A Greedy Path-Based Algorithm for Traffic Assignment

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March 13, 2023

Introduction

- The UE-TAP has been used as a standard tool to predict network flows.
- Path based algorithms solve this problem by storing and manipulating a set of paths for each O-D pair, which are iteratively generated by 'column generation'.
- Limitations:
  - storing and manipulating paths require a lot of memory
  - decomposing scheme by O-D pairs generate a large number of sub-problems.

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- Path based algorithms solve this problem by storing and manipulating a set of paths for each O-D pair, which are iteratively generated by 'column generation'.
- Limitations:
  - storing and manipulating paths require a lot of memory
  - decomposing scheme by O-D pairs generate a large number of sub-problems.
- RAM capacity no longer an issue!
- Path based algorithms are easier to understand and implement, compared to many other methods.

Overview

The paper presents a new path-based algorithm for the UE-TAP, which is more efficient than most modern algorithms.

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Notations

- $G(N, A)$ – strongly connected directed transportation network. N set of nodes, A set of links.
- $R, S \subseteq N$, collection of origin and destination nodes.
- d_{rs}, the travel demand between $r \in R$ and $s \in S$.
- H_{rs} – set of simple paths between r and s.
- (i, j) – link with head node i and tail node j. $t_{ij}(x_{ij})$ is the cost of traversing (i, j). The function $t_{ij}(\cdot)$ is strictly positive and monotonically increasing.
- For path $h \in H_{rs}$, the flow in h will be denoted by f_h.

Wardrop's Principle

Definition (UE-Condition)

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This can be formulated as an optimization problem.

$$\min z(f) = \sum_{(i,j)\in A} \int_0^{x_{ij}} t(\omega) d\omega$$

where

$$x_{ij} = \sum_{r\in R} \sum_{s\in S} \sum_{h\in H_{rs}} f_h \delta_{ij}^h$$

Subject to:

$$f_h \geq 0 \text{ and } \sum_{h\in H_{rs}} f_h = d_{rs} \quad \forall\, r \in R, s \in S$$

Approximation

- Solution to the above problem satisfies UE-condition.
- Will use quadratic approximation of the Beckman function as the objective function.
- Consider the case of a single O-D pair.
- Second order Taylor approximation at path flow solution g:

$$\hat{z}_{rs}(f) \approx z_{rs}(g) + \sum_{h \in H_{rs}} \left.\frac{\partial z_{rs}}{\partial f_h}\right|_g (f_h - g_h) + \sum_{h \in H_{rs}} \left.\frac{\partial^2 z_{rs}}{\partial f_h^2}\right|_g (f_h - g_h)^2$$

- Denote $\left.\frac{\partial z_{rs}}{\partial f_h}\right|_g =: v_h^g$ and $\left.\frac{\partial^2 z_{rs}}{\partial f_h^2}\right|_g =: s_h^g$.
- After removing constants, objective becomes:

$$\sum_{h \in H_{rs}} \left[(v_h^g - s_h^g g_h) f_h + \frac{1}{2} s_h^g f_h^2\right]$$

KKT Conditions

These constants can be calculated easily by taking the derivative of the Beckman function.

$$\left.\frac{\partial z_{rs}}{\partial f_h}\right|_g = \sum_{(i,j)\in A} \delta_{ij}^h t_{ij}(x_{ij}^g); \quad \left.\frac{\partial^2 z_{rs}}{\partial f_h^2}\right|_g = \sum_{(i,j)\in A} \delta_{ij}^h t'_{ij}(x_{ij}^g)$$

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The KKT conditions of the modified problem with the same constraints as before imply:

$$v_h^g + s_h^g(f_h - g_h) - w_{rs} \geq 0$$

and

$$f_h(v_h^g + s_h^g(f_h - g_h) - w_{rs}) = 0$$

Here the Lagrange multiplier (for the equality constraint), w_{rs} is interpreted as the least travel cost between r and s.

