8. $\sin(n+3)\theta$.

$u_n = \sin n\theta \rightarrow$ $u_0 = \sin 0 = 0$, $u_1 = \sin\theta$, $u_2 = \sin 2\theta$

$u_{n+3} = \sin(n+3)\theta$

$U(z) = Z(u_n)$
$= Z(\sin n\theta)$

$$U(z) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z(u_{n+3}) = z^3\left[U(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

$$Z(\sin(n+3)\theta) = z^3\left[\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} - 0 - \frac{\sin\theta}{z} - \frac{\sin 2\theta}{z^2}\right].$$

9. Prove that $Z(n u_n) = -z\frac{d}{dz}\{Z(u_n)\}$. Hence find $Z\left(\frac{n}{n+1}\right)$ given that $Z\left(\frac{1}{n+1}\right) = z\log\left(\frac{z}{z-1}\right)$.

By definition $\quad Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

differentiate on b.s. w.r.t z

$$\frac{d}{dz}Z(u_n) = \sum_{n=0}^{\infty} u_n (-n) z^{-n-1}$$

$$= -\sum_{n=0}^{\infty} n u_n z^{-n} z^{-1}$$

multiply both side by $(-z)$.

$$\frac{d}{dz}Z(u_n) = -\sum_{n=0}^{\infty} n u_n z^{-n}\cdot\frac{1}{z} \qquad \times(-z)$$

$$-z\frac{d}{dz}Z(u_n) = \sum_{n=0}^{\infty}(n u_n) z^{-n}$$

$$-z\frac{d}{dz}Z(u_n) = Z(n u_n)$$

$$\boxed{Z(n u_n) = -z\frac{d}{dz}Z(u_n).}$$

Taking $u_n = \frac{1}{n+1}$.

$$Z(n\,u_n) = -z\frac{d}{dz}Z(u_n)$$

$$Z\left(n\frac{1}{n+1}\right) = -z\frac{d}{dz}\left\{z\log\left(\frac{z}{z-1}\right)\right\}.$$

$$Z\left(\frac{n}{n+1}\right) = -z\frac{d}{dz}\left\{z\left[\log z - \log(z-1)\right]\right\}$$

$$Z\left(\frac{n}{n+1}\right) = -z\left\{1\left[\log z - \log(z-1)\right] + z\left[\frac{1}{z} - \frac{1}{z-1}\right]\right\}$$

$$= -z\left\{\log\frac{z}{z-1} + \left(1 - \frac{z}{z-1}\right)\right\}.$$

$$= -z\left\{\log\left(\frac{z}{z-1}\right) - \frac{1}{z-1}\right\}.$$

$$\frac{\cancel{z}-1-\cancel{z}}{z-1} = -\frac{1}{z-1}$$

$$Z\left(\frac{n}{n+1}\right) = -z\left\{\log\left(\frac{z}{z-1}\right) - \frac{1}{z-1}\right\}$$

Find. $Z(na^n)$

$$Z(n u_n) = -z\frac{d}{dz}Z(u_n)$$

$$Z(na^n) = -z\frac{d}{dz}Z(a^n)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-a}\right) = -z\left[\frac{(z-a).1 - z(1-0)}{(z-a)^2}\right]$$

$$= \frac{-z(-a)}{(z-a)^2} = \frac{az}{(z-a)^2}.$$

$$Z(na^n) = \frac{az}{(z-a)^2}$$

$$\boxed{Z(n2^n) = \frac{2z}{(z-2)^2}}$$

$$Z(n(-2)^n) = \frac{-2z}{(z+2)^2}.$$

If $U(z) = \dfrac{2z^2+3z+12}{(z-1)^4}$, find u_0, u_1, u_2, u_3.

Soln:

$$u_0 = \lim_{z\to\infty} U(z)$$

$$u_0 = \lim_{z\to\infty} \frac{2z^2+3z+12}{(z-1)^4} = \lim_{z\to\infty} \frac{\cancel{z^2}\left(2+\frac{3}{z}+\frac{12}{z^2}\right)}{z^{\cancel{4}\,2}\left(1-\frac{1}{z}\right)^4}$$

$$\cancel{u_0 = 2+0+0}$$

$$u_0 = 0+0+0 = \underline{\underline{0.}} \qquad 1/\infty = 0$$

$$u_1 = \lim_{z\to\infty} z\left[U(z) - u_0\right]$$

$$u_1 = \lim_{z\to\infty} z\left[\frac{2z^2+3z+12}{(z-1)^4} - 0\right]$$

$$= \lim_{z\to\infty} \frac{\cancel{z}\; z^2\left(2+\frac{3}{z}+\frac{12}{z^2}\right)}{z^{\cancel{4}\,1}\left(1-\frac{1}{z}\right)^4} = \frac{0+0+0}{(1-0)^4} = \underline{\underline{0.}}$$

$$u_2 = \lim_{z\to\infty} z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$

$$= \lim_{z\to\infty} z^2\left[\frac{2z^2+3z+12}{(z-1)^4} - 0 - \frac{0}{z}\right]$$

$$= \lim_{z\to\infty} \frac{\cancel{z^2}\;\cancel{z^2}\left[2+\frac{3}{z}+\frac{12}{z^2}\right]}{\cancel{z^4}\left(1-\frac{1}{z}\right)^4} = \frac{2+0+0}{(1-0)^4} = \underline{2.}$$

$$u_3 = \lim_{z\to\infty} z^3\left[U(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{2z^2+3z+12}{(z-1)^4} - 0 - 0 - \frac{2}{z^2}\right]$$

$$= \lim_{z\to\infty} z^3\left[\frac{(2z^2+3z+12)z^2 - 2(z-1)^4}{(z-1)^4 z^2}\right].$$

$$u_3 = \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+12z^2-2(z-1)^4}{(z-1)^4 z^2}\right]$$

