UNIT – I

RANDOM VARIABLES

Probability and Random Variables: Basic concepts of Probability

i) Random experiment :-

All possible outcomes of an experiment may be known in advance, the outcome of a particular performance of the experiment cannot be predicted, such an experiment is called a random experiment.

Example :-

Let a coin be tossed. Nobody knows what will exactly going to occur, but it is certain that either head or tail will occur.

ii) Sample space :

The collection of all possible outcomes or sample points of a random experiment is called a sample space. It is denoted by S.

Example :-

When a die is thrown, there are six sample points corresponding to six faces. The sample space may be represented by

$$S = \{1, 2, 3, 4, 5, 6\} \text{ and } n(S) = 6$$

iii) Mutually exclusive events (Disjoint or Incompatible events)

Two events A and B in a sample space are said to be mutually exclusive events if they cannot occur together, i.e., if one occur, the other cannot.

In otherwords, mutually exclusive events means that the happening of an event excludes the possibility of happening of the other event in the same trial.

NOTE:- If A and B are mutually exclusive then $A \cap B = \phi$

Example:-

When we toss a coin, either head or tail can be up, but both cannot be up at a time.

NOTE:- Mutually exclusive events are applicable for a single trial only.

iv) Equally likely events:-

Two events are said to be equally likely if either of them cannot be expected to occur in preference to the other

Example:-

Cast a uniform die, then all the six sample points are equally likely events, since any of them cannot be expected to turn up in preference to the other.

v) Favourable events:-

The number of outcomes favourable to an event in an experiment is the number of outcomes which entail the happening of the event.

Example:-

In tossing two coins, the cases favourable to the event of getting a head are HH, HT, TH.

vi) Exhaustive events:

- outcomes are said to be exhaustive when they include all possible outcomes

Example :-

In drawing 2 cards from a pack of 52 cards, the exhaus-tive number of cases is $52 C_2$

vii) Independence events

Two or more events are considered to be independent if the occurence or non-occurence of an event does not affect the occurence or non-occurence of the other.

Example :-

If ~~a~~ coin is thrown twice, the result of the second throw is in no way affected by the result of the first throw.

Mathematical definition of Probability :-

Let S be the sample space (the set of all possible outcomes which are assumed equally likely) and A be an event (a sub-set of S consisting of possible outcomes) associated with a random experiment. Let n(S) and n(A) be the number of elements of S and A.

Then the probability of event A occuring, denoted as P(A), is defined by

$$P(A) = \frac{n(A)}{n(S)} = \frac{\text{Number of cases favourable to A}}{\text{Exhaustive number of cases in S.}}$$

(or)

$$P(A) = \frac{\text{The number of ways of getting favourable events}}{\text{... in the sample space}}$$

Example :-

The probability of getting an event number in the die tossing experiment is $\frac{1}{2}$

$\because S = \{1, 2, 3, 4, 5, 6\}$; $E = \{2, 4, 6\}$; $n(S) = 6$; $n(E) = 3$

Axiomatic definition of Probability :-

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, denoted by P(A), is defined as a real number satisfying the following axioms.

i) $0 \leq P(A) \leq 1$

ii) $P(S) = 1$

iii) If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

iv) If $A_1, A_2, \dots A_n, \dots$ are a set of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_n \dots) = P(A_1) + P(A_2) + P(A_3) + \dots + P(A_n) + \dots$$

Standard Results on Probability :-

1. The probability of an impossible event is zero. i.e, $P(\phi) = 0$
2. The probability of the complement event $\bar{A}$ of A is $P(\bar{A}) = 1 - P(A)$
3. If A and B which are not disjoint events, then

i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

ii) $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

4: Two events A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

5. If A and B are dependent events in a sample space S,

then $P(A \cap B) = P(A) \cdot P(B/A)$, $P(A) \neq 0$

$= P(B) \cdot P(A/B)$, $P(B) \neq 0$

where $P(B/A)$ represents the conditional probability of B given A.

6. Conditional probability of A given B is defined by

$$P(A/B) = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) \neq 0$$

and

conditional probability of B given A is defined by

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \quad \text{if } P(A) \neq 0$$

1. From a bag containing 10 black and 12 white balls, a ball is drawn at random. What is the probability that it is black?

509, 52, 111

Let $A \rightarrow$ event selecting a black ball.

Total no of balls $\longrightarrow 10(B) + 12(W) = 22$ balls.

Total number of possible ways of choosing one ball

$$= 22C_1 = 22$$

Out of 10 black balls, the number of ways of choosing one black ball is $= 10C_1 = 10$

$\therefore$ Probability of getting a black ball $= \frac{10C_1}{22C_1} = \frac{10}{22} = 0.4545$

2. If atleast one child in a family of three children is a boy, what is the probability that all three are boys?

Let $A \rightarrow$ event selecting all three are boys.

The sample space $S = \{BBB, BBG, BGB, GBB, GGB, GBG, BGG\}$

$n(S) = 7$

$\therefore P(\text{all three are boys}) = \frac{7C_1}{7} = \frac{7}{7} = \frac{1}{7}$

3. Four cards are to dealt successively, at random and without replacement, from an ordinary deck of playing cards. Find the probability of receiving a spade, a heart, a diamond and a club in that order.

Out of 52 cards → 13 spades, 13 hearts, 13 diamonds, 13 clubs.

P(getting a spade out of 52 cards) $= \frac{13C_1}{52C_1} = \frac{13}{52}$

P(getting a ~~spad~~ heart as the second card) $= \frac{13C_1}{51C_1} = \frac{13}{51}$

P(getting a diamond as the third card) $= \frac{13C_1}{50C_1} = \frac{13}{50}$

P(getting a club as the fourth card) $= \frac{13C_1}{49C_1} = \frac{13}{49}$

∴ Required Probability $= \frac{13}{52} \times \frac{13}{51} \times \frac{13}{50} \times \frac{13}{49} = 0.0044$

4. Four persons are choosen at random from a group containing 3 men, 2 women and 4 children. Show that the chance of exactly two of them being children is $\frac{10}{21}$.

Total no of persons $= 3(M) + 2(W) + 4(C) = 9$

Four persons can be selected in ${}^9C_4$ ways.

Probability of selecting exactly two children and the remaining two from among 3 men and 2 women $= \frac{{}^4C_2 \times {}^5C_2}{{}^9C_4} = \frac{10}{21} = 0.476$

5. A is known to hit the target in 2 out of 5 shots. B is known to hit the target in 3 out of 4 shots. Find the probability of the target being hit when both try.

$P(A) = 2/5$; $P(B) = 3/4$

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

As A and B are independent $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$\therefore P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

$P(A \cup B) = \frac{2}{5} + \frac{3}{4} - \left(\frac{2}{5} \times \frac{3}{4}\right) = \frac{17}{20}$

6. Events A and B are such that $P(A+B) = \frac{3}{4}$, $P(AB) = 1/4$, $P(\bar{A}) = 2/3$. Find $P(B)$.

$P(A+B) = P(A \cup B)$
$P(AB) = P(A \cap B)$

$P(A) = 1 - P(\bar{A}) = 1 - 2/3 = 1/3$

$P(A+B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore P(B) = P(A+B) - P(A) + P(AB) = \frac{3}{4} - 1/3 + 1/4 = 2/3$

$\Rightarrow P(B) = 2/3$

7. If A and B are events with $P(A) = 3/8$, $P(B) = 1/2$ and $P(A \cap B) = 1/4$ find $P(A^c \cap B^c)$.

$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$

To find $P(A \cup B)$:

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 3/8 + 1/2 - 1/4 = 5/8$

$\therefore P(A^c \cap B^c) = 1 - 5/8 = 3/8.$

PROBABILITY THEOREMS

- Addition Theorem (For simultaneous trials)
  - Events are mutually exclusive
  - Events are partially overlapping
- Multiplication theorem (For consecutive trials)
  - Events are independent
  - Events are dependent.
- Baye's Theorem

Addition Theorem of Probability

i) If A and B are two mutually exclusive events of a random experiment, the probability of occurrence of the event 'AUB' is the sum of the probabilities of the events A and B. i.e., If $A \cap B = \phi$ then

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) \rightarrow \text{mutually exclusive.}$$

ii) If A and B are two events associated with a random experiment then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

NOTE: i) If the events have no outcome in common, i.e., they are mutually exclusive, then

$A \cap B = \phi$ and therefore

$$P(A \cap B) = P(\phi) = 0$$

Conditional Probability:

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Given two events A and B associated with the sample space of the same random experiment, the conditional probability $P(A|B)$ of the occurence of A knowing that event B has already occured is given by

$$P(A|B) = \frac{P(A\cap B)}{P(B)}, \quad P(B) \neq 0$$

Multiplication Theorem of Probability:

i) The probability of occurence of two independent events is equal to the product of their individual probabilities:

$$P(A\cap B) = P(A) \times P(B)$$

Remarks:

1. If the events E and F are independent then the complementary events $\bar{E}$ and $\bar{F}$ are also independent.

2. P { happening of atleast one of the events $E_1, E_2, \ldots E_n$ }

= 1 − P { none of the events $E_1, E_2, \cdots E_n$ happens }

equivalently

P { none of the given events happens }

= 1 − P { atleast one of them happens }

Baye's Theorem

Let $A_1, A_2, \ldots A_n$ be 'n' mutually exclusive and exhaustive events with $P(A_i) \neq 0$ for $i = 1, 2, \ldots n$. Let B be an event such that $B \subset \bigcup_{i=1}^{n} A_i$, $P(B) \neq 0$ then

$$P(A_i/B) = \frac{P(A_i)\, P(B/A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)}$$

Problems based on basic properties

1) A bag contains 10 white, 6 red, 4 black and 7 blue balls. 5 balls are drawn at random. What is the probability that 2 of them are red and one is black?

Total number of balls $= 10+6+4+7 = 27$

5 balls can be drawn from these 27 balls in $27C_5$ ways.

$$\frac{27!}{5! \times 22!} = \frac{23 \times 24 \times 25 \times 26 \times 27}{1 \times 2 \times 3 \times 4 \times 5} = 80730 \text{ ways.}$$

$\therefore$ Total number of Exhaustive events $= 80730$

2 red balls can be drawn from 6 red balls in $6C_2$ ways.

1 black ball can drawn from 4 black balls in $4C_1$ ways.

$\therefore$ The number of favourable cases $= 6C_2 \times 4C_1 = 15 \times 4 = 60$

$\therefore$ Required probability $= \dfrac{60}{80730}$

2) what is the probability of having a king and a queen when two cards are drawn from a pack of 52 cards?

Two cards are drawn from a pack of 52 cards in $52C_2$ ways.

1 queen can be drawn from 4 queen cards in $4C_1$ ways.

1 king can be drawn from 4 king cards in $4C_1$ ways.

P(drawing 1 queen and 1 king)

$$= \frac{4C_1 \times 4C_1}{52C_2} = \frac{16}{1326} = \frac{8}{663}$$

Problems based on conditional probability

1) A bag contains 3 red and 4 white balls. Two draws are made without replacement. what is the probability that both the balls are red.