More Notations

Note that

$$\hat{v}_h = v_h^g + s_h^g(f_h - g_h)$$

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Put

$$c_h^g := v_h^g - s_h^g g_h$$

then the conditions seen in previous slide for used paths ($\hat{H}_{rs}$) become

$$c_h^g + s_h^g f_h = w_{rs} := \bar{w}_{rs} \quad \forall h \in \hat{H}_{rs}$$

Path Flow Updates

Dividing the above expression by s_h^g and summing over all paths,

$$\bar{w}_{rs} = \frac{d_{rs} + \sum_{h \in \hat{H}_{rs}} (c_h^g / s_h^g)}{\sum_{h \in \hat{H}_{rs}} 1/s_h^g}$$

And the new flows are

$$f_h = \frac{\bar{w}_{rs} - c_h^g}{s_h^g}$$

The paths in $H_{rs} \setminus \hat{H}_{rs}$ recieve zero flow.

Algorithm – Single O-D Pair (Single Step)

Takes some initial path flow vector $\{g_h : h \in H_{rs}\}$ as input and outputs updated path flow vector. The steps involved are:

1. Calculate the constants v_h^g, s_h^g and c_h^g.
2. Sort paths based on increasing c_h^g.
3. Add the path with smallest c_h^g to $\hat{H}_{rs}$ and update $\bar{w}_{rs}$.
4. Keep adding subsequent paths until c_h^g exceeds $\bar{w}_{rs}$.
5. Update the flows.
6. The new path flow vector $\hat{H}_{rs}$ is given as output.

Algorithm – Single O-D Pair (Single Step)

1: **Input:** The current solution $\{g_h, \forall h \in H_{rs}\}$ is given.
2: **Initialization:** lines 3–9.
3: **for** each $h \in H_{rs}$ **do**
4: Compute v_h^g and s_h^g according to formulation (16) and formulation (17), respectively.
5: Compute c_h^g according to formulation (18).
6: **end for**
7: Sort all paths $h \in H_{rs}$ according to the increasing order of c_h^g, i.e., $H_{rs} = \{1, 2, 3, \ldots\}$ with $c_1^g \leqslant c_2^g \leqslant c_3^g \leqslant \ldots$.
8: Let $B = 1/(s_1^g d_{rs})$, $C = c_1^g/(s_1^g d_{rs})$ and $\bar{w}_{rs} = (1.0 + C)/B$.
9: Set $h = 2$, $\hat{H}_{rs} = \{1\}$.
10: **Main Loop:** lines 11–17.
11: **while** $h \leqslant |H_{rs}|$ and $c_h^g < \bar{w}_{rs}$ **do**
12: Set $C = C + c_h^g/(s_h^g d_{rs})$.
13: Set $B = B + 1/(s_h^g d_{rs})$.
14: Set $\bar{w}_{rs} = (1.0 + C)/B$.
15: Let $\hat{H}_{rs} = \hat{H}_{rs} \cup \{h\}$.
16: Set $h = h + 1$.
17: **end while**
18: **Flow Update:** lines 19–31.
19: **end for**
20: **for** each $h \in \hat{H}_{rs}$ **do**
21: Set $f_h = (\bar{w}_{rs} - c_h^g)/s_h^g$.
22: **for** each $h \in H_{rs} \backslash \hat{H}_{rs}$ **do**
23: Set $f_h = 0$.
24: **end for**
25: **for** each $h \in H_{rs}$ **do**
26: **if** $f_h \neq g_h$ **then**
27: Update the link flow by $x_{ij} = x_{ij} + (f_h - g_h), \forall (i,j) \in h$.
28: Update $t_{ij}(x_{ij})$ and $\frac{\partial t_{ij}(x_{ij})}{\partial x_{ij}}$ for each link $(i,j) \in h$.
29: **end if**
30: **end for**
31: Let $H_{rs} = \hat{H}_{rs}$.
32: **Output:** A new solution $\{f_h, \forall h \in H_{rs}\}$.

