Z - transforms

Definition: The Z - transform of a sequence u_n defined for discrete values $n=0,1,2,\dots$ (and $u_n=0$ for $n<0$) is denoted by $Z(u_n)$ and is defined as

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n} = U(z) \quad \text{— (1)}$$

$Z(u_n) = Z_T(u_n$

where U is a function of z.

Z - transform exists only when the infinite series in (1) is convergent.

Properties: (1) Linear Property.

If a and b are constants and u_n and v_n are the n^{th} terms of the sequence $\{u_n\}$ and $\{v_n\}$, then

$$Z(au_n + bv_n) = aZ(u_n) \pm bZ(v_n) \quad \text{— (i)}$$

proof. By definition $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

$$Z(au_n + bv_n) = \sum_{n=0}^{\infty} (au_n + bv_n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a u_n z^{-n} + \sum_{n=0}^{\infty} b v_n z^{-n}$$

$$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n} \quad \text{By defn}$$

$$Z(au_n + bv_n) = aZ(u_n) + bZ(v_n)$$

(2) Damping rule.

If $Z(u_n) = U(z)$, then. for a constant $a \neq 0$.

(i) $Z(a^{-n}u_n) = U(az)$

(ii) $Z(a^n u_n) = U(z/a)$

Proof: By definition $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

(i) $Z(a^{-n}u_n) = \sum_{n=0}^{\infty} a^{-n} u_n z^{-n}$

$= \sum_{n=0}^{\infty} u_n a^{-n} z^{-n}$

$= \sum_{n=0}^{\infty} u_n (az)^{-n}$

$$\boxed{Z(a^{-n}u_n) = U(az)}$$

$U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

$U(az) = \sum_{n=0}^{\infty} u_n (az)^{-n}$

(ii) $Z(a^n u_n) \overset{\text{By definition}}{=} \sum_{n=0}^{\infty} a^n u_n z^{-n}$

$= \sum_{n=0}^{\infty} u_n a^n z^{-n}$

$= \sum_{n=0}^{\infty} u_n \frac{z^{-n}}{a^{-n}}$

$= \sum_{n=0}^{\infty} u_n \left(\frac{z}{a}\right)^{-n}$

$$\boxed{Z(a^n u_n) = U\left(\frac{z}{a}\right)}$$

$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

$Z(a^n u_n) = \sum_{n=0}^{\infty} (a^n u_n) z^{-n}$

$U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

$U\left(\frac{z}{a}\right) = \sum_{n=0}^{\infty} u_n \left(\frac{z}{a}\right)^{-n}$

Exponential damping rule:

(i) $Z(e^{-an}u_n) = U(e^{a}z)$

(ii) $Z(e^{an}u_n) = U(e^{-a}z) = U\left(\frac{z}{e^{a}}\right)$.

(Proof change a to e^{a} in damping rule).

3. Shifting Rules

If $k>0$, then (i) $Z(u_{n-k}) = z^{-k} Z(u_n)$ or $\frac{1}{z^k} Z(u_n)$

(ii) $Z(u_{n+k}) = z^{k}\left\{ Z(u_n) - \sum_{m=0}^{k-1} u_m z^{-m} \right\}$.

By definition $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

Proof: (i) For $n \geq k$,

(definition for $(n<0)$ $u_n = 0$, $n = -k, -k+1, \ldots$

$u_{-k} = 0$ $(k>0)$

$m = n - k$

$n = k$ $m = k - k = 0$

$n = \infty$ $m = \infty - k = \infty$

$m: 0 \to \infty$)

$$Z(u_{n-k}) = \sum_{n=k}^{\infty} u_{n-k} z^{-n}$$

put $m = n-k$

$$= \sum_{m=0}^{\infty} u_m z^{-(m+k)}$$

$$= \sum_{m=0}^{\infty} u_m z^{-m} \cdot z^{-k}$$

$$= z^{-k} \sum_{m=0}^{\infty} u_m z^{-m}$$

$$= z^{-k} \sum_{n=0}^{\infty} u_n z^{-n}$$

($U(z) = \sum_{m=0}^{\infty} u_m z^{-m}$

$U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$)

$$= z^{-k} Z(u_n) = z^{-k} U(z)$$

$$Z(u_{n-k}) = z^{-k} U(z) \quad \text{(or)} \quad Z(u_{n-k}) = \frac{1}{z} U(z)$$

$$Z(u_{n-1}) = z^{-1} U(z) = \frac{1}{z} Z(u_n)$$

$$Z(u_{n-2}) = z^{-2} U(z) = \frac{1}{z^2} Z(u_n)$$

$$Z(u_{n-3}) = z^{-3} U(z) = \frac{1}{z^3} Z(u_n) \quad \text{and so on.}$$

(ii) $$Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} z^{-n}$$

put $m = n+k$

$n = 0$, $m = 0 + k = k$ $\qquad m: k \to \infty$

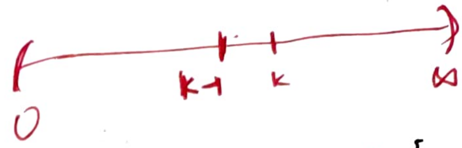
$n = \infty$, $m = \infty + k = \infty$

$$= \sum_{m=k}^{\infty} u_m z^{-(m-k)}$$

$$= \sum_{m=k}^{\infty} u_m z^{-m} z^{k}$$

$$Z(u_{n+k}) = z^k \sum_{m=k}^{\infty} u_m z^{-m}$$

$$= z^k \left\{ \sum_{m=0}^{\infty} u_m z^{-m} - \sum_{m=0}^{k-1} u_m z^{-m} \right\}$$



