UE-TAP

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Introduction

Setup and Notation

- $G(N, A)$ – strongly connected directed transportation network. N set of nodes, A set of links.
- $R, S \subseteq N$, collection of origin and destination nodes.
- d_{rs}, the travel demand between $r \in R$ and $s \in S$.
- H_{rs} – set of simple paths between r and s.
- (i, j) – link with head node i and tail node j. $t_{ij}(x_{ij})$ is the cost of traversing (i, j). The function $t_{ij}(\cdot)$ is strictly positive and monotonically increasing.
- For path $h \in H_{rs}$, the flow in h will be denoted by f_h.

Wardrop's Principle

Definition (UE-Condition)

For each O-D pair (r, s), every used path between r and s must have equal and minimal travel-time.

This can be formulated as an optimization problem.

$$\min z(f) = \sum_{(i,j)\in A} \int_0^{x_{ij}} t(\omega) d\omega$$

where

$$x_{ij} = \sum_{r\in R} \sum_{s\in S} \sum_{h\in H_{rs}} f_h \delta_{ij}^h$$

Subject to:

$$f_h \geq 0 \text{ and } \sum_{h\in H_{rs}} f_h = d_{rs} \quad \forall\, r \in R, s \in S$$

Approximation

- Solution to the above problem satisfies UE-condition.
- Will use quadratic approximation of the Beckman function as the objective function.
- Consider the case of a single O-D pair.
- Second order Taylor approximation at path flow solution g:

$$\hat{z}_{rs}(f) \approx z_{rs}(g) + \sum_{h \in H_{rs}} \frac{\partial z_{rs}}{\partial f_h}\bigg|_g (f_h - g_h) + \sum_{h \in H_{rs}} \frac{\partial^2 z_{rs}}{\partial f_h^2}\bigg|_g (f_h - g_h)^2$$

- Denote $\frac{\partial z_{rs}}{\partial f_h}\big|_g =: v_h^g$ and $\frac{\partial^2 z_{rs}}{\partial f_h^2}\big|_g (f_h - g_h)^2 =: s_h^g$.
- After removing constants, objective becomes:

$$\sum_{h \in H_{rs}} \left[(v_h^g - s_h^g g_h) f_h + \frac{1}{2} s_h^g f_h^2 \right]$$

KKT Conditions

These constants can be calculated easily by taking the derivative of the Beckman function.

$$\left.\frac{\partial z_{rs}}{\partial f_h}\right|_g = \sum_{(i,j)\in A} \delta_{ij}^h t_{ij}(x_{ij}); \quad \left.\frac{\partial^2 z_{rs}}{\partial f_h^2}\right|_g = \sum_{(i,j)\in A} \delta_{ij}^h t'_{ij}(x_{ij})$$

The KKT conditions of the modified problem with the same constraints as before imply:

$$v_h^g + s_h^g(f_h - g_h) - w_{rs} \geq 0$$

and

$$f_h(v_h^g + s_h^g(f_h - g_h) - w_{rs}) = 0$$

Here the Lagrange multiplier (for the equality constraint), w_{rs} is interpreted as the least travel cost between r and s.

More Notations

Note that

$$\hat{v}_h = v_h^g + s_h^g (f_h - g_h)$$

is a first order approximation of the path travel cost.
Put

$$c_h^g := v_h^g - s_h^g g_h$$

then the conditions seen in last slide for used paths $(\hat{H}_{rs})$ become

$$c_h^g + s_h^g f_h = w_{rs} := \bar{w}_{rs} \quad \forall h \in \hat{H}_{rs}$$

Path Flow Updates

Dividing the above expression by s_h^g and summing over all paths,

$$\bar{w}_{rs} = \frac{d_{rs} + \sum_{h \in \hat{H}_{rs}} (c_h^g / s_h^g)}{\sum_{h \in \hat{H}_{rs}} 1/s_h^g}$$

And the new flows are

$$f_h = \frac{\bar{w}_{rs} - c_h^g}{s_h^g}$$

The paths in $H_{rs} \setminus \hat{H}_{rs}$ recieve zero flow.

Algorithm – Single Step Path Flow Update

Takes some initial path flow vector $\{g_h : h \in H_{rs}\}$ as input and outputs updated path flow vector. The steps involved are:

1. Calculate the constants v_h^g, s_h^g and c_h^g.
2. Sort paths based on increasing c_h^g.