Initialization

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Greedy Path-Based Algorithm

The steps involved are:

0. Initialization: assign all passengers to shortest path between O-D pairs and set H_{rs} to contain the shortest path between r and s.
1. Perform flow adjustment on H_{rs} using the previous algorithm for each r, s.
2. (Inner loop) Update flows on O-D pairs with larger deviation:
   - If $\max v_h - \min v_h \geq RG^{k-1}/2$, for an O-D pair, H_{rs} is passed to the previous algorithm again.
   - If this update is not being done for any O-D pair (or if the iteration exceeds a limit), break from the loop.
3. Push the shortest path between each O-D pairs to H_{rs} and repeat from step 1 until convergence.

Greedy Path-Based Algorithm

1: **Initialize**: lines 2–7
2: **for** each O-D pair (r, s) **do**
3: Compute the shortest path $\hat{h}$ from r to s.
4: Assign all d_{rs} to $\hat{h}$ and push it into H_{rs}.
5: **end for**
6: Update link flows by $x_{ij} = \sum_{r\in R}\sum_{s\in S}\sum_{h\in H_{rs}} f_h \delta_{ij}^h, \quad \forall (i,j) \in A$.
7: Update the link cost $t_{ij}(x_{ij})$ and its derivative $\frac{\partial t_{ij}(x_{ij})}{\partial x_{ij}}$ for each link $(i,j) \in A$.

8: **Main Loop**: lines 9–34
9: **for** each $r \in R$ **do**
10: Compute the shortest path tree from r to all its destinations S_r.
11: **for** each $s \in S_r$ **do**
12: Build the shortest path $\hat{h}$ from r to s.
13: Push $\hat{h}$ into H_{rs} if $\hat{h} \notin H_{rs}$; otherwise delete $\hat{h}$.
14: Perform path flow adjustment for H_{rs} by Algorithm 1.
15: **end for**
16: **end for**

17: **Inner Loop**: lines 18–34
18: Set $l = 0, MaxI = 1000, FC = 0$.
19: **while** l<$MaxI$ **do**
20: Let $l = l + 1$ and $FC = 0$.
21: **for** each O-D pair (r, s) **do**
22: **if** $l\%100 = 0$ **then**
23: Compute $\Delta_{rs} = \max\{v_h\} - \min\{v_h\}, \forall h \in H_{rs}$.
24: **end if**
25: **if** $\Delta > RG^{k-1}/2.0$ **then**
26: Let $FC = FC + 1$.
27: Perform path flow adjustment for H_{rs} by Algorithm 1.
28: Update x_{ij}, $t_{ij}(x_{ij})$ and $\frac{\partial t_{ij}(x_{ij})}{\partial x_{ij}}$ for $\forall (i,j) \in h, \forall h \in H_{rs}$.
29: **end if**
30: **end for**
31: **if** $FC = 0$ **then**
32: Break the inner loop.
33: **end if**
34: **end while**