P(drawing a red ball in the first draw) $= \frac{3C_1}{7} = \frac{3}{7}$

$$P(A) = \frac{3}{7}$$

P(drawing a red ball in the second draw given that first ball is red) $= \frac{2C_1}{6} = \frac{2}{6}$

$$P(B|A) = 2/6$$

$$\therefore P(AB) = P(A) \times P(B|A)$$

$$P(AB) = 3/7 \times \frac{2}{6} = 1/7$$

2) A bag contains 5 white balls and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are black.

P(drawing black ball in the first draw) $= \frac{3C_1}{8} = 3/8$

$P(A) = 3/8$

P(drawing the second black ball given that first ball drawn is black) $= \frac{2C_1}{7} = \frac{2}{7}$

$P(B|A) = \frac{2}{7}$

$\therefore$ P(both balls drawn are black) $= P(A) \times P(B|A)$

$= 3/8 \times \frac{2}{7} = 6/56$

$P(AB) = \frac{3}{28}$

3) Find the probability of drawing a queen and a king from a pack of cards in two consecutive draws, the cards drawn not being replaced.

P(drawing a queen card) $= \frac{4}{52} = P(A)$

P(drawing a king card) $= \frac{4}{51} = P(B|A)$

$P(AB) = P(A) \cdot P(B|A)$

$P(AB) = \frac{4}{52} \times \frac{4}{51} = \frac{4}{663}$

Random Variable :-

A random variable is a rule that assigns a numerical value to each possible outcome of an experiment.

Example :-

If a coin is tossed twice then the sample space is $S = \{HH, HT, TH, TT\}$. Let X be a random variable that denotes the number of heads. Therefore the out-come assigns the numerical values as follows.

for HH, the value of $X = 2$

HT, the value is $X = 1$

TH, the value is $X = 1$

TT, the value of $X = 0$

Definition :-

A random variable is a function that assigns a real number $X(s)$ to every element $s \in S$, where S is the sample space corresponding to a random Experiment.

Types of Random Variable :-

i) Discrete random variable

ii) continuous random variable.

Discrete random Variable :-

A random variable X is discrete if it assumes only finite number of values (OR) countably infinite number of possible discrete values.

Example:-

1. The number of students who are absent for a particular period.

24

2. The number shown when a die is thrown.

Continuous Random Variable:-

If X is a random variable which can take all values (i.e; infinite number of values) in an interval, then X is called a continuous random variable.

Example:-

1. The number of typographical errors in a book.
2. The operating time between two failures of a computer.

DISCRETE RANDOM VARIABLE

PROBABILITY FUNCTION

1. If X is a discrete RV which take the values $x_1, x_2, x_3 \ldots\ldots$ such that $P(X = x_i) = P_i$, the P_i is called the probability function or probability mass function or point probability function, provided P_i satisfy the following conditions.

i) $P_i \geq 0$, for all i and

ii) $\sum_i P_i = 1$

The collection of pairs $(x_i, P(x_i)), i = 1, 2, \ldots$ is called the probability distribution of X and $P(X_i)$ is called the probability mass function of X.

CONTINUOUS RANDOM VARIABLE

PROBABILITY DENSITY FUNCTION (pdf)

1. Let X be a continuous random variable. The function $f(x)$ is called the probability density function (p.d.f) of the random variable X if it satisfies the following conditions.

i) $f(x) \geq 0$, for all x

ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

PROPERTIES:-

i) $P(a \leq X \leq b) = \int_a^b f(x)\,dx$; $f(x) \to$ p.d.f

ii) For any specified value x_0,

$$P(X = x_0) = 0 \quad \because \int_{x_0}^{x_0} f(x)\,dx = 0$$

iii) $$P(a \leq X \leq b) = P(a < X \leq b)$$
$$= P(a \leq X < b)$$
$$= P(a < X < b)$$

CUMULATIVE DISTRIBUTION FUNCTION (c.d.f)

2. If X is an Random variable, discrete or continuous, then $F(x) = P(X \leq x)$ is called the cumulative distribution function of X or distribution function of X and denoted by $F(x)$.

$$F(x) = \sum_{\substack{j \\ x_j \leq x}} P_j$$

CUMULATIVE DISTRIBUTION FUNCTION (c.d.f)

$$F(x) = P(-\infty < X \leq x)$$

$$= \int_{-\infty}^{x} f(x)\,dx$$

PROPERTIES of the cdf ($F(x)$)

1. $F(x)$ is a non-decreasing function of x i.e.,

If $x_1 < x_2$ then $F(x_1) \leq F(x_2)$.

$var(x) = E(x^2) - E(x)$

$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

S.D of $x = \sqrt{var(x)}$

variance (σ^2)

$= var(x) = E(x^2) - [E(x)]^2$

$E(x^2) = \sum_i x_i^2 P(x_i)$

2. $F(-\infty) = 0$ and $F(\infty) = 1$

3. If X is a discrete random variable taking values $x_1, x_2 \dots$

S.D $= \sqrt{var(x)}$

where $x_1 < x_2 < x_3 < \dots < x_{i-1} < x_i < \dots$ then

$$P(X = x_i) = F(x_i) - F(x_{i-1})$$

4. If X is a continuous random variable then

$\frac{d}{dx}(F(x)) = f(x)$, at all points where $F(x)$ is differentiable

MATHEMATICAL EXPECTATION (OR) EXPECTATION:

DISCRETE

Let X be a discrete RV

X:	x_1,	x_2	x_3	....	x_n
P:	P_1	P_2	P_3	....	P_n

Then

$$E(X) = \sum_{i=1}^{n} x_i P_i = x_1 P_1 + x_2 P_2 + \dots x_n P_n$$

& $\sum P_i = 1$

CONTINUOUS

Let X be a continuous RV with p.d.f $f(x)$ then

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

Here

$E(x)$ = mean value or stochastic average or ensemble average of x,

$\therefore \quad E(x) = \bar{x}$

MOMENTS : Importance of Moments

Though mean or expectation is a significant number representing the behaviour of a random variable, it is not adequate to describe the behaviour of a random variable completely. It is possible to reconstruct the probability density function of a random variable from its various moments.

DEFINITION :- Moments about the origin (Raw moments) :-

The n^{th} moment about the origin of a random variable x is defined as the expected value of the n^{th} power of x.

For discrete random variable x, the n^{th} moment is defined as

$$E[x^n] = \sum_i x_i^n P_i = \mu_n' \; ; \; n \geq 1$$

For continuous random variable x, the n^{th} moment is defined as

$$E[x^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx = \mu_n' \; ; \; n \geq 1$$

DEFINITION :- Moments about the mean (Central moment) :-

The n^{th} central moment of a discrete random variable x is its moment about the mean value $\bar{x}$ and is defined as

$$E[(x-\bar{x})^n] = \sum_i (x_i - \bar{x})^n P_i = \mu_n$$

For continuous random variable, the n^{th} moment about the mean is defined as

$$\mu_n = \int_{-\infty}^{\infty} (x-\bar{x})^n f(x)\,dx$$

For a random variable, we can find the various n^{th} order moments about origin and mean. These numerical values give all information about a random variable.

The second moment about the mean is called Variance and represented as σ_X^2.

$$\therefore \mu_2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\,dx = \sigma_X^2 = \text{Variance}$$

The positive square root σ_X of the Variance is called the standard deviation.

$$\therefore \text{variance} = (\text{standard deviation})^2$$

Relation b/w Moments about origin and moments about mean:-

NOTE:- $\mu_n = \mu_n' - nC_1 \mu_{n-1}' \mu + nC_2 \mu_{n-2}' \mu^2 + \cdots\cdots$

$$\text{Variance} = \sigma_X^2 = E\left[(x-\bar{x})^2\right]$$

$$= E\left[x^2 - 2x\bar{x} + \bar{x}^2\right] \qquad \because E(ax+b) = aE(x)+b$$

$$= E(x^2) - 2\bar{X}E(X) + \bar{X}^2 \qquad \because E(X) = \bar{X}$$

$$= \overline{x^2} - 2\bar{X}\bar{X} + \bar{X}^2$$

$$\sigma_X^2 = \overline{x^2} - \bar{X}^2$$

$$\sigma_X^2 = \mu_2' - (\mu_1')^2$$

Origin	D.RV	C.RV	
Mean			

ii) Properties of Expectation:-

* $E(a) = a$
* $E(X+Y) = E(X) + E(Y)$
* $E(aX+b) = aE(X)+b$
* $E(X-\bar{X}) = 0$
* $\left[E(XY)^2\right] \leq E(X^2)\,E(Y^2)$

Schwartz inequality

iii)

$$P(X<a) = \int_{-\infty}^{a} f(x)\,dx$$

$$P(X>a) = \int_{a}^{\infty} f(x)\,dx$$

$$P(a<X<b) = \int_{a}^{b} f(x)\,dx$$

Properties of variance:-

i) $var(X) \geq 0$

ii) $E(X^2) \geq [E(X)]^2$

iii) $var(b) = 0$: $b \to$ const

iv) $var(aX \pm b) = a^2 var(X)$

v) $var(aX \pm bY) = a^2 var(X) + b^2 var(Y)$

If X is the number of Red balls

Problems based on Probability distribution function:-

3 balls are choosen at random

1. A box contains 4 white and 3 Red balls, Find the probability distribution of number of red balls in 3 drawn with replacement from the box.

W	R
4	3

Let X denotes member of red balls. Then X can take the values 0, 1, 2, 3. ∴ The probability distribution function is given by

$$Prob = \frac{\text{favourable case}}{\text{Total no of exhaustive}}$$

X	0	1	2	3
P(X)	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$

& $\sum P(X) = 1$

i) $P(X=0)$ = Probability of getting no red balls. $= P(WWW)$

$$= \frac{4C_1}{7C_1} \times \frac{4C_1}{7C_1} \times \frac{4C_1}{7C_1}$$

$$P(X=0) = \frac{4}{7} * \frac{4}{7} * \frac{4}{7} = \frac{64}{343}$$

ii) $P(X=1)$ = Probability of getting one red ball

$= P(WWR) + P(WRW) + P(RWW)$

$= P(WWR)$ or $P(WRW)$ or $P(RWW)$

$$= \left(\frac{4}{7} \times \frac{4}{7} \times \frac{3}{7}\right) + \left(\frac{4}{7} \times \frac{3}{7} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{4}{7} \times \frac{4}{7}\right)$$

$$P(X=1) = \frac{48}{343} + \frac{48}{343} + \frac{48}{343} = \frac{144}{343}$$

iii) $P(X=2)$ = Probability of getting two red balls.

$= P(RRW \text{ or } RWR \text{ or } WRR)$

$= \left(\frac{3}{7} \times \frac{3}{7} \times \frac{4}{7}\right) + \left(\frac{3}{7} \times \frac{4}{7} \times \frac{3}{7}\right) + \left(\frac{4}{7} \times \frac{3}{7} \times \frac{3}{7}\right)$

$P(X=2) = 3\left(\frac{36}{343}\right) = \frac{108}{343}$

iv) $P(X=3)$ = Probability of getting all red balls

$= P(RRR) = \left(\frac{3}{7} \times \frac{3}{7} \times \frac{3}{7}\right) = \frac{27}{343}$

∴ The probability distribution of x is

X	0	1	2	3
P(X)	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$.

