Studies on Analytical Solution for Two-dimensional Non-Linear Convection-Reaction-Diffusion Equations

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BONAFIDE CERTIFICATE

This is to certify that the dissertation entitled "**Studies on Analytical Solution for Two-dimensional Non-Linear Convection-Reaction-Diffusion Equations**" submitted by **Ms. ANJITHA.K.M**
(Reg. No. : CB.SC.P2MAT19001), for the award of **Degree of Master of Science in Mathematics** is a bonafied record of the work carried out by her under my guidance and supervision at Department of Mathematics, Amrita School of Engineering, Coimbatore.

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DECLARATION

I, **Ms.Anjitha.K.M (Register Number-CB.SC.P2MAT19001)**,hereby declare that this dissertation entitled **Studies on Analytical Solution for Two-dimensional Non-Linear Convection-Reaction-Diffusion Equations**, is the record of the original work done by me under the guidance of **Dr. Prakash P**, Department of Mathematics, Amrita School of Engineering, Coimbatore. To the best of knowledge this work has not formed the basis for the award of any degree/diploma/associateship/fellowship/or a similar award to any candidate in any University.

Place: Coimbatore **Signature of the Student**

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Date:25-06-2021

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Anjitha.K.M

1 List of symbols

$\hat{\mathcal{K}}[u], G[u]$	Non-linear differential operators
x_1, x_2	Independent space variables
t	Independent time variable
$m_i, i = 1, 2$	Order of the nonlinear differential operators $\hat{\mathcal{K}}, G$
W_n	Linear space
n	Dimension of linear space W_n
D	Derivative operator
$h_i(x_j)\ i = 1, \cdots, n, j = 1, 2$	Linearly independent functions
$\alpha_i, \alpha_i \in \mathbb{R}^n$	Coefficients of linearly independent functions $h_i(x_j)$
ψ_i	Expansion coefficient of $\hat{\mathcal{K}}[u]$ and $G[u]$
$L[u]$	Linear ODEs
$L[\hat{\mathcal{K}}[u]]$	Invariant condition of linear space W_n
$C_i, i = 1, 2, 3, 4$	Real arbitrary constants

Abstract

This dissertation gives a detailed study for finding the exact solutions of scalar and coupled nonlinear partial differential equations (PDEs) using the invariant subspace method. Also, we have shown that how to extend the invariant subspace method of (1+1)-dimensional PDEs to (2+1)-dimensional PDEs. In particular, this systematic study mainly investigates how to find exact solutions for the two-dimensional convection-reaction-diffusion equation using the obtained invariant subspaces. The applicability and effectiveness of this method have been illustrated through the various types of nonlinear PDEs. The obtained exact solutions can be expressed in terms of trigonometric, polynomial and exponential functions.

2 Introduction

This dissertation deals with the study for finding the exact solutions of non-linear two-dimensional convection-reaction-diffusion(CRD) equation. We consider the CRD equation in the following form

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} B_i(u)\frac{\partial u}{\partial x_i} + C(u), \tag{2.1}$$

where the functions $A_i(u)$ $(i = 1, 2)$ denote the process of diffusion (the movement of particles from a higher concentration region to a lower concentration region), $B_i(u)$ $(i = 1, 2)$ denote convection (the process of energy transfer by particle movement), and $C(u)$ denotes the reaction (change caused by particle interaction or time change) of a system with space variables x_1 and x_2. The above equation (2.1) has been widely used in various areas of science and engineering, such as heat transfer processes, adsorption in the porous medium, chemical reactions, population dynamics, lubrication and viscosity of fluids, the turbulence of heterogeneous physical systems, amount of contamination transported through groundwater, and so on.

We know that the derivation of the exact solution is an important task because it will help to understand the qualitative features and behaviour of the system of the complex phenomena. For this reason, in recent years, many mathematician and physicists have paid much attention to develop the analytical and numerical methods for solving nonlinear ODEs and PDEs such as the homotopy analysis method, differential transformation method, iteration method, Lie group analysis method, invariant subspace method and so on. Among those methods, recent investigations have shown that the invariant subspace method is a very effective and powerful mathematical tool to derive exact solutions for nonlinear scalar and coupled nonlinear PDEs. This method is popularly known as the generalized separation of the variable method. The invariant subspace method was originally introduced by Galaktionov and Svirshchevski [1] and many others [2**?** **?** –5] for nonlinear PDEs. However, finding exact solutions for higher dimensions nonlinear PDEs very complicated. Hence the main aim of this work is to give a systematic investigation for finding the exact solutions of two-dimensional CRD equation using the invariant subspace method. In addition, the special types of two-dimesional CRD equation are discussed and derived their exact solutions.

3 Invariant subspace method to one-dimensional scalar PDE

Let us consider a scalar PDE [5] with one space variable $x \in \mathbb{R}$, and a time variable $t > 0$ in the form

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \hat{\mathcal{K}}(x, u, \partial_x u, \cdots, \partial_x^m u), \tag{3.1}$$

where $\hat{\mathcal{K}}$ is an ordinary differential operator of order m, and $\hat{\mathcal{K}}(.)$ is a sufficiently given smooth function and $\partial_x^i u = \frac{\partial^i u}{\partial x^i}$, $i = 1, 2, \cdots, m$.
Let $\{h_i(x), i = 1, \cdots, n\}$, $n \geq 1$ be a linearly independent set. The linear space W_n is defined as

$$W_n = \mathrm{Span}\{h_1(x), h_2(x), \cdots, h_n(x)\} = \left\{ \sum_{i=1}^{n} \alpha_i h_i \middle| \alpha_i \in \mathbb{R}, i = 1, 2, \ldots, n \right\}. \tag{3.2}$$

The linear space W_n is said to be invariant under the given differential operator $\hat{\mathcal{K}}[u]$ if $\hat{\mathcal{K}}[W_n] \subseteq W_n, i.e, \hat{\mathcal{K}}[u] \in W_n, \forall u \in W_n$, where,

$$\hat{\mathcal{K}}[u] = \hat{\mathcal{K}}(x, u, \partial_x u, \cdots, \partial_x^m u), u_i = \frac{\partial^i u}{\partial x^i}, i = 1, 2 \ldots, n.$$

If the given differential operator $\hat{\mathcal{K}}$ admits an invariant subspace W_n, then there exists functions $\Psi_1, \Psi_2, \ldots, \Psi_n$ such that

$$\hat{\mathcal{K}}[\sum_{i=1}^{n} \alpha_i h_i(x)] = \sum_{i=1}^{n} \Psi_i(\alpha_1, \alpha_2, \cdots, \alpha_n) h_i(x), \text{for } \alpha_i \in \mathbb{R},$$

where $\{\Psi_i\}$ are the expansion coefficients of $\hat{\mathcal{K}}[u] \in W_n$ in the basis $\{h_i\}$, $i = 1, 2, \cdots, n$. Let the linear space W_n be an invariant under the given nonlinear differential operator $\hat{\mathcal{K}}$. Then the given equation (3.1) admits the solution in the form

$$u(x,t) = \sum_{i=1}^{n} \alpha_i(t) h_i(x),$$

where the coefficients satisfy the following system of ordinary differential equations (ODEs)

$$\frac{d\alpha_i}{dt} = \psi_i(\alpha_1(t), ..., \alpha_n(t)), i = 1, ..., n.$$

The above system of ODEs may be solved by the well-known analytical method.

