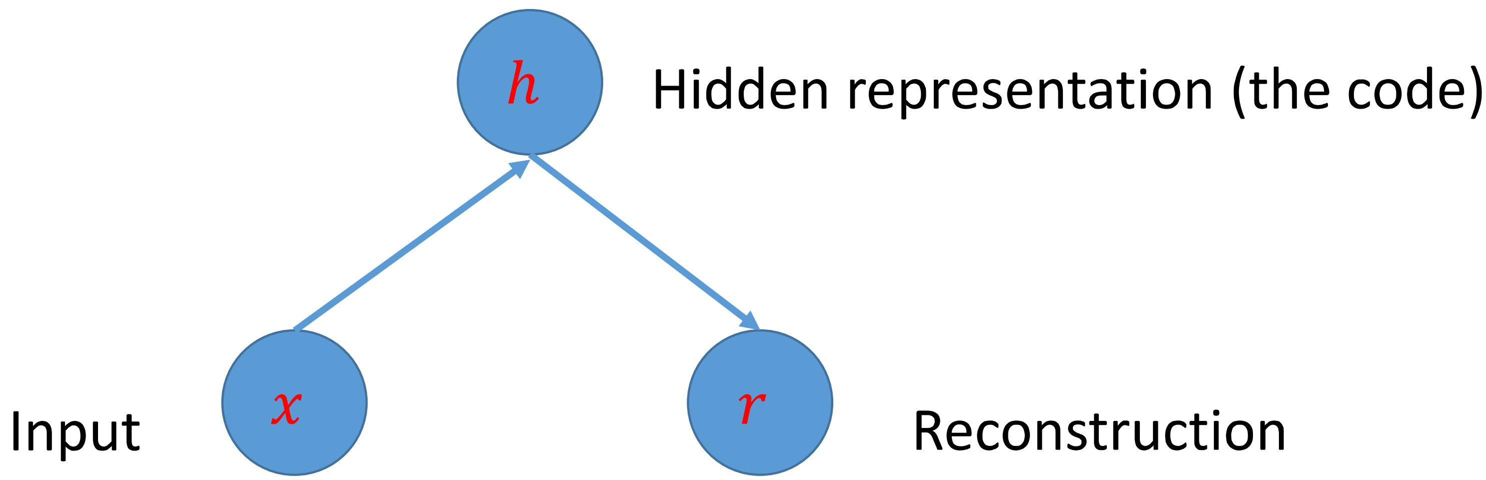
Instructor: Yingyu Liang

Autoencoder

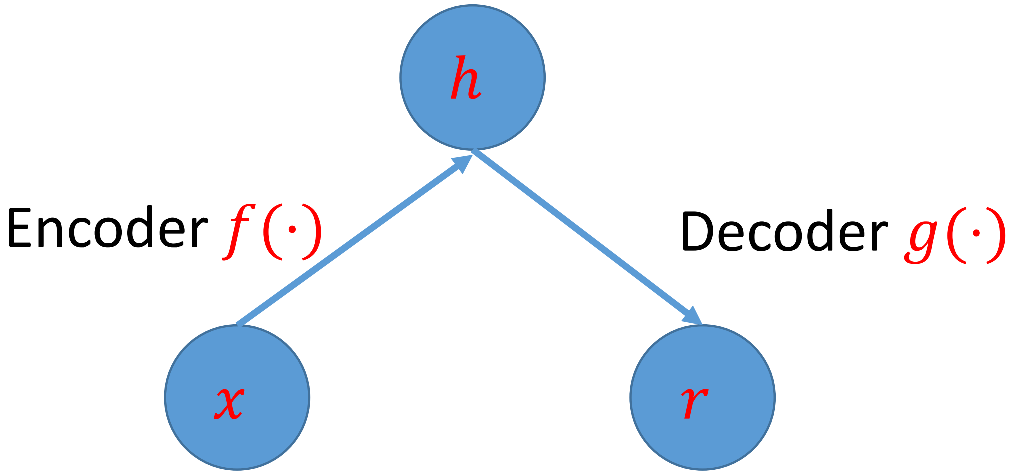
Autoencoder

- Neural networks trained to attempt to copy its input to its output
- Contain two parts:
  - Encoder: map the input to a hidden representation
  - Decoder: map the hidden representation to the output