2. A shipment of 6 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If x is the number of defective sets purchased by the hotel, find the probability distribution of x.

There are only 2 defective sets in a lot. Let X denotes number of defective sets.

∴ X takes the values 0, 1, 2

$P(X=0)$ = Probability for no defective set $= P(GGG)$

$= \frac{4C_1}{6C_1} \times \frac{3C_1}{5C_1} \times \frac{2C_1}{4C_1}$ (OR) $= \frac{4C_3}{6C_3} = \frac{1}{5}$

$= \frac{4}{6} \times \frac{3}{5} \times \frac{2}{4} = \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{3}}{5} \times \frac{1}{\cancel{2}}$

$P(X=0) = \frac{1}{5}$

ii) $P(X=1) = P(GGD,\ GDG,\ DGG)$

$$= P(GGD) + P(GDG) + P(DGG)$$

$$= \left(\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\right) + \left(\frac{4}{6}\times\frac{2}{5}\times\frac{3}{4}\right) + \left(\frac{2}{6}\times\frac{4}{5}\times\frac{3}{4}\right)$$

$$= 3\left(\frac{4}{6}\times\frac{3}{5}\times\frac{2}{4}\right) \qquad \text{(or)} \quad \frac{4C_2 \cdot 2C_1}{6C_3} = \frac{3}{5}$$

$$P(X=1) = \frac{3}{5}$$

iii) $P(X=2) = P(GDD,\ DDG,\ DGD)$

$$= \left(\frac{4}{6}\times\frac{2}{5}\times\frac{1}{4}\right) + \left(\frac{2}{6}\times\frac{1}{5}\times\frac{4}{4}\right) + \left(\frac{2}{6}\times\frac{4}{5}\times\frac{1}{4}\right)$$

$$P(X=2) = 3\left(\frac{1}{15}\right) = \frac{1}{5} \qquad \text{(or)} \quad \frac{2C_2 \cdot 4C_1}{6C_3} = \frac{4}{20} = \frac{1}{5}$$

$\therefore$ The probability distribution of X is given by

X	0	1	2
$P(X)$	$\frac{1}{5}$	$3/5$	$1/5$

and $\Sigma P(X) = 1$.

$\frac{10}{12}$ $\frac{2}{12}$

3. A lot containing 10 items, 2 of which are defective, 3 items are chosen at random. If X is the number of defectives found, obtain the probability distribution of X, when the items are chosen i) without replacement ii) with replacement.

A.W ①

Case (i):- Without replacement

Let X denotes number of defective items. Then X takes the values 0, 1, 2

i) $P(X=0) = P(\text{no defective item}) = P(GGG)$

$$= \frac{10}{12}\times\frac{9}{11}\times\frac{8}{10} = \frac{6}{11} \quad (\text{or}) \quad \frac{10C_3}{12C_3} = \frac{6}{11}$$

ii) $P(X=1) = P(\text{one defective item}) = P(DGG)+P(GDG)+P(GGD)$

$$= \left(\frac{2}{12}\times\frac{10}{11}\times\frac{9}{10}\right)+\left(\frac{10}{12}\times\frac{2}{11}\times\frac{9}{10}\right)+\left(\frac{10}{12}\times\frac{9}{11}\times\frac{2}{10}\right)$$

$$= \cancel{3}\left(\frac{1}{\cancel{6}_2}\times\frac{\cancel{10}}{11}\times\frac{9}{\cancel{10}}\right) = \frac{9}{22} \quad (\text{or}) \quad \frac{10C_2 \cdot 2C_1}{12C_3}$$

iii) $P(X=2) = P(2 \text{ defective item})$

$$= \frac{10C_1 \times 2C_2}{12C_3} = \frac{1}{22}$$

$\therefore$ The probability distribution of X is given by

X	0	1	2
$P(X)$	$\frac{6}{11}$	$\frac{9}{22}$	$\frac{1}{22}$

and $\Sigma P(X) = 1$

Case (ii) With replacement:-

i) $P(X=0) = P(\text{no defective}) = P(GGG) = \frac{10}{12}\times\frac{10}{12}\times\frac{10}{12} = \cancel{3}\left(\frac{5}{\cancel{6}}\right) = \frac{\cancel{5}}{2}$

$(\text{Or}) \quad \frac{10C_3}{12C_3}$

ii) $P(X=1) = P(\text{one defective}) = P(DGG)+P(GDG)+P(GGD)$

$$= \left(\frac{2}{12}\times\frac{10}{12}\times\frac{10}{12}\right)+(\ )+(\ )$$

$$= \cancel{3}\left(\frac{1}{\cancel{6}_2}\times\frac{5}{6}\times\frac{5}{6}\right) = \frac{25}{72}$$

iii) $P(X=2) = P(DDG)+P(GDD)+P(DGD)$

$$= \left(\frac{2}{12}\times\frac{2}{12}\times\frac{10}{12}\right)+(\ )+(\ )$$

$$= \cancel{3}\left(\frac{1}{\cancel{6}_2}\times\frac{1}{6}\times\frac{5}{6}\right) = \frac{5}{72}$$

4. If the random variable X takes the values 1, 2, 3 and 4 such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$, find the probability distribution function of X. and cumulative distribution of X.

Sol:

X:	1	2	3	4
$P(x)$:	$k/2$	$k/3$	k	$k/5$

Given $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4) = k$ (say)

$\Rightarrow$ $2P(X=1) = k \Rightarrow P(1) = k/2$; $P(X=2) = k/3$; $P(X=3) = k$

and $P(X=4) = k/5$

To find k:-

We know that $\Sigma P(x) = 1$

$$\Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\Rightarrow \frac{15k + 10k + 30k + 6k}{30} = 1$$

$$\Rightarrow \frac{61k}{30} = 1 \Rightarrow k = \frac{30}{61}$$

$\therefore$ The probability distribution of X is

X:	1	2	3	4
P:	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Cumulative distribution of X:-

The cdf is defined as

$$F(x) = P(X \le x)$$

$F(1)\ P[X \le 1] = 0$

$F(2) = P[X \le 2] = \frac{15}{61}$

when $x < 1$, $F(x) = 0$

when $1 \le x < 2$, $F(x) = P(X=1) = \frac{15}{61}$

when $2 \le x < 3$, $F(x) = P(X=1) + P(X=2) = \frac{25}{61}$

$F(3) - P(X \le 3) = \frac{55}{61}$

when $3 \le x < 4$: $F(x) = P(X=1) + P(X=2) + P(X=3) = \frac{55}{61}$

when $x \ge 4$: $F(x) = P(X=1) + P(X=2) + P(X=3) + P(X=4) = 1$.

5. Given $f(x) = C(x-2)^2$; where $x = 1, 2, 3 \ldots 6$. Find the value of c such that $f(x)$ is probability density function. Find also mean and its variance.

Given $f(x) = C(x-2)^2$

x :	1	2	3	4	5	6
$f(x)$:	C	0	C	$4C$	$9C$	$16C$

$\because f(x)$ is density function $\Rightarrow$ Total probability $= 1$

$\Rightarrow \quad C + C + 4C + 9C + 16C = 1 \Rightarrow C = \frac{1}{31}$

Mean: $E(X) = \sum x_i p_i$

$$E(X) = \sum_{i=1}^{6} x_i p_i = x_1 p_1 + x_2 p_2 + \cdots + x_6 p_6$$

$$E(X) = \frac{1}{31} + \frac{3}{31} + \frac{16}{31} + \frac{45}{31} + \frac{96}{31}$$

$$E(X) = \frac{161}{31}$$

Variance: $\sigma^2 : E(X^2) - [E(X)]^2$

$$E(X^2) = \sum_{i=1}^{6} x_i^2 p_i = C + 9C + 64C + 225C + 576C = \frac{875}{31}$$

$$\therefore \text{Variance} = E(X^2) - [E(X)]^2 = \frac{875}{31} - \left(\frac{161}{31}\right)^2 = \frac{875}{31} - \frac{25921}{961}$$

$$= 28.226 - 26.973$$

$$\sigma^2 = 1.253$$

6. A continuous RV X that can assume any value between $x=2$ and $x=5$ has a density function given by $f(x) = k(1+x)$. Find $P(X<4)$.

By the property of pdf, $\int f(x)\,dx = 1$

$$\therefore \int_2^5 k(1+x)\,dx = 1$$

$$k\left(x+\frac{x^2}{2}\right)_2^5 = 1$$

$$k\left[\left(5+\frac{25}{2}\right)-(2+2)\right] = 1 \Rightarrow k\left[\frac{35}{2}-4\right] = 1$$

$$\Rightarrow \frac{27k}{2} = 1 \Rightarrow \boxed{k = \frac{2}{27}}$$

NOW $P(X<4) = \int_2^4 k(1+x)\,dx = P(2<X<4)$

$$= \frac{2}{27}\left(x+\frac{x^2}{2}\right)_2^4 = \frac{2}{27}\left(\frac{2x+x^2}{2}\right)_2^4$$

$$= \frac{1}{27}\left\{(8+16)-(4+4)\right\}$$

$$P(X<4) = \frac{16}{27}$$

7. A continuous RV X has a pdf $f(x) = kx^2e^{-x}$; ~~otherwise~~; $x>0$. Find k, mean and variance.

find mean & variance.

③ ii) $f(x) = Kxe^{-\lambda x}$; $x>0, \lambda>0$

$= 0$ otherwise.