Note that the linear space W_n is the space of solution of linear homogenous n-th order ODE

$$L[y] = y^{(n)} + a_1 y^{(n-1)}(x) + \cdots + a_n y(x) = 0, \tag{3.3}$$

where the constants $a_i, i = 1, 2 \cdots, n$ are to be determined. Then the invariance condition of the linear space W_n takes the form

$$L(\hat{\mathcal{K}}[u]) \mid_{L(u)=0} = 0.$$

This leads to an over determined system for the coefficients of $a_i, (i = 1, 2, \cdots, n)$ of (3.3) which gives the description of all invariant spaces of order n.

Theorem 3.1. *If a linear subspace W_n is invariant under a nonlinear ordinary differential opearator $\hat{\mathcal{K}}[u] = \hat{\mathcal{K}}(x, u, \partial_x u, \cdots, \partial_x^m u)$ of order m, then $n \leq 2m + 1$.*

The applicability and effectiveness of this method can be illustrated through the following given problems that are discussed in [5].

Example 1. *Let us consider the generalized nonlinear convection-reaction-diffusion equation [4]*

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] = A(u)u_{xx} + A_u(u)(u_x)^2 + B(u)u_x + C(u). \tag{3.4}$$

The linear space is defined as

$$W_n = \{y \mid L[y] = y^n + a_{n-1}y^{n-1} + \cdots + a_1 y' + a_0 y = 0\}$$

$$= Span\{h_1(x), \cdots, h_n(x)\},$$

where, $y^m = \frac{d^m}{dx^m}, m \in \mathbb{N}$ and $h_1(x), \cdots, h_n(x)$ are linearly independent solutions of n-th order linear ODEs. The invariant condition takes the form

$$L\left(\hat{\mathcal{K}}[u]\right) \mid_{L(u)=0} = D^n\hat{\mathcal{K}} + a_{n-1}D^{n-1}\hat{\mathcal{K}} + \cdots + a_0\hat{\mathcal{K}} \mid_{L(u)=0} = 0, \tag{3.5}$$

where, $D^r = \frac{d^r}{dx^r}, r = 1, 2, \cdots, n$ where $a_{n-1}, \cdots, a_0$ are constants which are to be determined.

By the maximal dimension theorem we have,$n \leq 5$..Therefore we get $n = \{2, 3, 4, 5\}$ Now for $n = 2$ our linear space W_n becomes a two dimensional invariant subspace W_2 given as :

$$W_2 = \{y \mid L[y] = y'' + a_1 y' + a_0 y = 0\} = Span\{h_1(x), h_2(x)\}$$

.

Therefore (3.5) becomes

$$D^2\hat{\mathcal{K}} + a_1 D\hat{\mathcal{K}} + a_0\hat{\mathcal{K}} \mid_{L(u)=0} = 0$$

. Applying this condition we get,
$-5a_1A_{uu}(u_x)^3+(-3a_0+4a_1^2)A_u(u_x)^2-2a_1B_u(u_x)^2-3a_0B_u(u_x)u+a_0C+A_{uuu}(u_x)^4+C_{uu}(u_x)^2+3a_0^2A_uu^2-a_0uC_u+B_{uu}(u_x)^3-6a_0uA_{uu}(u_x)^2+7a_0a_1uA_uu_x=0.$
This provides the total classification of two dimensional invariant subspaces.

Case1: Let us consider a two dimensional invariant subsapce $W_2=\{y \mid y''+a_1y'+a_0y=0\}$.Whenever $a_1\neq 0$ and $a_0=\frac{2}{9}a_1^2$,we obtain $W_2=Span\{e^{\frac{-2}{3}a_1x},e^{\frac{-1}{3}a_1x}\}$ is invariant if $A(u)=c_1u+c_0$, $B(u)=\frac{7}{3}a_1c_1u+d_0$ and $C(u)=\frac{2}{3}c_1a_1^2u^2+k_1u$, since,

$$G[A_1e^{\frac{-2}{3}a_1x}+A_2e^{\frac{-1}{3}a_1x}]=[\frac{1}{9}a_1^2c_1A_2^2+\frac{4}{9}c_0a_1^2A_1-\frac{2}{3}d_0a_1A_1+k_1A_1]e^{\frac{-2}{3}a_1x_1}$$
$$+[\frac{1}{9}c_0a_1^2A_2-\frac{1}{3}d_0a_1A_2+k_1A_2]e^{\frac{-1}{3}a_1x}\in W_2$$

Thus , we obtain the exact solution of (3.4) as follows

$$u(x,t)=A_1(t)e^{\frac{-2}{3}a_1x}+A_2(t)e^{\frac{-1}{3}a_1x}$$

where,$A_1(t),A_2(t)$ which satisfies the system of ODEs:

$$\frac{dA_1}{dt}=\frac{1}{9}a_1^2c_1A_2^2+\frac{4}{9}c_0a_1^2A_1-\frac{2}{3}d_0a_1A_1+k_1A_1$$

$$\frac{dA_2}{dt}=\frac{1}{9}c_0a_1^2A_2-\frac{1}{3}d_0a_1A_2+k_1A_2.$$

Solving this system of ODEs we get $A_1(t)=(-\frac{a_1^2c_1c^2e^{\frac{-1}{9}t(2a_1^2c_0-9k_1)}}{2a_1^2c_0-9k_1}+a)e^{\frac{1}{9}(4a_1^2c_0-6a_1d_0+9k_1)t}$ and $A_2(t)=be^{\frac{1}{9}(a_1^2c_0-3a_1d_0+9k_1)t}$. Therefore, the exact solution of the generalised CRD equation is :

$$u(x,t)=\left(-\frac{a_1^2c_1c^2e^{\frac{-1}{9}t(2a_1^2c_0-9k_1)}}{2a_1^2c_0-9k_1}+a\right)e^{\frac{1}{9}(4a_1^2c_0-6a_1d_0+9k_1)t-\frac{2}{3}a_1x}+be^{\frac{1}{9}(a_1^2c_0-3a_1d_0+9k_1)t-\frac{1}{3}a_1x},$$

where, a,c $\in\mathbb{R}$.
Let us see one more example which is also a two dimensional invariant subspace.
Case2: Now ,let us consider $W_2=\{y \mid y''+a_1y'+a_0y=0\}$.
Whenever $a_1=a_0=0$,we get a two dimensional polynomial subspace, $W_2=Span\{1,x\}$.