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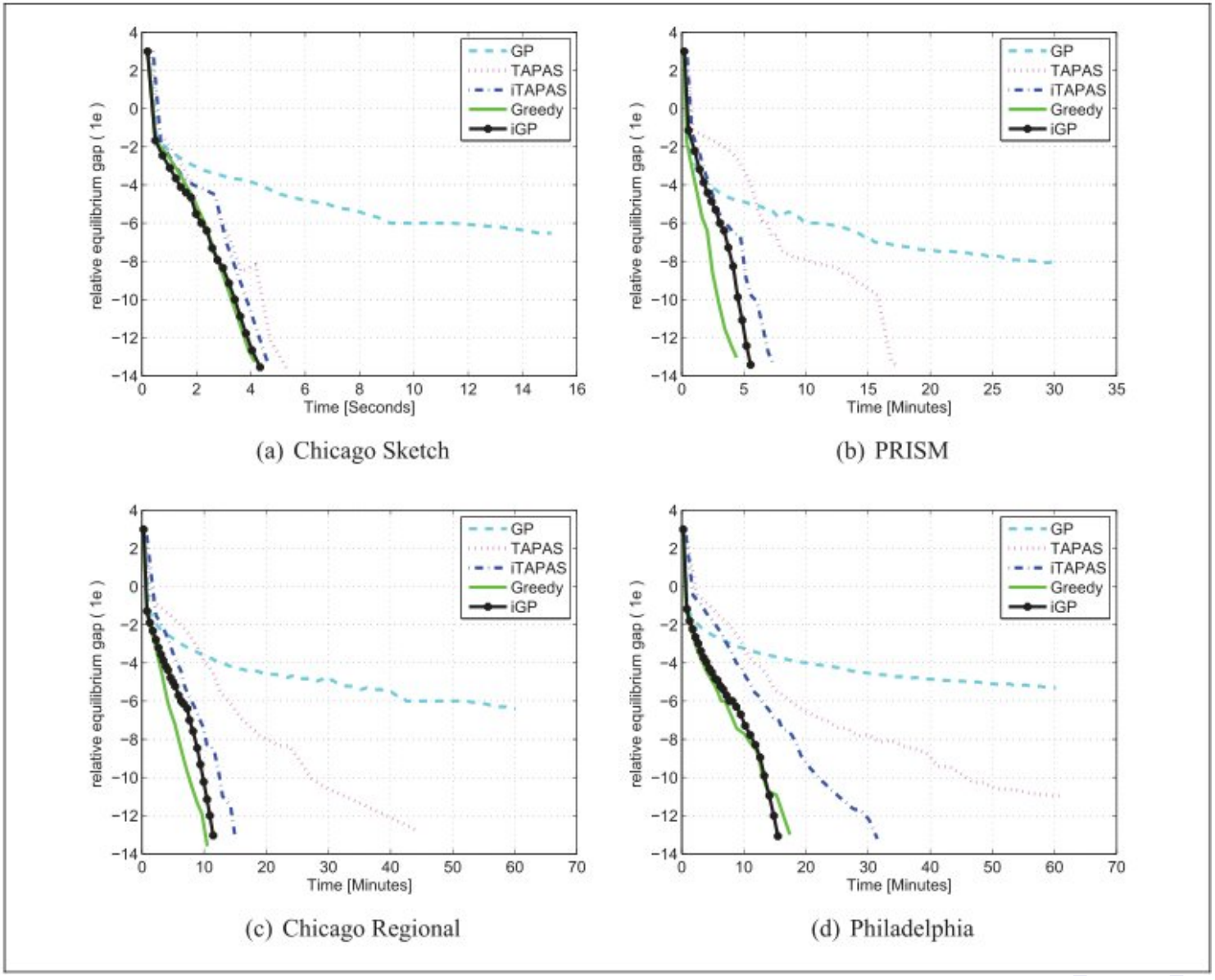
Results

