10K Feet View of Universality

PH 567 Presentation

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Today's Topics

What is Universality?

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- For eg: All gases behave similarly at high temperature and low pressure regardless of the type of molecule

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- Lyapunov Exponents
- Feigenbaum Constants

Bifurcation Points

- At a bifurcation point, a small change of the parameter value of the system may cause:
  - a large change in the number or stability of the equilibrium points of the system,
  - the emergence of limit cycles (oscillations),
  - or chaos to emerge from an attracting orbit

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  - the emergence of limit cycles (oscillations),
  - or chaos to emerge from an attracting orbit
- The bifurcation point itself does not represent any dynamical change of the system, but rather a qualitative change of its behavior as one or more parameters crosses a critical threshold.

Bifurcation Diagram

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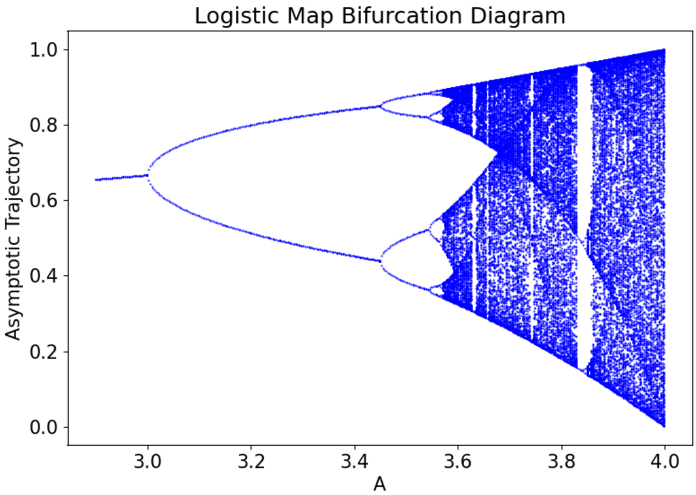
- For a one dimensional map the bifurcation parameter is shown on the horizontal axis of the plot and the vertical axis shows the set of values of the iterated function visits asymptotically from some initial condition.

Example Bifurcation Diagram



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  where λ is the Lyapunov exponent and n is the number of iterations.
- The Lyapunov exponent is positive for chaotic systems, zero for non-chaotic systems (or bifurcation points), and negative for systems that converge to fixed points.

Lyapunov Exponents

- For a one dimensional map $x_{n+1} = f(x_n)$, the Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \tag{2}$$

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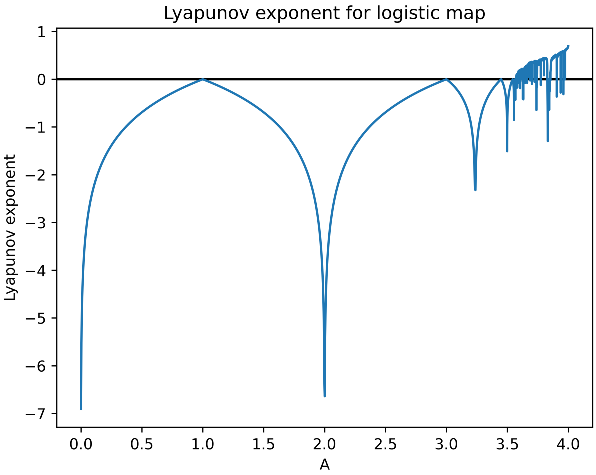
  where f is the map and x_i is the ith iterate of the map.
- Typically, the lyapunov exponent is zero at the bifurcation points, and highly negative (Read: $-\infty$) at the superstable points.