We obtain (3.5) as

$$D^2G\mid_{L(u)=0}=0.$$

This is invariant under the given differential operator G if A(u) ,B(u) and C(u) takes the specific forms :
$i)A(u)=c_1u+c_0, B(u)=d_1u+d_0, C(u)=k_1u+k_0,$
$ii)A(u)=c_2u^2+c_1u+c_0, B(u)=d_1u+d_0, C(u)=k_1u+k_0.$
Similarly proceeding as in case1 discussed above we get the exact solution for case(i) of (3.4) is as follows:

$$\begin{aligned}
&u(x,t)=\frac{1}{(ak_1e^{-k_1t}-d_1)}+\\
&\left[-ak_0e^{-k_1t}+k_0d_1t+c-\left(\frac{2k_0d_1}{k_1}+\frac{c_1k_1}{d_1}-d_0\right)ln(e^{k_1t}d_1-ak_1)\right.\\
&\left.+\left(\frac{2d_1k_0}{k_1}-d_0\right)ln(ak_1e^{-k_1t}-d_1)+k_1x\right],\ a,c\in\mathbb{R}.
\end{aligned}$$

Now, we can see that the invariant subspace methodin (1+1) dimension has been extended to coupled case.

4 Invariant subspace method to coupled PDE in (1+1) dimension

Let us consider a system of two coupled PDEs [5] with two independent variables , one space variable x and one time variable t in the form:

$$\frac{\partial u_1}{\partial t}=\hat{\mathcal{K}}_1(x,u_1,u_2,u_1^{(1)},u_2^{(1)}\cdots,u_1^{(m_1)},u_2^{(m_2)}), \tag{4.1}$$

$$\frac{\partial u_2}{\partial t}=\hat{\mathcal{K}}_2(x,u_1,u_2,u_1^{(1)},u_2^{(1)}\cdots,u_1^{(m_1)},u_2^{(m_2)}). \tag{4.2}$$

the order of non linear operators $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$ are m_1th and m_2th order and $u_q^{(j)}=\frac{\partial^j u_q(x,t)}{\partial x^j}, j=1,\cdots,m_q, q=1,2..$ They satisfies the following equations :

$$\left(\frac{\partial\hat{\mathcal{K}}_1}{\partial u_1^{(m_1)}}\right)^2+\left(\frac{\partial\hat{\mathcal{K}}_1}{\partial u_2^{(m_1)}}\right)^2\neq 0 \tag{4.3}$$

$$\left(\frac{\partial\hat{\mathcal{K}}_2}{\partial u_1^{(m_2)}}\right)^2+\left(\frac{\partial\hat{\mathcal{K}}_2}{\partial u_2^{(m_2)}}\right)^2\neq 0, \tag{4.4}$$

$$\left(\frac{\partial\hat{\mathcal{K}}_1}{\partial u_2}\right)^2+\left(\frac{\partial\hat{\mathcal{K}}_1}{\partial u_2^{(1)}}\right)^2+\cdots+\left(\frac{\partial\hat{\mathcal{K}}_1}{\partial u_2^{(m_1)}}\right)^2\neq 0 \tag{4.5}$$

$$\left(\frac{\partial\hat{\mathcal{K}}_2}{\partial u_1}\right)^2+\left(\frac{\partial\hat{\mathcal{K}}_2}{\partial u_1^{(1)}}\right)^2+\cdots+\left(\frac{\partial\hat{\mathcal{K}}_2}{\partial u_1^{(m_2)}}\right)^2\neq 0, \tag{4.6}$$

and

$$\frac{\partial^2\hat{\mathcal{K}}_q}{\partial u_r^{(l)}\partial u_p^{(s)}}\neq 0 \tag{4.7}$$

for some $p,q,r\in\{1,2\}$ and $l,s\in\{0,1,2,\cdots,k_q\}$.

Let W be a linear subspace, $W^1_{n_1}\times W^2_{n_2}$ where

$$\begin{aligned}W^q_{nq}&=\mathrm{Span}\{h^q_1(x),\cdots,h^q_{n_q}(x)\}\\&\equiv\{\sum_{j=1}^{n_q}\alpha^q_jh^q_j(x)\mid\alpha^q_j=constants,\ q=1,2,j=1,\cdots,n_q\},\end{aligned}$$

where $h^q_1(x),\cdots,h^q_{n_q}(x)$ are linearly independent functions. If the vector operator $\mathbb{K}=(\hat{\mathcal{K}}_1,\hat{\mathcal{K}}_2)$ satisfies the condition $\mathbb{K}:W^1_{n_1}\times W^2_{n_2}\to W^q_{n_q},q=1,2,$ i.e, $\hat{\mathcal{K}}_q:W^1_{n_1}\times W^2_{n_2}\to W^q_{n_q},q=1,2$. Therefore,

$$\hat{\mathcal{K}}_q\left[\sum_{j=1}^{n_1}\alpha^1_jh^1_j(x),\sum_{j=1}^{n_2}\alpha^2_jh^2_j(x)\right]=\sum_{j=1}^{n_q}\Psi^q_j(\alpha^1_1,\cdots,\alpha^1_{n_1},\alpha^2_1,\cdots,\alpha^2_{n_2})h^q_j(x).$$

If the vector operator $\mathbb{K}$ admits the subspace W, then the two -coupled PDEs (4.1) admits an exact solution of the form

$$u_q(x,t)=\sum_{j=1}^{n_q}\alpha^q_j(t)h^q_j(x),q=1,2,$$

with $\alpha^q_j(t)$ satisfying the following system of ODEs,

$$\frac{d\alpha^q_j(t)}{dt}=\Psi^q_j(\alpha^1_1(t),\cdots,\alpha^1_{n_1}(t),\alpha^2_1(t),\cdots,\alpha^2_{n2}(t)),j=1,\cdots,n_q,q=1,2.$$