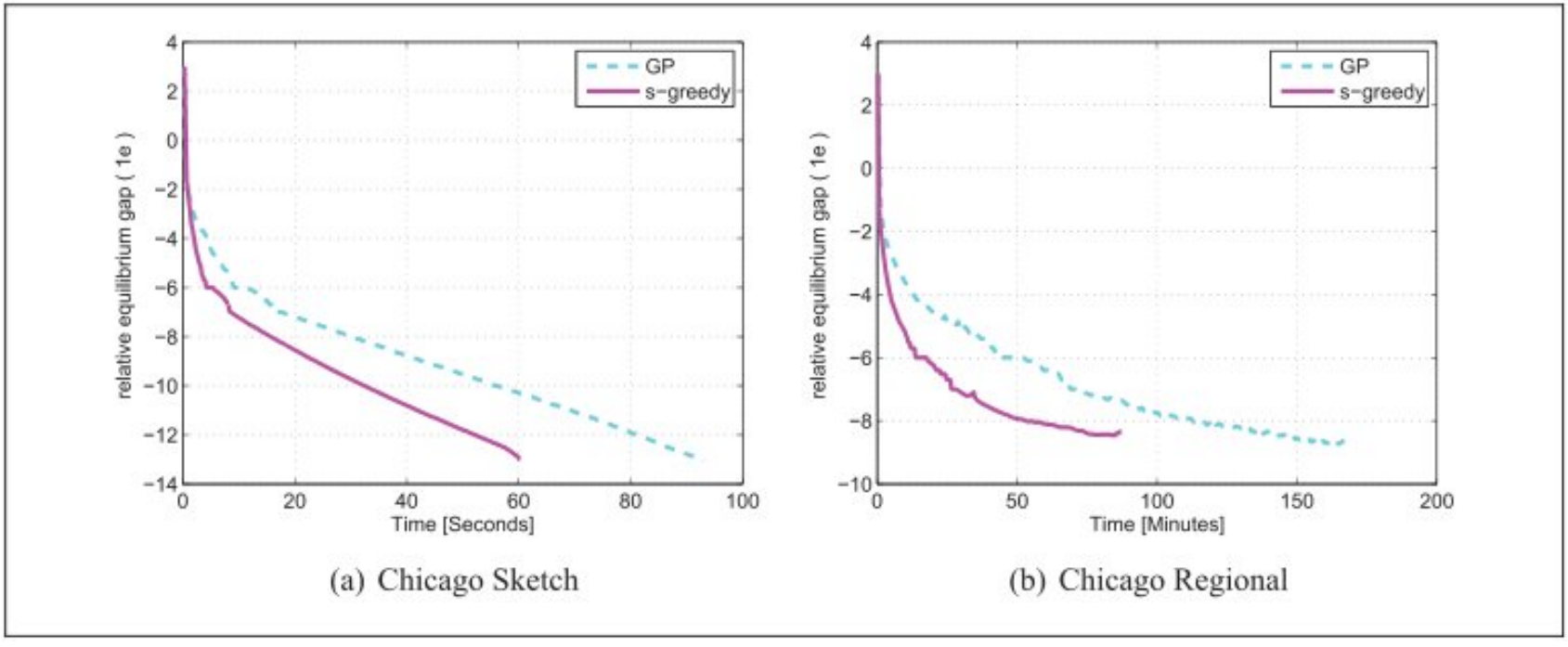
- Numerical results were produced on Windows 10 workstation with $3.3GHz$ Intel Xenon processor and $16GB$ RAM using C++.
- The greedy algorithm performs better than GP, TAPAS, iTAPAS in all the experiments.
- iGP performs almost as good as the greedy method.

Results


GP
TAPAS
iTAPAS
Greedy
iGP
relative equilibrium gap (1e)
Time [Seconds]
(a) Chicago Sketch
Time [Minutes]
(b) PRISM
(c) Chicago Regional
(d) Philadelphia

Results

Even without the inner loop, the greedy algorithm performs better than GP.



(a) Chicago Sketch

(b) Chicago Regional

Results – Memory Usage

Table 2. Path Information for Different Algorithms

Networks	Information	GP	greedy	iGP
Chicago Regional	path number	1,985,744	1,991,055	1,920,298
	memory size	489 M	491 M	473 M
Philadelphia	path number	1,712,502	1,709,818	1,376,289
	memory size	583 M	562 M	437 M

Note: M = megabyte.

The path-based algorithms generates less than 2 million used paths, which consumes less than 600 megabytes of memory.

Conclusion

- Frequency of path flow adjustments must be higher than that of column generation.
- More flow adjustment must be done on less converged O-D pairs.
- Do not try to get high precision results for sub-problems in the early iterations.