$$\left(\cancel{u_0} + \cancel{\frac{u_1}{z}} + \cancel{\frac{u_2}{z^2}} + \dots + \cancel{\frac{u_{k-1}}{z^{k-1}}} + \frac{u_k}{z^k} + \dots \infty\right) - \left(\cancel{u_0} + \cancel{\frac{u_1}{z}} + \cancel{\frac{u_2}{z^2}} + \dots + \cancel{\frac{u_{k-1}}{z^{k-1}}}\right)$$

$$= \frac{u_k}{z^k} + \frac{u_{k+1}}{z^{k+1}} + \dots \infty$$

$$= \sum_{m=k}^{\infty} u_m z^{-m}$$

$$Z(u_{n+k}) = z^k \left\{ \sum_{m=0}^{\infty} u_m z^{-m} - \sum_{m=0}^{k-1} u_m z^{-m} \right\}$$

$$\boxed{Z(u_{n+k}) = z^k \left[U(z) - \sum_{m=0}^{k-1} u_m z^{-m} \right]}$$

$$Z(u_{n+1}) = z\left[U(z) - u_0\right] \quad \text{(or)} \quad z\left[\underset{U(z)}{Z(u_n)} - u_0\right]$$

$$Z(u_{n+2}) = z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$

$$Z(u_{n+3}) = z^3\left[U(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right], \text{ and so on.}$$

5. Initial value property: If $Z(u_n) = U(z)$, then

$$\lim_{z\to\infty} U(z) = u_0.$$

Proof: By defn $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$

$$U(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \dots$$

$$U(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots,$$

$$\lim_{z\to\infty} U(z) = \lim_{z\to\infty}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots\right]$$

$$\lim_{z\to\infty} U(z) = \lim_{z\to\infty} u_0 + \lim_{z\to\infty}\frac{u_1}{z} + \lim_{z\to\infty}\frac{u_2}{z^2} + \lim_{z\to\infty}\frac{u_3}{z^3} + \dots,$$

$$= u_0 + 0 + 0 + 0 + \dots + 0 = u_0.$$

Note: Similarly, (i) $\lim_{z\to\infty} \{z[U(z)-u_0]\} = u_1$

(ii) $\lim_{z\to\infty} \left\{ z^2\left[U(z)-u_0-\frac{u_1}{z}\right]\right\} = u_2$

(iii) $\lim_{z\to\infty} \left\{ z^3\left[U(z)-u_0-\frac{u_1}{z}-\frac{u_2}{z^2}\right]\right\} = u_3$ and so on.

6. Final - value Property:

If $Z(u_n) = U(z)$, then $\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)U(z)$.

Proof: consider

$$Z(u_{n+1}-u_n) = Z(u_{n+1}) - Z(u_n)$$

$$= z\left[U(z)-u_0\right] - U(z)$$

(shifting prop)

$$= zU(z) - zu_0 - U(z)$$

$$Z(u_{n+1}) - Z(u_n) = (z-1)U(z) - zu_0$$

$$\sum_{n=0}^{\infty} u_{n+1} z^{-n} - \sum_{n=0}^{\infty} u_n z^{-n} = (z-1)U(z) - zu_0$$

$$(z-1)U(z) = zu_0 + \sum_{n=0}^{\infty} u_{n+1} z^{-n} - \sum_{n=0}^{\infty} u_n z^{-n}$$

$$= zu_0 + \sum_{n=0}^{\infty} (u_{n+1}-u_n) z^{-n}$$

$$= \cancel{zu_0 + (u_1 - u_0}$$

$$\lim_{z\to 1} (z-1)U(z) = \lim_{z\to 1} zu_0 + \lim_{z\to 1} \sum_{n=0}^{\infty} (u_{n+1}-u_n) z^{-n}.$$

$$= u_0 + \sum_{n=0}^{\infty} (u_{n+1}-u_n) \cdot 1$$

$$\lim_{z\to 1} ~~(z-1)U(z)~~ = u_0 \cancel{+ (u_1 + u_2 + u_3 + \cdots +} u_n$$

$$\lim_{z\to 1} (z-1)U(z) = u_0 + \lim_{N\to\infty} \sum_{n=0}^{N} (u_{n+1} - u_n)$$

$$= u_0 + \lim_{N\to\infty} \left(\cancel{u_1} - u_0 + \cancel{u_2} - \cancel{u_1} + u_3 - \cancel{u_2} + \cdots + u_{N+1} - u_N \right)$$

$$= u_0 + \lim_{N\to\infty} \left[-u_0 + u_{N+1} \right]$$

$$= \cancel{u_0} - \cancel{u_0} + \lim_{N\to\infty} u_{N+1}$$

$$\lim_{z\to 1} (z-1)U(z) = \lim_{n\to\infty} u_n$$

$$\boxed{\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)U(z)}$$

Convolution: ~~Theorem~~

Let u_n and v_n be the n^{th} terms of the sequence $\{u_n\}$ and $\{v_n\}$ respectively. Then the sum $u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \cdots + u_{n-1} v_1 + u_n v_0$ is called the convolution of the sequences $\{u_n\}$ and $\{v_n\}$ and is denoted by $u_n * v_n$.