Example Lyapunov Exponent Plot



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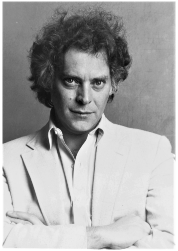

Figure: Mitchell Feigenbaum, 1987. Photo by Ingbert Grüttner

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- The difference between the bifurcation points decreased at the same geometric rate δ
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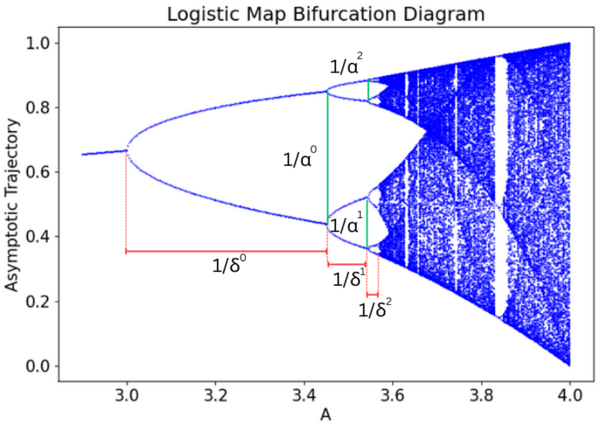
A thorough understanding of the simple Logistic Map is sufficient for describing both qualitatively, and to a large extent, quantitatively, the period-doubling route to chaos for all systems in the family.

Feigenbaum's Constants Visualized



Today's Topics

Overview

We will look at four different one dimensional maps:

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- Biquadratic Map

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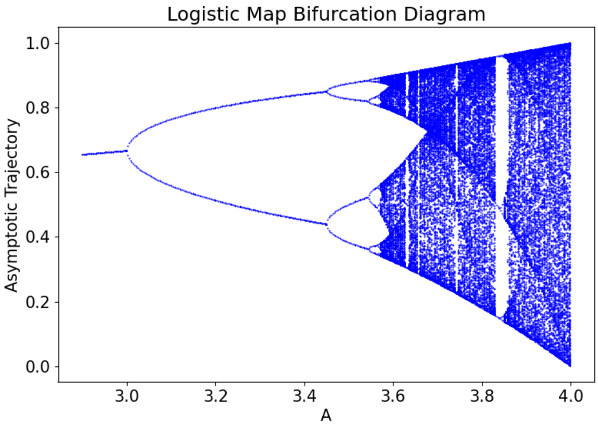
- It was popularized by the biologist Robert May as a discrete-time demographic model.
- The logistic map is written as

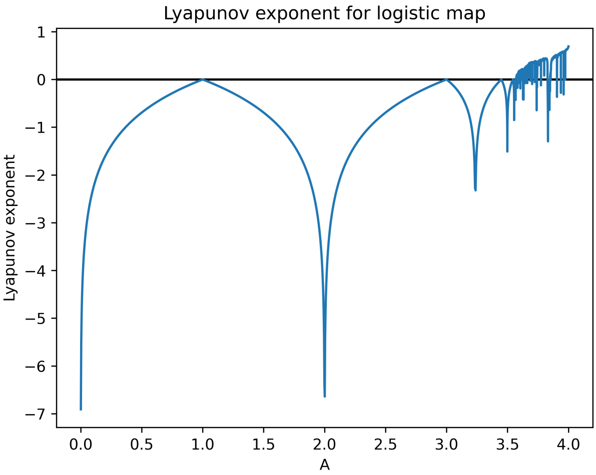
$$x_{n+1} = Ax_n(1 - x_n) \tag{3}$$

where x_n denotes the ratio of existing population to the maximum possible population at discrete time n, and A denotes a parameter measuring the rate of growth.