Assume that $W^q_{n_q}=\mathrm{Span}\{h^q_1(x),\cdots,h^q_{n_q}(x)\}$ is generated by solutions of the linear n_qth order ODEs

$$L_q[y_q]=y_q^{(n_q)}+a^q_{n_q-1}(x)y_q^{(n_q-1)}+\cdots+a^q_1(x)y_q^{(1)}+a^q_0(x)y_q=0,\ q=1,2.$$

It follows that the invariant conditions for the subspace W with respect to the operator $\mathbb{K}$ are

$$L_q[\hat{\mathcal{K}}_q[u_1,u_2]]\mid_{[H_1]\cap[H_2]}=0,q=1,2,$$

where we denote $[H_q]$ the equation $L_q[u_q] = 0$ and its differential consequences with respect to x_1.

Theorem 4.1. *[5] Assume that* $(4.1) - (4.7)$*,* $m_1 \geq m_2$*,with* $\mathbb{K} = (\hat{\mathcal{K}}_1, \hat{\mathcal{K}}_2)$ *as a vector operator,then* $n_1 - n_2 \leq m_2, n_1 \leq 2(m_1 + m_2) + 1$ *, if the vector operator* $\mathbb{K}$ *acknoweledges the invariant subspace* $W^1_{n_1} \times W^2_{n_2} (n_1 \geq n_2 > 0)$ *.*

Theorem 4.2. *[5]Without loss of generality, assume that* $m_1 \geq m_2$*,with* $\hat{\mathcal{K}}_2 \in \mathbb{K}$ *a nonlinear differential operator and* $\mathbb{K}$ *a really coupled* (4.5)*. Then* $n_2 - n_1 \leq m_1, n_2 \leq 2(m_1 + m_2) + 1$ *holds when the operator* $\mathbb{K}$ *permits the invariant subspace* $W^1_{n_1} \times W^2_{n_2} (0 < n_1 < n_2)$*.*

Let us take a look at an example.

Example 2. *Let us consider nonlinear system of dispersive evolution equations [2]*

$$u_t = \hat{\mathcal{K}}_1 = (u_{xxx} + \alpha_1 v v_x)_x + \alpha_2 v^2, v_t = \hat{\mathcal{K}}_2 = u_{xxx} + \beta_1 u + \beta_2 v, \tag{4.8}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2$ are constants.

Let

$$u_x = \frac{\partial u}{\partial x}, v_x = \frac{\partial v}{\partial x}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

and so on.

After substituting for $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$, collecting the coefficients and further simplifying , we obtainthe system of over determined equations given as:

$$\begin{aligned}
&(v_x)^2 : 7\alpha_1 b_1^2 + \alpha_1 a_0 + 2\alpha_2 - 4\alpha_1 b_0 - 3\alpha_1 a_1 b_1 = 0, \\
&v v_x : 12\alpha_1 b_0 b_1 - \alpha_1 a_0 b_1 - 4\alpha_1 a_1 b_0 + 2\alpha_2 a_1 - \alpha_1 b_1^3 - 2\alpha_2 b_1 + \alpha_1 a_1 b_1^2 = 0, \\
&v^2 : 4\alpha_1 b_0^2 + \alpha_2 a_0 - \alpha_1 b_0 b_1^2 + \alpha_1 a_1 b_0 b_1 - \alpha_1 a_0 b_0 - 2\alpha_2 b_0 = 0, \\
&u_x : a_1^4 - 3a_0 a_1^2 + a_0^2 - \beta_1 a_1 - a_1^3 b_1 + 2a_0 a_1 b_1 + \beta_1 b_1 + a_1^2 b_0 - a_0 b_0 = 0, \\
&u : a_0 a_1^3 - 2a_0^2 a_1 - \beta_1 a_0 - a_0 a_1^2 b_1 + a_0^2 b_1 + a_0 a_1 b_0 + \beta_1 b_0 = 0.
\end{aligned}$$

Solving this system of equations using Maple, we obtain many invariant subspaces corresponding to the system of evolution equation. For example ,let us consider the system :

$$u_t = (u_{xxx} + \alpha_1 v v_x)_x - \frac{1}{8}\alpha_1 a_1^2 v^2, v_t = u_{xxx} + \frac{8}{27}a_1^3 u + \beta_2 v, \tag{4.9}$$

which admits an invariant subspace,

$$W_2^1 \times W_2^2 = Span\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{2}{3}a_1 x}\} \times Span\{e^{-\frac{1}{3}a_1 x}, e^{-\frac{1}{6}a_1 x}\}. \tag{4.10}$$

The exact solution of (5.2) will be in the form:

$$u = P_1(t)e^{-\frac{1}{3}a_1 x} + P_2(t)e^{-\frac{2}{3}a_1 x},$$

$$v = Q_1(t)e^{-\frac{1}{3}a_1 x} + Q_2(t)e^{-\frac{1}{6}a_1 x},$$

where the coefficients $P_1(t), P_2(t), Q_1(t)$ and $Q_2(t)$ are to be determined.After substituting in (5.2) and solving we get the required exact solution of (5.2) .

5 Invariant subspace method to two-dimensional coupled PDE

In this section, we present the extension of the invariant subspace method for coupled system of nonlinear PDE in (1+2) dimension that are discussed in [6]. Let us consider a coupled system of PDE in (1+2) dimension, that is , two space variables x_1 and x_2 and one time variable t, is given by

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} = \hat{\mathcal{K}}_1[u_1, u_2] \\
&\equiv \hat{\mathcal{K}}_1\left(x_1, x_2, u_1, u_2, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_2}, \frac{\partial^2 u_1}{\partial x_1 \partial x_2}, \frac{\partial^2 u_2}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^{m_1} u_2}{\partial x_1^{r_1}\partial x_2^{s_1}}\right) \\
&\frac{\partial u_2}{\partial t} = \hat{\mathcal{K}}_2[u_1, u_2] \\
&\equiv \hat{\mathcal{K}}_2\left(x_1, x_2, u_1, u_2, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_2}, \frac{\partial^2 u_1}{\partial x_1 \partial x_2}, \frac{\partial^2 u_2}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^{m_2} u_2}{\partial x_1^{r_2}\partial x_2^{s_2}}\right),
\end{aligned}$$

where $\hat{\mathcal{K}}_q[u_1, u_2], q = 1, 2$ are m_qth order of nonlinear differential operators, and $m_q = r_q + s_q, q = 1, 2$. Let $W_{n_1,n_2} = W_{n_1}^1 \times W_{n_2}^2$ be a linear space, that is defined as

$$\begin{aligned}
W_{n_q}^q &= \text{Span}\{\phi_1^q(x_1, x_2), \phi_2^q(x_1, x_2), \cdots, \phi_{n_q}^q(x_1, x_2)\} \\
&= \{\sum_{j=1}^{n_q} \alpha_j^q \phi_j^q(x_1, x_2) \mid \alpha_j^q \in \mathbb{R}, j = 1, \cdots, n_q\}, q = 1, 2.
\end{aligned} \tag{5.1}$$