Thus, by definition

$$u_n * v_n = u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \cdots + u_{n-1} v_1 + u_n v_0 = \sum_{k=0}^{n} u_k v_{n-k}$$

$$\boxed{u_n * v_n = v_n * u_n}$$

Convolution theorem: $Z(u_n * v_n) = Z(u_n) \cdot Z(v_n)$

Proof: $Z(u_n) \cdot Z(v_n) = \left(\sum_{n=0}^{\infty} u_n z^{-n}\right)\left(\sum_{n=0}^{\infty} v_n z^{-n}\right)$

$= \left(u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \dots\right) \cdot \left(v_0 + v_1 z^{-1} + v_2 z^{-2} + v_3 z^{-3} + \dots\right)$

$= u_0 v_0 + (u_0 v_1 + u_1 v_0) z^{-1} + (u_0 v_2 + u_1 v_1 + u_2 v_0) z^{-2}$
$+ (u_0 v_3 + u_1 v_2 + u_2 v_1 + u_3 v_0) z^{-3} + \dots,$

$= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \dots + u_{n-1} v_1 + u_n v_0) z^{-n}$

$= \sum_{n=0}^{\infty} (u_n * v_n) z^{-n}$

$= Z\left(\sum_{k=0}^{n} u_k v_{n-k}\right)$

$$\boxed{Z(u_n) \cdot Z(v_n) = Z(u_n * v_n)}$$

Z - Transforms of some common functions.

1. Transforms of a^n and $(-a)^n$.

By defn $Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$

$Z(a^n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$

~~$Z(a^n) = \frac{1}{1-az}$~~ $Z(a^n) = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$

$Z(a^n) = 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots,$ $= \frac{1}{1 - a/z}$

$= \left(1 - \frac{a}{z}\right)^{-1}$

$1 + x + x^2 + x^3 + \dots$
$= (1-x)^{-1}$
$= \frac{1}{1-x}$
by binomial

$$Z(a^n) = \frac{1}{\frac{z-a}{z}} = \frac{z}{z-a}$$

$1 + x + x^2 + x^3 + \dots,$

$= (1-x)^{-1} = \frac{1}{1-x}$

by binomial

here $x = \frac{a}{z}$.

$$\boxed{Z(a^n) = \frac{z}{z-a}}$$

$$Z((-1)^n a^n) = \boxed{Z(-a^{-n}) = \frac{z}{z+a}.}$$

(change a to $-a$)

$a = 1$

$$Z(1^n) = Z(1) = \frac{z}{z-1}$$

$$\boxed{Z(1) = \frac{z}{z-1}}$$

$$Z(2) = 2Z(1) = 2 \cdot \frac{z}{z-1}$$

$$Z(2^n) = \frac{z}{z-2}, \quad Z(3^n) = \frac{z}{z-3}$$

$$Z((-2)^{n}) = \frac{z}{z+2}, \quad Z((-3)^{n}) = \frac{z}{z+3}.$$

② Transform of e^{an} and e^{-an}

WKT $Z(a^n) = \frac{z}{z-a}$

$$Z(e^{an}) = Z((e^a)^n) = \frac{z}{z-e^a}$$

$$Z(e^{-an}) = Z((e^{-a})^n) = \frac{z}{z-e^{-a}}$$

~~$Z(a^{-n}) = \frac{z}{z+a}$~~

~~$Z(e^{-an}) = Z((e^{a})^{-n})$~~

~~$Z(e^{-an}) = \frac{z}{z+e^{a}}$~~

3. Transform of $\cos n\theta$

$$\cos n\theta = \frac{e^{in\theta} + e^{-in\theta}}{2}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$Z(\cos n\theta) = \frac{1}{2} Z\left(e^{in\theta} + e^{-in\theta}\right)$$

$$= \frac{1}{2}\left[Z\left(e^{in\theta}\right) + Z\left(e^{-in\theta}\right)\right]$$

$$= \frac{1}{2}\left[Z\left((e^{i\theta})^n\right) + Z\left((e^{-i\theta})^n\right)\right]$$

$$= \frac{1}{2}\left[\frac{z}{z - e^{i\theta}} + \frac{z}{z - e^{-i\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{1}{z - e^{i\theta}} + \frac{1}{z - e^{-i\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z - e^{-i\theta} + z - e^{i\theta}}{(z - e^{i\theta})(z - e^{-i\theta})}\right]$$

$$e^{i\theta} \cdot e^{-i\theta} = e^0 = 1$$

$$= \frac{z}{2}\left[\frac{2z - (e^{i\theta} + e^{-i\theta})}{z^2 - ze^{i\theta} - ze^{-i\theta} + e^0}\right]$$

$$= \frac{z}{2}\left[\frac{2z - 2\cos\theta}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1}\right]$$

$$= \frac{z}{\cancel{2}}\left[\frac{\cancel{2}(z - \cos\theta)}{z^2 - z\,2\cos\theta + 1}\right]$$

$$= \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$\boxed{Z(\cos n\theta) = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}}$$

$$Z\left(\cos \frac{n\pi}{2}\right) = \frac{z(z - \cancel{\cos\frac{\pi}{2}}^{\,0})}{z^2 - 2z\cancel{\cos\frac{\pi}{2}}_{\,0} + 1}$$

$$Z\left(\cos \frac{n\pi}{2}\right) = \frac{z^2}{z^2 + 1}$$

4. Transform of $\sin n\theta$

$$\sin n\theta = \frac{e^{in\theta} - e^{-in\theta}}{2i}$$

$$Z(\sin n\theta) = \frac{1}{2i}\left[Z(e^{in\theta}) - Z(e^{-in\theta})\right]$$

$$= \frac{1}{2i}\left[Z((e^{i\theta})^n) - Z((e^{-i\theta})^n)\right]$$

$$= \frac{1}{2i}\left[\frac{z}{z-e^{i\theta}} - \frac{z}{z-e^{-i\theta}}\right] = \frac{z}{2i}\left[\frac{1}{z-e^{i\theta}} - \frac{1}{z-e^{-i\theta}}\right]$$

$$= \frac{z}{2i}\left[\frac{\cancel{z}-e^{-i\theta}-\cancel{z}+e^{i\theta}}{(z-e^{i\theta})(z-e^{-i\theta})}\right]$$

