10K Feet View of Universality

PH 567 Presentation

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Objectives of This Talk

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The goal of this talk is to discuss:

What is Universality?

Universality of Feigenbaum Constants

and to answer an important question:

Is Feigenbaum constant the same for all 1D Maps?

Today's Topics

What is Universality?

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- All gases behave similarly at high temperature and low pressure regardless of the type of molecule
- Critical exponents show universality in wide range of systems, including ferro-magnets, fluids, superconductors

Today's Topics

The Tools

There are a few properties that we can use to compare and contrast between different systems:

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- Bifurcation Points

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- Bifurcation Diagrams

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- Bifurcation Points
- Bifurcation Diagrams
- Lyapunov Exponents

Bifurcation Points

- At a bifurcation point, a small change of the parameter value of the system may cause:
  - a large change in the number or stability of the equilibrium points of the system,
  - the emergence of limit cycles (oscillations),
  - or chaos to emerge from an attracting orbit

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  - a large change in the number or stability of the equilibrium points of the system,
  - the emergence of limit cycles (oscillations),
  - or chaos to emerge from an attracting orbit
- The bifurcation point itself does not represent any dynamical change of the system, but rather a qualitative change of its behavior as one or more parameters crosses a critical threshold.

Bifurcation Diagram

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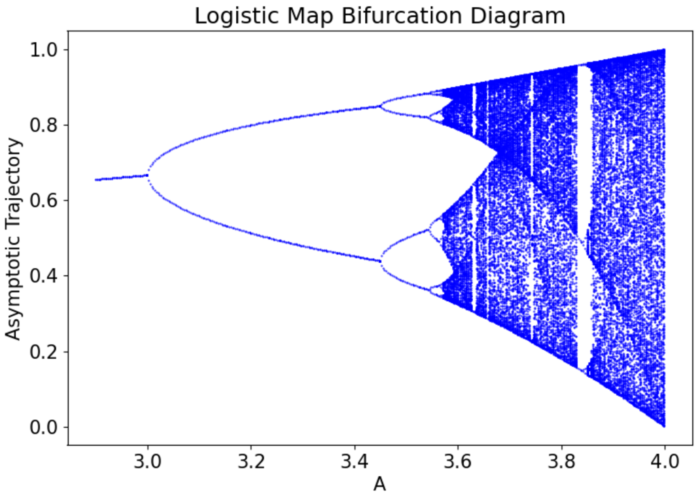
- For a one dimensional map the bifurcation parameter is shown on the horizontal axis of the plot and the vertical axis shows the set of values of the iterated function visited asymptotically from some initial condition.

Example Bifurcation Diagram



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  where λ is the Lyapunov exponent and n is the number of iterations.
- The Lyapunov exponent is positive for chaotic systems, zero for non-chaotic systems (or bifurcation points), and negative for systems that converge to fixed points.

Lyapunov Exponents

- For a one dimensional map $x_{n+1} = f(x_n)$, the Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \tag{2}$$

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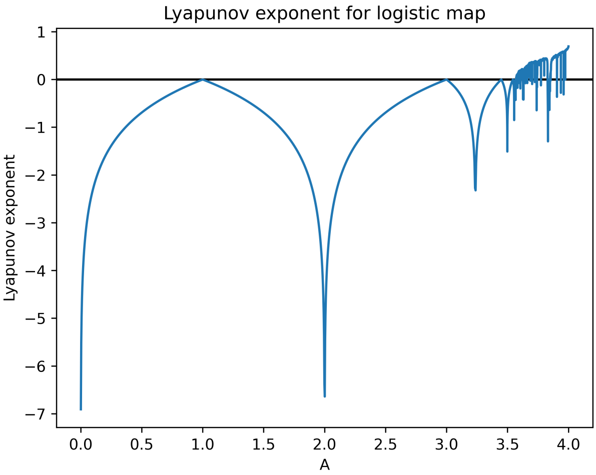
- Typically, the lyapunov exponent is zero at the bifurcation points, and highly negative (Read: $-\infty$) at the superstable points.

Example Lyapunov Exponent Plot



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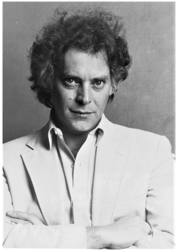

Figure: Mitchell Feigenbaum, 1987. Photo by Ingbert Grüttner

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- The logistic map and the sine map are just two examples of this large universality class. The great significance of this result is that the global details of the dynamical system do not matter.