The differential operator $\hat{\mathcal{K}} = (\hat{\mathcal{K}}_1, \hat{\mathcal{K}}_2)$ admits invariant subspace W_{n_1,n_2} if $\hat{\mathcal{K}}_q : W^1_{n_1} \times W^2_{n_2} \to W^q_{n_q}, q = 1, 2$.

Therefore,

$$\begin{aligned}
&\hat{\mathcal{K}}_q \left[\sum_{j=1}^{n_1} \alpha^1_j \phi^1_j(x_1, x_2), \sum_{j=1}^{n_2} \alpha^2_j \phi^2_j(x_1, x_2) \right] \\
&= \sum_{j=1}^{n_q} \psi^q_j(\alpha^1_1, \cdots, \alpha^1_{n_1}, \alpha^2_1, \cdots, \alpha^2_{n_2}) \phi^q_j(x_1, x_2).
\end{aligned}$$

There are two special cases of invariant subspaces for the differential operator $\hat{\mathcal{K}}$. For $q = 1, 2$, let $W^q_{k_q}$ and $W^q_{n_q}$ be represent

$$\begin{aligned}
W^q_{k_q} &= \mathrm{Span}\{h^q_1(x_1), \cdots, h^q_{k_q}(x_1)\}, \\
W^q_{n_q} &= \mathrm{Span}\{p^q_1(x_2), \cdots, p^q_{n_q}(x_2)\},\ q = 1, 2.
\end{aligned}$$

Assume that $W^q_{k_q}$ and $W^q_{n_q}$ are the space of solutions of the following linear ODEs

$$\begin{aligned}
L^{x_1}[w_q(x_1)] &= \frac{d^{k_q} w_q}{dx_1^{k_q}} + a_{k_q-1} \frac{d^{k_q-1} w_q}{dx_1^{k_q-1}} + \cdots + a_1 \frac{dw_q}{dx_1} + a_0 w_q = 0, \\
L^{x_2}[z_q(x_2)] &= \frac{d^{n_q} z_q}{dx_2^{n_q}} + b_{n_q-1} \frac{d^{n_q-1} z_q}{dx_2^{n_q-1}} + \cdots + b_1 \frac{dz_q}{dx_2} + b_0 z_q = 0.
\end{aligned}$$

We consider two kinds of invariant subspaces corresponding to $\hat{\mathcal{K}}$.

Type 1: Let $W_{k_1 n_1, k_2 n_2} = W^1_{k_1 n_1} \times W^2_{k_2 n_2}$

$$\begin{aligned}
W^q_{k_q n_q} =& \mathrm{Span}\{h^q_1(x_1)p^q_1(x_2), h^q_1(x_1)p^q_2(x_2), \cdots, h^q_1(x_1)p^q_{n_q}(x_2), h^q_2(x_1)p^q_1(x_2), \\
& \cdots, h^q_{k_q}(x_1)p^q_{n_q}(x_2)\} \\
=& \{\sum_{ij} \alpha^q_{ij} h^q_i(x_1) p^q_j(x_2) \mid \alpha^q_{ij} \in \mathbb{R}, i = 1, \cdots, k_q, j = 1, \cdots, n_q\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\hat{\mathcal{K}}_q \left[\sum_{ij} \alpha^1_{ij} h^1_i(x_1) p^1_j(x_2), \sum_{ij} \alpha^2_{ij} h^2_i(x_1) p^2_j(x_2) \right] \\
&= \sum_{ij} \psi^q_{ij}(\alpha^1_{11}, \cdots, \alpha_{k_1 n_1}, \alpha^2_{11}, \cdots, \alpha^2_{k_2 n_2}) h^q_i(x_1) p^q_j(x_2).
\end{aligned}$$

Thus, the exact solution of the system of PDEs will be in the form

$$u_q(t, x_1, x_2) = \sum_{i=1}^{k_q} \left(\sum_{j=1}^{n_q} \alpha^q_{ij}(t) p^q_j(x_2) \right) h^q_i(x_1) = \sum_{j=1}^{n_q} \left(\sum_{i=1}^{k_q} \alpha^q_{ij}(t) h^q_i(x_1) \right) p^q_j(x_2).$$

Clearly, we can see that $L^{x_1}[u_q] = 0$ and $L^{x_2}[u_q] = 0, q = 1, 2$.

For Type 1, the invariant condition takes the form $L^{x_1}[\hat{\mathcal{K}}_q[u_1, u_2]] \mid_{[H^{x_1x_2}]} = 0$, and $L^{x_2}[\hat{\mathcal{K}}_q[u_1, u_2]] \mid_{[H^{x_1x_2}]} = 0, q = 1, 2$, where $[H^{x_1x_2}]$ consists of $\{\cap_{q=1}^{2} L^{x_1}[u_q] = 0\} \cap \{\cap_{q=1}^{2} L^{x_2}[u_q] = 0\}$, and their differential consequences.