$$= \frac{z}{2i}\left[\frac{e^{i\theta}+e^{-i\theta}}{z^2-ze^{-i\theta}-ze^{i\theta}+e^{0}}\right]$$

$$= \frac{z}{\cancel{2i}}\,\frac{\cancel{2i}\sin\theta}{z^2-z(e^{i\theta}+e^{-i\theta})+1}$$

$$\boxed{Z(\sin n\theta) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}}$$

$$Z\left(\sin \frac{n\pi}{2}\right) = \frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1}$$

$$\boxed{Z\left(\sin\frac{n\pi}{2}\right) = \frac{z}{z^2+1}}$$

5. Transform of $\cosh n\theta$.

$$\cosh\theta = \frac{e^{\theta}+e^{-\theta}}{2}$$

$$\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2}$$

$$Z(\cosh n\theta) = Z\left(\frac{e^{n\theta}+e^{-n\theta}}{2}\right)$$

$$Z(\cosh n\theta) = \frac{1}{2}\left[Z(e^{n\theta}) + Z(e^{-n\theta})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}}\right]$$

$$\boxed{Z(\cosh n\theta) = \frac{z(z-\cosh\theta)}{z^2-2z\cosh\theta+1}}$$

6. Transform of $\sinh n\theta$.

$$Z(\sinh n\theta) = \frac{1}{2} Z\left(e^{n\theta} - e^{-n\theta}\right)$$

$$= \frac{1}{2}\left[Z(e^{n\theta}) - Z(e^{-n\theta})\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}}\right]$$

(simplify)

$$\boxed{Z(\sinh n\theta) = \frac{z\sinh\theta}{z^2 - 2z\cosh\theta + 1}}$$

7. Transform of n^p, p is a positive integer

Let p be a positive integer.

$$Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$$

Differentiating on both side w.r.t z, we get

$$\frac{d}{dz} Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p-1}(-n)\, z^{-n-1}$$

$$= \sum_{n=0}^{\infty} (-n)\, n^{p-1} z^{-n} z^{-1}$$

$$= z^{-1} \sum_{n=0}^{\infty} -n^{p} z^{-n}$$

multiply both side by $\times -z$.

$$\frac{d}{dz} Z(n^{p-1}) = -\frac{1}{z} \sum_{n=0}^{\infty} n^{p} z^{-n}$$

$$-z\frac{d}{dz} Z(n^{p-1}) = \sum_{n=0}^{\infty} n^{p} z^{-n}$$

By definition.

$$-z\frac{d}{dz} Z(n^{p-1}) = Z(n^{p})$$

$$\boxed{Z(n^p) = -z\frac{d}{dz}Z(n^{p-1})} \quad — (*)$$

This is a recurrence relation from which $Z(n^p)$ can be computed if $Z(n^{p-1})$ is known before hand.

For $p=1$ in $(*)$ the relation gives

$$Z(n) = -z\frac{d}{dz}Z(1)$$

$$= -z\frac{d}{dz}\left(\frac{z}{z-1}\right)$$

$$= -z\left[\frac{(z-1).1 - z(1-0)}{(z-1)^2}\right]$$

$$\boxed{Z(n) = \frac{z}{(z-1)^2}}$$

For $p=2$,

$$Z(n^2) = -z\frac{d}{dz}Z(n)$$

$$= -z\frac{d}{dz}\left(\frac{z}{(z-1)^2}\right)$$

$$= -z\left[\frac{(z-1)^2.1 - z\,2(z-1)}{[(z-1)^2]^2}\right]$$

$$= -z\left[\frac{(z-1)^2 - 2z(z-1)}{(z-1)^4}\right]$$

$$= -z\,\cancel{(z-1)}\left[\frac{z-1-2z}{(z-1)^{\cancel{4}\,3}}\right] = -z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$\boxed{Z(n^2) = \frac{z(z+1)}{(z-1)^3}}$$

For $P=3$,

$$Z(n^3) = -z\frac{d}{dz}Z(n^2)$$

$$Z(n^3) = -z\frac{d}{dz}\left[\frac{z(z+1)}{(z-1)^3}\right] = -z\frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left[\frac{(z-1)^3\cdot(2z+1) - (z^2+z)\cdot 3(z-1)^2}{[(z-1)^3]^2}\right]$$

$$= -z(z-1)^{\cancel{2}}\left[\frac{(2z+1)(z-1) - 3(z^2+z)}{(z-1)^{\cancel{6}\,4}}\right]$$

$$= -\cancel{z(z-1)^2}\left[\right. = -z\left[\frac{2z^2 \cancel{-} 2z + z - 1 - 3z^2 - 3z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-4z-1}{(z-1)^4}\right]$$

$$\boxed{Z(n^3) = \frac{z(z^2+4z+1)}{(z-1)^4}}$$

For $p=4$,

$$Z(n^4) = -z\frac{d}{dz}\left\{Z(n^3)\right\}$$

$$= -z\frac{d}{dz}\left\{\frac{z(z^2+4z+1)}{(z-1)^4}\right\}$$

$z(z^2+4z+1)$

$\underline{z^3+4z^2+z}$

$\frac{d}{dz}$ "

$$= -z\left[\frac{(3z^2+8z+1)(z-1)^4 - \cancel{4z}\,(z^3+4z^2+z)\,4(z-1)^3}{(z-1)^8}\right]$$

$$= \cancel{(z-1)^3}(-z)\left[\frac{(3z^2+8z+1)(z-1) - 4z(z^2+4z+1)}{(z-1)^{\cancel{8}\,5}}\right]$$

$$\boxed{Z(n^4) = \frac{z(z^3+11z^2+11z+1)}{(z-1)^5}}$$

Similarly find $Z(n^5), Z(n^6)\ldots$ etc.

