Linear Regression

Machine Learning Decal

Hosted by Machine Learning at Berkeley

Agenda

Background

Linear Algebra Perspective

Optimization via Gradient Descent

Probabilistic Perspective

Background

Where are we?

	Continuous	Discrete
Supervised	**Regression**	Classification
Unsupervised	Dimensionality Reduction	Clustering

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- How does a computer make predictions?

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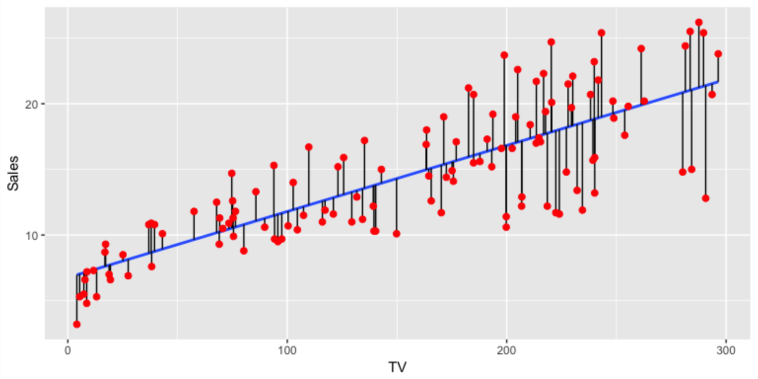
    - (Economics) Okun's law

Example of Linear Regression



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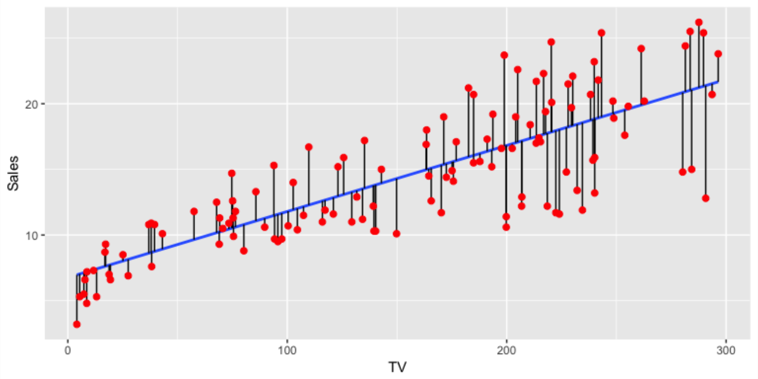

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Minimizing the sum of the square of those errors!

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- We introduce $x_0 = 1$ for simplicity so that:

$$h_\theta(x) = \sum_{i=1}^{p} \theta_i^T x_i = \theta^T x$$

Warm-up: predicting house prices

- Suppose we have the following data about houses:

Price	# of Square Feet	# of Bedrooms
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- Our linear model has the form:

$$h_\theta(sqft) = \theta_0 + \theta_1 sqft$$

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Linear Algebra Perspective

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- Our predictions for test points are then

$$\hat{y} = h_{\hat{\theta}}(x)$$

- We can rewrite linear regression as

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1p} \\ 1 & x_{21} & x_{22} & \dots & x_{2p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{np} \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_p \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_n \end{pmatrix}$$

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- Here, we're using capital letters to represent matrices, and arrows to represent vectors

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- That is,

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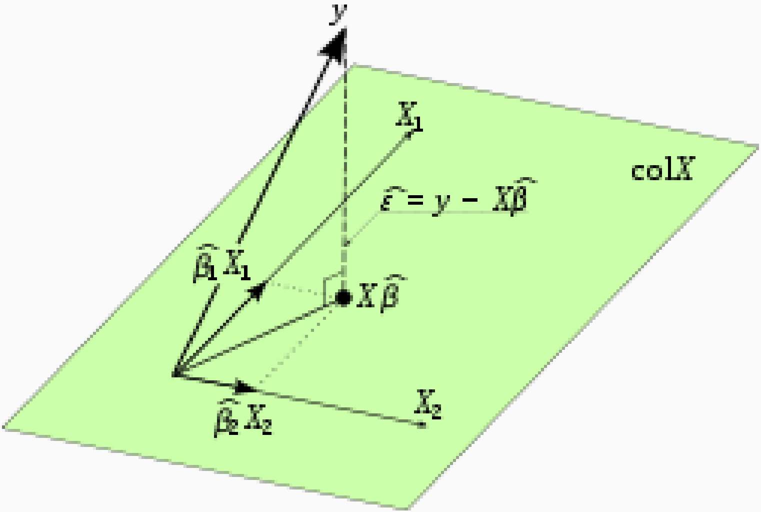
Geometric Interpretation



Projection of y on the features of X

Estimation (least squares) via vector calculus

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$$= argmin_{\vec{\theta}}(\vec{y} - X\vec{\theta})^T(\vec{y} - X\vec{\theta})$$

$$= argmin_{\vec{\theta}}\vec{y}^T\vec{y} - 2\vec{y}X\vec{\theta} + \vec{\theta}^TX^TX\vec{\theta}$$