Type 2 : Consider the linear space $W_{k_1+n_1, k_2+n_2} = W^1_{k_1+n_1} \times W^2_{k_2+n_2}$ as

$$\begin{aligned} W^q_{k_q+n_q} =& \mathrm{Span}\{h^q_1(x_1), \cdots, h^q_1(x_1), p^q_1(x_2), p^q_2(x_2), \cdots, p^q_{n_q}(x_2)\} \\ =& \{\sum_{i=1}^{k_q} \alpha^q_i h^q_i(x_1) + \sum_{j=1}^{n_q} \beta^q_j p^q_j(x_2) \mid \alpha^q_i, \beta^q_j \in \mathbb{R}, i = 1, \cdots, k_q, j = 1, \cdots, n_q\}, \end{aligned}$$

where $q = 1, 2$. Thus, we have

$$\begin{aligned} &\hat{\mathcal{K}}_q \left[\sum_{i=1}^{k_1} \alpha^1_i h^1_i(x_1) + \sum_{j=1}^{n_1} \beta^1_j p^1_j(x_2), \sum_{i=1}^{k_2} \alpha^2_i h^2_i(x_1) + \sum_{j=1}^{n_2} \beta^2_j p^2_j(x_2) \right] \\ &= \sum_{i=1}^{k_q} \psi^q_i(.) h^q_i(x_1) + \sum_{j=1}^{n_q} \psi^q_j(.) p^q_j(x_2), \end{aligned}$$

where $\psi^q(.) = \psi^q(\alpha^1_1, \cdots, \alpha^1_{k_1}, \alpha^2_1, \cdots, \alpha^2_{k_2}, \beta^1_1, \cdots, \beta^1_{n_1}, \beta^2_1, \cdots, \beta^2_{n_2})$, $q = 1, 2$. Thus, $u(t, x, y) = (u_1(t, x, y), u_2(t, x, y))$ will be the solution of system of PDEs such that for $q = 1, 2$

$$u_q(t, x_1, x_2) = \sum_{i=1}^{k_q} K^q_i(t) h^q_i(x_1) + \sum_{j=1}^{n_q} L^q_j(t) p^q_j(x_2).$$

Clearly,

$$L^{x_1}[u_q] = 0, L^{x_2}[u_q] = 0, \text{and } \frac{\partial^2 u_q}{\partial x_1 \partial x_2} = 0, q = 1, 2.$$

For Type 2 the invariant condition is of the form
$L^{x_1}[\hat{\mathcal{K}}_q[u_1, u_2]] \mid_{[H^{x_1x_2}]} = 0, L^{x_2}[\hat{\mathcal{K}}_q[u_1, u_2]] \mid_{[H^{x_1x_2}]} = 0$ and $\left[\frac{\partial^2 \hat{\mathcal{K}}_q[u_1,u_2]}{\partial x_1 \partial x_2}\right] \mid_{[H^{x_1x_2}]} = 0, q = 1, 2$, where $[H^{x_1x_2}]$ denotes $\{\cap_{q=1}^{2} L^{x_1}[u_q] = 0\} \cap \{\cap_{q=1}^{2} L^{x_2}[u_q] = 0\} \cap \{\cap_{q=1}^{2} \frac{\partial^2 u_q}{\partial x_1 \partial x_2} = 0\}$, and their differential consequences.

The Effectiveness of the method has been illustrated through the following problem.

Example 3. *Let us consider a two dimensional system of nonlinear coupled diffusion equation[6],*

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= \hat{\mathcal{K}}_1[u_1, u_2] \equiv \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \mu_1 u_2 \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) + \mu_2 u_2^2, \\ \frac{\partial u_2}{\partial t} &= \hat{\mathcal{K}}_2[u_1, u_2] \equiv \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \mu_3 u_1 \left(\frac{\partial^2 u_2}{\partial x_1^2} + \frac{\partial^2 u_2}{\partial x_2^2} \right) + \mu_5 u_2. \end{aligned}$$

Type 1 invariant subspace is defined as

$$W_{k_1n_1,k_2n_2} = W^1_{k_1n_1} \times W^2_{k_2n_2}.$$

Let us consider $W_{4,4} = W^1_{2.2} \times W^2_{2.2}$. Then the invariant condition takes the form

$$\left(\frac{d^2\hat{\mathcal{K}}_1}{dx_1^2} + a_1\frac{d\hat{\mathcal{K}}_1}{dx_1} + a_0\hat{\mathcal{K}}_1\right)|_{[H^{x_1x_2}]} = 0,$$
$$\left(\frac{d^2\hat{\mathcal{K}}_1}{dx_2^2} + c_1\frac{d\hat{\mathcal{K}}_1}{dx_2} + c_0\hat{\mathcal{K}}_1\right)|_{[H^{x_1x_2}]} = 0,$$
$$\left(\frac{d^2\hat{\mathcal{K}}_2}{dx_1^2} + b_1\frac{d\hat{\mathcal{K}}_2}{dx_1} + b_0\hat{\mathcal{K}}_2\right)|_{[H^{x_1x_2}]} = 0,$$
$$\left(\frac{d^2\hat{\mathcal{K}}_2}{dx_2^2} + d_1\frac{d\hat{\mathcal{K}}_2}{dx_2} + d_0\hat{\mathcal{K}}_2\right)|_{[H^{x_1x_2}]} = 0,$$

where $[H^{x_1x_2}]$ contains $\{\cap_{q=1}^2 L^{x_1}[u_q] = 0\} \cap \{\cap_{q=1}^2 L^{x_2}[u_q] = 0\}$ and their differential consequences. Substituting the values of $\hat{\mathcal{K}}_1$ and $\hat{\mathcal{K}}_2$, and equating the coefficients to zero, we get some system of algebraic equations and by solving those equations we obtain different parametric constraints also various values of $a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$. For example, let us consider the invariant subspace as

$$\begin{aligned}
&W^1_{2.2} \times W^2_{2.2}\\
&= Span\{e^{x_2\sqrt{-a_1^2-\mu_3}}, e^{-x_2\sqrt{-a_1^2-\mu_3}}, \frac{-e^{-a_1x_1}}{a_1}e^{x_2\sqrt{-a_1^2-\mu_3}}, \frac{-e^{-a_1x_1}}{a_1}e^{-x_2\sqrt{-a_1^2-\mu_3}}\}\\
&\times Span\{e^{x_2\sqrt{-a_1^2-\mu_3}}, x_1e^{x_2\sqrt{-a_1^2-\mu_3}}, e^{-x_2\sqrt{-a_1^2-\mu_3}}, x_1e^{-x_2\sqrt{-a_1^2-\mu_3}}\}.
\end{aligned}$$

The obtained exact solution of (5.2) is as follows

$$\begin{aligned}
u_1(t,x_1,x_2) =& P_1(t)e^{x_2\sqrt{-a_1^2-\mu_3}} + P_2(t)e^{-x_2\sqrt{-a_1^2-\mu_3}}\\
&+ P_3(t)\frac{-e^{-a_1x_1}}{a_1}e^{x_2\sqrt{-a_1^2-\mu_3}} + P_4(t)\frac{-e^{-a_1x_1}}{a_1}e^{-x_2\sqrt{-a_1^2-\mu_3}},\\
u_2(t,x_1,x_2) =& P_5(t)e^{x_2\sqrt{-a_1^2-\mu_3}} + P_6(t)x_1e^{x_2\sqrt{-a_1^2-\mu_3}}\\
&+ P_7(t)e^{-x_2\sqrt{-a_1^2-\mu_3}} + P_8(t)x_1e^{-x_2\sqrt{-a_1^2-\mu_3}},
\end{aligned}$$

where the functions $P_i(t)$, $i = 1, \cdots, 8$ are to be determined.