Results: Bifurcation Plot



Results: Lyapunov Exponent Plot



Logistic Map: Calculation of Feigenbaum Constant δ

Bifurcation points: $x^* = 3, 3.449, 3.544, \ldots$

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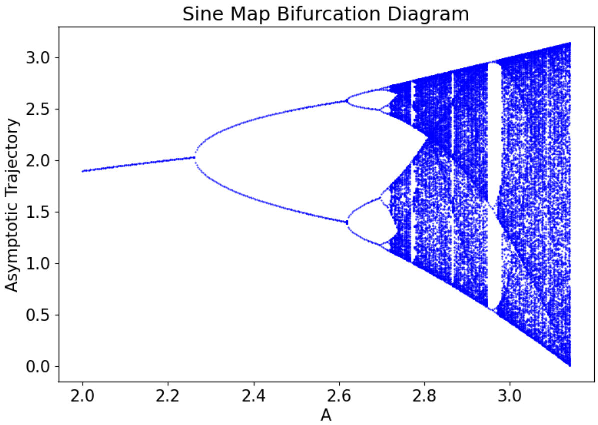
$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$
$$\delta_2 = \frac{A_2 - A_1}{A_3 - A_2} = 4.751$$
$$\delta_3 = \frac{A_3 - A_2}{A_4 - A_3} = 4.655$$

Today's Topics

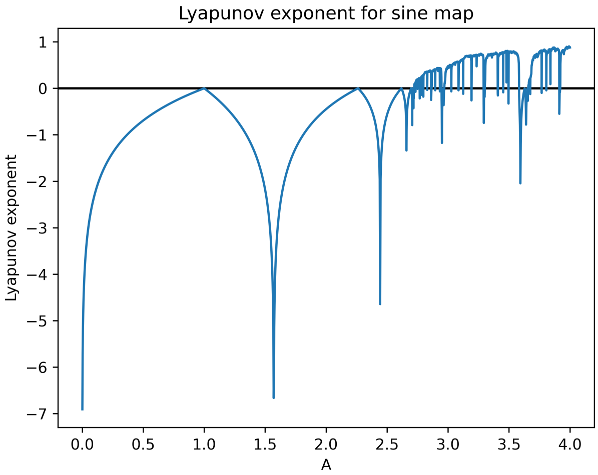
Sine Map: Background

$$x_{n+1} = f(x_n) \text{ , where } A > 0,\ x \in [0, 1]$$
$$f(x) = A\ sin(x)$$
$$f'(x) = A\ cos(x)$$

Results: Bifurcation Plot



Results: Lyapunov Exponents Plot

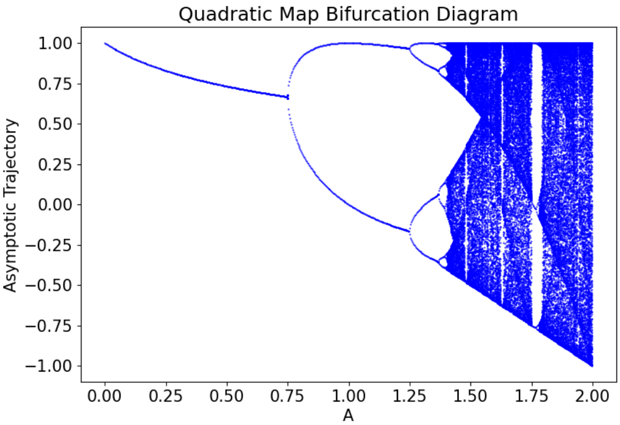


Today's Topics

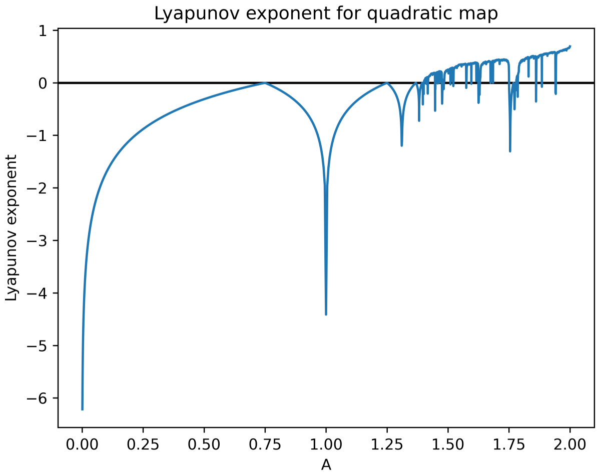
Quadratic Map: Background

$$
\begin{aligned}
x_{n+1} &= f(x_n) \text{ , where} \\
f(x) &= 1 - Ax^2 \\
f'(x) &= -2Ax
\end{aligned}
$$