How to minimize a function? Take the derivative!

$$\frac{\partial L}{\partial \vec{\theta}} = 2X^TX\vec{\theta} - 2X^T\vec{y} = 0$$

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- For a one unit increase in x_{ik}, we expect y_i to, **on average** increase by $\hat{\theta}_k$

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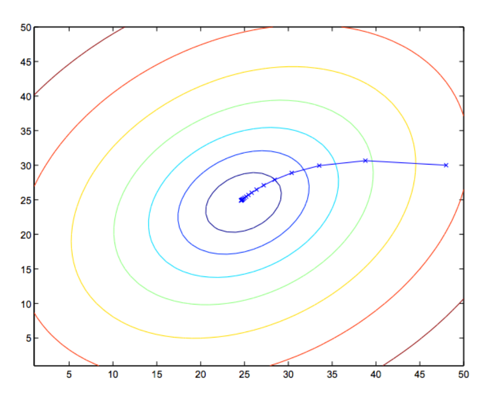

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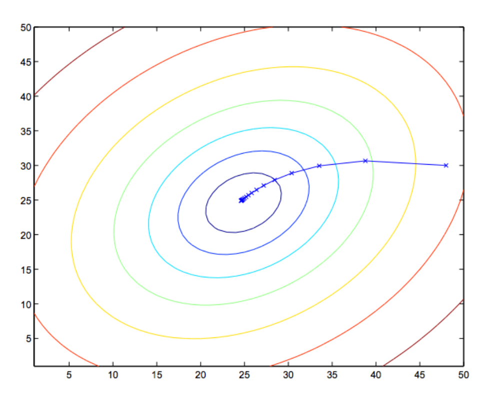

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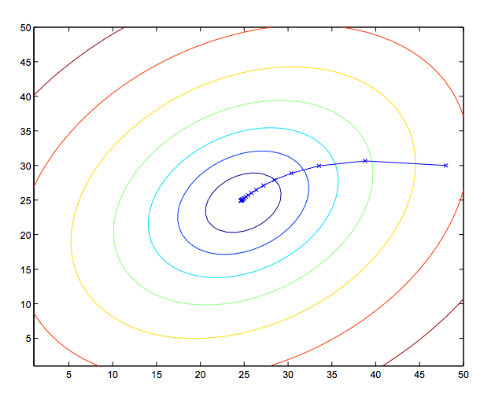

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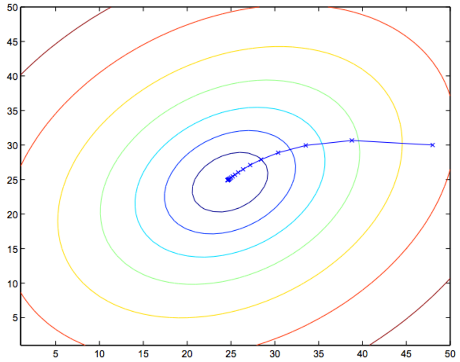

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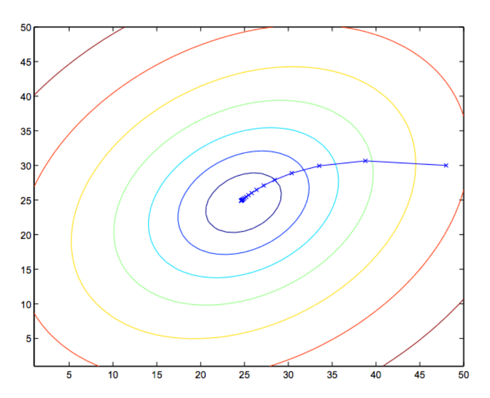
  - Keep doing this until $L(\theta)$ reaches its minimum



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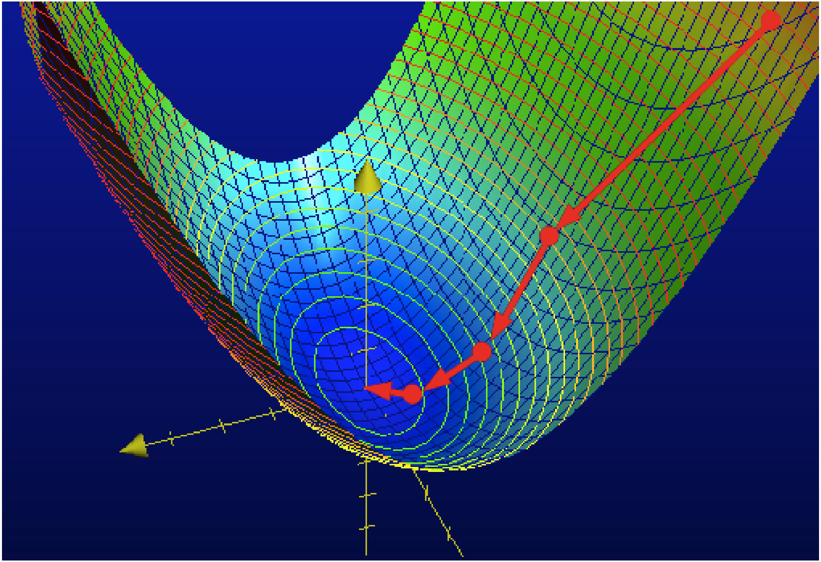
- ϵ is called the **learning rate**