Autoencoder



Autoencoder



$$h = f(x), r = g(h) = g(f(x))$$

Why want to copy input to output

- Not really care about copying

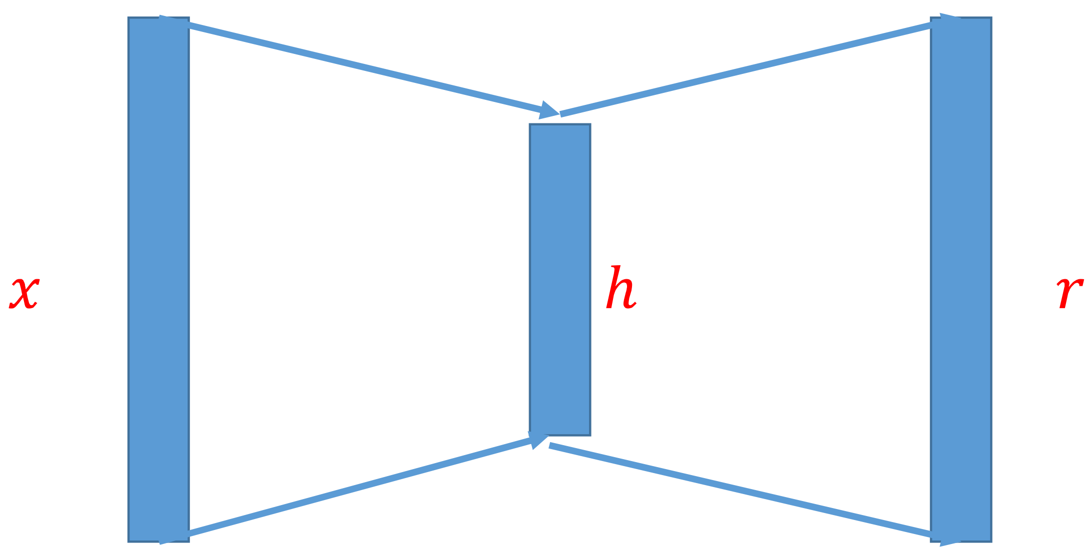
- Interesting case: NOT able to copy exactly but strive to do so
- Autoencoder forced to select which aspects to preserve and thus hopefully can learn useful properties of the data

- Historical note: goes back to (LeCun, 1987; Bourlard and Kamp, 1988; Hinton and Zemel, 1994).

Undercomplete autoencoder

- Constrain the code to have smaller dimension than the input
- Training: minimize a loss function

$$L(x,r) = L(x,g(f(x)))$$



Undercomplete autoencoder

- Constrain the code to have smaller dimension than the input
- Training: minimize a loss function

$$L(x, r) = L(x, g(f(x)))$$

- Special case: f, g linear, L mean square error
- Reduces to Principal Component Analysis

Undercomplete autoencoder

- What about nonlinear encoder and decoder?

- Capacity should not be too large
- Suppose given data $x_1, x_2, \dots, x_n$
  - Encoder maps x_i to i
  - Decoder maps i to x_i
- One dim h suffices for perfect reconstruction