$(a-b)^3 = a^3 - b^3 - 3ab(a-b)$

$= a^3 - b^3 - 3a^2b + 3ab^2$

$\cancel{(z-1)^3 = z^3 - 1}$

$$(z-1)^4 = (z-1)^3(z-1) = \left(z^3 - 1 - 3z^2\cdot 1 + 3z\cdot 1^2\right)(z-1)$$

$$= z^4 - z - 3z^3 + 3z^2 - z^3 + 1 + 3z^2 - 3z$$

$$(z-1)^4 = z^4 - 4z^3 + 6z^2 - 4z + 1$$

$$u_3 = \lim_{z\to\infty} z^3\left[\frac{2z^4+3z^3+12z^2-2(z^4-4z^3+6z^2-4z+1)}{(z-1)^4 z^2}\right]$$

$$u_3 = \lim_{z\to\infty} z^3\left[\frac{\cancel{2z^4}+3z^3+\cancel{12z^2}-\cancel{2z^4}+8z^3-\cancel{12z^2}+8z-2}{(z-1)^4 z^2}\right]$$

$$u_3 = \lim_{z\to\infty} z^3\left[\frac{11z^3+8z-2}{z^4(1-\frac{1}{z})^4 z^2}\right]$$

$$= \lim_{z\to\infty} \cancel{z^3}\ \cancel{z^3}\left[\frac{11+\frac{8}{z^2}-\frac{2}{z^3}}{\cancel{z^6}(1-\frac{1}{z})^4}\right]$$

$$= \frac{11+0-0}{(1-0)^4} = 11$$

$$\boxed{u_3 = 11}$$

$$\boxed{u_0 = 0,\quad u_1 = 0,\quad u_2 = 2\quad \&\quad u_3 = 11}$$

If $Z(u_n) = \dfrac{2z^2+3z+4}{(z-3)^3}$, ~~show~~ find u_0, u_1, u_2, u_3.

Soln: $u_0 = 0$, $u_1 = 2$, $u_2 = 21$, and $u_3 = 139$.

If $U(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$, find u_0, u_2, u_2 & u_3.

Soln: $u_0 = 0$, $u_1 = 0$, $u_2 = 2$ and $u_3 = 13$.

(10) If $u_n = \dfrac{z}{z-1} + \dfrac{z}{z^2+1}$, show that $Z(u_{n+2}) = \dfrac{z(z^2-z+2)}{(z-1)(z^2+1)}$.

$$Z(u_{n+2}) = z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] \quad \text{— } (*)$$

$$U(z) = Z(u_n) = \left(\frac{z}{z-1} + \frac{z}{z^2+1}\right)$$

$$u_0 = \lim_{z\to\infty} U(z) = \lim_{z\to\infty} \frac{z}{z-1} + \frac{z}{z^2+1}$$

$$= \lim_{z\to\infty}\left[\frac{\cancel{z}}{\cancel{z}(1-\frac{1}{z})} + \frac{\cancel{z}}{z^{\cancel{2}}(1+\frac{1}{z^2})}\right]$$

$$= \frac{1}{(1-0)} + \frac{0}{(1+0)} = 1.$$

$$u_1 = \lim_{z\to\infty} z[U(z) - u_0] = \lim_{z\to\infty} z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right]$$

$$u_1 = \lim_{z\to\infty} z\left[\frac{z(z^2+1) + z(z-1) - (z-1)(z^2+1)}{(z-1)(z^2+1)}\right]$$

$$u_1 = \lim_{z\to\infty} z\left[\frac{\cancel{z^3} + \cancel{z} + z^2 - \cancel{z} - \cancel{z^3} + z^2 - z + 1}{(z-1)(z^2+1)}\right]$$

$$u_1 = \lim_{z\to\infty} z\left[\frac{2z^2 - z + 1}{(z-1)(z^2+1)}\right] = \lim_{z\to\infty} \frac{\cancel{z}\, z^{\cancel{2}}\left[2 - \frac{1}{z} + \frac{1}{z^2}\right]}{\cancel{z}(1-\frac{1}{z})\, \cancel{z^2}(1+\frac{1}{z^2})}$$

$$= \frac{2-0+0}{(1-0)(1+0)} = 2.$$

(*) becomes

$$Z(u_{n+2}) = z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$$

$$-(z^3-z^2+z-1)z \qquad -2(z^3-z^2+z-1)$$

$$= z^{2}\left[\frac{z(z^2+1)z + z(z-1)z - (z-1)(z^2+1)z - 2(z-1)(z^2+1)}{\cancel{z}(z-1)(z^2+1)}\right]$$

$$= z\left[\frac{\cancel{z^4} + \cancel{z^2} + z^3 - z^2 - \cancel{z^4} + z^3 - z^2 + z - 2z^3 + 2z^2 - 2z + 2}{(z-1)(z^2+1)}\right]$$

$$\boxed{Z(u_{n+2}) = \frac{z(z^2 - z + 2)}{(z-1)(z^2+1)}}$$

If $U(z) = Z(u_n) = \frac{z}{z-1} - \frac{z}{z^2+1}$, find $Z(u_{n+2})$.