By the property of pdf, $\int f(x)\,dx = 1$

$$\Rightarrow \int_0^\infty kx^2e^{-x}\,dx = 1$$

$$\Rightarrow k\int_0^\infty x^2e^{-x}\,dx = 1 \Rightarrow k\left[x^2\left(\frac{e^{-x}}{-1}\right)-(2x)(e^{-x})+2(-e^{-x})\right]_0^\infty$$

$$\Rightarrow k\left[0-(-2)\right] = 1 \Rightarrow k = \frac{1}{2}$$

$K = \lambda^2$

$E(X) = \frac{2}{\lambda}$

$E(X^2) = 6/\lambda^2$

$Var = 2/\lambda^2$

Mean: $E(x) = \int x f(x) dx$

Variance: $\sigma^2 = V(x) = E(x^2) - [E(x)]^2$

$\int_0^\infty e^{-x} x^{n-1} dx = \sqrt{n} ; \sqrt{n+1} = n!$

$\int_0^\infty e^{-ax} x^n dx = \frac{\sqrt{n+1}}{a^{n+1}}$

$$Mean = E(x) = \frac{1}{2} \int_0^\infty x^3 e^{-x} dx$$

$$= \frac{1}{2} \left\{ x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - 6(e^{-x}) \right\}_0^\infty$$

$$E(x) = \frac{1}{2} \times 6 = 3$$

$$E(x^2) = \int x^2 f(x) dx = \frac{1}{2} \int_0^\infty x^4 e^{-x} dx$$

$$= \frac{1}{2} \left\{ x^4(-e^{-x}) - (4x^3)(e^{-x}) + (12x^2)(-e^{-x}) - (24x)(e^{-x}) + 24(-e^{-x}) \right\}_0^\infty$$

$$= \frac{1}{2}(24) = 12$$

$$\therefore variance = E(x^2) - [E(x)]^2 = 12 - 9 = 3$$

8. A continuous RV X has a pdf $f(x) = 3x^2$, $0 \le x \le 1$. Find a and b such that i) $P(X \le a) = P(X > a)$ and ii) $P(X > b) = 0.05$.

i) $P(X \le a) = P(X > a)$

$$\therefore \int_0^a 3x^2 dx = \int_a^1 3x^2 dx$$

ie, $a^3 = 1 - a^3$

$2a^3 = 1 \Rightarrow a^3 = \frac{1}{2}$

$\Rightarrow a = 0.7937$

ii) $P(X > b) = 0.05$

$$\int_b^1 3x^2 dx = 0.05$$

$1 - b^3 = 0.05$

$b^3 = 1 - 0.05$

$b^3 = 0.95$

$b = 0.9830$

$\therefore a = 0.7937$

$b = 0.9830$

9. $dF = y_0 e^{-|x|} dx$; $-\infty \le x \le \infty$. P.T $y_0 = \frac{1}{2}$; $\mu' = mean = 0$; $S.D = \sigma = \sqrt{2}$

9. The distribution function of a RV X is given by $F(x) = 1-(1+x)e^{-x}$, $x \geq 0$. Find the density function, mean and variance of X.

By the property of $F(x)$, the pdf $f(x)$ is given by

$$f(x) = \frac{d}{dx}[F(x)].$$

The given cdf is continuous for $x \geq 0$

$$\therefore \quad f(x) = (1+x)e^{-x} - e^{-x} = xe^{-x}, \quad x \geq 0.$$

Mean $= E(X) = \int x f(x)\,dx$

$$E(X) = \int_0^\infty x^2 e^{-x}\,dx = \left\{x^2(-e^{-x}) - (2x)(e^{-x}) + 2(-e^{-x})\right\}_0^\infty$$

$$E(X) = 2$$

Variance : $V(X) = \sigma^2 = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_0^\infty x^3 e^{-x}\,dx = 6$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

10. If
$$f(x) = \begin{cases} 0 & ; \text{ for } x < 2 \\ \dfrac{3+2x}{18} & ; \text{ for } 2 \leq x \leq 4 \\ 0 & ; \text{ if } x > 4 \end{cases}$$

Prove that $f(x)$ is a pdf. Find its mean and Variance.

To prove $f(x)$ is pdf: $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

ie, $$\int_{-\infty}^{2} f(x)\,dx + \int_2^4 f(x)\,dx + \int_4^\infty f(x)\,dx = 1$$

Consider $$\int_2^4 f(x)\,dx = \frac{1}{18}\int_2^4 (3+2x)\,dx = \frac{1}{18}\left\{3x + x^2\right\}_2^4$$

$$= \frac{1}{18}\left\{(12+16) - (6+4)\right\}$$

To find Mean :-

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{2}^{4} x \times \frac{1}{18}(3+2x)\, dx$$

$$= \frac{1}{18}\int_{2}^{4} (3x+2x^2)\, dx \quad = \frac{1}{18}\left[\frac{3x^2}{2} + \frac{2x^3}{3}\right]_2^4$$

$$= \frac{1}{18}\left\{\left(24 + \frac{128}{3}\right) - \left(\frac{6}{\cancel{2}} + \frac{16}{3}\right)\right\}$$

$$= \frac{1}{18}\left\{(24-6) + \left(\frac{128}{3} - \frac{16}{3}\right)\right\}$$

$$= \frac{1}{18}\left\{\overset{18}{\cancel{21}} + \frac{112}{3}\right\} = \frac{1}{18}\left\{\frac{\overset{54}{\cancel{63}} + 112}{3}\right\}$$

$$E(X) = \frac{166}{54}$$

To find Variance :- $V(X) = \sigma^2 = E(X^2) - [E(X)]^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx \quad = \frac{1}{18}\int_2^4 (3x^2 + 2x^3)\, dx$$

$$= \frac{1}{18}\left[x^3 + \frac{x^4}{2}\right]_2^4 \quad = \frac{1}{36}\left[2x^3 + x^4\right]_2^4$$

$$= \frac{1}{18\times 2}\left[(128 + 256) - (16+16)\right]$$

$$E(X^2) = \frac{1}{18\times 2}\left[384 - 32\right] = \frac{352}{18\times 2} = \frac{176}{18}$$

$$\therefore \sigma^2 = V(X) = \frac{176}{18} - \left(\frac{166}{54}\right)^2 = \frac{\cancel{9504} - 2988}{\cancel{972}} = \frac{6516}{972}$$

$$= \frac{176}{18} - \frac{27556}{2916} = \frac{513216 - 496008}{52488}$$

$$\sigma^2 = \frac{17208}{52488} = 0.3279$$

11. If $f(x) = \frac{k}{1+x^2}$; $0 \le x \le \infty$ is a pdf. Find i) k ii) $P(X>2)$

Given $f(x)$ is pdf $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\Rightarrow \int_0^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$\Rightarrow k\left[\tan^{-1} x\right]_0^{\infty} = 1$$

$$k \cdot \frac{\pi}{2} = 1 \Rightarrow k = \frac{2}{\pi}$$

$P(X>2)$: $\int_2^{\infty} f(x)\,dx$

$$P(X>2) = \int_2^{\infty} \frac{2}{\pi} \cdot \frac{1}{1+x^2}\,dx$$

$$= \frac{2}{\pi}\left\{\tan^{-1} x\right\}_2^{\infty} = \frac{2}{\pi}\left\{\frac{\pi}{2} - \tan^{-1}(2)\right\}$$

12. A random variable X has the following probability distribution

x:	0	1	2	3	4	5	6	7
$P(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

. Find

i) the value of k ii) $P(1.5 < X < 4.5 \,/\, X>2)$ and iii) The smallest value of λ for which $P(X \le \lambda) > \frac{1}{2}$.

$10k^2 + 9k - 1 = 0$

$(10k-1)(k+1)$

We know that $\sum P(x) = 1$.

$$\Rightarrow 10k^2 + 9k = 1 \Rightarrow (10k-1)(k+1) = 0$$

$-1 \,|\, 10 \quad 9 \quad -1$

$\quad\; 0 \quad -10 \quad \boxed{0}$

$\quad\; 10 \quad -1$

$$\Rightarrow k = \frac{1}{10} \text{ or } -1$$

$(m+1)(10m-1) = 0$

The value $k=-1$ which makes some $P(x)$ negative, which is meaningless.

$m=-1$ $\quad 10m = 1$

$m = \frac{1}{10}$

$$\therefore k = \frac{1}{10}$$

$\therefore$ The actual distribution is

x:	0	1	2	3	4	5	6	7
$P(x)$:	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{17}{100}$

ii) $P(1.5 < X < 4.5 / X > 2)$

$$P(1.5 < X < 4.5 / X > 2) = P(A/B), \text{ say}$$

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P(X > 2)}$$

$$= \frac{P[X = 2, \text{ or } 3, \text{ or } 4 \cap X = 3 \text{ or } 4 \text{ or } 5 \text{ or } 6 \text{ or } 7]}{P(X>2)}$$

$$= \frac{P(X=3) + P(X=4)}{P(X=3) + \cdots + P(X=7)}$$

$$P(1.5 < X < 4.5 / X > 2) = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{7}$$

The smallest value of λ for which $P(X \le \lambda) > \frac{1}{2}$

By trials: $P(X \le 0) = 0$; $P(X \le 1) = \frac{1}{10}$; $P(X \le 2) = \frac{3}{10}$

$$P(X \le 3) = 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{1}{2}$$

$$P(X \le 4) = \frac{1}{2} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5} = 0.8 > 0.5$$

$\therefore$ The smallest value of λ for which $P(X \le \lambda) > \frac{1}{2}$ is 4.

13. The life in hours of a certain radio tube has the probability density function given by 68,

$$f(x) = \begin{cases} \frac{100}{x^2}, & \text{for } x \ge 100 \\ 0, & \text{for } x < 100 \end{cases}$$

Find i) the probability that none of three tubes in a given radio ~~tube~~ set will have to be replaced during the first 150 Hrs,

ii) the probability that all the three original tubes would have been replaced during the first 150 Hrs.

iii) distribution function.

Let x be the life time of the tube.

$P(x \geq 150)$ = Probability that a tube has life time more than 150 Hrs.

= Probability that none of the tubes will have to be replaced.

$$= \int_{150}^{\infty} f(x)\,dx = \int_{150}^{\infty} \frac{100}{x^2}\,dx = 100\left(\frac{-1}{x}\right)_{150}^{\infty}$$

$$P(x \geq 150) = 100\left[0 + \frac{1}{150}\right] = \frac{100}{150} = \frac{2}{3}$$

Probability that a tube need not to be replaced $= 2/3$

Probability that a tube has to be replaced $= 1 - 2/3 = 1/3$

$\therefore$ Probability that none of three valves need not to be replaced $= 2/3 \times 2/3 \times 2/3 = \frac{8}{27}$

& $\therefore$ Probability that all ~~the~~ three tubes has to be replaced $= \cancel{1 - \frac{8}{27}} = \cancel{\frac{19}{27}} = \frac{1}{3} \times 1/3 \times 1/3$

$= \frac{1}{27}$

Distribution Function:- $\int_{-\infty}^{x} f(x)\,dx = F(x)$

If $x < 100$: $f(x) = 0 \Rightarrow F(x) = 0$.