Visualizing gradient descent



Deriving the update rule

- Let's start with the case where we only have one training example (x, y)

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\nabla_\theta L(\theta) &= \nabla_\theta \frac{1}{2}(h_\theta(x) - y)^2 \\
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- For a single training example, the update rule is:

$$
\theta \leftarrow \theta - \epsilon(y^1 - h_\theta(x^1))x^1
$$

Gradient descent for linear regression

- For n training examples:

$$\theta_j \leftarrow \theta_j - \epsilon \sum_{i}^{n} (h_\theta(x^i) - y^i) x_j^i$$

for $j = 1, \ldots, p$ and $i = 1, \ldots, n$

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- If the prediction $h_\theta(x^i)$ is close the actual y^i then the parameters θ shouldn't need much changing

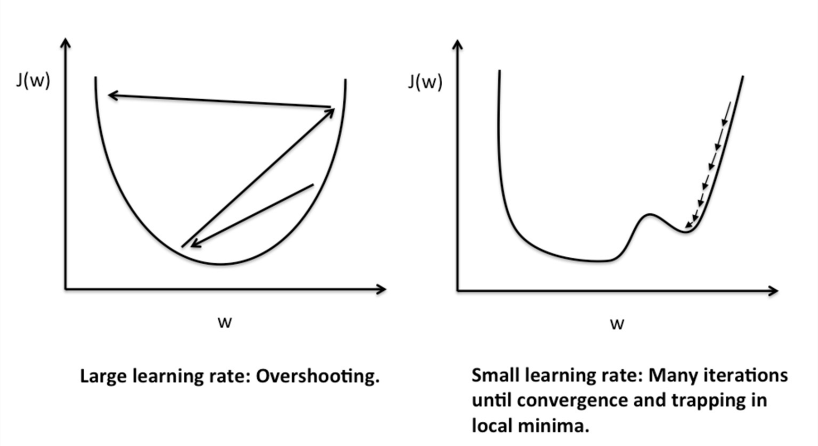
Stochastic gradient descent for linear regression

While $L(\theta)$ is not minimized:

For $i = i, \ldots, n$:

$$\theta_j \leftarrow \theta_j - \epsilon(h_\theta(x^i) - y^i)x_j^i \quad \text{(for each j)}$$

Choosing the learning rate



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- We can find the optimal θ by solving the **normal equations** as covered previously

Probabilistic Perspective

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- We need to know what those assumptions are, where they come from, and what to do when they fall apart

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- In order to reason this process more rigorously, we can construct a "toy universe" to analyze our models

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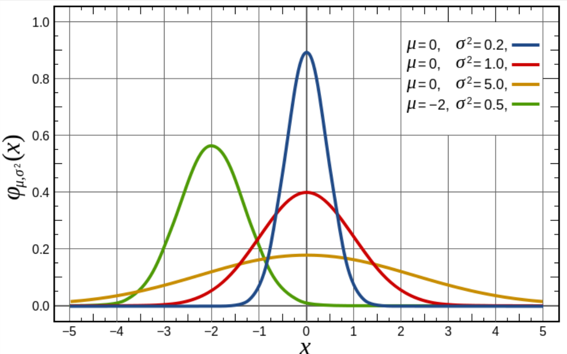

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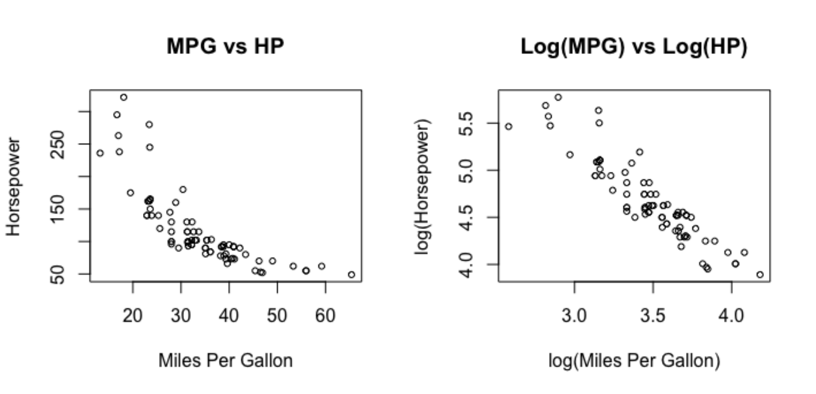
  - Outliers and/or high leverage points may contribute to this issue

Example of the beauty of a log transform



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Questions?