Table of Z-transform

u_n	$U(z) = Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$
1	$\frac{z}{z-1}$
$(-1)^n$	$\frac{z}{z+1}$
a^n	$\frac{z}{z-a}$
$(-a)^n$	$\frac{z}{z+a}$
e^{an}	$\frac{z}{z-e^{a}}$
$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$
$\sin n\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}$
$\cosh n\theta$	$\frac{z(z-\cosh\theta)}{z^2-2z\cosh\theta+1}$
$\sinh n\theta$	$\frac{z\sinh\theta}{z^2-2z\cosh\theta+1}$
n	$\frac{z}{(z-1)^2}$
n^2	$\frac{z(z+1)}{(z-1)^3}$
n^3	$\frac{z(z^2+4z+1)}{(z-1)^4}$
n^4	$\frac{z(z^3+11z^2+11z+1)}{(z-1)^5}$

Find Z-transform of ~~the~~ following

(i) $u_n = \frac{1}{n}$. $n > 0$.

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$\log(1-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$Z\left(\frac{1}{n}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}$$

$$x = \left(\frac{-1}{z}\right)$$

$$= z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \frac{z^{-4}}{4} + \cdots \quad \text{—— (i)}$$

$$= -\left(\frac{-1}{z}\right) + \frac{\left(\frac{-1}{z}\right)^2}{2} - \frac{\left(\frac{-1}{z}\right)^3}{3} + \frac{\left(\frac{-1}{z}\right)^4}{4} + \cdots$$

$$= -\left\{\left(\frac{-1}{z}\right) - \frac{\left(\frac{-1}{z}\right)^2}{2} + \frac{\left(\frac{-1}{z}\right)^3}{3} - \frac{\left(\frac{-1}{z}\right)^4}{4} + \cdots\right\}$$

$$\log m^n = n \log m$$
$$\log\left(1-\frac{1}{z}\right)^{-1} = -\log\left(1-\frac{1}{z}\right)$$

$$= -\log\left(1-\frac{1}{z}\right),$$

$$= \log\left(1-\frac{1}{z}\right)^{-1} = \log\left(\frac{z-1}{z}\right)^{-1}$$

$$= \log\frac{z}{z-1}$$

(ii) For $n \geq 0$.

$$u_n = \left(\frac{1}{n+1}\right)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$Z(u_n) = Z\left[\frac{1}{n+1}\right) = \sum_{n=1}^{\infty} \frac{1}{n+1} z^{-n}$$

$$= 1 + \frac{z^{-1}}{2} + \frac{z^{-2}}{3} + \frac{z^{-3}}{4} + \cdots$$

$$= z\left\{z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \frac{z^{-4}}{4} + \cdots\right\}$$

$$= z\log\left(\frac{z}{z-1}\right),$$ (see previous problem (i))

(iii) For $n > 0$

$$Z\left(\frac{1}{n(n+1)}\right) = Z\left(\frac{n+1-n}{n(n+1)}\right) = Z\left(\frac{1}{n} - \frac{1}{n+1}\right)$$
$$= Z\left(\frac{1}{n}\right) - Z\left(\frac{1}{n+1}\right)$$

$$Z\left(\frac{1}{n(n+1)}\right) = Z\left(\frac{1}{n}\right) - Z\left(\frac{1}{n+1}\right)$$

$$= \log\left(\frac{z}{z-1}\right) - z\log\left(\frac{z}{z-1}\right) = (1-z)\log\left(\frac{z}{z-1}\right).$$

(iv) For $n \geq 0$

$$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{z^{-1}}{1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots,$$

$$= 1 + \frac{(1/z)}{1} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots$$

$$= e^{1/z}$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots,$$

here $x = 1/z$

$$Z\left(\frac{1}{n!}\right) = e^{1/z}$$

Find Z-transform of the following

(i) $\left(\frac{1}{2}\right)^n + \left(-\frac{1}{3}\right)^n$

$$Z\left(\left(\tfrac{1}{2}\right)^n + \left(-\tfrac{1}{3}\right)^n\right) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}}$$

$$= \frac{2z}{2z-1} + \frac{3z}{3z+1}.$$

(ii) a^{2n+3}

$$Z(a^{2n+3}) = Z(a^{2n}\cdot a^3) = a^3 Z(a^{2n}) = a^3 Z((a^2)^n)$$

$$= a^3 \frac{z}{z-a^2}.$$

(iii) $(-a)^{3n-2}$

$$Z((-a)^{3n-2}) = Z\left((-a)^{3n}(-a)^{-2}\right) = (-a)^{-2} Z\left([(-a)^3]^n\right)$$

$$= \frac{1}{(-a)^2} \cdot \frac{z}{z+a^3}.$$

Find Z- transform of the following:

(i) $(2n-1)^2$ (ii) $(n+\frac{1}{3})^3$.

$(a-b)^2$

$(2n-1)^2 = 4n^2-4n+1$

$Z((2n-1)^2) = 4Z(n^2) - 4Z(n) + Z(1)$

$= 4.\frac{z(z+1)}{(z-1)^3} - 4.\frac{z}{(z-1)^2} + \frac{z}{z-1}$

$(a+b)^3 = a^3+b^3+3ab^2+3a^2b$

(ii) $Z((n+\frac{1}{3})^3) = Z(n^3+\frac{1}{9}+3n\frac{1}{9}+3n^2.\frac{1}{3})$

$= Z(n^3+\frac{1}{9}+\frac{n}{3}+n^2)$

$= Z(n^3)+\frac{1}{9}Z(1)+\frac{1}{3}Z(n)+Z(n^2)$

$= \frac{z(z^2+4z+1)}{(z-1)^4}+\frac{1}{9}\frac{z}{z-1}+\frac{1}{3}\frac{z}{(z-1)^2}+\frac{z(z+1)}{(z-1)^3}$.