If $x \geq 100$: $f(x) = \frac{100}{x^2} \Rightarrow F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{+100}^{x} f(x)\,dx$

$$= \left[-\frac{100}{x}\right]_{100}^{x} = 1 - \frac{100}{x}$$

$$\therefore F(x) = \begin{cases} 0 & \text{for } x < 100 \\ 1 - \frac{100}{x} & \text{for } x \geq 100 \end{cases}$$

14. The p.d.f of a continuous R.V is $f(x) = ke^{-|x|}$. Find K and F. Also find its mean and standard deviation.

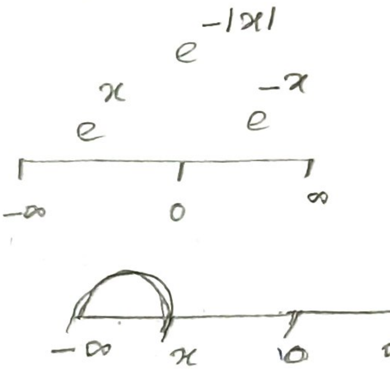
(2)

Given $f(x)$ is pdf $\Rightarrow \int_{-\infty}^{\infty} f(x)dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} ke^{-|x|} dx = 1 \Rightarrow 2k\int_{0}^{\infty} e^{-|x|} dx = 1$$

$$\Rightarrow 2k \int_0^{\infty} e^{-x} dx = 2k\left(-e^{-x}\right)_0^{\infty} = 1$$

$$\Rightarrow 2k = 1 \Rightarrow k = 1/2$$



TO find $F(x)$:-

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

$$\therefore \text{Given } f(x) = ke^{-|x|} = \begin{cases} ke^{x} & ; -\infty < x < 0 \\ ke^{-x} & ; 0 < x < \infty \end{cases}$$

For $x \leq 0$, $F(x) = \int_{-\infty}^{x} \frac{1}{2} e^{x} dx = \frac{1}{2}\left(e^{x}\right)_{-\infty}^{x} \Rightarrow \frac{1}{2}e^{x}$

For $x > 0$, $F(x) = \int_{-\infty}^{0} \frac{1}{2} e^{x} dx + \int_0^{x} \frac{1}{2} e^{-x} dx$

$$F(x) = \frac{1}{2}\left[2 - e^{-x}\right]$$

$\int_{x}^{x} f(x)dx$

$\int_{x}^{\infty} + \int_0^{x}$

$$\text{Mean} = E(x) = \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\mu_1' = \int_{-\infty}^{\infty} x \cdot \frac{1}{2} e^{-|x|}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x e^{-|x|}\,dx \quad \text{: integrand is odd.}$$

$$\Rightarrow E(x) = \mu_1' = 0.$$

$$\mu_2' = E(x^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{2} \cdot e^{-|x|}\,dx$$

$$= \int_0^{\infty} x^2 e^{-|x|}\,dx$$

$$= \int_0^{\infty} x^2 e^{-x}\,dx = \left[x^2(-e^{-x}) - (2x)(e^{-x}) + 2(-e^{-x})\right]_0^{\infty}$$

$$\mu_2' = 2.$$

$$\therefore \sigma^2 = E(x^2) - [E(x)]^2 = 2$$

$$\sigma = S.D = \sqrt{2}.$$

15. The probability of success of an event is 'p' and that of failure is 'q'. Find the expected number of trials to get the first success. ~~in each trial with constant prob of success.~~

Given: $p \to$ Probability for success

$q \to$ Probability for failure & $p+q=1$

Let x denotes the number of trials to get first success.

$E(x) = ? = \sum x_i p_i$

Let x takes the values $1, 2, 3, 4 \ldots \infty$

X	:	1	2	3	4	5	..... ∞
$P(X)$	:	p	qp	q^2p	q^3p	q^4p	.....

continued on page *

$\therefore E(x) = 1 \cdot p + 2 \cdot pq + 3 \cdot q^2 p + 4 \cdot q^3 \cdot p + \dots$

Page *

$= p[1 + 2q + 3q^2 + 4q^3 + \dots]$

$E(x) = p[1-q]^{-2} = \frac{p}{p^2} = \frac{1}{p}$

16. A coin is tossed, find the probability of expected number of getting head.

Prob(getting head) $= \frac{1}{2}$

$E(x) = \frac{1}{1/2} = 2$

17. Find the expected number of getting '6' when a die is thrown.

Prob(getting '6') $= 1/6$

$E(x) = \frac{1}{1/6} = 6.$

18. The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the function

$$f(x) = \begin{cases} A e^{-x/5} & ; x \geq 0 \\ 0 & ; \text{otherwise.} \end{cases}$$

a) Find the value of A that makes $f(x)$ a p.d.f

b) What is the prob that the number of minutes that she will talk over the phone is

i) More than 10 minutes

ii) less than 5 minutes

iii) b/w 5 and 10 minutes.

a) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for a valid pdf

$\Rightarrow \int_0^{\infty} A e^{-x/5}\,dx = 1 \Rightarrow -A(5)\left[e^{-x/5}\right]_0^{\infty} = -5A\left[e^{-\infty} - e^{0}\right] = 1$

$\Rightarrow A = 1/5$

$\therefore \quad f(x) = \frac{1}{5} e^{-x/5}$ for $x \geq 0$

0 otherwise.

The Probability that the lady talks more than 10 min over the Phone is given by

$$P(X > 10) = \int_{10}^{\infty} \frac{1}{5} e^{-x/5} dx = \frac{1}{5}\left[\frac{-e^{-\frac{x}{5}}}{1/5}\right]_{10}^{\infty}$$

$$P(X > 10) = -[e^{-\infty} - e^{-2}] = \frac{1}{e^2}$$

ii) The Probability that the lady talks less than 5 min over the Phone is given by

$$P(X<5) = 1 - P(X \geq 5) = 1 - \int_{5}^{\infty} \frac{1}{5} e^{-x/5} dx = 1 - e^{-1} = 1 - \frac{1}{e}$$

iii) The Probability that the lady talks between 5 and 10 min is given by

$$P(5<X<10) = \int_{5}^{10} \frac{1}{5} e^{-x/5} dx = e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2} = \frac{e-1}{e^2}$$

* Three cards are drawn successively with replacement from a well shuffled pack of 52 cards and x denotes the number of spade cards. Find the Prob. distribution of x.

Let x denotes number of spade cards and take the values $x = 0, 1, 2, 3$.

no of spade = 13

$$P(X=0) = P(\text{no spade}) = P(ccc) = \frac{39}{52} \times \frac{39}{52} \times \frac{39}{52}$$

$$P(X=1) = P(scc) \text{ or } P(ccs) \text{ or } P(csc) = 3\left(\frac{13}{52} \times \frac{39}{52} \times \frac{39}{52}\right)$$

$$P(X=2) = P(ssc) \text{ or } P(scs) \text{ or } P(css) = 3\left(\frac{13}{52} \times \frac{13}{52} \times \frac{39}{52}\right)$$

$$P(X=3) = P(sss) = \frac{13}{52} \times \frac{13}{52} \times \frac{13}{52}$$

1. Let x be a random variable such that $P(X=-2) = P(X=-1) = P(X=1) = P(X=2)$ and $P(X<0) = P(X=0) = P(X>0)$. Determine the probability mass function of x and distribution function of x.

(A) Let $P(X=-2) = P(X=-1) = P(X=1) = P(X=2) = a$

Then $P(X<0) = P(X=0) = P(X>0) = 2a$

The probability distribution is

X:	-2	-1	0	1	2
$P(x)$:	a	a	$2a$	a	a

$\therefore 6a = 1 \Rightarrow a = 1/6$

$\therefore$ The probability mass function of x and distribution function of x are given by

$P(X=0) = P(X<0)$
$= P(X=-2 \text{ or } P(X=-1))$
$= a+a$
$= 2a$

x:	-2	-1	0	1	2
$P(x)$:	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x) = P(X \le x)$:	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

2. $P(X=x) = \frac{x}{15}$; $x = 1, 2, 3, 4, 5$. find (i) $P(X=1 \text{ or } X=2)$ (ii) $P(\frac{1}{2} < X < 5/2 \mid X>1)$

The probability mass function of x is

x:	1	2	3	4	5
$P(X=x)$:	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

i) $P(X=1 \text{ or } X=2) = P(X=1) + P(X=2) = 3/15 = 1/5$

ii) $P(\frac{1}{2} < X < 5/2 \mid X>1) = \dfrac{P(\frac{1}{2} < X < 5/2 \cap X>1)}{P(X>1)} = \dfrac{P(X=2)}{1-P(X\le 1)}$

$= \dfrac{P(X=2)}{1-P(X=1)} = \dfrac{2/15}{1-1/15} = \frac{2}{15} \times \frac{15}{14} = \frac{1}{7}$

3. The probability function of an infinite discrete distribution is given by $P[X=j] = \frac{1}{2^j}$; $j = 1, 2, \ldots \infty$. Find the mean and variance of the distribution. Also find $P[X \text{ is even}]$, $P[X \geq 5]$ and $P[X \text{ is divisible by } 3]$

Given: $P[X=j] = \frac{1}{2^j}$

$$x_j : \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots$$

$$P(x_j) = \tfrac{1}{2} \quad (\tfrac{1}{2})^2 \quad (\tfrac{1}{2})^3 \quad (\tfrac{1}{2})^4 \quad (\tfrac{1}{2})^5$$

$$\text{Mean} = E(X) = \sum_{j=1}^{\infty} x_j P(x_j)$$

$$= 1 \cdot \tfrac{1}{2} + 2 (\tfrac{1}{2})^2 + 3 (\tfrac{1}{2})^3 + \ldots$$

$$= \tfrac{1}{2} \{ 1 + 2(\tfrac{1}{2}) + 3(\tfrac{1}{2})^2 + \ldots \}$$

$$E(X) = \tfrac{1}{2} \{ (1-x)^{-2} \} = \tfrac{1}{2} \{ (1-\tfrac{1}{2})^{-2} \} = \tfrac{1}{2} \times 4 = 2$$

Formula:-

$$(1-x)^{-3} = \tfrac{1}{2} \{ 1.2 + 2.3x + 3.4x^2 + \ldots \}$$

$$E(X^2) = \sum_{j=1}^{\infty} x_j^2 P(x_j) = \sum_{j=1}^{\infty} x_j(x_j+1) P(x_j) - \sum_{j=1}^{\infty} x_j P(x_j)$$

$$= \{ 1.2.(\tfrac{1}{2}) + 2.3(\tfrac{1}{2})^2 + 3.4(\tfrac{1}{2})^3 + \ldots \} - 2$$

$$= \tfrac{1}{2} \{ 1.2 + 2.3(\tfrac{1}{2}) + 3.4(\tfrac{1}{2})^2 + \ldots \} - 2$$

$$E(X^2) = (1-\tfrac{1}{2})^{-3} - 2 = (\tfrac{1}{2})^{-3} = 8 - 2 = 6$$

$$\therefore \text{variance} = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

$P[X \text{ is even}]$: $P(X=2) + P(X=4) + P(X=6) + \ldots$

$$= (\tfrac{1}{2})^2 + (\tfrac{1}{2})^4 + (\tfrac{1}{2})^6 + \ldots$$

$$= \cancel{(\tfrac{1}{2})^2} \; \tfrac{1}{4} + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^3 + \ldots$$

$$= (1-\tfrac{1}{4})^{-1} - 1 = (\tfrac{3}{4})^{-1} - 1 = \tfrac{4}{3} - 1 = \tfrac{1}{3}$$

$$P[X \geq 5] = P[X=5] + P[X=6] + \ldots$$

$$= \left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^6 + \left(\tfrac{1}{2}\right)^7 + \cdots\cdots$$

$$= \left(\tfrac{1}{2}\right)^5 \left\{1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots\cdots \right\}$$

$$= \left(\tfrac{1}{2}\right)^5 \left\{1 - \tfrac{1}{2}\right\}^{-1} = \left(\tfrac{1}{2}\right)^5 \times 2 = \left(\tfrac{1}{2}\right)^4 = \tfrac{1}{16}$$

P[x is divisible by 3] or P[x is multiple of 3]:-

$$= P[x=3] + P[x=6] + \cdots$$

$$= \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^6 + \left(\tfrac{1}{2}\right)^9 + \cdots\cdots$$

$$= \tfrac{1}{8} + \left(\tfrac{1}{8}\right)^2 + \left(\tfrac{1}{8}\right)^3 + \cdots\cdots$$

$$= \left(1 - \tfrac{1}{8}\right)^{-1} - 1 = \left(\tfrac{7}{8}\right)^{-1} = \tfrac{8}{7} - 1 = \tfrac{1}{7}$$

$\tfrac{1}{2}\left[1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots\right]$

$\tfrac{1}{2}\left[1 - \tfrac{1}{2}\right]^{-1} = \tfrac{1}{2} \cdots$

* Suppose X is a discrete random variable such that $P(X=0) = 1 - P(X=1)$ and $E(X) = 3\,var(X)$. Find $P(X=0)$ and $P(X=1)$.