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- A thorough understanding of the simple logistic map is sufficient for describing both qualitatively and, to a large extent, quantitatively the period-doubling route to chaos in a wide variety of nonlinear dynamical systems.
- In fact, this universality class extends beyond one-dimensional maps to nonlinear dynamical systems described by more realistic physical models corresponding to two-dimensional maps, systems of ordinary differential equations, and even partial differential equations.

Overview

We will look at three different one dimensional maps:

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- Logistic Map

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- Logistic Map
- Sine Map
- Quadratic Map

Today's Topics

Logistic Map: Background

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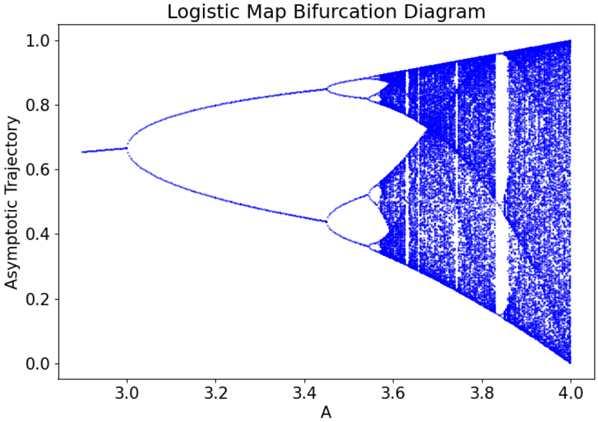
- The map was popularized in a seminal 1976 paper by the biologist Robert May, in part as a discrete-time demographic model analogous to the logistic equation first created by Pierre François Verhulst.
- The logistic map is written

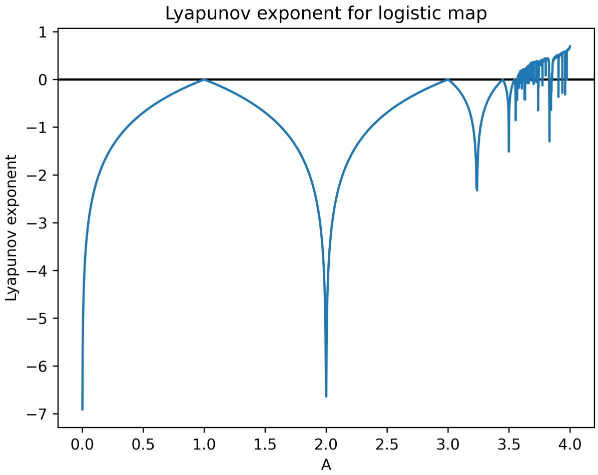
$$x_{n+1} = Ax_n(1 - x_n) \tag{3}$$

where x_n denotes the ratio of existing population to the maximum possible population at discrete time n, and A denotes a parameter measuring the rate of growth.

Results: Bifurcation Plot



Results: Lyapunov Exponent Plot



Logistic Map: Calculation of Feigenbaum Constant δ

Bifurcation points: $x^* = 3, 3.449, 3.544, \ldots$

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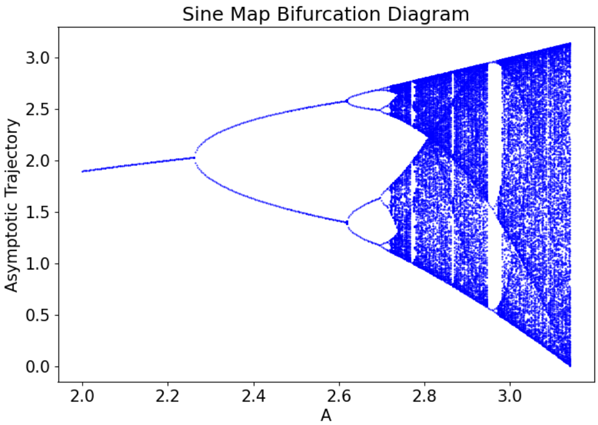
$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$
$$\delta_2 = \frac{A_2 - A_1}{A_3 - A_2} = 4.751$$
$$\delta_3 = \frac{A_3 - A_2}{A_4 - A_3} = 4.655$$