Find Z - transform of the following

(i) $(\cos\theta + i\sin\theta)^n$ (ii) $\sin(4n+6)$

$(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$Z((\cos\theta+i\sin\theta)^n) = Z(\cos n\theta) + iZ(\sin n\theta)$

$= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$.

$\cos 4n.\sin 6$

(ii) $Z(\sin(4n+6)) = Z(\sin 4n.\cos 6 + \cos 4n\sin 6)$

$= \cos 6\, Z(\sin 4n) + \sin 6\, Z(\cos 4n)$

$= \cos 6 \frac{z\sin 4}{z^2-2z\cos 4+1} + \sin 6 \frac{z\sin 4}{z^2-2z\cos 4+1}$

Find z-transforms of the following

(i) $3n - 4\sin\frac{n\pi}{4} - 5a^2$

$Z(\sin n\theta) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$

$3Z(n) - 4Z\left(\sin\frac{n\pi}{4}\right) - 5a^2 Z(1)$

$= 3\frac{z}{(z-1)^2} - 4.\frac{z\sin\frac{\pi}{4}}{z^2 - 2z\cos\frac{\pi}{4} + 1} - 5a^2\frac{z}{z-1}$

$= \frac{3z}{(z-1)^2} - \frac{4z\frac{1}{\sqrt{2}}}{z^2 - 2z\frac{1}{\sqrt{2}} + 1} - \frac{5a^2 z}{z-1}.$

(ii) $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$\cos\frac{\pi}{2} = 0$

$Z\left(\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)\right) = Z\left(\cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}\right)$

$= \cos\frac{\pi}{4} Z\left(\cos\frac{n\pi}{2}\right) - \sin\frac{\pi}{4} Z\left(\sin\frac{n\pi}{2}\right)$

$= \frac{1}{\sqrt{2}}\left[\frac{z(z - \cancelto{0}{\cos\frac{\pi}{2}})}{z^2 - 2z\cancelto{0}{\cos\frac{\pi}{2}} + 1}\right] - \frac{1}{\sqrt{2}}\left[\frac{z\sin\frac{\pi}{2}}{z^2 - 2z\cancelto{0}{\cos\frac{\pi}{2}} + 1}\right]$

$= \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$

$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}$

(iii) $\cosh\left(\frac{n\pi}{2} + \theta\right)$

$Z\left(\cosh\left(\frac{n\pi}{2} + \theta\right)\right) = Z\left(\frac{e^{\frac{n\pi}{2}+\theta} + e^{-\left(\frac{n\pi}{2}+\theta\right)}}{2}\right)$

$= \frac{1}{2}\left[Z\left(e^{\theta + n\pi/2} + e^{-(\theta + n\pi)/2}\right)\right]$

$= \frac{1}{2}\left[Z\left(e^{\theta}.e^{n\pi/2} + e^{-\theta}.e^{-\frac{n\pi}{2}}\right)\right]$

$$= \frac{1}{2}\left[e^{\theta} \cdot \frac{z}{z - e^{\pi/2}} + e^{-\theta} \cdot \frac{z}{z - e^{-\pi/2}} \right].$$

$$= \frac{z}{2}\left[\frac{e^{\theta}(z - e^{-\pi/2}) + e^{-\theta}(z - e^{\pi/2})}{(z - e^{\pi/2})(z - e^{-\pi/2})} \right]$$

$$= \frac{z}{2} \; \frac{z(e^{\theta} + e^{-\theta}) - \{ e^{\theta - \pi/2} + e^{-(\theta - \pi/2)} \}}{z^2 - z(e^{\pi/2} + e^{-\pi/2}) + 1}$$

$$= \frac{z}{2} \cdot \frac{2z\cosh\theta - 2\cosh(\theta - \pi/2)}{z^2 - 2z\cosh(\pi/2) + 1} = \frac{z^2\cosh\theta - z\cosh(\theta - \pi/2)}{z^2 - 2z\cosh(\pi/2) + 1}.$$

Find the z-transforms of the following:

① na^n

WKT ~~$Z(\ldots)$~~ =

Damping rule.

$Z(na^n)$.

$$Z(u_n a^{n}) = U(z/a) \quad \text{and} \quad Z(u_n a^{-n}) = U(az)$$

$$Z(na^n) = U(z/a) \qquad Z(u_n) = U(z)$$

$$\text{Since } Z(n) = \frac{z}{(z-1)^2} = U(z)$$

$$Z(na^n) = \frac{z/a}{(z/a - 1)^2}$$

$$= \frac{z/a}{\frac{(z-a)^2}{a^2}} = \frac{az}{(z-a)^2}$$

$$\boxed{Z(na^n) = \frac{az}{(z-a)^2}}$$

② na^{n-1}

$$\text{Since } Z(n) = \frac{z}{(z-1)^2} = U(z)$$

$$Z(na^{n-1}) = Z\left(na^n \frac{1}{a}\right) = \frac{1}{a} Z(na^n)$$

$$= \frac{1}{a} \frac{az}{(z-a)^2} = \frac{z}{(z-a)^2}.$$

(vi) $n^2 a^n$

WKT $Z(u_n a^n) = U(z/a)$ Damping rule.