(refer previous problem)

$u_0 = 1$, $u_1 = 0$. apply this

$$Z(u_{n+2}) = z^2\left[U(z) - u_0 - \frac{u_1}{z}\right].$$

Exercise: Find z-transform of the following

1. $3n + 5\sin\frac{n\pi}{4} + \cos\frac{n\pi}{2}$
2. $5n^2 + 4\cos\frac{n\pi}{4} + \sin\frac{n\pi}{2}$
3. $(3n-2)^3$, $\cos(2n-3)\theta$.
4. $n a^{-n}$, $n^2 a^{-n}$, $a^n \sinh n\theta$, $a^{-n}\cosh n\theta$.

$\sin(n+1)\theta$, $(n+1)^3$, $\frac{1}{(n+1)!}$, $\cos(n+3)\theta$, $\cos(n-3)\theta$.

Inverse Z-Transforms.

If $Z(u_n) = U(z)$, then $u_n = Z^{-1}(U(z))$. this u_n is called inverse z-transform of $U(z)$.

$$Z^{-1}(U(z)) = u_n.$$

since $Z(a^n) = \frac{z}{z-a}$, $\quad Z^{-1}\left(\frac{z}{z-a}\right) = a^n$

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}, \quad Z^{-1}\left(\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right) = \sin n\theta$$

Table of Inverse Z-Transforms.

$U(z) = Z(u_n)$	$u_n = Z^{-1}(U(z))$
$\frac{z}{z-a}$	a^n
$\frac{z}{z+a}$	$(-a)^n$
$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$	$\cos n\theta$
$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$	$\sin n\theta$
$\frac{z}{(z-1)^2}$	n
$\frac{z(z+1)}{(z-1)^3}$	n^2
$\frac{z(z^2+4z+1)}{(z-1)^4}$	n^3
$e^{1/z}$	$\frac{1}{n!}$
$\frac{z}{z-1}$	1

$\frac{z^2}{z^2+1}$	$\cos\frac{n\pi}{2}$
$\frac{z}{z^2+1}$	$\sin\frac{n\pi}{2}$

Find the inverse Z-transforms of the following:

(i) $\frac{1}{z-a}$ (ii) $\frac{1}{(z-a)^2}$ (iii) $\frac{z}{(z-a)^2}$ (iv) $\frac{z^2}{(z-a)^2}$.

$$\frac{1}{z-a} = \frac{1}{z(1-a/z)} = \frac{1}{z}\left(1-a/z\right)^{-1}$$

$(1-x)^{-1} = 1+x+x^2+x^3+\cdots,$

$$= \frac{1}{z}\left\{1+a/z+\left(\frac{a}{z}\right)^2+\left(\frac{a}{z}\right)^3+\cdots,\right\}$$

$$= \frac{1}{z}\sum_{n=1}^{\infty}\left(\frac{a}{z}\right)^{n-1} = \sum_{n=1}^{\infty}\left(a^{n-1}\right)z^{-n}$$

$\frac{1}{z}\,\frac{a^{n-1}}{z^{n-1}}$

$\frac{1}{z}\cdot a^{n-1}z\,z^{-n}$

$$\frac{1}{z-a} = Z\left(a^{n-1}\right), \text{ for } n\geq 1$$

$$\therefore \boxed{z^{-1}\left(\frac{1}{z-a}\right) = a^{n-1}}, \text{ for } n\geq 1.$$

$(1-x)^{-2} = 1+2x+3x^2+4x^3+\cdots,$

(ii) $$\frac{1}{(z-a)^2} = \frac{1}{z^2(1-a/z)^2} = \frac{1}{z^2}\left(1-a/z\right)^{-2}$$

$$= \frac{1}{z^2}\left\{1+2\left(\frac{a}{z}\right)+3\left(\frac{a}{z}\right)^2+4\left(\frac{a}{z}\right)^3+\cdots,\right\}$$

$$= \frac{1}{z^2}\sum_{n=1}^{\infty} n\left(\frac{a}{z}\right)^{n-1} = \sum_{n=1}^{\infty}\frac{na^{n-1}}{z^{n+1}}$$

$\frac{1}{z^2}\cdot\frac{1}{z^{n-1}}$

$\frac{1}{z^{n+1}}$

$$\frac{1}{(z-a)^2} = \sum_{n=2}^{\infty}\left\{(n-1)\,a^{n-2}\right\}z^{-n}, \text{ for } n\geq 2.$$

Therefore, $$z^{-1}\left(\frac{1}{(z-a)^2}\right) = (n-1)\,a^{n-2}, \text{ for } n\geq 2.$$

(iii) $$\frac{z}{(z-a)^2} = \frac{z}{z^2(1-a/z)^2} = \frac{1}{z(1-a/z)^2}$$

$$= \frac{1}{z}\left(1-a/z\right)^{-2}$$

$$= \frac{1}{z}\left\{1+2\left(a/z\right)+3\left(a/z\right)^2+4\left(a/z\right)^3+\cdots\right\}$$

$$= \frac{1}{z}\cdot\sum_{n=1}^{\infty} n\left(\frac{a}{z}\right)^{n-1} = \sum_{n=1}^{\infty}\left(na^{n-1}\right)z^{-n}$$

$$\frac{z}{(z-a)^2} = Z\left(na^{n-1}\right)$$

$$\therefore \boxed{z^{-1}\left(\frac{z}{(z-a)^2}\right) = na^{n-1}}$$

(iv) $\frac{z^2}{(z-a)^2} = \frac{\cancel{z^2}}{\cancel{z^2}(1-\frac{a}{z})^2} = \left(1-\frac{a}{z}\right)^{-2}$

$= 1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + 4\left(\frac{a}{z}\right)^3 + \cdots$