Given: $P(X=0) = 1 - P(X=1) \Rightarrow P(X=0) + P(X=1) = 1$.

$\therefore$ x takes values 0 and 1 only.

Let $P(X=0) = a$ then $P(X=1) = 1-a$

Given: $E(X) = \sum x_i P(x_i) = 3\,var(X)$

ie, $\sum x_i P(x_i) = 3\left[\sum x_i^2 P(x_i) - \left[\sum x_i P(x_i)\right]^2\right]$

x_i	$P(x_i)$	$x_i P(x_i)$	x_i^2	$x_i^2 P(x_i)$
0	a	0	0	0
1	1-a	1-a	1	1-a
		$\boxed{E(X) = 1-a}$		$E(X^2) = 1-a$

$$var(X) = E(X^2) - [E(X)]^2$$

$$= (1-a) - (1-a)^2 = 1-a-(1+a^2-2a) = 1-a-1-a^2+2a$$

$$= a - a^2$$

Now $E(X) = 3\,var(X)$.

$$1-a = 3(a-a^2) \Rightarrow 1-a = 3a(1-a) \Rightarrow 3a = 1$$

$$\boxed{a = \tfrac{1}{3}}$$

$\therefore \; P(X=0) = a = 1/3$

$P(X=1) = 1 - 1/a = 1 - 1/3 = 2/3$

Moments and moment generating Function

★ Moments about the origin (Raw moments)

★ Moments about the mean (central moments)

	Discrete case	continuous case
★ Moments about the origin - μ_r'	$\mu_r' = E[x^r]$ $= \sum_{i=1}^{n} x_i^r P(x_i)$	$\mu_r' = E[x^r]$ $= \int_{-\infty}^{\infty} x^r f(x) dx$
Moments about mean - μ_r	$\mu_r = \sum_{i=1}^{n} (x_i - \bar{x})^r P(x_i)$	$\mu_r = \int_{-\infty}^{\infty} (x-\bar{x}) f(x) dx$
Moments about any point "a"	$\mu_r' = \sum_{i=1}^{n} (x_i - A)^r P(x_i)$	$\mu_r' = \int_{-\infty}^{-\infty} (x-A)^r f(x) dx$
MGF $(M_x t)$	$M_x(t) = E[e^{tx}]$ $= \sum_{i=1}^{n} e^{tx} P(x_i)$	$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$

Note:

$$M_x(t) = E[e^{tx}] = E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \cdots\right]$$

$$= 1 + \frac{t}{1!} E(x) + \frac{t^2}{2!} E(x^2) + \cdots$$

$$= 1 + \frac{t}{1!} \mu_1' + \frac{t^2}{2!} \mu_2' + \frac{t^3}{3!} \mu_3' + \cdots$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

Hence $\mu_r' = r^{th}$ moment about the origin

$= \text{coeff of } \frac{t^r}{r!} \text{ in } M_X(t)$

★ $\mu_r' = \frac{d^r}{dt^r} M_X(t) \Big|_{t=0}$

Relation between μ_r and μ_r': [First four moments about mean]

1. $\mu_1 = 0$ (always)

2. $\mu_2 = \mu_2' - (\mu_1')^2 \rightarrow$ variance

3. $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

4. $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$E[e^{cxt}]$

Note:

★

1. Mean $= \mu_1'$; variance $= \mu_2$

2. If μ_1' is the first moment about a point $x=A$ then mean $= A+\mu_1'$ i.e., $\boxed{\bar{x} = A+\mu_1'}$

Properties of MGF:

1. Let $Y = aX+b$ where X is a r.v with MGF $M_X(t)$ then $M_Y(t) = e^{bt} M_X(at)$

2. $M_{cX}(t) = M_X(ct)$; $c \rightarrow$ constant.

3. If x and y are independent random variables then $M_{x+y}(t) = M_x(t)\, M_y(t)$

Proof (i):

$$M_y(t) = E[e^{yt}] = E\left[e^{(ax+b)t}\right] = E\left[e^{axt}e^{bt}\right]$$

$$M_y(t) = e^{bt} E[e^{axt}] = e^{bt} M_x(at)$$

Proof (iii):

$$M_{x+y}(t) = E\left[e^{(x+y)t}\right] = E\left[e^{xt}e^{yt}\right] = E[e^{tx}]\,E[e^{ty}]$$

$\because$ x & y are independent

$$\boxed{M_{x+y}(t) = M_x(t) \cdot M_y(t)}$$

1\. If the p.d.f of a RV X is defined by $f(x) = \begin{cases} ce^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ compute the first four moments about mean. Hence find mean and variance of X.

Solution:- To find c:-

Given f(x) is a p.d.f $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^{\infty} ce^{-ax}\,dx = 1 \Rightarrow c\left[-\frac{e^{-ax}}{a}\right]_0^{\infty} = 1 \Rightarrow \frac{c}{a} = 1 \Rightarrow c = a$$

$$\therefore f(x) = ae^{-ax}$$

To find $\mu_1, \mu_2, \mu_3, \mu_4,$ first we have to find $\mu_1', \mu_2', \mu_3', \mu_4'$

Now $E[x^n] = \mu_n' = \int_{-\infty}^{\infty} x^n f(x)\,dx$

$$= \int_0^{\infty} x^n . ae^{-ax}\,dx = a\int_0^{\infty} e^{-ax} x^n\,dx = \frac{n!}{a^{n+1}} \cdot a = \frac{n!}{a^n}$$

By gamma integral, we have

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1}\,dx \text{ and } \Gamma n+1 = n! = \int_0^{\infty} e^{-x} x^n\,dx.$$

~~Now~~

$$\cancel{\mu_n' = \frac{a^{n-1}}{a^{n-1}}\int_0^{\infty} e^{-ax} \cdot ax^n\,dx} = \cancel{\frac{1}{a^{n-1}}\int_0^{\infty} e^{-ax}(ax)^n\,dx} = \cancel{\Gamma n+1}$$

Put $ax = t \Rightarrow a = \frac{t}{x}$ or $x = t/a$

$a\,dx = dt$

$$\therefore E[x^n] = a\int_0^{\infty} e^{-t}\left(\frac{t}{a}\right)^n \frac{dt}{a} = \frac{1}{a^n}\int_0^{\infty} e^{-t} t^n\,dt = \frac{\Gamma n+1}{a^n}$$

$$E[x^n] = \frac{n!}{a^n} = \mu_n'$$

put $n=1$: $\mu_1' = \frac{1}{a}$

put $n=2$: $\mu_2' = \frac{2}{a^2}$

put $n=3$: $\mu_3' = \frac{6}{a^3}$

put $n=4$: $\mu_4' = \frac{24}{a^4}$

To find the moments about Mean: [μ_n]

$\mu_1 = 0$ (always).

$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{2}{a^2} - \frac{1}{a^2} = \frac{1}{a^2}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$

$= \frac{6}{a^3} - 3\left(\frac{2}{a^2}\right)\left(\frac{1}{a}\right) + \frac{2}{a^3}$

$= \frac{6}{a^3} - \frac{6}{a^3} + \frac{2}{a^3}$

$\mu_3 = \frac{2}{a^3}$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= \frac{24}{a^4} - 4\left(\frac{6}{a^3}\right)\left(\frac{1}{a}\right) + 6\left(\frac{2}{a^2}\right)\left(\frac{1}{a^2}\right) - \frac{3}{a^4}$

$= \frac{24}{a^4} - \frac{24}{a^4} + \frac{12}{a^4} - \frac{3}{a^4}$

$\mu_4 = \frac{9}{a^4}$

∴ Moments about the Mean are

$\mu_1 = 0$: $\mu_2 = \frac{1}{a^2}$: $\mu_3 = \frac{2}{a^3}$: $\mu_4 = \frac{9}{a^4}$

MEAN $= E(X) = \mu_1' = \frac{1}{a}$

VARIANCE $= \mu_2 = \frac{1}{a^2}$

A continuous R.V X has p.d.f

$f(x) = kx^2 e^{-x}$; $x \geq 0$.

Find k, r^{th} raw moment, mean and Variance.

$k = \frac{1}{2}$

$\mu_r' = \frac{(r+2)!}{2}$

$= \frac{1}{2}\Gamma(r+3)$

Mean $= \mu_1' = 3$

$\mu_2' = 12$.

Variance $= \mu_2' - (\mu_1')^2$

Variance $= 12 - 9 = 3$.

2. For the distribution defined by the pdf $f(x) = \begin{cases} x & ; 0 \le x \le 1 \\ 2-x & ; 1 \le x \le 2 \\ 0 & ; \text{otherwise} \end{cases}$ compute the r^{th} moment about origin. Hence deduce the first four moments about mean.

Solution:- To find r^{th} moment:-

$$E[x^r] = \mu_r' = \int_{-\infty}^{\infty} x^r f(x)\,dx = \int_0^1 x^r \cdot x\,dx + \int_1^2 x^r(2-x)\,dx$$

$$= \left[\frac{x^{r+2}}{r+2}\right]_0^1 + \left[\frac{2x^{r+1}}{r+1}\right]_1^2 - \left[\frac{x^{r+2}}{r+2}\right]_1^2$$

$$= \frac{1}{r+2} + \frac{2}{r+1}\left[2^{r+1}-1\right] - \frac{1}{r+2}\left[2^{r+2}-1\right]$$

$$= \frac{1}{r+2} + \frac{2^{r+2}}{r+1} - \frac{2}{r+1} - \frac{2^{r+2}}{r+2} + \frac{1}{r+2}$$

$$= \frac{2}{r+2} + 2^{r+2}\left[\frac{1}{r+1} - \frac{1}{r+2}\right] - \frac{2}{r+1}$$

$$= \frac{2}{r+2} + 2^{r+2}\left[\frac{r+2-r-1}{(r+1)(r+2)}\right] - \frac{2}{r+1}$$

$$= \frac{2}{r+2} + \frac{2^{r+2}}{(r+1)(r+2)} - \frac{2}{r+1}$$

$$= \frac{2r+2-2r-4}{(r+1)(r+2)} + \frac{2^{r+2}}{(r+1)(r+2)}$$

$$\mu_r' = \frac{2^{r+2}-2}{(r+1)(r+2)} \; ; \; r = 1, 2, 3 \ldots$$

Moments about origin:-

$\mu_1' = 6/6 = 1 \; ; \; \mu_2' = \frac{16-2}{12} = \frac{7}{6} \; ; \; \mu_3' = \frac{3}{2}$ and

$\mu_4' = \frac{31}{15}$

Moments about Mean:-

$\mu_1 = 0$ (always)

$\mu_2 = \mu_2' - (\mu_1')^2 = \frac{7}{6} - 1 = \frac{1}{6}$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = \frac{3}{2} - \frac{7}{2} - 2 = \underline{\underline{-4}}$ Check $= 0$

$\mu_4 = \mu_4' - 4\mu_3' \cdot \mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$\mu_4 = \frac{31}{15} - \frac{14}{3} + 9 - 3 = \frac{51}{15} = \frac{1}{15}$ check

3. The first four moments of a distribution about the value 4 of the variables are $-1.5, 17, -30$ and 108. Find the mean, moments about the mean. Hence find the first four moments about $x = 2$.