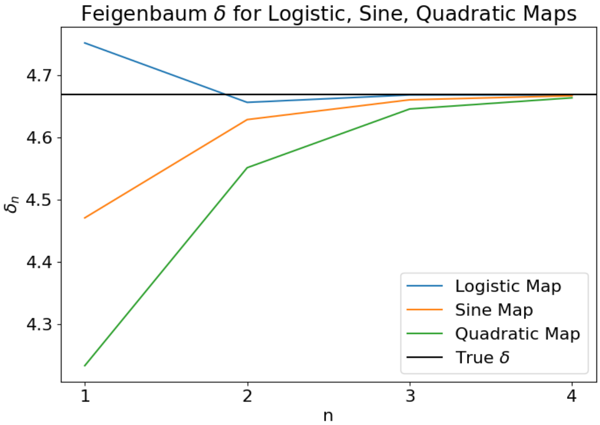
Results: Bifurcation Diagram Plot



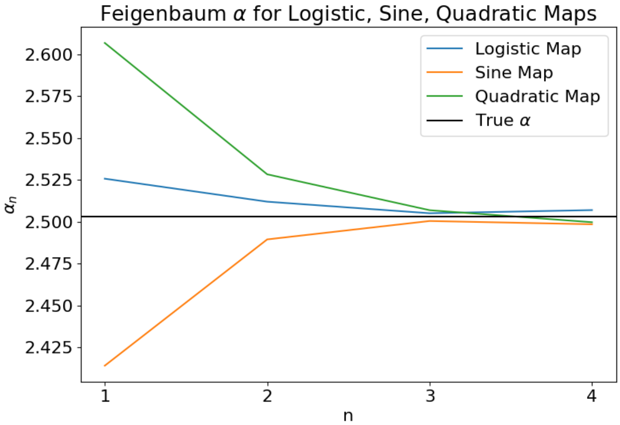
Results: Lyapunov Exponent Plot



Comparison: Feigenbaum Constant δ



Comparison: Feigenbaum Constant α



Today's Topics

A Quick Question...

Is everyone convinced that Feigenbaum constants are universal?

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Let's try another example!

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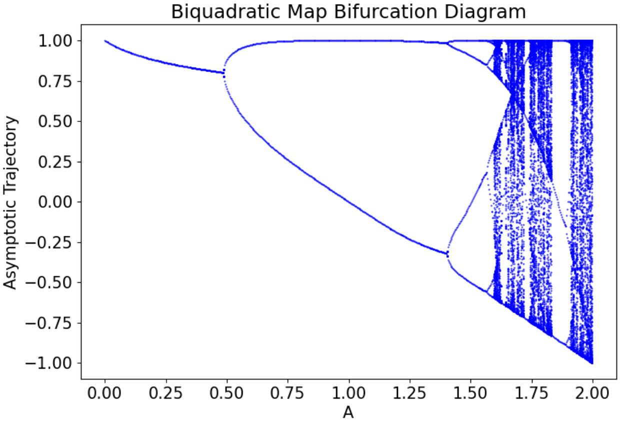
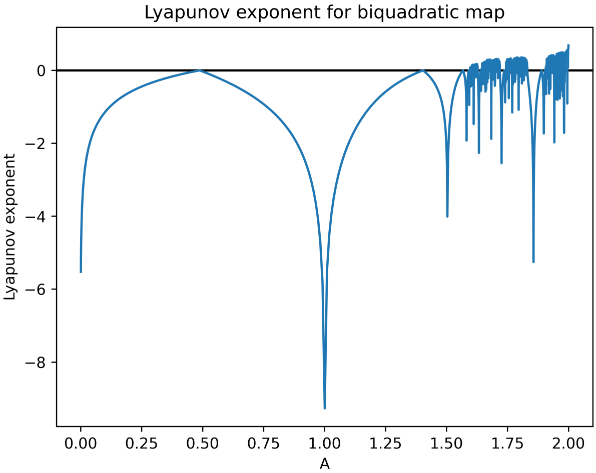
Bi-Quadratic Map: Background

