

Introduction to Data Science - 1MS041

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Recall from last time

- A random variable is a function from the sample space to a number (or vector).
- A distribution function is $F(x) = \mathbb{P}(X \leq x)$. If X is multivariate we call the Joint Distribution Function (JDF), for $X = (X_1, \dots, X_n)$ and $x = (x_1, \dots, x_n)$

$$F(x) = F(X \leq x) = \mathbb{P}(X_1 \leq x_1; \dots, X_n \leq x_n).$$

- A discrete random variable takes discrete values, i.e. $0, 1, 2, 3, \dots$. The probability mass function is defined as $f(x) = \mathbb{P}(X = x)$.
- A random variable is called continuous if the distribution function F can be written as

$$F(x) = \int_{-\infty}^x f(v) dv$$

for a piecewise continuous function f . f is called the density function.

Compare and contrast

Discrete	Continuous
$F(x) = \sum_{x_i \leq x} f(x_i)$	$F(x) = \int_{-\infty}^x f(v)dv$
$F(b) - F(a) = \sum_{a < x_i \leq b} f(x_i)$	$F(b) - F(a) = \int_a^b f(x)dx$
$\mathbb{P}(X = x) = f(x)$	$\mathbb{P}(X = x) = 0$
$\sum_x f(x) = 1$	$\int_{-\infty}^{\infty} f(x)dx = 1.$

Joint Distribution Function

Definition (JDF)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability triple and let X be a \mathbb{R}^m valued RV. Then the **joint distribution function (JDF)** or **joint cumulative distribution function (JCDF)**, $F_X(x) : \mathbb{R}^m \rightarrow [0, 1]$ is defined as

$$\begin{aligned} F_X(x) &= \mathbb{P}(\cap_{i=1}^m (X_i \leq x_i)) = \mathbb{P}(X_1 \leq x_1, \dots, X_m \leq x_m) \\ &= \mathbb{P}(\{\omega : X_1(\omega) \leq x_1, \dots, X_m(\omega) \leq x_m\}), \end{aligned}$$

where $X = (X_1, \dots, X_m)$ and each $X_i \in \mathbb{R}$, and $x = (x_1, \dots, x_m) \in \mathbb{R}^m$.

See example in notebook.

Marginal

Consider a JDF of two random variables X, Y , $F_{X,Y}$. The marginal distribution for X is defined as

$$F_X(x) := F_{X,Y}((x, \infty)) = \mathbb{P}(X \leq x, Y \leq \infty) = \mathbb{P}(X \leq x).$$

Simply put

The marginal distribution for X is what we get when ignoring the value of Y .

Recall

We say that two events A and B are **independent** if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

or equivalently

$$\mathbb{P}(A \mid B) = \mathbb{P}(A).$$

Independence

Definition (Independence of Two RVs)

Consider an \mathbb{R}^2 -valued RV $X := (X_1, X_2)$. Then the \mathbb{R} -valued RVs X_1 and X_2 are said to be independent or independently distributed if and only if

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \mathbb{P}(X_1 \leq x_1)\mathbb{P}(X_2 \leq x_2)$$

or equivalently,

$$F_{X_1, X_2}(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2),$$

for any pair of real numbers $(x_1, x_2) \in \mathbb{R}^2$.

Independence

Consider two random variables X_1, X_2 and let $X = (X_1, X_2)$ be a random variable.

Discrete	Continuous
Joint Distribution Function	Joint Distribution Function
Joint Probability Mass Function	Joint Probability Density Function
$F_{X_1, X_2}(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)$	$F_{X_1, X_2}(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)$
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- We say that the sequence is **identically distributed** if all the marginal distributions are the same, i.e. $F_{X_i} = F_{X_j}$ for all pairs.
- We say that the sequence is **independent and identically distributed** i.i.d. (or IID) if they are both independent and identically distributed.

Conditional

Definition

Let (X, Y) be a \mathbb{R}^2 valued random variable, and let $A \subset \mathbb{R}$ be a Borel ("think interval") set such that $\mathbb{P}(Y \in A) > 0$ then define the conditional distribution function of X given that $Y \in A$ as

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If Y is a discrete random variable and $f_Y(y) > 0$ the above definition is well defined for $A = \{y\}$ and we can write

$$F_{X|Y}(x | y) := \frac{\mathbb{P}(X \leq x, Y = y)}{\mathbb{P}(Y = y)}.$$

Conditional

Definition (Conditional PDF or PMF)

Let (X, Y) be a \mathbb{R}^2 valued RV. Then the **conditional probability mass / density function** is defined as

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Lemma

Let (X, Y) be a \mathbb{R}^2 valued RV. Then

$$f_{X|Y}(x | y)f_Y(y) = f_{X,Y}(x, y)$$

where the left hand side is interpreted as 0 if $f_Y(y) = 0$.