[NOTE:- If μ_1' is the first moment about a point $x = A$. then Mean $= A + \mu_1'$ ie., $\bar{x} = A + \mu_1'$.]

Let $\mu_r' = r^{th}$ moment about $x = 4 \Rightarrow A = 4$.

Now $\mu_1' = -1.5$; $\mu_2' = 17$; $\mu_3' = -30$; $\mu_4' = 108$.

$\therefore$ Mean $= \bar{x} = 4 + \mu_1' = 4 - 1.5 = 2.5$

Moments about Mean:-

$\mu_1 = 0$; $\mu_2 = \mu_2' - (\mu_1')^2 = 17 - (-1.5)^2 = 14.75$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3 = -30 - 3(17)((-1.5)) + 2(-1.5)^3 = 39.7$

$\mu_3 = 39.75$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$

$= 108 - 4(-30)(-1.5) + 6(17)(-1.5)^2 - 3(-1.5)^4$

$\mu_4 = 142.3125$

Moments about the point : $x=2$:

$A=2$: $\mu_1' = \bar{x} - A = 2.5 - 2 = 0.5$

$\mu_2' = \mu_2 + (\mu_1')^2 = 14.75 + (0.5)^2 = 15$

$\mu_3' = \mu_3 + 3\mu_2'\mu_1' - 2(\mu_1')^3$

$= 39.75 + 3(15)(0.5) - 2(0.5)^3$

$\mu_3' = 62$

$\mu_4' = \mu_4 + 4\mu_3'\mu_1' - 6\mu_2'(\mu_1')^2 + 3(\mu_1')^4$

$= 142.3125 + 4(62)(0.5) - 6(15)(0.5)^2 + 3(0.5)^4$

$\mu_4' = 244.$

Hence moments about $x=2$ are

$\mu_1' = 0.5$; $\mu_2' = 15$; $\mu_3' = 62$; $\mu_4' = 244.$

Moment generating functions :- ; Moments about the origin

$M_X(t) = E(e^{tX}) = E\left(1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \dots\right)$

$= 1 + \frac{t}{1!}\mu_1' + \frac{t^2}{2!}\mu_2' + \frac{t^r}{r!}\mu_r'$

Hence r^{th} moment about the origin

$\mu_r' = $ coeff of $\frac{t^r}{r!}$ in $M_X(t)$

NOTE :-

We can prove

$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r'$

$\therefore \mu_r' = r^{th}$ moment about the origin

$=$ coeff of $\frac{t^r}{r!}$ in $M_X(t)$

$\mu_r' = \frac{d^r}{dt^r}\left[M_X(t)\right]_{t=0}$

1. Find the M.G.F of a R V X whose moments are given by $\mu_r' = (r+1)!\, 2^r$.

We know that

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} (r+1)!\, 2^r$$

$$= \sum_{r=0}^{\infty} (r+1)(2t)^r$$

$$= 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + \cdots$$

$$M_x(t) = (1-2t)^{-2} = \frac{1}{(1-2t)^2}$$

2. X is a discrete random variable with probability function $P(x) = \frac{1}{K^x}$; $x = 1, 2, \ldots$ (K = constant). Find its i) M.G.F ii) Mean and iii) Variance.

Solution:-

We know that, for the case of discrete random variable

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} P(x) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{K^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{K}\right)^x$$

$$= \frac{e^t}{K}\left[1 + \frac{e^t}{K} + \left(\frac{e^t}{K}\right)^2 + \cdots\right]$$

$$= \frac{e^t}{K}\left[1 - \frac{e^t}{K}\right]^{-1} = \frac{e^t}{K}\left[\frac{K-e^t}{K}\right]^{-1}$$

$$M_x(t) = \frac{e^t}{K} \cdot \frac{K}{K-e^t} = \frac{e^t}{K-e^t}$$

We know that

$$\mu_1' = \frac{d}{dt}[M_x(t)]_{t=0} = \frac{d}{dt}\left[\frac{e^t}{k-e^t}\right]_{t=0}$$

$$= \left[\frac{(k-e^t)(e^t) - e^t(-e^t)}{(k-e^t)^2}\right]_{t=0}$$

$$\mu_1' = \left[\frac{ke^t}{(k-e^t)^2}\right]_{t=0} \Rightarrow \frac{k}{(k-1)^2} = \text{Mean}$$

$$\mu_2' = \frac{d^2}{dt^2}[M_x(t)]_{t=0} = \left[\frac{(k-e^t)^2 ke^t - ke^t \cdot 2\cdot(k-e^t)(-e^t)}{(k-e^t)^4}\right]_{t=0}$$

$$\mu_2' = \frac{(k-1)^2 k + 2k(k-1)}{(k-1)^4} = \frac{(k-1)k+2k}{(k-1)^3} = \frac{k(2+k-1)}{(k-1)^3}$$

$$\mu_2' = \frac{k(k+1)}{(k-1)^3}$$

$$\text{Variance} = \mu_2' - (\mu_1')^2 = \frac{k(k+1)}{(k-1)^3} - \frac{k^2}{(k-1)^4} = \frac{k(k^2-1)-k^2}{(k-1)^4}$$

$$\text{Variance} = \frac{k^3-k^2-k}{(k-1)^4}$$

3. A random variable X has probability function $P(x) = \frac{1}{2^x}$, $x = 1, 2, 3 \ldots$ Find the M.G.F, Mean and Variance.

Moment generating function:-

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} P(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} \cdot \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2}\left[1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \cdots\right] = \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1} = \frac{e^t}{2}\left[\frac{2-e^t}{2}\right]^{-1}$$

$$M_x(t) = \frac{e^t}{2} \times \frac{2}{2-e^t} = \frac{e^t}{2-e^t}$$

MEAN:-

$$\mu_1' = \frac{d}{dt}[M_x(t)]_{t=0} = \frac{d}{dt}\left[\frac{e^t}{2-e^t}\right]_{t=0} = \left[\frac{(2-e^t)e^t + e^{2t}}{(2-e^t)^2}\right]_{t=0}$$

$$\mu_1' = \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0} = 2 \Rightarrow \text{Mean} = 2$$

VARIANCE:- : $\mu_2' - (\mu_1')^2$

$$\therefore \mu_2' = \frac{d^2}{dt^2}[M_x(t)]_{t=0} = \frac{d}{dt}\left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)^2 2e^t - 2e^t\, 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= \left[\frac{(2-e^t)2e^t + 4e^{2t}}{(2-e^t)^3}\right]_{t=0}$$

$$\mu_2' = \frac{(2-1)2+4}{(2-1)^3} = 6.$$

$$\therefore \text{variance} = \mu_2' - (\mu_1')^2 = 6-4 = 2.$$

4. If a random variable x has the moment generating function $M_x(t) = \frac{2}{2-t}$, determine the variance of x.

$$M_x(t) = \frac{2}{2-t} = \frac{1}{\left(1-\frac{t}{2}\right)} = \left[1-\frac{t}{2}\right]^{-1}$$

$$M_x(t) = 1 + \frac{t}{2} + \left(\frac{t}{2}\right)^2 + \dots \infty$$

The coefficient of $\frac{t}{1!}$ in this equation $= E(X) = \frac{1}{2}$

The coefficient of $\frac{t^2}{2!} = E(x^2) = \frac{1}{2}$

$$\therefore \text{Variance} = E(x^2) - [E(x)]^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

5. The Probability mass function of a discrete random variable x is given by $P(X=x) = \frac{1}{2^x}$; $x = 1, 2, 3 \ldots\ldots$. Find the moment generating function, mean, variance and also $P(x$ is even$)$, $P(X \geq 5)$ and $P(x$ is divisible by $3)$.

Solution :- $P(x$ is even$)$:

$P(x \text{ is even}) = P(X=2) + P(X=4) + \ldots\ldots$

$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots\ldots$

$= \frac{1}{2^2}\left[1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \ldots\right]$

$P(x \text{ is even}) = \cancel{\frac{1}{4}\left[1-\left(\frac{1}{2}\right)\right]^{-1} = \frac{1}{4}\left[\frac{1}{2}\right]^{-1} = \frac{2}{4} = \frac{1}{2}}$

$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$

$= \frac{1}{4} \cdot \frac{1}{1-\frac{1}{4}}$

$P(x \text{ is even}) = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$

$\frac{1}{4}\left[1 + \left(\frac{1}{2^2}\right) + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^2}\right)^3 + \right]$

$\left(1-\frac{1}{4}\right)^{-1} = \frac{4}{3} \times \frac{1}{4} = \frac{1}{3}$

$P(X \geq 5)$:-

$P(X \geq 5) = P(X=5) + P(X=6) + P(X=7) + \ldots\ldots$

$= \frac{1}{2^5} + \frac{1}{2^6} + \frac{1}{2^7} + \ldots\ldots$ $(a + ar + ar^2 + \ldots)$

$a = \frac{1}{2^5}$; $r = \frac{1}{2}$

$= \frac{1}{2^5} \cdot \frac{1}{1-\frac{1}{2}}$

$= \frac{1}{2^5} \times 2 = \frac{1}{2^4}$

$P(X \geq 5) = \frac{1}{16}$

$P(x \text{ is divisible by } 3)$:-

$P(x \text{ is divisible by } 3) = P(x=3) + P(x=6) + P(x=9) + \cdots$

$= \frac{1}{2^3} + \frac{1}{2^6} + \frac{1}{2^9} + \cdots$

Here $a = \frac{1}{2^3}$; $r = \frac{1}{2^3}$

$= \frac{a}{1-r} = \frac{1}{8} \times \frac{1}{1-1/8}$

$P(x \text{ is divisible by } 3) = \frac{1}{8} \times \frac{8}{7} = \frac{1}{7}$

$\frac{1}{2^3}\left[1 + \left(\frac{1}{2^3}\right) + \left(\frac{1}{2^3}\right)^2 + \right]$

$= \frac{1}{8} \times (1 - 1/8)^{-1}$

6. Find the M.G.F of the distribution defined by $dF = \frac{1}{2} e^{-|x|} dx$, $-\infty < x < \infty$. Hence find the variance.