$$x_{n+1} = f(x_n) \text{ , where}$$
$$f(x) = 1 - Ax^4$$
$$f'(x) = -4Ax^3$$

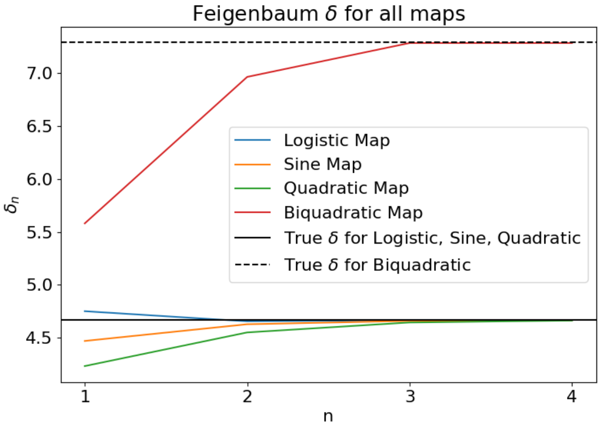
Results: Bifurcation Diagram Plot



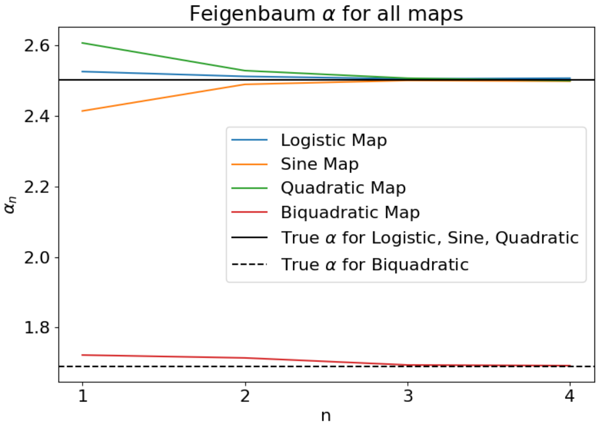
Results: Lyapunov Exponent Plot



Comparing δ: Bi-Quadratic Map v/s Quadratic Maps



Comparing α: Bi-Quadratic Map v/s Quadratic Maps



Ambiguity?

Why is this constant different from the Feigenbaum constant?

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Is it still the Feigenbaum constant?

Yes, but for a different family of functions!

General Case

$$x_{n+1} = 1 - A|x_n|^r, \text{ where } \mu > 0$$

Then the Feigenbuam constants for different values of r[1] are:

r	δ	α
3	5.9679687038...	1.9276909638...
4	7.2846862171...	1.6903029714...
5	8.3494991320...	1.5557712501...
6	9.2962468327...	1.4677424503...

[1] Keith Briggs. "A Precise Calculation of the Feigenbaum Constants". In: *Mathematics of Computation* 57.195 (1991), pp. 435–439.

Today's Topics

Summary and Conclusion

- Feigenbaum studied one dimensional maps and defined two constants δ and α related to the period doubling route to chaos
- Two systems belonging to the same family share the same Feigenbaum Constants
- There are also systems which do not share these constants; they belong to different families

Summary and Conclusion

Period doubling routes to chaos are observed in many real-world systems:

- Convection currents in Liquid Mercury[2]
- Chemical Oscillators[3]
- Lasers[4]

Knowing about the properties which these vastly different systems share brings us one step closer to having a theory of everything :).

[2] Libchaber et. al. "Period doubling cascade in mercury, a quantitative measurement". In: *Journal de Physique Lettres* 43 (1982).

[3] Doona et. al. "Period-doubling route to chaos in the chlorite-thiocyanate chemical oscillator". In: *The Journal of Physical Chemistry* 98 (1994), pp. 513–517.

[4] Tarroja et. al. "Period-doubling route to chaos in a standing-wave laser: A comparison between theory and experiment". In: *Optics Communications* 84.3 (1991), pp. 162–168.