6 Invariant subspace method to two-dimensional scalar PDE

In this section, we explain how to extend the invariant subspace method to scalar PDE.

Theorem 6.1. *Consider a PDE in (2+1) dimension as in the form*

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}\left(x_1, x_2, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial^m u}{\partial x_1^{m_1} \partial x_2^{m_2}}\right) \tag{6.1}$$

and if

$$W_n = Span\{h_k(x_1, x_2) : k \in \{1.2, ..., n\}\} \tag{6.2}$$

is a invariant subspace under the operator $\hat{\mathcal{K}}[u(x_1, x_2, t)]$, *where* $\frac{\partial u}{\partial t}$ *denotes the derivative of* u *with respect to* $t, m_1 + m_2 = m$ *and*

$$\{h_k(x_1, x_2) : k \in \{1.2, ..., n\}\} \tag{6.3}$$

is a linearly independent set, then there exists exact solution for the given PDE in separation of variables form

$$u(x_1, x_2, t) = \sum_{k=1}^{n} \alpha_k(t) h_k(x_1, x_2), \tag{6.4}$$

where $\{\alpha_k(t) : k \in \{1, 2, .., n\}\}$ *is given by the solution of ODE*

$$\frac{d\alpha_k(t)}{dt} = \Psi_k(\alpha_1(t), \alpha_2(t), ..., \alpha_n(t)) \quad \forall\ k \in \{1, 2, .., n\}, \tag{6.5}$$

where $\Psi_k(\alpha_1, ..., \alpha_n)$ *is given by the coefficients of* $\hat{\mathcal{K}}[u(x_1, x_2, t)]$ *when expanded with respect to the basis in* (6.3).

proof 1. *Consider a invariant subspace* W_n *under the differential operator* $\hat{\mathcal{K}}[u(x_1, x_2, t)]$ *and let* $u(x_1, x_2, t) = \sum\limits_{k=1}^{n} \alpha_k(t) h_k(x_1, x_2)$.
If we substitute $u(x_1, x_2, t)$ *in operator* $\hat{\mathcal{K}}$ *in* (6.1), *then by expansion of coefficients we have*

$$\hat{\mathcal{K}}[u(x_1, x_2, t)] = \sum_{k=1}^{n} \Psi_k(\alpha_1(t), \alpha_2(t), ...\alpha_n(t)) h_k(x_1, x_2). \tag{6.6}$$

Also by definition of partial derivatives we have

$$\frac{\partial u(x_1, x_2, t)}{\partial t} = \sum_{k=1}^{n} \frac{d\alpha_k(t)}{dt} h_k(x_1, x_2). \tag{6.7}$$

Consider the given PDE

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u(x_1, x_2, t)], \\
\Rightarrow &\frac{\partial u}{\partial t} - \hat{\mathcal{K}}[u(x_1, x_2, t)] = 0, \\
\Rightarrow &\sum_{k=1}^{n} [\frac{d\alpha_k(t)}{dt} - \Psi_k(\alpha_1(t), \alpha_2(t), ..., \alpha_n(t))] h_k(x_1, x_2) = 0.
\end{aligned}$$

Since set (6.3) *is a linearly independent set of functions, thus as required we have that the given PDE reduces to the system in* (6.5)

$$\frac{d\alpha_k(t)}{dt} = \Psi_k(\alpha_1(t), \alpha_2(t), ...\alpha_n(t)) \quad \forall\ k \in \{1, 2, ..n\}.$$

6.1 Estimation of invariant subspaces

In this section, our aim is to find the exact solution of CRD equation in (2+1) dimension, here we are in interested in two special kinds of invariant subspaces admitted by given (2+1) dimensional PDE. For that purpose, assume that the linearly independent sets of functions

$$W_{x_1} = \{h_i(x_1) : i \in \{1, 2, ..n_1\}\}, \quad W_{x_2} = \{p_j(x_2) : j \in \{1, 2, ..n_2\}\}$$

spans the solution spaces of homogeneous linear ODEs respectively as follows

$$\begin{aligned} L_{x_1}[y] \equiv & \frac{d^{n_1}y}{dx_1^{n_1}} + c_{n_1-1}\frac{d^{n_1-1}y}{dx_1^{n_1-1}} + c_{n_1-2}\frac{d^{n_1-2}y}{dx_1^{n_1-2}} + ... + c_0 y = 0, \\ L_{x_2}[y] \equiv & \frac{d^{n_2}y}{dx_2^{n_2}} + k_{n_2-1}\frac{d^{n_2-1}y}{dx_2^{n_2-1}} + k_{n_2-2}\frac{d^{n_2-2}y}{dx_2^{n_2-2}} + ... + k_0 y = 0. \end{aligned} \tag{6.8}$$

Type I: Consider the subspace of the kind

$$W_n = Span\{h_i(x_1)p_j(x_2) : i \in \{1, 2, ..n_1\}, j \in \{1, 2, ..n_2\}\}, \quad n = n_1 n_2. \tag{6.9}$$

The linear space W_n is admitted by $\hat{\mathcal{K}}$ as in (6.1) if it satisfies the the invariance condition

$$u \in Span(W_{x_1}) \ \cap \ Span(W_{x_2}) \Rightarrow \hat{\mathcal{K}}[u] \in Span(W_{x_1}) \ \cap \ Span(W_{x_2}) \tag{6.10}$$

and their differential consequences with respect to x_1 and x_2.

One of the important consequences of this kind of Type I invariant subspaces is that existence of $1 \in W_{x_1}\, or\, W_{x_2}\, or\, 1 \in W_{x_1} \ \cap \ W_{x_2}$ determines the dimension of the space, given by $\dim(W_n) = n_1 n_2$ .The reason behind the first two cases is that from(6.10), we get that

$1 \in W_{x_1} \ \Rightarrow c_0 = 0\ and\ 1 \in W_{x_2} \ \Rightarrow k_0 = 0$ then

$W_n = Span\{p_1(x_2), \ldots, p_{n_2}(x_2), h_2(x_1)p_1(x_2), \ldots, h_{n_1}(x_1)p_{n_2}(x_2)\}\ if\ 1 = h_1(x_1) \in W_{x_1}$

$W_n = Span\{h_1(x_1), \ldots, h_{n_1}(x_1), h_1(x_1)p_2(x_2), \ldots, h_{n_1}(x_1)p_{n_2}(x_2)\}\ if\ 1 = p_1(x_2) \in W_{x_2}$.