Regularization

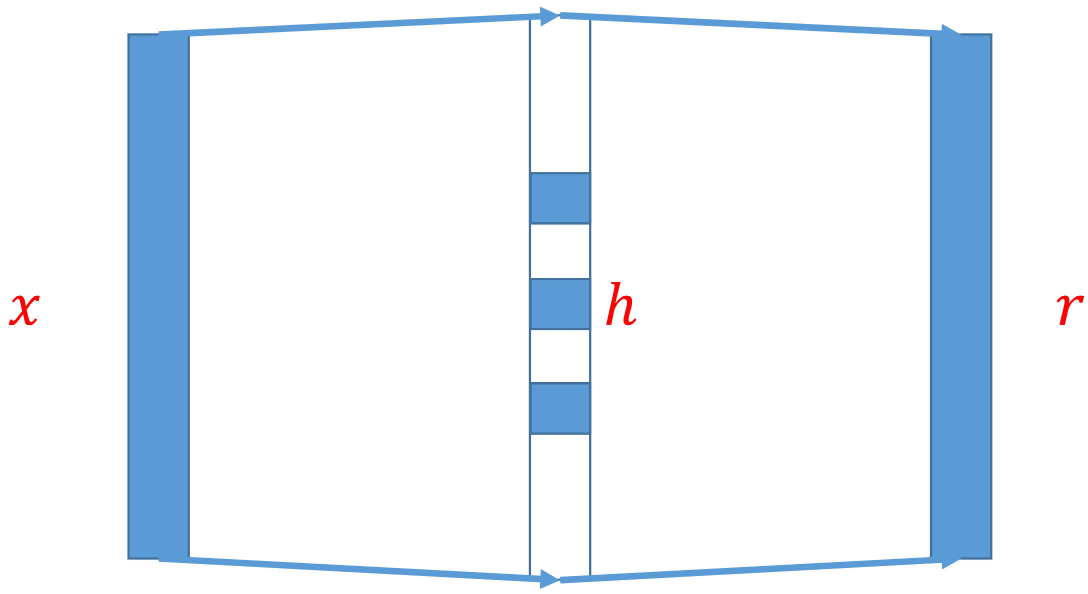
- Typically NOT
  - Keeping the encoder/decoder shallow or
  - Using small code size

- Regularized autoencoders: add regularization term that encourages the model to have other properties
  - Sparsity of the representation (sparse autoencoder)
  - Robustness to noise or to missing inputs (denoising autoencoder)
  - Smallness of the derivative of the representation

Sparse autoencoder

- Constrain the code to have sparsity
- Training: minimize a loss function

$$L_R = L(x, g(f(x))) + R(h)$$



Probabilistic view of regularizing h

- Suppose we have a probabilistic model $p(h, x)$
- MLE on x

$$\log p(x) = \log \sum_{h'} p(h', x)$$

- ☹ Hard to sum over h'

Probabilistic view of regularizing h

- Suppose we have a probabilistic model $p(h, x)$
- MLE on x

$$\max \log p(x) = \max \log \sum_{h'} p(h', x)$$

- Approximation: suppose $h = f(x)$ gives the most likely hidden representation, and $\sum_{h'} p(h', x)$ can be approximated by $p(h, x)$

Probabilistic view of regularizing h

- Suppose we have a probabilistic model $p(h, x)$
- Approximate MLE on $x, h = f(x)$

$$\max \log p(h, x) = \max \underbrace{\log p(x|h)}_{\text{Loss}} + \underbrace{\log p(h)}_{\text{Regularization}}$$

Sparse autoencoder

- Constrain the code to have sparsity
- Laplacian prior: $p(h) = \frac{\lambda}{2}\exp(-\frac{\lambda}{2}|h|_1)$

- Training: minimize a loss function

$$L_R = L(x, g(f(x))) + \lambda|h|_1$$

Denoising autoencoder

- Traditional autoencoder: encourage to learn $g\left(f(\cdot)\right)$ to be identity

- Denoising : minimize a loss function

$$L(x,r) = L(x, g\left(f(\tilde{x})\right))$$

where $\tilde{x}$ is $x + noise$

Boltzmann machine

Boltzmann machine

- Introduced by Ackley *et al.* (1985)

- General "connectionist" approach to learning arbitrary probability distributions over binary vectors
- Special case of energy model: $p(x) = \frac{\exp(-E(x))}{Z}$

