10K Feet View of Universality

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Today's Topics

1. The Theory
   - What is Universality?
   - The Tools
2. Universality in One Dimensional Maps
   - Feigenbaum's Discovery
   - Logistic Map
   - Sine Map
   - Quadratic Map

What is Universality?

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- All gases behave similarly at high temperature and low pressure regardless of the type of molecule
- Critical exponents show universality in wide range of systems, including ferro-magnets, fluids, superconductors

The Tools

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- Lyapunov Exponents

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- The bifurcation point itself does not represent any dynamical change of the system, but rather a qualitative change of its behavior as one or more parameters crosses a critical threshold.

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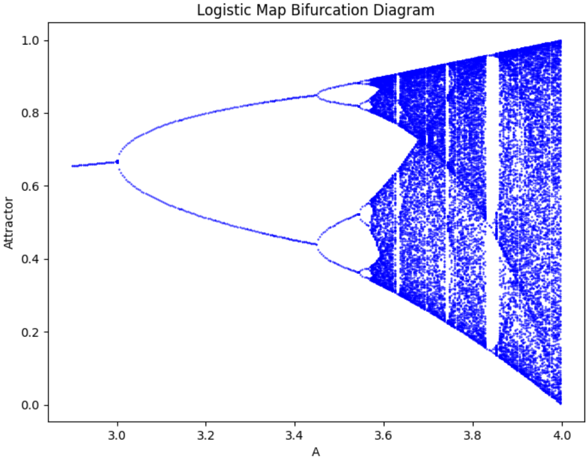
- For a one dimensional map the bifurcation parameter r is shown on the horizontal axis of the plot and the vertical axis shows the set of values of the logistic function visited asymptotically from almost all initial conditions.

Example Bifurcation Diagram



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  where λ is the Lyapunov exponent and n is the number of iterations.
- The Lyapunov exponent is positive for chaotic systems, zero for non-chaotic systems, and negative for systems that converge to fixed points.

Lyapunov Exponents

- For a one dimensional map $x_{n+1} = f(x_n)$, the Lyapunov exponent is given by

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \left| f'(x_i) \right| \tag{2}$$

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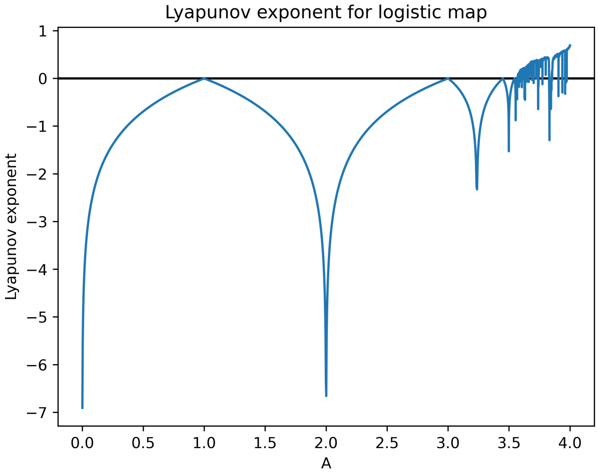
  where f is the map and x_i is the ith iterate of the map.
- Typically, the lyapunov exponent is zero at the bifurcation points, and highly negative (Read: $-\infty$) at the superstable points.

Example Lyapunov Exponent Plot



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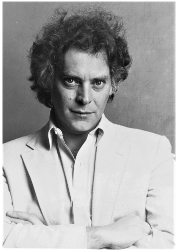

Figure: Mitchell Feigenbaum, 1987. Photo by Ingbert Grüttner

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- This observation ultimately led to a rigorous proof, using the mathematical methods of the renormalization group borrowed from the theory of critical phenomena, that these geometrical ratios were universal numbers that would apply to the quantitative description of any period-doubling sequence generated by nonlinear maps with a single quadratic extremum.
- The logistic map and the sine map are just two examples of this large universality class. The great significance of this result is that the global details of the dynamical system do not matter.

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- In fact, this universality class extends beyond one-dimensional maps to nonlinear dynamical systems described by more realistic physical models corresponding to two-dimensional maps, systems of ordinary differential equations, and even partial differential equations.