Today's Topics

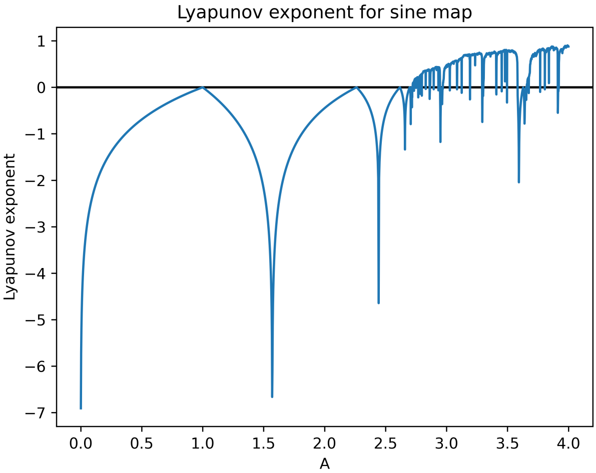
Sine Map: Background

$$x_{n+1} = f(x_n) \text{ , where } A > 0,\ x \in [0,1]$$
$$f(x) = A\ sin(x)$$
$$f'(x) = A\ cos(x)$$

Results: Bifurcation Plot



Results: Lyapunov Exponents Plot



Results: Calculation of Feigenbaum Constant δ

Bifurcation points: $x^* = 2.261, 2.617, \ldots$

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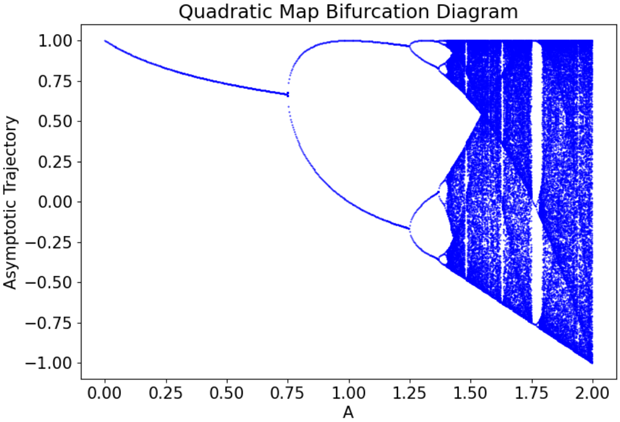
$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$
$$\delta_2 = \frac{A_2 - A_1}{A_3 - A_2} = 4.470$$
$$\delta_3 = \frac{A_3 - A_2}{A_4 - A_3} = 4.627$$

Today's Topics

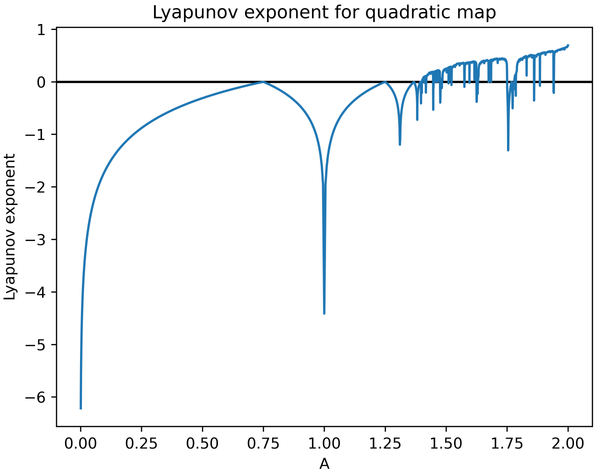
Quadratic Map: Background

$$x_{n+1} = f(x_n) \text{ , where}$$
$$f(x) = A - x^2$$
$$f'(x) = -2x$$

Results: Bifurcation Diagram Plot



Results: Lyapunov Exponent Plot



Results: Feigenbaum Constant δ

Bifurcation points: $x^* = 0.749, 1.249, \ldots$

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$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$