Functions of RVs

Definition (Expectation of a function of a RV)

The **Expectation** of a function $g(X)$ of a random variable X is defined as:

$$\mathbb{E}(g(X)) := \int g(x)dF(x) = \begin{cases} \sum g(x)f(x) & \text{discrete} \\ \int_{-\infty}^{\infty} g(x)f(x)dx & \text{continuous} \end{cases}$$

provided $\mathbb{E}(g(X))$ exists, i.e., $\int |g(x)|dF(x) < \infty$.

Moments

The simplest form of functions of interest is powers. The common statistical quantities that we often look at is the so called central moments

$$p\text{:th central moment: } \mathbb{E}[(X - \mathbb{E}[X])^p]$$

where $p = 1, 2, 3, \dots$. There is also the p :th standardized moment

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- $p = 2$ central moment: Variance (Measures spread)
- $p = 3$ standardized moment: Skewness (Measures lopsidedness)
- $p = 4$ standardized moment: Kurtosis (Measure heavy tailedness)

Sample versions

For each of the moments we can use data to try to estimate it, let's consider the sample variance of X_1, \dots, X_n an i.i.d. sequence of random variables

$$\frac{1}{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

L^p , not all random variables are created equal

- Not all random variables have a finite variance. Ex. the Pareto distribution
- Not all random variables have a finite expectation. Ex. The Cauchy distribution.
- If the p :th moment of a random variable X exists we will say that $X \in L^p(\mathbb{P})$. (We will mostly be working with $p = 1$ or $p = 2$).

Properties of the expectation

1. If $X \in L^1(\mathbb{P})$ is an \mathbb{R} valued RV and $\alpha \in \mathbb{R}$, then

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3. If $X, Y \in L^2(\mathbb{P})$ are independent \mathbb{R} valued RV, then

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3. If $X, Y \in L^2(\mathbb{P})$ are independent \mathbb{R} valued RV, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

4. $A \subset \mathbb{R}$

$$\mathbb{E}[\mathbf{1}_A(X)] = \mathbb{P}(X \in A)$$

Conditional expectation

We can also construct conditional expectations, like

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But it also allows for the following very useful property

Theorem (The tower property)

Let (X, Y) be a \mathbb{R}^2 valued RV. Then

$$\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X].$$

We introduced a new notation, namely $\mathbb{E}[X \mid Y]$, what is this?
Denote $g(y) = \mathbb{E}[X \mid Y = y]$, then define

$$\mathbb{E}[X \mid Y] := g(Y).$$

Note

By our definition, $\mathbb{E}[X \mid Y]$ is now another random variable. I.e. we have averaged out everything w.r.t. X but Y is still not averaged over.

Transformations

Consider a random variable X with distribution function F . Also consider another function $g : \mathbb{R} \rightarrow \mathbb{R}$, what is the distribution function of $g(X)$?

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- For any set A , $g^{[-1]}(A^c) = (g^{[-1]}(A))^c$

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- $g^{[-1]}(\mathbb{Y}) = \mathbb{X}$
- For any set A , $g^{[-1]}(A^c) = (g^{[-1]}(A))^c$
- For any collection of sets $\{A_1, A_2, \dots\}$,

$$g^{[-1]}(A_1 \cup A_2 \cup \dots) = g^{[-1]}(A_1) \cup g^{[-1]}(A_2) \cup \dots \quad .$$

Consequently,

$$\mathbb{P}_g(A) = P(g(X) \in A) = P(X \in g^{[-1]}(A)) \quad (1)$$

For a discrete random variable X with probability mass function f_X we can obtain the probability mass function f_Y of $Y = g(X)$ as follows:

$$\begin{aligned} f_Y(y) &= \mathbb{P}(Y = y) = \mathbb{P}(Y \in \{y\}) \\ &= P(g(X) \in \{y\}) = P\left(X \in g^{[-1]}(\{y\})\right) \\ &= P\left(X \in g^{[-1]}(y)\right) = \sum_{x \in g^{[-1]}(y)} f_X(x) = \sum_{x \in \{x: g(x)=y\}} f_X(x) . \end{aligned}$$

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This gives the formula:

$$f_Y(y) = \mathbb{P}(Y = y) = \sum_{x \in g^{[-1]}(y)} f_X(x) = \sum_{x \in \{x: g(x)=y\}} f_X(x) . \quad (2)$$

Transformations of continuous random variables

In the continuous context, we have to really care about a function not being 1 – 1 for instance. For more information, see the lecture notes.