Overview

We will look at three different one dimensional maps:

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Logistic Map: Background

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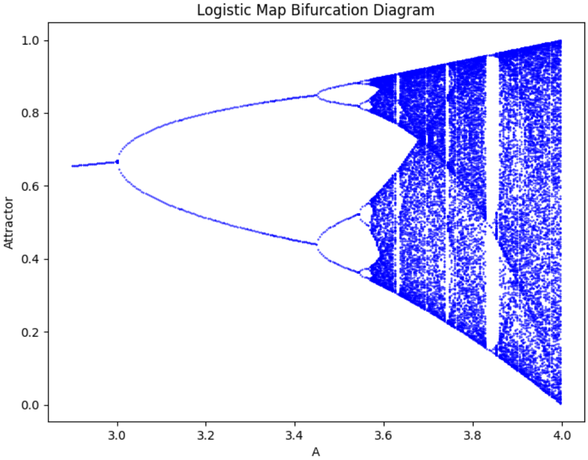
- The map was popularized in a seminal 1976 paper by the biologist Robert May, in part as a discrete-time demographic model analogous to the logistic equation first created by Pierre François Verhulst.
- The logistic map is written

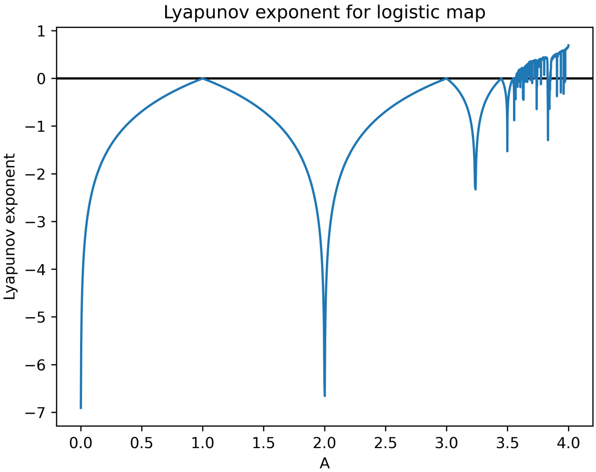
$$x_{n+1} = Ax_n(1 - x_n) \tag{3}$$

where x_n denotes the ratio of existing population to the maximum possible population at discrete time n, and A denotes a parameter measuring the rate of growth.

Results: Bifurcation Plot



Results: Lyapunov Exponent Plot



Logistic Map: Calculation of Feigenbaum Constant δ

Bifurcation points: $x^* = 3, 3.449, 3.544, \ldots$

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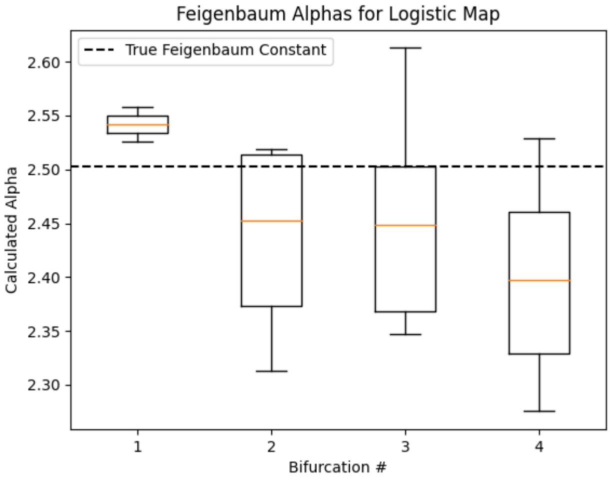
$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$

