PRINCIPLES OF POINT ESTIMATION

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MATHEMATICS FOR COMPUTER SCIENCE ENGINEERS

MSE for Bernoulli, Binomial and Poisson Distribution:

Computer Science and Engineering

Topics to be covered...

- ✓MSE for Bernoulli Distribution.
- ✓MSE for Binomial distribution.
- ✓MSE for Poisson distribution.

MSE for Bernoulli Distribution

Derivation: MSE for Bernoulli with one variable

Define the problem

We have a random variable X with:

$$X \sim \text{Bernoulli}(p).$$

That means:

$$P(X = 1) = p, \quad P(X = 0) = 1 - p.$$

We want to estimate the unknown parameter p based on **only this single observation**.

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MSE for Bernoulli Distribution

$$\hat{p} = X.$$

$$\text{Bias}(\hat{p}) = E[\hat{p}] - p.$$

$$E[\hat{p}] = E[X].$$

For a Bernoulli random variable:

$$E[X] = p.$$

So:

$$\text{Bias}(\hat{p}) = p - p = 0.$$

The estimator $\hat{p} = X$ is unbiased.

MSE for Bernoulli Distribution

$$\mathrm{Var}(\hat{p}) = \mathrm{Var}(X).$$

For a Bernoulli random variable:

$$\mathrm{Var}(X) = p(1-p).$$

The Mean Squared Error is defined by:

$$\mathrm{MSE}(\hat{p}) = E\left[(\hat{p}-p)^2\right].$$

There is an important identity:

$$\mathrm{MSE}(\hat{p}) = \mathrm{Var}(\hat{p}) + \left(\mathrm{Bias}(\hat{p})\right)^2.$$

Since Bias=0:

$$\mathrm{MSE}(\hat{p}) = \mathrm{Var}(\hat{p}) = p(1-p).$$

Problem

Let $X \sim \text{Bin}(n, p)$ where p is unknown. Find the MSE of $\hat{p} = X/n$

Solution: As we learnt in the Sample Proportion of Binomial Distribution,

Expectation (to get the bias).

$E[X] = np$ for a $\text{Bin}(n, p)$ variable, so

$$E[\hat{p}] = E\left[\frac{X}{n}\right] = \frac{E[X]}{n} = \frac{np}{n} = p.$$

Hence the bias is

Bias(p^)=E[p^]−p=p−p=0,

so $\hat{p}$ is unbiased

Variance.

$\text{Var}(X) = np(1-p)$ for $X \sim \text{Bin}(n, p)$. Thus

$$\text{Var}(\hat{p}) = \text{Var}\left(\frac{X}{n}\right) = \frac{\text{Var}(X)}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}.$$

Problem

$$MSE_{\hat{\theta}} = (Bias\,of\ \hat{\theta})^2 + Var(\hat{\theta}) = 0 + p(1-p)/n$$

$$MSE_{\hat{\theta}} = p(1-p)/n$$

Note: When bias is 0. MSE will be equal to variance.

Problem

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Determine the MSE of the estimator $\hat{\mu} = \bar{X}$ of the parameter μ of the Poisson(μ) distribution.

Problem

$$X_i \sim Poi(\mu)$$

$$E(x_i) = \mu, \quad Var(x_i) = \mu$$

$$MSE(\hat{\mu}) = Var(\hat{\mu}) + Bias^2(\hat{\mu})$$

$$Bias(\hat{\mu}) = E(\hat{\mu}) - \mu = E(\bar{X}) - \mu = E\left(\frac{1}{n}\sum x_i\right) - \mu$$

$$= \frac{1}{n}\sum E(x_i) - \mu = \frac{1}{n} \cdot n\mu - \mu = \mu - \mu = 0$$

$$Var(\hat{\mu}) = Var(\bar{X}) = Var\left(\frac{1}{n}\sum x_i\right) = \frac{1}{n^2}\sum Var(x_i)$$

$$= \frac{1}{n^2} \cdot n \cdot \mu = \frac{\mu}{n}$$

Problem

$$MSE(\hat{\mu}) = Var(\hat{\mu}) + Bias^2(\hat{\mu})$$

$$MSE(\hat{\mu}) = \frac{\mu}{n} + o^2$$

$$MSE(\hat{\mu}) = \frac{\mu}{n}$$

Problem

Let $X \sim \text{Binomial}(n = 50, p = 0.4)$.

Define estimator:

$$T = 0.9\hat{p} + 0.05.$$

Find MSE of T.

$$\hat{p} = X/50, \quad E[\hat{p}] = p = 0.4.$$

$$E[T] = 0.9 \times 0.4 + 0.05 = 0.36 + 0.05 = 0.41.$$

True parameter being estimated is $p = 0.4$.

$$\text{Bias} = 0.41 - 0.4 = +0.01.$$

$$\text{Var}(\hat{p}) = \frac{p(1-p)}{n} = \frac{0.4 \times 0.6}{50} = 0.24/50 = 0.0048.$$

$$\text{Var}(T) = (0.9)^2 \times 0.0048 = 0.81 \times 0.0048 = 0.003888.$$

$$\text{MSE} = 0.003888 + (0.01)^2 = 0.003888 + 0.0001 = 0.003988.$$

Problem

$X_1, \ldots, X_{30}$ from Poisson with mean $\lambda = 4$.

Estimator:

$$T = \bar{X}^2.$$

Find $E[T]$ and bias.

$\bar{X}$ is unbiased: $E[\bar{X}] = 4$.

$$E[\bar{X}^2] = \mathrm{Var}(\bar{X}) + (E[\bar{X}])^2.$$

$$\mathrm{Var}(\bar{X}) = \frac{\lambda}{n} = 4/30 = 0.1333.$$

$$E[\bar{X}^2] = 0.1333 + 16 = 16.1333.$$

True value we estimate is $\lambda^2 = 16$.

$$\mathrm{Bias} = 16.1333 - 16 = 0.1333.$$

Problem

Let $X \sim \text{Bernoulli}(p = 0.4)$.

Define estimator:

$$T = 0.5X + 0.3.$$

Find MSE of T.

Solution:

$$E[T] = 0.5E[X] + 0.3 = 0.5 \times 0.4 + 0.3 = 0.2 + 0.3 = 0.5.$$

True value $p = 0.4$:

$$\text{Bias} = 0.5 - 0.4 = 0.1.$$

$$\text{Var}(T) = (0.5)^2\text{Var}(X) = 0.25 \times p(1-p).$$

$$p(1-p) = 0.4 \times 0.6 = 0.24.$$

$$\text{Var}(T) = 0.25 \times 0.24 = 0.06.$$

$$\text{MSE} = 0.06 + (0.1)^2 = 0.06 + 0.01 = 0.07.$$

Problem: Do it itself.

$X \sim \text{Bernoulli}(p = 0.6)$.

Define estimator:

$$T = 1 - X.$$

Compute MSE of T.

$X \sim \text{Binomial}(n = 100, p = 0.3)$.

Estimator:

$$T = 0.85\hat{p} + 0.05.$$

Compute MSE.

$X_1, \ldots, X_{10}$ from Poisson with mean $\lambda = 3$.

Define estimator:

$$\tilde{\lambda} = \bar{X} + 0.5.$$

Find MSE of $\tilde{\lambda}$.

THANK YOU

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