$= \sum_{n=0}^{\infty} (n+1)\left(\frac{a}{z}\right)^n$

$= \sum_{n=0}^{\infty} (n+1)\, a^n \, z^{-n}$

$\frac{z^2}{(z-a)^2} = Z\left[(n+1)a^n\right)$

$\therefore \quad \boxed{Z^{-1}\left(\frac{z^2}{(z-a)^2}\right) = (n+1)a^n}$

Find inverse Z-transform of the following

$\left| \log\left(1+\frac{1}{z}\right)^{-1} \right.$

$\log m^n = n \log m$

$= -1 \log(1+\frac{1}{z}$

(i) $\log \frac{z}{z+1}$

$\log \frac{z}{z+1} = \log \frac{\cancel{z}}{\cancel{z}(1+\frac{1}{z})} = -\log\left(1+\frac{1}{z}\right)$

$= -\left\{ \frac{1}{z} - \frac{(\frac{1}{z})^2}{2} + \frac{(\frac{1}{z})^3}{3} - + \cdots \right\}$

$= \frac{-1}{z} + \frac{(\frac{1}{z})^2}{2} - \frac{(\frac{1}{z})^3}{3} + \cdots$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n}$

$\log \frac{z}{z+1} = Z\left(\frac{(-1)^n}{n}\right)$, for $n \geq 1$

Therefore $Z^{-1}\left\{\log \frac{z}{z+1}\right\} = \frac{(-1)^n}{n}$, for $n \geq 1$.

(ii) $z \log \frac{z}{z+1}$

$z \log \frac{z}{z+1} = z\left[\frac{-1}{z} + \frac{(\frac{1}{z})^2}{2} - \frac{(\frac{1}{z})^3}{3} - \frac{(\frac{1}{z})^4}{4} + - \cdots\right]$

$= -1 + \frac{1}{2}\left(\frac{1}{z}\right) - \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} z^{-n}$

$z \log \frac{z}{z+1} = Z\left(\frac{(-1)^{n+1}}{n+1}\right)$

$$Z^{-1}\left(z\log\frac{z}{z+1}\right) = \frac{(-1)^{n+1}}{n+1}.$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(iii) $\frac{1}{z}e^{1/z}$

$$\frac{1}{z}e^{1/z} = \frac{1}{z}\left[1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots\right]$$

$$= \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \frac{1}{2!}\left(\frac{1}{z}\right)^3 + \frac{1}{3!}\left(\frac{1}{z}\right)^4 + \cdots$$

$$\frac{1}{z}e^{1/z} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}z^{-n} = Z\left(\frac{1}{(n-1)!}\right)$$

$$\boxed{Z^{-1}\left(\frac{1}{z}e^{1/z}\right) = \frac{1}{(n-1)!}}$$ for $n \geq 1$

(iv) $z(e^{1/z} - 1)$

$$ze^{1/z} = z\left[1 + \frac{1}{z} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \frac{(1/z)^4}{4!} + \cdots\right]$$

$$ze^{1/z} = z + 1 + \frac{1}{2!}\left(\frac{1}{z}\right) + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots$$

$$ze^{1/z} - z = 1 + \frac{1}{2!}\left(\frac{1}{z}\right) + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots$$

$$\cancel{= Z}\left(= \sum_{n=0}^{\infty}\frac{1}{(n+1)!}z^{-n}\right.$$

$$ze^{1/z} - z = Z\left(\frac{1}{(n+1)!}\right)$$

$$\boxed{Z^{-1}\left(ze^{1/z} - z\right) = \frac{1}{(n+1)!}}$$

Obtain the Inverse Z-transform of the following

① $\dfrac{z}{(z-1)(z+2)}$

$U(z) = \dfrac{z}{(z-1)(z+2)}$, $\dfrac{U(z)}{z} = \dfrac{1}{(z-1)(z+2)}$

$$\frac{U(z)}{z} = \frac{1}{(z-1)(z+2)} = \frac{A}{z-1} + \frac{B}{z+2} \quad -(*)$$

$$1 = A(z+2) + B(z-1)$$

Put $z=-2$	Put $z=1$
$1 = 0 + (-3)B$	$1 = 3A + 0$
$\boxed{B = -\frac{1}{3}}$	$\boxed{A = \frac{1}{3}}$

(*) becomes

multiplying by $\times z$

$$\frac{U(z)}{z} = \frac{1}{3}\cdot\frac{1}{z-1} - \frac{1}{3}\cdot\frac{1}{z+2}$$

$$U(z) = \frac{1}{3}\cdot\frac{z}{z-1} - \frac{1}{3}\frac{z}{z+2}$$

$$U(z) = \frac{1}{3}Z(1) - \frac{1}{3}Z\left((-2)^n\right)$$

$$U(z) = Z\left(\frac{1}{3} - \frac{1}{3}(-2)^n\right)$$

$$Z^{-1}(U(z)) = \frac{1}{3} - \frac{1}{3}(-2)^n$$

$$\boxed{u_n = \frac{1}{3} - \frac{1}{3}(-2)^n} = \frac{1}{3}\left[1 - (-2)^n\right].$$

② $\dfrac{5z}{(z-2)(3z-1)}$

Divide both side by z.