$$\delta_2 = \frac{A_2 - A_1}{A_3 - A_2} = 4.751$$

$$\delta_3 = \frac{A_3 - A_2}{A_4 - A_3} = 4.655$$

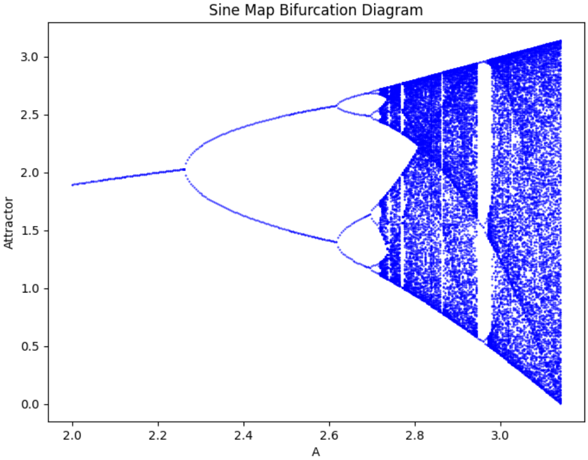
Results: Feigenbaum Constant α



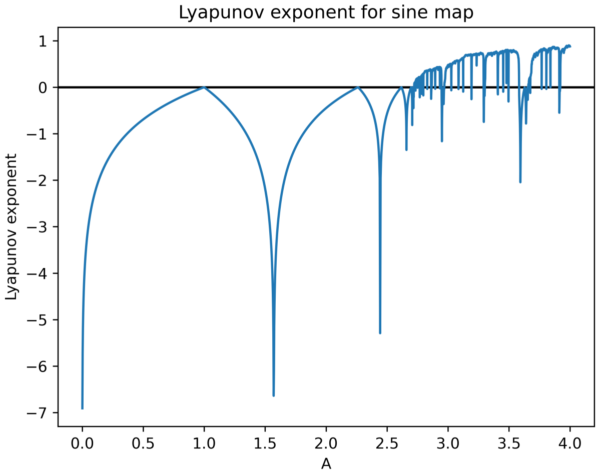
Sine Map: Background

$$x_{n+1} = f(x_n)\text{ , where}$$
$$f(x) = A\,sin(x)$$
$$f'(x) = A\,cos(x)$$

Results: Bifurcation Plot



Results: Lyapunov Exponents Plot



Results: Calculation of Feigenbaum Constant δ

Bifurcation points: $x^* = 2.261, 2.617, \ldots$

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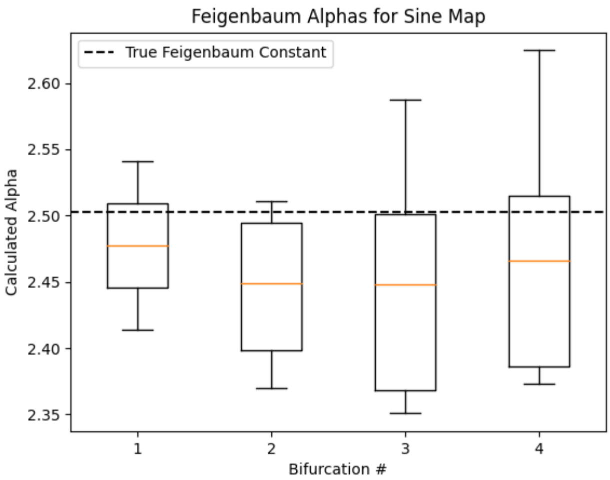
$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$
$$\delta_2 = \frac{A_2 - A_1}{A_3 - A_2} = 4.470$$
$$\delta_3 = \frac{A_3 - A_2}{A_4 - A_3} = 4.627$$

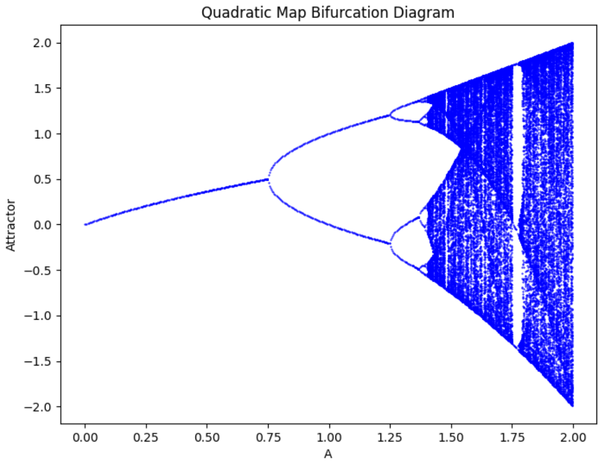
Results: Feigenbaum Constant α



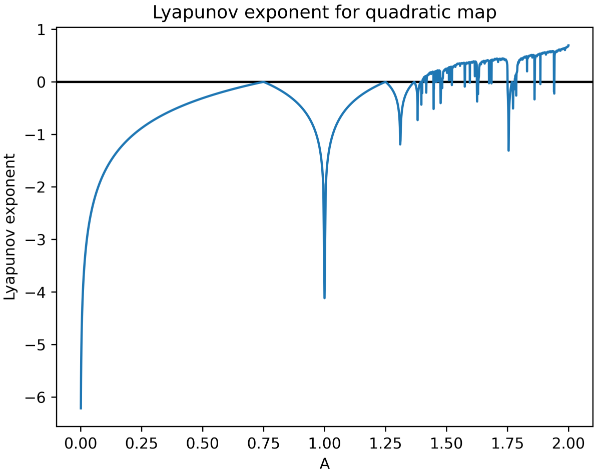
Quadratic Map: Background

$$x_{n+1} = f(x_n)\text{ , where}$$
$$f(x) = A - x^2$$
$$f'(x) = -2x$$

Results: Bifurcation Diagram Plot



Results: Lyapunov Exponent Plot



Results: Feigenbaum Constant δ

Bifurcation points: $x^* = 0.749, 1.249, \ldots$

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$$\delta_n = \frac{A_n - A_{n-1}}{A_{n+1} - A_n}$$
$$\delta_2 = \frac{A_2 - A_1}{A_3 - A_2} = 4.233$$
$$\delta_3 = \frac{A_3 - A_2}{A_4 - A_3} = 4.550$$

Results: Feigenbaum Constant α