$$\delta_2 = \frac{A_2 - A_1}{A_3 - A_2} = 4.233$$

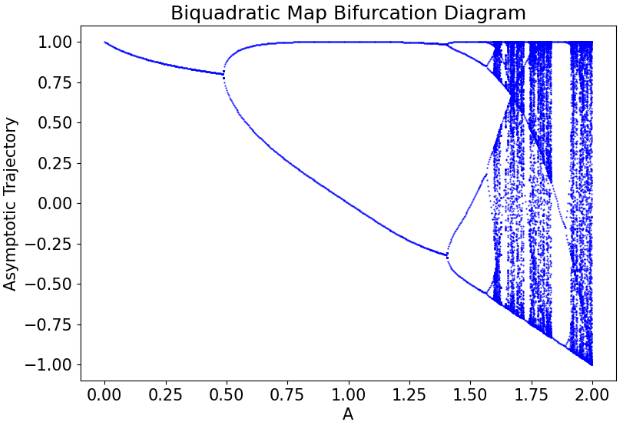
$$\delta_3 = \frac{A_3 - A_2}{A_4 - A_3} = 4.550$$

Today's Topics

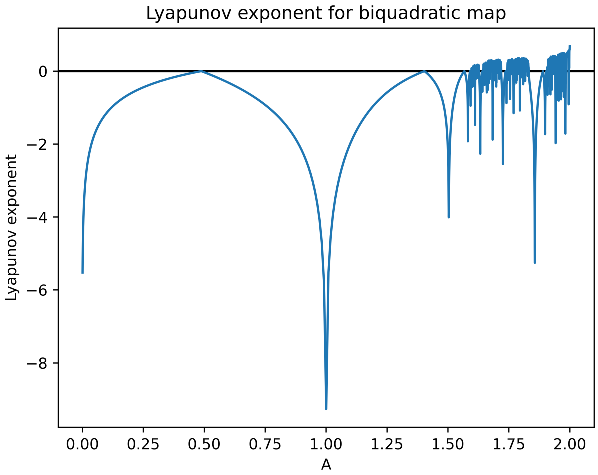
Bi-Quadratic Map: Background

$$x_{n+1} = f(x_n) \text{ , where}$$
$$f(x) =$$
$$f'(x) =$$

Results: Bifurcation Diagram Plot



Results: Lyapunov Exponent Plot



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Ambiguity?

Why is this constant different from the Feigenbaum constant?

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Because the degree of equation is different

Is it still Feigenbaum constant?

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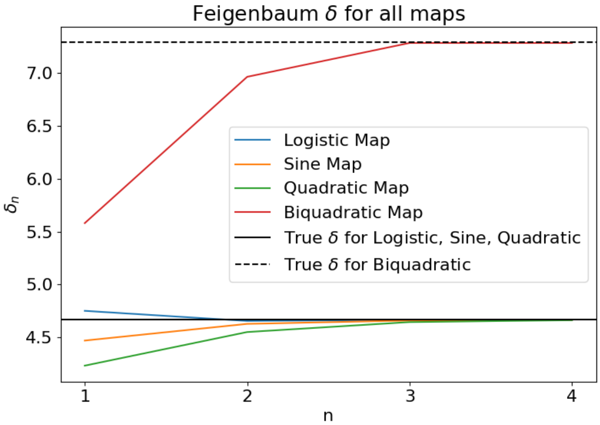
Because the degree of equation is different

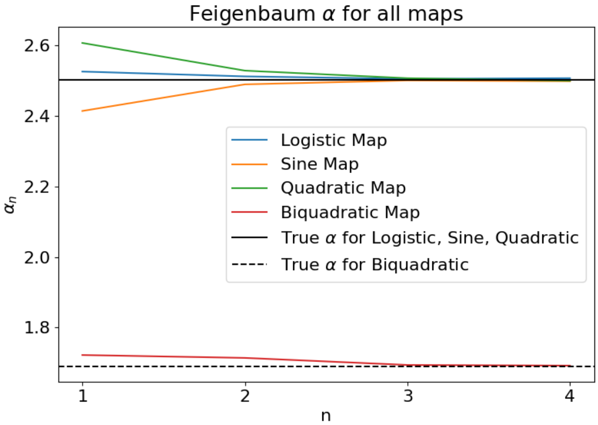
Is it still Feigenbaum constant?

Yes, but for a different degree of function

Comparing δ: Bi-Quadratic Map v/s Single Maxima Maps



Comparing α: Bi-Quadratic Map v/s Single Maxima Maps



General Case

$$x_{n+1} = 1 - \mu|x_n|^n, \text{ where } \mu > 0$$

Then the Feigenbuam constants for different values of r are:

feigenbaum_for_different_r.png