$$U(z) = \frac{5z}{(z-2)(3z-1)}$$

$$\frac{U(z)}{z} = \frac{5}{(z-2)(3z-1)} = \frac{A}{z-2} + \frac{B}{3z-1} \quad -(*)$$

$$\frac{5}{(2-z)(3z-1)} = \frac{A}{2-z} + \frac{B}{3z-1}$$

$$5 = A(3z-1) + B(2-z)$$

Put $z=\frac{1}{3}$

$$5 = 0 + B\left(2-\tfrac{1}{3}\right)$$

$$\cancel{5} = \frac{\cancel{5}}{3} B$$

$$\boxed{B=3}$$

Put $z=2$

$$\cancel{5} = A(\cancel{5}) + 0$$

$$\boxed{A=1}$$

(*) becomes

$$\frac{U(z)}{z} = \frac{1}{2-z} + \frac{3}{3z-1}$$ multiply b.s by z

$$U(z) = \frac{z}{2-z} + 3\frac{z}{3z-1}$$

$$= \frac{z}{-(z-2)} + \cancel{3}\frac{z}{\cancel{3}(z-\frac{1}{3})}$$

$$= -\frac{z}{z-2} + \frac{z}{z-\frac{1}{3}}$$

$$= -Z(2^n) + Z\left(\left(\tfrac{1}{3}\right)^n\right)$$

$$U(z) = Z\left(-2^n + \left(\tfrac{1}{3}\right)^n\right)$$

$$Z^{-1}(U(z)) = -2^n + \left(\tfrac{1}{3}\right)^n$$

$$\boxed{u_n = -2^n + \left(\tfrac{1}{3}\right)^n}$$

3 $\dfrac{z(2z+3)}{(z+2)(z-4)}$

$$U(z) = \frac{z(2z+3)}{(z+2)(z-4)}$$

b.s. $\div z$

$$\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)}$$

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$$\frac{U(z)}{z} = \frac{2z+3}{(z+2)(z-4)} = \frac{A}{z+2} + \frac{B}{z-4} \quad (*)$$

$$2z+3 = A(z-4) + B(z+2)$$

Put $z=4$

$$11 = 0 + 6B$$

$$B = 11/6$$

put $z=-2$

$$-1 = -6A + 0$$

$$1/6 = A.$$

(*) becomes

multiplying by z

$$\frac{U(z)}{z} = \frac{1}{6}\cdot\frac{1}{z+2} + \frac{11}{6}\cdot\frac{1}{z-4}$$

$$U(z) = \frac{1}{6}\cdot\frac{z}{z+2} + \frac{11}{6}\,\frac{z}{z-4}$$

$$U(z) = \frac{1}{6}\,Z((-2)^n) + \frac{11}{6}\,Z(4^n)$$

$$U(z) = Z\left(\frac{1}{6}(-2)^n + \frac{11}{6}4^n\right)$$

$$\boxed{u_n = Z^{-1}(U(z)) = \frac{1}{6}(-2)^n + \frac{11}{6}4^n}$$

(4) $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$, $\quad U(z) = \dfrac{\overset{z(4z-2)}{4z^2-2z}}{z^3-5z^2+8z-4}$ $\quad \div z$

$$\frac{U(z)}{z} = \frac{4z-2}{z^3-5z^2+8z-4}$$

z^3-5z^2+8z-4

1 is a root

$1-5+8-4 = 0.$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$1z^2-4z+4$

$z^3-5z^2+8z-4 = (z-1)(z^2-4z+4)$

$$\frac{U(z)}{z} = \frac{4z-2}{(z-1)(\underset{a^2-2ab+b^2=(a-b)^2}{z^2-4z+4})}$$

$$\frac{U(z)}{z} = \frac{4z-2}{(z-1)(z-2)^2}$$

$$\frac{U(z)}{z} = \frac{4z-2}{(z-1)(z-2)^2} = \frac{A}{z-1} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2} \longrightarrow (*)$$

LCM $(z-1)(z-2)^2$

$$4z-2 = A(z-2)^2 + B(z-1)(z-2) + C(z-1)$$

Put $z=2$

$6 = 0+0+C$

$\boxed{C=6}$

Put $z=1$

$2 = A+0+0$

$\boxed{A=2}$

Put $z=0$

$-2 = 4A+2B-C$

$-2 = 4.2+2B-6$

$2B = -2+6-8$

$2B = -4$

$\boxed{B=-2}$

(*) becomes

$$\frac{U(z)}{z} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2}$$

$$U(z) = 2\frac{z}{z-1} - 2\frac{z}{z-2} + 6\frac{z}{(z-2)^2}$$

$$U(z) = 2\frac{z}{z-1} - 2\frac{z}{z-2} + 3\frac{2z}{(z-2)^2}$$

WKT.

$$Z(na^n) = \frac{az}{(z-a)^2}.$$

$$Z(n2^n) = \frac{2z}{(z-2)^2}$$

$$U(z) = 2Z(1) - 2Z(2^n) + 3Z(n2^n)$$

$$U(z) = Z(2.1 - 2\,2^n + 3n2^n)$$

$$Z^{-1}(U(z)) = 2 - 2.2^n + 3n2^n$$

$$\boxed{u_n = 2 - 2^{n+1} + 3n2^n}$$

(5) $$\frac{3z^2+2z}{(5z-1)(5z+2)} = \frac{z(3z+2)}{(5z-1)(5z+2)}$$

$$\frac{U(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)}$$

$$\frac{U(z)}{z} = \frac{A}{5z-1} + \frac{B}{5z+2} \quad -(*)$$

$3z+2 = A(5z+2) + B(5z-1)$ 40.