$f(x) =$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{tx} e^{-|x|} dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{tx} e^{+x} dx + \int_{0}^{\infty} e^{tx} e^{-x} dx\right]$$

$$= \frac{1}{2}\left[\int_{-\infty}^{0} e^{(1+t)x} dx + \int_{0}^{\infty} e^{-(1-t)x} dx\right]$$

$$= \frac{1}{2}\left[\left(\frac{e^{(t+1)x}}{t+1}\right)_{-\infty}^{0} + \left(-\frac{e^{-(1-t)x}}{1-t}\right)_{0}^{\infty}\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+t} + \frac{1}{1-t}\right] \Rightarrow M_x(t) = \frac{1}{1-t^2}$$

Now $M_x(t) = \frac{1}{1-t^2} = (1-t^2)^{-1} = 1 + t^2 + t^4 + \cdots$

$1 + 2\frac{t^2}{2!} + \cdots$

$\mu_1' = 0$; $\mu_2' = \text{coeff } \frac{t^2}{2!} = 2! = 2$

$\therefore$ variance $= \mu_2' - (\mu_1')^2 = 2$

$\therefore$ Mean $= 0$; Variance $= 2$

7. Find the M.G.F of X whose p.d.f is given by $f(x) = 1-|x|$. $-1<x<1$

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-1}^{0} e^{tx}(1+x)\,dx + \int_{0}^{1} e^{tx}(1-x)\,dx$$

$$= \left[(1+x)\cdot\frac{e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_{-1}^{0} + \left[\frac{e^{tx}}{t}(1-x) + \frac{e^{tx}}{t^2}\right]_0^1$$

$$= \frac{1}{t}\cancel{e^t} - \frac{1}{t^2} + \frac{e^{-t}}{t^2} + \cancel{\frac{e^t}{t^2}} - \overset{1}{e}$$

$$= \cancel{\frac{1}{t}} - \frac{1}{t^2} + \frac{e^{-t}}{t^2} + \cancel{\frac{e^t}{t}}(0) + \frac{e^t}{t^2} - \cancel{\frac{1}{t}} - \frac{1}{t^2}$$

$$M_X(t) = \frac{e^t + e^{-t} - 2}{t^2}$$

8. The random variable X assumes the value x with the probability $P(X=x) = q^{x-1}p$; $x = 1,2,3\ldots$ Find the M.G.F of X and its mean and variance.

Solution:-

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} P(x) = E[e^{tx}]$$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} e^{tx} q^x$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (e^t q)^x$$

$$= \frac{p}{q}\left[e^t q + (e^t q)^2 + (e^t q)^3 + \cdots\right]$$

$$= e^t p\left[1 + e^t q + (e^t q)^2 + \cdots\right]$$

$$M_X(t) = e^t p\,[1 - e^t q]^{-1} \Rightarrow \quad M_X(t) = \frac{pe^t}{1-e^t q}$$

* The PDF of a random variable X follows the following probability law

$$f(x) = \frac{1}{2\theta} e^{-\frac{|x-\theta|}{\theta}} \quad ; -\infty<x<\infty.$$

Find the MGF of X, E(X) & Var(X).

Ans: $M_X(t) = \dfrac{e^{t\theta}}{1-t^2\theta^2}$

$E(X) = \theta$

$Var(X) = 2\theta^2$

$$\text{Mean} = \mu_1' = \frac{d}{dt}\left[M_X(t)\right]_{t=0}$$

$$= \frac{d}{dt}\left(\frac{pe^t}{1-qe^t}\right)_{t=0}$$

$$= \left.\frac{(1-qe^t)\,pe^t + (pe^t)\,qe^t}{(1-qe^t)^2}\right|_{t=0}$$

$$= \left.\frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2}\right|_{t=0}$$

$$\text{Mean} = \left.\frac{pe^t + 0}{(1-qe^t)^2}\right|_{t=0} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p} \qquad \because \; p+q=1, \; p = 1-q$$

$$\text{Variance}: \mu_2' - (\mu_1')^2$$

$$\mu_2' = \left.\frac{d^2}{dt^2} M_X(t)\right|_{t=0} = \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left.\frac{(1-qe^t)^2\,pe^t - (pe^t)\left(2(1-qe^t).(-qe^t)\right)}{(1-qe^t)^4}\right|_{t=0}$$

$$= \left.\frac{(1-qe^t)\,pe^t + 2\,pqe^{2t}}{(1-qe^t)^3}\right|_{t=0}$$

$$= \left.\frac{pe^t - pqe^{2t} + 2pqe^{2t}}{(1-qe^t)^3}\right|_{t=0}$$

$$\mu_2' = \left.\frac{pe^t + pqe^{2t}}{(1-qe^t)^3}\right|_{t=0} = \frac{p+pq}{(1-q)^3} = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

$$\therefore \text{Variance} = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

9. Find the M.G.F of a random variable X whose p.d.f is defined by

$$f(x) = \begin{cases} x & ; 0 \le x \le 1 \\ 2-x & ; 1 \le x \le 2 \\ 0 & ; \text{otherwise} \end{cases}$$

; Hence find mean and variance of x.

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_0^1 x e^{tx}\,dx + \int_1^2 e^{tx}(2-x)\,dx + 0.$$

$$= \left[x\left(\frac{e^{tx}}{t}\right) - \frac{e^{tx}}{t^2}\right]_0^1 + \left[(2-x)\left(\frac{e^{tx}}{t}\right) + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \frac{\cancel{x.e^t}}{t} - \frac{e^t}{t^2} + \frac{1}{t^2} + \frac{e^{2t}}{t^2} - \frac{\cancel{e^t}}{t} - \frac{e^t}{t^2}$$

$$= \frac{e^t(x-1)}{t} = \frac{e^{2t} - 2e^t + 1}{t^2} = \frac{e^t \cdot (1-e^t)^2}{t^2}$$

$$M_X(t) = \frac{(1-e^t)^2}{t^2}$$

Mean: $\mu_1' = \frac{d}{dt} M_X(t) \Big|_{t=0}.$

$$= \frac{d}{dt}\left[\frac{(1-e^t)^2}{t^2}\right]_{t=0} = \frac{t^2\, 2(1-e^t)(-e^t) - (1-e^t)^2 . 2t}{t^4}\Bigg|_{t=0}$$

$$= \frac{-2te^t(1-e^t) - 2(1-e^t)^2}{t^3}\Bigg|_{t=0}.$$

$$= \frac{-2te^t + 2te^{2t} - e^{2t} + 2e^t - 1}{t^3}\Bigg|_{t=0}$$ dr & Nr $\to 0$.

Apply L'Hospital rule:

Now $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\dots$

$$\mu_1' = \frac{d}{dt}\left[\left(\frac{1-e^t}{t}\right)^2\right]_{t=0}.$$

expanding $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\cdots$

$$\therefore \mu_1' = \frac{d}{dt}\left[\left(\frac{1-1-\frac{t}{1!}-\frac{t^2}{2!}-\frac{t^3}{3!}-\cdots}{t}\right)^2\right]_{t=0}$$

$$= \frac{d}{dt}\left[-\frac{t\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\cdots\right)}{t}\right]^2_{t=0}$$

$$= \frac{d}{dt}\left[\left[1+\frac{t}{2!}+\frac{t^2}{3!}+\cdots\right]^2\right]_{t=0}$$

$$= \left[2\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}+\cdots\right)\left(\frac{1}{2}+\frac{2t}{3!}+\frac{3t^2}{4!}+\cdots\right)\right]_{t=0}$$

$$= 2(1)\left(\frac{1}{2}\right) = 1.$$

$\therefore \mu_1' = \text{Mean} = 1$

To find variance : $\mu_2' - (\mu_1')^2$

$$\mu_2' = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0}.$$

$$= \frac{d}{dt}\left[2\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}+\cdots\right)\left(\frac{1}{2}+\frac{2t}{3!}+\frac{3t^2}{4!}+\cdots\right)\right]$$

$$= \left[2\left(1+\frac{t}{2!}+\frac{t^2}{3!}+\frac{t^3}{4!}+\cdots\right)\left(\frac{2}{3!}+\frac{6t}{4!}+\cdots\right)\right.$$
$$\left. + \left(\frac{1}{2}+\frac{2t}{3!}+\frac{3t^2}{4!}+\cdots\right)\left(2\left(\frac{1}{2!}+\frac{2t}{3!}+\frac{3t^2}{4!}+\cdots\right)\right)\right]_{t=0}$$

$$= 2\left[\left(1 \times \frac{2}{3!}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right)\right]$$

$$\mu_2' = 2\left[\frac{1}{3}+\frac{1}{4}\right] = 2\left(\frac{7}{12}\right) = \frac{7}{6}$$

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$\therefore$ Variance $= \frac{7}{6} - 1 = \frac{1}{6} = \sigma^2$

10. Find the Probability distribution of total number of heads obtained in 2 tosses of a coin. Hence find M.G.F, mean + Variance.

Let a coin be tossed 2 times. The sample space is

~~S = {HHHH, HHHT, HHTH, HTHH, THHH, TTHH, TTTH, TTHT, TTTT}~~

$S = \{HH, HT, TH, TT\}$; Let X denotes the number of heads,

The possible values of X are 0, 1, 2.

$P(X=0) = P(TT) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$P(X=1) = P(HT)$ or $P(TH) = \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2}\right) = \frac{2}{4}$

$P(X=2) = P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore$ The probability distribution of X is

X :	0	1	2
$P(x)$:	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

0	1	2	3	4
TTTT	HTTT	HHTT	THHH	HHHH
	THTT	THHT	HTHH	
	TTHT	TTHH	HHTH	
	TTTH	HTHT	HHHT	
		THTH		
		HTTH		

To find Moment generating function:-

$$M_X(t) = \sum_{x=0}^{2} e^{tx} P(x) = \frac{1}{4} + \frac{2}{4} e^{t} + \frac{1}{4} e^{2t}$$

$$M_X(t) = \frac{1}{4}\left[1 + 2e^{t} + e^{2t}\right]$$

0	1	2	3	4
$\frac{1}{16}$	$4/16$	$6/16$	$3/16$	$1/16$

$$M_X(t) = \frac{1}{16}\left[1 + 4e^{t} + 6e^{2t} + 4e^{3t} + e^{4t}\right]$$

$$\text{Mean} = \frac{d}{dt} M_X(t)\Big]_{t=0} = \mu_1'$$

$$\mu_1' = \frac{1}{4}\left[2e^{t} + 2e^{2t}\right]_{t=0} = \frac{1}{4}[4] = 1$$

$E(X) = \mu_1' = \frac{d}{dt} M_X(t)|_{t=0} = 2$

$E(X^2) = 5$; $V(X) = 1$

$$\mu_2' = \frac{d^2}{dt^2}\left[M_X(t)\right]_{t=0} = \frac{1}{4}\left[2e^{t} + 4e^{2t}\right]_{t=0} = \frac{1}{4}[6] = \frac{3}{2}$$

$$\therefore \text{Variance} = \sigma^2 = \mu_2' - (\mu_1')^2 = \frac{3}{2} - 1 = \frac{1}{2}$$