10K Feet View of Universality

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Today's Topics

What is Universality?

- Universality is the observation that there are properties for a large class of systems that are independent of the dynamical details of the system

The Tools

There are a few properties that we can use to compare and contrast between different systems:

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- Bifurcation Points

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- Lyapunov Exponents

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- The bifurcation point itself does not represent any dynamical change of the system, but rather a qualitative change of its behavior as one or more parameters crosses a critical threshold.

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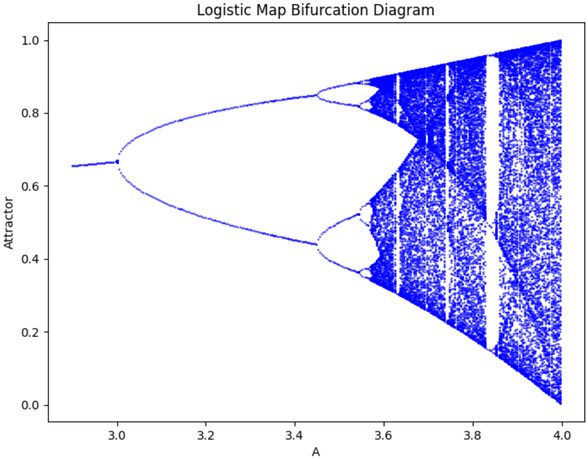
- For a one dimensional map the bifurcation parameter r is shown on the horizontal axis of the plot and the vertical axis shows the set of values of the logistic function visited asymptotically from almost all initial conditions.

Example Bifurcation Diagram



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  where λ is the Lyapunov exponent and n is the number of iterations.
- The Lyapunov exponent is positive for chaotic systems, zero for non-chaotic systems, and negative for systems that converge to fixed points.

Lyapunov Exponents

- For a one dimensional map $x_{n+1} = f(x_n)$, the Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \tag{2}$$

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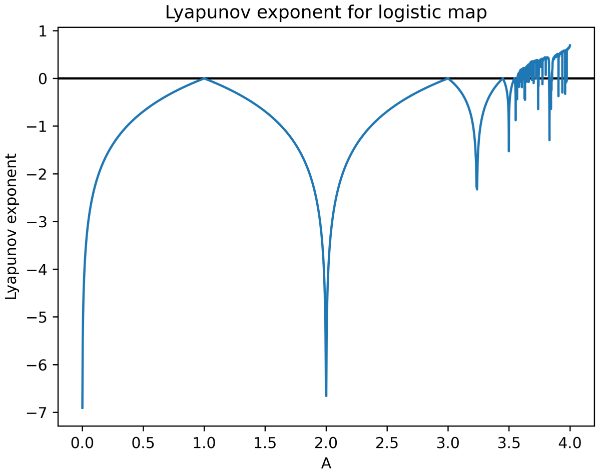
- Typically, the lyapunov exponent is zero at the bifurcation points, and highly negative (Read: $-\infty$) at the superstable points.

Example Lyapunov Exponent Plot



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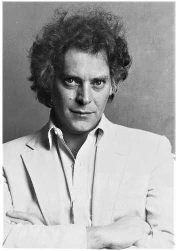

Figure: Mitchell Feigenbaum, 1987. Photo by Ingbert Grüttner

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- This observation ultimately led to a rigorous proof, using the mathematical methods of the renormalization group borrowed from the theory of critical phenomena, that these geometrical ratios were universal numbers that would apply to the quantitative description of any period-doubling sequence generated by nonlinear maps with a single quadratic extremum.

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- This observation ultimately led to a rigorous proof, using the mathematical methods of the renormalization group borrowed from the theory of critical phenomena, that these geometrical ratios were universal numbers that would apply to the quantitative description of any period-doubling sequence generated by nonlinear maps with a single quadratic extremum.
- The logistic map and the sine map are just two examples of this large universality class. The great significance of this result is that the global details of the dynamical system do not matter.

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- In fact, this universality class extends beyond one-dimensional maps to nonlinear dynamical systems described by more realistic physical models corresponding to two-dimensional maps, systems of ordinary differential equations, and even partial differential equations.

Overview

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- Quadratic Map

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- The logistic map is written

$$x_{n+1} = Ax_n(1 - x_n) \tag{3}$$

  where x_n denotes the ratio of existing population to the maximum possible population at discrete time n, and A denotes a parameter measuring the rate of growth.