Put $z = -\frac{2}{5}$

$$3\left(-\frac{2}{5}\right)+2 = A\left(5\cdot\left(-\frac{2}{5}\right)+2\right) + B\left(5\cdot\left(-\frac{2}{5}\right)-1\right)$$

$$-\frac{6}{5}+2 = B(-3)$$

$$\frac{-6+10}{5} = -3B$$

$$4/5 = -3B$$

$$B = -\frac{4}{15}$$

Put $z = \frac{1}{5}$

$$3\cdot\frac{1}{5}+2 = A\left(5\cdot\frac{1}{5}+2\right) + B\left(5\cdot\frac{1}{5}-1\right)$$

$$\frac{3+10}{5} = 3A$$

$$\frac{13}{5} = 3A$$

$$\boxed{\frac{13}{15} = A}$$

(*) becomes.

$$\frac{U(z)}{z} = \frac{13}{15}\cdot\frac{1}{5z-1} - \frac{4}{15}\cdot\frac{1}{5z+2}$$

$$U(z) = \frac{13}{15}\cdot\frac{z}{5z-1} - \frac{4}{15}\cdot\frac{z}{5z+2}$$

$$U(z) = \frac{13}{15}\,\frac{z}{5(z-1/5)} - \frac{4}{15}\cdot\frac{z}{5(z+2/5)}$$

$$U(z) = \frac{13}{75}\,\frac{z}{z-1/5} - \frac{4}{75}\cdot\frac{z}{z+2/5}$$

$$U(z) = \frac{13}{75}\,Z\left[\left(\frac{1}{5}\right)^n\right] - \frac{4}{75}\cdot Z\left(\left(-\frac{2}{5}\right)^n\right)$$

$$U(z) = Z\left(\frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left(-\frac{2}{5}\right)^n\right)$$

$$\boxed{u_n = Z^{-1}(U(z)) = \frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left(-\frac{2}{5}\right)^n}$$

$$Z(a^n) = \frac{z}{z-a}$$

$$Z((-a)^n) = \frac{z}{z+a}$$

Difference equations.

An equation u_n can be expressed interms of u_{n+1}, u_{n+2}, ... u_{n-2}, u_{n-1} etc. are called the difference equation.

The process of determining u_n, from such an equation are called the solving the difference equation.

Example: $u_{n+2} - 4u_{n+1} + u_n = 2^n$

$$u_{n+3} + 2u_{n+2} - 6u_{n+1} - 5u_n = 0.$$

$$u_{n+2} - 6u_n = 0.$$

1. Solve the difference equation using Z-transforms.

(i) $u_{n+2} + u_n = 0$, given $u_0 = 0$, $u_1 = 2$.

Soln: $u_{n+2} + u_n = 0.$

Take Z-transform on both side

$$Z(u_{n+2}) + Z(u_n) = Z(0)$$

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] + Z(u_n) = 0.$$

Shifting rule

$$Z(u_{n+2}) = z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$

$$Z(u_{n+1}) = z\left[U(z) - u_0\right]$$

$$Z(u_n) = U(z)$$

given $u_0 = 0$, $u_1 = 2$

$$z^2\left[U(z) - 0 - \frac{2}{z}\right] + U(z) = 0$$

$$z^2 U(z) - 2z + U(z) = 0$$

$$(z^2+1)\,U(z) - 2z = 0$$

$$U(z) = \frac{2z}{z^2+1}$$

$$U(z) = 2\,Z\left(\sin n\frac{\pi}{2}\right)$$

$$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$Z\left(\sin\frac{n\pi}{2}\right) = \frac{z}{z^2+1}$$

$$U(z) = 2\,Z\left(\sin \frac{n\pi}{2}\right)$$

$$Z^{-1}(U(z)) = 2 \sin \frac{n\pi}{2}$$

$$\boxed{u_n = 2 \sin \frac{n\pi}{2}}$$

(2) Solve the difference equation $u_{n+2} + u_n = 0$, for $u_0 = 1, u_1 = 2$. using Z-transforms.

$$u_{n+2} + u_n = 0.$$

Take z-transform on b.s.

$$Z(u_{n+2}) + Z(u_n) = Z(0)$$

shifting prop.

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] + Z(u_n) = 0.$$

given $u_0 = 1$, $u_1 = 2$

$$z^2\left[U(z) - 1 - \frac{2}{z}\right] + U(z) = 0$$

$$z^2 U(z) - z^2 - 2z + U(z) = 0$$

$$(z^2+1)\,U(z) - z^2 - 2z = 0$$

$$(z^2+1)\,U(z) = z^2 + 2z$$

$$(z^2+1)\,U(z) = z^2 + 2z$$

$$U(z) = \frac{z^2+2z}{z^2+1}$$

$$U(z) = \frac{z^2}{z^2+1} + 2\,\frac{z}{z^2+1}$$

(3) Solve the difference equation $u_{n+2} - 3u_{n+1} + 2u_n = 0$ given $u_0 = 0$, $u_1 = -1$

Sol:

$$Z(u_{n+2}) - 3Z(u_{n+1}) + 2Z(u_n) = Z(0)$$

$$z^2\left[U(z)-u_0-\frac{u_1}{z}\right]-3z\left[U(z)-u_0\right]+2Z(u_n)=Z(0)$$

Shifting rule   Shifting rule.

given $u_0=0,\ u_1=-1$

$$z^2\left[U(z)-0+\frac{1}{z}\right]-3z\left[U(z)-0\right]+2U(z)=0.$$

$$z^2U(z)+z-3zU(z)+2U(z)=0$$

$$(z^2-3z+2)U(z)=-z$$

Divide both side by z

$\div z$

$$U(z)=\frac{-z}{z^2-3z+2}$$

$$\frac{U(z)}{z}=\frac{-1}{z^2-3z+2}$$

2, −2, −1

$$\frac{U(z)}{z}=\frac{-1}{(z-1)(z-2)}$$

z^2-3z+2

$z^2-2z-z+2$

$z(z-2)-1(z-2)$

$(z-2)(z-1)$

$$\frac{U(z)}{z}=\frac{-1}{(z-1)(z-2)}=\frac{A}{z-1}+\frac{B}{z-2} \quad (*)$$

$$-1=A(z-2)+B(z-1)$$

Put $z=2$

$-1=0+B$

$\boxed{B=-1}$

Put $z=1$

$-1=-A$

$\boxed{A=1}$

$(*)$ becomes

multiply b s by $\times z$

$$\frac{U(z)}{z}=\frac{1}{z-1}-\frac{1}{z-2}$$

$$U(z)=\frac{z}{z-1}-\frac{z}{z-2}$$

$$U(z)=Z(1)-Z(2^n)$$

$$U(z)=Z(1-2^n)$$

$$Z^{-1}(U(z))=1-2^n$$

$$\boxed{u_n=1-2^n}$$

4. Solve the difference equation

$u_{n+2} + 4u_{n+1} + 4u_n = 7$ given $u_0 = 1$ and $u_1 = 2$

Take z-transform on b.s.