$Z(n^2 a^n) = U(z/a)$

$U(z) = Z(u_n)$

$U(z) = Z(n^2) = \dfrac{z(z+1)}{(z-1)^3}$

$U(z/a) = \dfrac{\frac{z}{a}\left(\frac{z}{a}+1\right)}{\left(\frac{z}{a}-1\right)^3}$

$= \dfrac{\frac{z(z+a)}{a^{\cancel{2}}}}{\frac{(z-a)^3}{a^{\cancel{3}\,1}}}$

$= \dfrac{az(z+a)}{(z-a)^3}$

$Z(n^2 a^n) = \dfrac{az(z+a)}{(z-a)^3}.$

(v) $a^{-n}\cos n\theta.$

WKT Damping rule.

$Z(u_n a^{-n}) = U(az)$

$Z(\cos n\theta\, a^{-n}) = U(az)$

$Z(u_n) = U(z)$

$Z(\cos n\theta) = U(z)$

$\dfrac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} = U(z)$

$Z(\cos n\theta\, a^{-n}) = \dfrac{az(az-\cos\theta)}{(az)^2-2az\cos\theta+1}$

(vi) $a^n \sin n\theta$

$Z(u_n a^n) = U(z/a)$

$Z(\sin n\theta\, a^n) = U(z/a)$

$U(z) = Z(u_n)$

$U(z) = Z(\sin n\theta)$

$= \dfrac{z\sin\theta}{z^2-2z\cos\theta+1}.$

$Z(\sin n\theta\, a^n) = \dfrac{\frac{z}{a}\sin\theta}{\left(\frac{z}{a}\right)^2-2\frac{z}{a}\cos\theta+1}$

$= \dfrac{\frac{z}{\cancel{a}}\sin\theta}{\frac{z^2-2az\cos\theta+a^2}{a^{\cancel{2}\,1}}} = \dfrac{az\sin\theta}{z^2-2az\cos\theta+a^2}.$

(Excmne)

⑥ $Z(a^n \cosh n\theta)$.

$$Z(\cosh n\theta\, a^n) = \frac{\frac{z}{a}\left(\frac{z}{a} - \cosh\theta\right)}{\left(\frac{z}{a}\right)^2 - 2\frac{z}{a}\cosh\theta + 1}$$

$$= \frac{z(z - a\cosh\theta)}{z^2 - 2az\cosh\theta + a^2}.$$

⑦ $\frac{a^n}{n!}$

$Z(u_n a^n) = U(z/a)$

$$Z\left(\frac{1}{n!}a^n\right) = U\left(z/a\right)$$

$$Z\left(\frac{1}{n!}a^n\right) = e^{1/(z/a)} = e^{a/z}.$$

$U(z) = Z(u_n)$

$U(z) = Z\left(\frac{1}{n!}\right)$

$U(z) = e^{1/z}$

⑧ $a^{-n} e^{bn}$

$Z(u_n a^{-n}) = U(az)$

$$Z(e^{bn} a^{-n}) = U(az)$$

$\cancel{Z(e^{bn}}$

$$Z(e^{bn} a^{-n}) = \frac{az}{az - e^b}.$$

$U(z) = Z(u_n)$

$U(z) = Z(e^{bn})$

$= Z\left((e^b)^n\right)$

$U(z) = \frac{z}{z - e^b}$

Find Z-transform of the following.

① $n e^{an}$ ⑪ $n^2 e^{n\theta}$

$Z(u_n a^n) = U(z/a)$ | $a = e^a$

$$Z(n e^{an}) = \frac{z/e^a}{\left(\frac{z}{e^a} - 1\right)^2}$$

$$= \frac{e^a z}{(z - e^a)^2}$$ //

$U(z) = Z(u_n)$

$U(z) = Z(n)$

$= \frac{z}{(z-1)^2}$

$Z(n^2) = \frac{z(z+1)}{(z-1)^3}$

$Z(u_n a^n) = U\left(\frac{z}{a}\right) = U\left(\frac{z}{e^\theta}\right) = U(z e^{-\theta})$,

⑪ $$Z(n^2 e^{n\theta}) = Z\left(n^2 (e^\theta)^n\right) = \frac{(e^{-\theta} z)(e^{-\theta} z + 1)}{(e^{-\theta} z - 1)^3}$$ //

Prove that $Z(nu_n) = -z \frac{d}{dz}\{Z(u_n)\}$ and hence find

$Z(n\cos n\theta)$

$Z(n\sin n\theta)$.

By definition

$$Z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$$

Differentiate on b.s. w.r.t z

$$\frac{d}{dz} Z(u_n) = \sum_{n=0}^{\infty} u_n (-n) z^{-n-1}$$

$$\frac{d}{dz} Z(u_n) = -\sum_{n=0}^{\infty} n u_n z^{-n} \cdot z^{-1}$$

$$\frac{d}{dz} Z(u_n) = -\frac{1}{z}\sum_{n=0}^{\infty} n u_n z^{-n}$$

multiply by b.sides by $(-z)$

$$-z\frac{d}{dz} Z(u_n) = \sum_{n=0}^{\infty} (n u_n) z^{-n}$$

By defn

$$-z\frac{d}{dz} Z(u_n) = Z(nu_n)$$

$$\boxed{Z(nu_n) = -z\frac{d}{dz}\{Z(u_n)\}}$$

$Z(n\cos n\theta)$

$$Z(n\cos n\theta) = -z\frac{d}{dz} Z(\cos n\theta)$$

$$= -z\frac{d}{dz}\left\{\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}\right\}$$

$$= -z\frac{d}{dz}\left\{\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right\}$$

$$Z(n\cos n\theta) = -z\left[\frac{(z^2-2z\cos\theta+1)(2z-\cos\theta)-(z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$Z(n\cos n\theta) = -z\left[\frac{\cancel{2z^3} - 4z^2\cos\theta + 2z - z^2\cos\theta + 2z\cos^2\theta - \cos\theta) - \cancel{2z^3} + 2z^2\cos\theta + 2z^2\cos\theta - 2z\cos^2\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$\boxed{Z(n\cos n\theta) = \frac{z(z^2\cos\theta - 2z + \cos\theta)}{(z^2-2z\cos\theta+1)^2}}$$