Boltzmann machine

- Energy model:

$$p(x) = \frac{\exp(-E(x))}{Z}$$

- Boltzmann machine: special case of energy model with

$$E(x) = -x^T U x - b^T x$$

where U is the weight matrix and b is the bias parameter

Boltzmann machine with latent variables

- Some variables are not observed

$$x = (x_v, x_h), \qquad x_v \text{ visible}, \ x_h \text{ hidden}$$

$$E(x) = -x_v^T R x_v - x_v^T W x_h - x_h^T S x_h - b^T x_v - c^T x_h$$

- Universal approximator of probability mass functions

Maximum likelihood

- Suppose we are given data $X = (x_v^1, x_v^2, \dots, x_v^n)$
- Maximum likelihood is to maximize

$$\log p(X) = \sum_i \log p(x_v^i)$$

where

$$p(x_v) = \sum_{x_h} p(x_v, x_h) = \sum_{x_h} \frac{1}{Z} \exp(-E(x_v, x_h))$$

- $Z = \sum \exp(-E(x_v, x_h))$: partition function, difficult to compute

Restricted Boltzmann machine

- Invented under the name *harmonium* (Smolensky, 1986)
- Popularized by Hinton and collaborators to *Restricted Boltzmann machine*

Restricted Boltzmann machine

- Special case of Boltzmann machine with latent variables:

$$p(v, h) = \frac{\exp(-E(v, h))}{Z}$$

where the energy function is

$$E(v, h) = -v^T W h - b^T v - c^T h$$

with the weight matrix W and the bias b, c

- Partition function

$$Z = \sum_v \sum_h \exp(-E(v, h))$$

Restricted Boltzmann machine

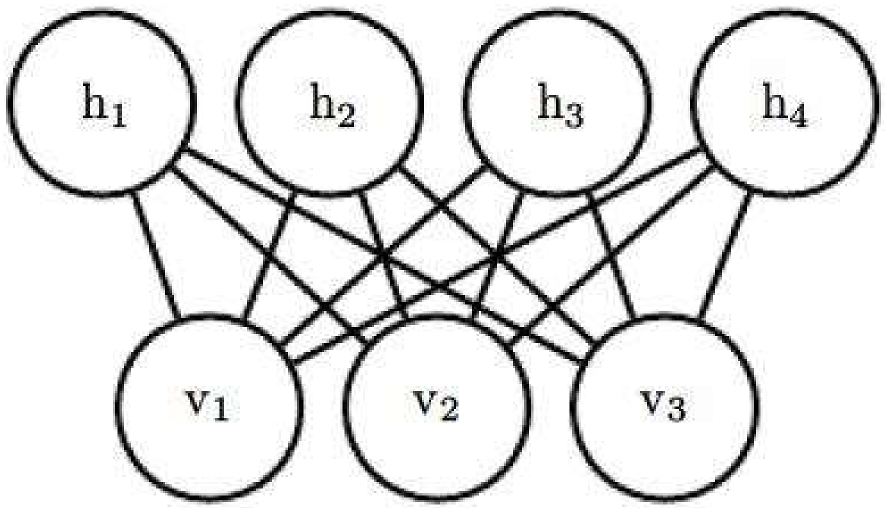


Figure from *Deep Learning*,
Goodfellow, Bengio and Courville

Restricted Boltzmann machine

- Conditional distribution is factorial

$$p(h|v) = \frac{p(v,h)}{p(v)} = \prod_j p(h_j|v)$$

and

$$p\left(h_j = 1|v\right) = \sigma\left(c_j + v^T W_{:,j}\right)$$

is logistic function

Restricted Boltzmann machine

- Similarly,

$$p(v|h) = \frac{p(v,h)}{p(h)} = \prod_i p(v_i|h)$$

and

$$p(v_i = 1|h) = \sigma\left(b_i + W_{i,:}h\right)$$

is logistic function

Deep Boltzmann machine

- Special case of energy model. Take 3 hidden layers and ignore bias:

$$p(v, h^1, h^2, h^3) = \frac{\exp(-E(v, h^1, h^2, h^3))}{Z}$$

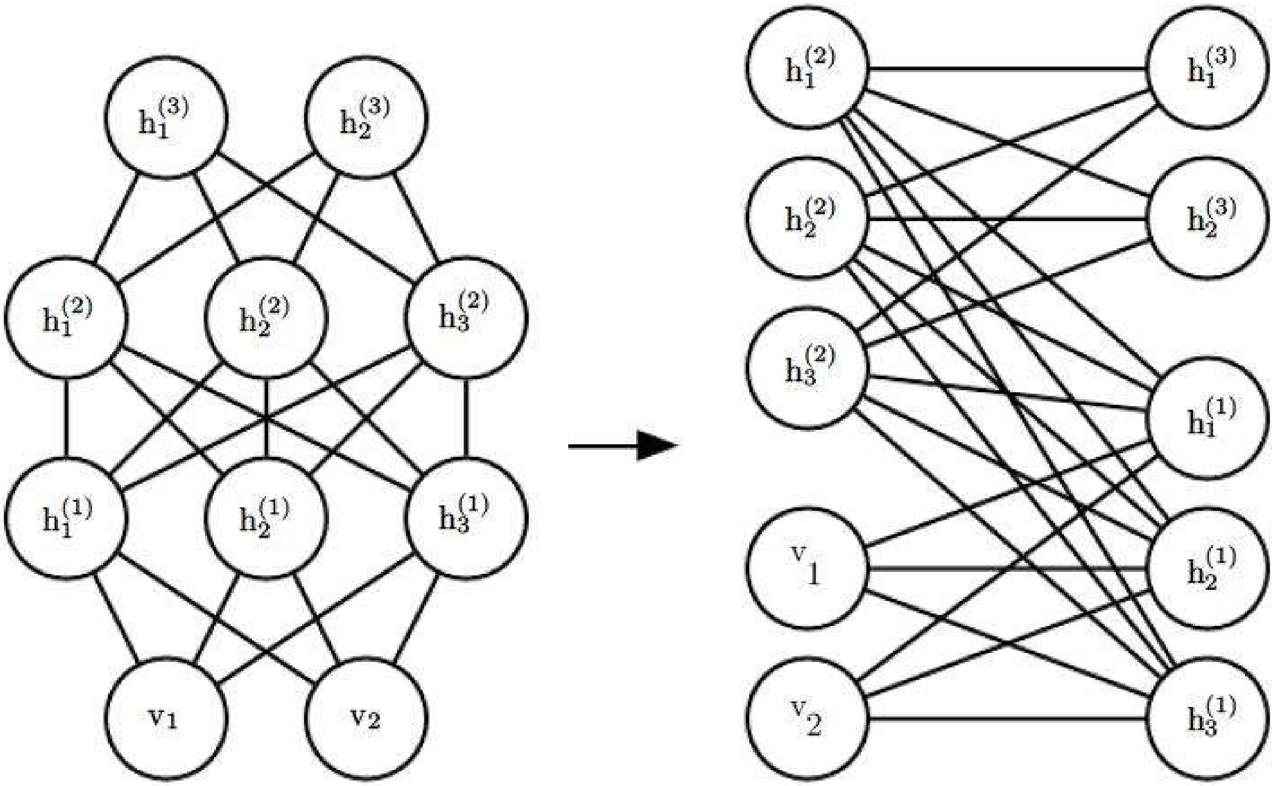
- Energy function

$$E(v, h^1, h^2, h^3) = -v^T W^1 h^1 - (h^1)^T W^2 h^2 - (h^2)^T W^3 h^3$$

  with the weight matrices W^1, W^2, W^3
- Partition function

$$Z = \sum_{v, h^1, h^2, h^3} \exp(-E(v, h^1, h^2, h^3))$$

Deep Boltzmann machine



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