$$Z(u_{n+2}) + 4Z(u_{n+1}) + 4Z(u_n) = 7Z(1)$$

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] + 4z\left[U(z) - u_0\right] + 4U(z) = 7 \cdot \frac{z}{z-1}$$

given $u_0 = 1, u_1 = 2$

$$z^2\left[U(z) - 1 - \frac{2}{z}\right] + 4z\left[U(z) - 1\right] + 4U(z) = \frac{7z}{z-1}$$

$$z^2U(z) - z^2 - 2z + 4zU(z) - 4z + 4U(z) = \frac{7z}{z-1}$$

$$(z^2 + 4z + 4)U(z) - z^2 - 6z = \frac{7z}{z-1}$$

$$(z+2)^2U(z) = z^2 + 6z + \frac{7z}{z-1}$$

$$U(z) = \frac{z^2+6z}{(z+2)^2} + \frac{7z}{(z-1)(z+2)^2}$$

b.s by $\div z$

$$\cancel{U(z) = \frac{(z^2+6z)(z-1) + 7z}{(z+2)^2(z-1)}}$$

$$\frac{U(z)}{z} = \frac{z+6}{(z+2)^2} + \frac{7}{(z-1)(z+2)^2}$$

$$\frac{U(z)}{z} = \frac{(z+6)(z-1) + 7}{(z-1)(z+2)^2}$$

$$\frac{U(z)}{z} = \frac{z^2 + 6z - z - 6 + 7}{(z-1)(z+2)^2}$$

$$\frac{U(z)}{z} = \frac{z^2+5z+1}{(z-1)(z+2)^2}$$

$$\frac{U(z)}{z} = \frac{z^2+5z+1}{(z-1)(z+2)^2} = \frac{A}{z-1} + \frac{B}{z+2} + \frac{C}{(z+2)^2} \quad -(*)$$

$$z^2+5z+1 = A(z+2)^2 + B(z-1)(z+2) + C(z-1)$$

Put $z=1$

$1+5+1$

$7 = 9A + 0 + 0$

$$\boxed{A = 7/9}$$

Put $z=-2$

$4-10+1$

$-5 = 0 + 0 - 3C$

$$\boxed{5/3 = C}$$

Put $z=0$

$1 = 4A - 2B - C$

$1 = 4 \cdot 7/9 - 2B - 5/3$

$2B = \frac{28}{9} - \frac{5}{3} - 1$

$2B = \frac{28-15-9}{9} = \frac{4}{9}$

$$\boxed{B = \frac{2}{9}}$$

(*) becomes.

multiply both side by $\times z$

$$\frac{U(z)}{z} = \frac{7}{9}\cdot\frac{1}{z-1} + \frac{2}{9}\cdot\frac{1}{z+2} + \frac{5}{3}\cdot\frac{1}{(z+2)^2}$$

$$U(z) = \frac{7}{9}\cdot\frac{z}{z-1} + \frac{2}{9}\cdot\frac{z}{z+2} + \frac{5}{3}\cdot\frac{2z}{2(z+2)^2}$$

$$U(z) = \frac{7}{9}\,Z(1) + \frac{2}{9}\,Z((-2)^n) + \frac{5}{\cancel{6}}\,Z(n(-2)^n)$$

$Z(na^n) = \frac{az}{(z-a)^2}$

$Z(na^{-n}) = \frac{az}{(z+a)^2}$

$$\boxed{u_n = Z^{-1}(U(z)) = \frac{7}{9}\cdot 1 + \frac{2}{9}(-2)^n + \frac{5}{6}\,n(-2)^n}$$

5. Solve the difference equation $u_{n+2} - 3u_{n+1} + 2u_n = 1$. using z-transform.

$$u_{n+2} - 3u_{n+1} + 2u_n = 1$$

Taking z-transform on both side

$$Z(u_{n+2}) - 3Z(u_{n+1}) + 2Z(u_n) = Z(1)$$

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] - 3z\left[U(z) - u_0\right] + 2Z(u_n) = \frac{z}{z-1}$$

$$z^2U(z) - z^2u_0 - zu_1 - 3zU(z) + 3zu_0 + 2U(z) = \frac{z}{z-1}$$

$$(z^2-3z+2)U(z) = z^2u_0 + zu_1 - 3zu_0 + \frac{z}{z-1}$$

$$(z-2)(z-1)U(z) = z^2u_0 + zu_1 - 3zu_0 + \frac{z}{z-1}$$

$$U(z) = \frac{z^2u_0 + zu_1 - 3zu_0 + \frac{z}{z-1}}{(z-2)(z-1)}$$

$$U(z) = \frac{z\left(zu_0 + u_1 - 3u_0 + \frac{1}{z-1}\right)}{(z-2)(z-1)}$$

$$\frac{U(z)}{z} = \frac{zu_0 + (u_1 - 3u_0) + \frac{1}{z-1}}{(z-2)(z-1)}$$

$$\frac{U(z)}{z} = \frac{z(z-1)u_0 + (z-1)(u_1-3u_0) + 1}{(z-2)(z-1)^2}$$

$$= \frac{u_0z^2 - zu_0 + zu_1 - 3zu_0 - u_1 + 3u_0 + 1}{(z-1)^2(z-2)}$$

$$\frac{U(z)}{z} = \frac{u_0z^2 + (u_1-4u_0)z + (3u_0 - u_1 + 1)}{(z-1)^2(z-2)}$$

say. (*)

$$\frac{U(z)}{z} = \frac{u_0z^2 + (u_1-4u_0)z + (3u_0-u_1+1)}{(z-1)^2(z-2)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z-2}$$