$Z(n u_n) = -z\frac{d}{dz}Z(u_n)$

$$Z(n\sin n\theta) = -z\frac{d}{dz}Z(\sin n\theta)$$

$$= -z\frac{d}{dz}\left\{\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right\}$$

$$= -z\left[\frac{(z^2-2z\cos\theta+1)\sin\theta - z\sin\theta(2z-2\cos\theta+0)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\left[\frac{z^2\sin\theta - 2z\cos\theta\sin\theta + \sin\theta - 2z^2\sin\theta + 2z\sin\theta\cos\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\frac{\left[-z^2\sin\theta + \sin\theta\right]}{(z^2-2z\cos\theta+1)^2}$$

$$\boxed{Z(n\sin n\theta) = \frac{z(z^2-1)\sin\theta}{(z^2-2z\cos\theta+1)^2}}$$

Find the Z-transforms of the following:

① $(n+1)^2$

Take $u_n = n^2$

$u_{n+1} = (n+1)^2$

$u_n = n^2$

$U(z) = Z(u_n) = Z(n^2) = \frac{z(z+1)}{(z-1)^3}$

By shifting property.

$Z(u_{n+1}) = z\left[U(z) - u_0\right]$

$U(z) = Z(u_n)$

$Z((n+1)^2) = z\left[\frac{z(z+1)}{(z-1)^3} - 0\right]$

$Z((n+1)^2) = \frac{z^2(z+1)}{(z-1)^3}$ //

(2) $(n-1)^3$.

Taking $u_n = n^3$

$u_{n-1} = (n-1)^3$

shifting prop.

$$Z(u_{n-1}) = \frac{1}{z} Z(u_n) = \frac{1}{z} Z(n^3)$$

$$Z((n-1)^3) = \frac{1}{\cancel{z}} \frac{\cancel{z}(z^2+4z+1)}{(z-1)^4} = \frac{z^2+4z+1}{(z-1)^4}.$$

(3) $(n+2)^2$.

Taking $u_n = n^2$

$u_{n+2} = (n+2)^2$

$$Z(u_{n+2}) = z^2\left\{U(z) - u_0 - \frac{u_1}{z}\right\}$$

$$Z((n+2)^2) = z^2\left\{\frac{z(z+1)}{(z-1)^3} - 0 - \frac{1}{z}\right\}$$

$u_n = n^2$,

$u_0 = 0^2 = 0$, $u_1 = 1^2 = 1$.

$U(z) = Z(u_n)$

$U(z) = Z(n^2)$

$= \frac{z(z+1)}{(z-1)^3}$

(4) $\frac{1}{n+2}$

taking $u_n = \frac{1}{n+1}$, $u_0 = \frac{1}{1+0} = 1$.

$$u_{n+1} = \frac{1}{(n+1)+1} = \frac{1}{n+2}$$

$$Z(u_{n+1}) = z\{U(z) - u_0\}$$

$$Z\left(\frac{1}{n+2}\right) = z\left\{z \log \frac{z}{z-1} - 1\right\}$$

$U(z) = Z(u_n) = Z\left(\frac{1}{n+1}\right)$

$= z \log \frac{z}{z-1}$

(5) $\frac{1}{n+3}$.

$u_n = \frac{1}{n+1}$

$u_{n+2} = \frac{1}{n+2+1} = \frac{1}{n+3}$

$u_0 = \frac{1}{0+1} = 1$

$u_1 = \frac{1}{1+1} = \frac{1}{2}$.

$$Z(u_{n+2}) = z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$
$$= z^2\left[z\log\frac{z}{z-1} - 1 - \frac{1}{2z}\right].$$

$U(z) = Z(u_n)$
$U(z) = Z\left(\frac{1}{n+1}\right)$
$= z\log\frac{z}{z-1}$

(5) $\frac{1}{(n+2)!}$

$u_n = \frac{1}{n!}$

$u_0 = \frac{1}{0!} = \frac{1}{1} = 1.$
$u_1 = \frac{1}{1!} = 1$

$u_{n+2} = \frac{1}{(n+2)!}$

$U(z) = Z(u_n)$
$U(z) = Z\left(\frac{1}{n!}\right)$
$= e^{1/z}.$

$$Z(u_{n+2}) = z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$
$$Z\left(\frac{1}{(n+2)!}\right) = z^2\left[e^{1/z} - 1 - \frac{1}{z}\right].$$

(6) $\cos(n+1)\theta.$

$u_n = \cos n\theta$, $u_0 = \cos n0 = \cos 0 = 1$

$u_{n+1} = \cos(n+1)\theta.$ (shifting rule)

$U(z) = Z(u_n)$
$U(z) = Z(\cos n\theta)$
$= \frac{z(z-\cos\theta)}{z^2 - 2z\cos\theta + 1}$

$$Z(u_{n+1}) = z\left[U(z) - u_0\right]$$
$$= z\left[\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - 1\right].$$

(7) $\cos(n-2)\theta.$

$u_n = \cos n\theta$

$u_{n-2} = \cos(n-2)\theta.$ $\to$ shifting prop.

$Z(u_n) = U(z)$
$U(z) = Z(\cos n\theta)$
$= \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

$$Z(u_{n-2}) = \frac{1}{z^2}Z(u_n) = \frac{1}{z^2}\cdot\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$
$$= \frac{1}{z}(z-\cos\theta)\Big/z^2-2z\cos\theta+1.$$