Then $u_0z^2 + (u_1-4u_0)z + (3u_0-u_1+1) = A(z-1)(z-2) + B(z-2) + C(z-1)^2$

Taking $z=1$

$$u_0 + (u_1 - 4u_0) + 3u_0 - u_1 + 1 = 0 - B + 0$$

$$1 = -B \Rightarrow \boxed{B = -1}$$

put $z = 2$

$$4u_0 + (u_1 - 4u_0)2 + (3u_0 - u_1 + 1) = 0 + 0 + C.$$

$$4u_0 + 2u_1 - 8u_0 + 3u_0 - u_1 + 1 = C$$

$$\boxed{u_1 - u_0 + 1 = C}$$

put $z = 0.$

$$u_0 0^2 + (u_1 - 4u_0)0 + 3u_0 - u_1 + 1 = 2A - 2B + C$$

$$3u_0 - u_1 + 1 = 2A - 2B + C$$

$$2A = 2B - C + 3u_0 - u_1 + 1$$

$$2A = 2.(-1) - (u_1 - u_0 + 1) + 3u_0 - u_1 + 1$$

$$= -2 - u_1 + u_0 - \cancel{1} + 3u_0 - u_1 + \cancel{1}$$

$$2A = 4u_0 - 2u_1 - 2$$

$$\cancel{2}A = \cancel{2}(2u_0 - u_1 - 1)$$

$$\therefore \boxed{A = 2u_0 - u_1 - 1}$$

multiply both side by z

(*) becomes

$$\frac{U(z)}{z} = (2u_0 - u_1 - 1)\frac{1}{z-1} + \frac{-1}{(z-1)^2} + (u_1 - u_0 + 1)\frac{1}{z-2}$$

$$U(z) = (2u_0 - u_1 - 1)\frac{z}{z-1} - \frac{z}{(z-1)^2} + (1 - u_0 + u_1)\frac{z}{z-2}$$

$$U(z) = (2u_0 - u_1 - 1)\,Z(1) - Z(n) + (1 - u_0 + u_1)\,Z(2^n)$$

Take inverse Z-transform

$$Z^{-1}(U(z)) = (2u_0 - u_1 - 1).1 - n + (1 - u_0 + u_1)\,2^n$$

$$\boxed{u_n = 2u_0 - u_1 - 1 - n + (1 - u_0 + u_1)\,2^n}$$

Solve $u_{n+2} + 6u_{n+1} + 9u_n = 2^n$, given $u_0 = u_1 = 0$.

$$Z(u_{n+2}) + 6Z(u_{n+1}) + 9Z(u_n) = Z(2^n)$$

$$z^2\left[U(z) - u_0 - \frac{u_1}{z}\right] + 6z\left[U(z) - u_0\right] + 9U(z) = \frac{z}{z-2}$$

given $u_0 = 0, u_1 = 0$.

$$z^2[U(z) - 0 - 0] + 6z[U(z) - 0] + 9U(z) = \frac{z}{z-2}$$

$$(z^2 + 6z + 9)U(z) = \frac{z}{z-2}$$

$$U(z) = \frac{z}{(z-2)(z^2+6z+9)} = \frac{z}{(z-2)(z+3)^2}$$

$$U(z) = \frac{z}{(z-2)(z+3)^2} \quad \div z$$

$$\frac{U(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2} \quad \text{—(*)}$$

$$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$$

put $z = -3$

$1 = 0 + 0 - 5C$

$\boxed{C = -\frac{1}{5}}$

put $z = 2$

$1 = 25A + 0 + 0$

$\boxed{A = \frac{1}{25}}$

put $z = 0$.

$1 = 9A - 6B - 2C$

$1 = 9 \cdot \frac{1}{25} - 6B - 2\left(-\frac{1}{5}\right)$

$6B = \frac{9}{25} + \frac{2}{5} - 1$

$6B = \frac{9 + 10 - 25}{25} = \frac{-6}{25}$

$\boxed{B = -\frac{1}{25}}$

(*) becomes

multiply both sides by $\times z$

$$\frac{U(z)}{z} = \frac{1}{25} \cdot \frac{1}{z-2} - \frac{1}{25} \cdot \frac{1}{z+3} - \frac{1}{5} \cdot \frac{1}{(z+3)^2}$$

$$U(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \cdot \frac{z}{z+3} - \frac{1 \cdot 3}{5 \cdot 3} \frac{z}{(z+3)^2}$$

$$U(z) = \frac{1}{25} Z(2^n) - \frac{1}{25} Z((-3)^n) - \frac{1}{15} Z(n(-3)^n)$$

$$\boxed{u_n = Z^{-1}(U(z)) = \frac{1}{25} 2^n - \frac{1}{25}(-3)^n - \frac{1}{15} n(-3)^n}$$

$Z(na^n) = \frac{az}{(z-a)^2}$

$Z(n(-a)^n) =$