Similarly if $1 \in W_{x_1} \ \cap \ W_{x_2}$ then

$W_n = Span\{h_1(x_1), \ldots, h_{n_1}(x_1), p_1(x_2), \ldots, p_{n_2}(x_2), ...h_1(x_1)p_1(x_2), \ldots, h_{n_1}(x_1)p_{n_2}(x_2)\}$.

In all the above cases, we get invariant subspaces of dimension $n_1 n_2$.

Type II: Consider the subspace of the kind

$$W_n = Span\{h_i(x_1)\,, p_j(x_2)\ : i \in \{1,2,..n_1\},\, j \in \{1,2,..n_2\}\}, \quad n = n_1 + n_2. \tag{6.11}$$

The linear space W_n is admitted by $\hat{\mathcal{K}}$ as in (6.1) if it satisfies the invariance condition

$$\begin{aligned} u \in Span(W_{x_1}) \cap Span(W_{x_2}), \quad & and \quad \frac{\partial^2 u(x_1,x_2,t)}{\partial x_1 \partial x_2} = 0 \\ \Longrightarrow \hat{\mathcal{K}}[u] \in Span(W_{x_1}) \cap Span(W_{x_2}), \quad & and \quad \frac{\partial^2 \hat{\mathcal{K}}[u(x_1,x_2,t)]}{\partial x_1 \partial x_2} = 0 \end{aligned} \tag{6.12}$$

and their differential consequences with respect to x_1 and x_2.

As similar to type I, here too existence of unit 1 in the spanning sets $W_{x_1},\ W_{x_2}$ gives some important information about the dimension of invariant subspace which plays a important role in determining the space itself.

If $1 \in W_{x_1}$ then $c_0 = 0$ thus $W_n = \{1, h_2(x_1) \ldots h_{n_1}(x_1), p_1(x_2), \ldots p_{n_2}(x_2)\}$

If $1 \in W_{x_2}$ then $k_0 = 0$ thus $W_n = \{1, h_1(x_1) \ldots h_{n_1}(x_1), p_2(x_2), \ldots p_{n_2}(x_2)\}$

which implies dimension is equal to $n_1 + n_2$ in both cases.

And if $1 \in W_{x_1} \cap W_{x_2}$ then $c_0 = 0 = k_0$, then $W_n = \{1, h_2(x_1) \ldots h_{n_1}(x_1), p_2(x_2), \ldots p_{n_2}(x_2)\}$ where without loss of generality we chose $h_1(x_1) = 1 = p_1(x_2)$ which gives dimension is $n_1 + n_2 - 1$.

7 Invariant subspaces for (2+1)-dimensional CRD equation

In this section, we deal with the study for finding the invariant subspaces for the well known family of generalized (2+1)-dimensional non-linear convection-reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} B_i(u)\frac{\partial u}{\partial x_i} + C(u), \tag{7.1}$$

where the functions $A_i(u)$, $B_i(u)$, $C(u)$

$i \in \{1,2\}$ denotes the process of diffusion, convection and reaction of a phenomenon under consideration. We explain how to find the invariant subspaces for the generalised (2+1)-dimensional non-linear CRD equation with different types of nonlinearities. The linear space is defined as a solution space of linear homogeneous ODEs as

$$Span(W_{x_1}) = \{y : L_{x_1}[y] \equiv \frac{d^{n_1}y}{dx_1^{n_1}} + a_{n_1-1}\frac{d^{n_1-1}y}{dx_1^{n_1-1}} + a_{n_1-2}\frac{d^{n_1-2}y}{dx_1^{n_1-2}} + ... + a_0 y = 0\} \tag{7.2}$$

$$Span(W_{x_2}) = \{y : L_{x_2}[y] \equiv \frac{d^{n_2}y}{dx_2^{n_2}} + b_{n_2-1}\frac{d^{n_2-1}y}{dx_2^{n_2-1}} + b_{n_2-2}\frac{d^{n_2-2}y}{dx_2^{n_2-2}} + ... + b_0 y = 0\} \tag{7.3}$$

Now we wish to find invariant subspace of Type I and Type II as mentioned in (6.9) and (6.11) with the invariance condition given in (6.10) and (6.12) for the operator $\hat{\mathcal{K}}[u]$ given in (7.1).

The invariance condition for subspaces of kind **Type I** under the operator $\hat{\mathcal{K}}[u]$ can be expressed as

$$\begin{aligned} &L_{x_1}[u] \equiv \frac{d^{n_1}u}{dx_1^{n_1}} + ... + a_0 u = 0, L_{x_2}[u] \equiv \frac{d^{n_2}u}{dx_2^{n_2}} + ... + b_0 u = 0 \\ \Longrightarrow &L_{x_1}[\hat{\mathcal{K}}[u]] \equiv \frac{d^{n_1}\hat{\mathcal{K}}[u]}{dx_1^{n_1}} + ... + a_0\hat{\mathcal{K}}[u] = 0, L_{x_2}[\hat{\mathcal{K}}[u]] \equiv \frac{d^{n_2}\hat{\mathcal{K}}[u]}{dx_2^{n_2}} + ... + b_0\hat{\mathcal{K}}[u] = 0 \end{aligned} \tag{7.4}$$

The invariance condition for subspaces of kind **Type II** under the operator $\hat{\mathcal{K}}[u]$ can be expressed as

$$\begin{aligned} &L_{x_1}[u] \equiv \frac{d^{n_1}u}{dx_1^{n_1}} + ... + a_0 u = 0, L_{x_2}[u] \equiv \frac{d^{n_2}u}{dx_2^{n_2}} + ... + b_0 u = 0 \quad and \quad \frac{\partial^2 u(x_1, x_2, t)}{\partial x_1 \partial x_2} = 0 \\ \Longrightarrow &L_{x_1}[\hat{\mathcal{K}}[u]] \equiv \frac{d^{n_1}\hat{\mathcal{K}}[u]}{dx_1^{n_1}} + ... + a_0\hat{\mathcal{K}}[u] = 0, L_{x_2}[\hat{\mathcal{K}}[u]] \equiv \frac{d^{n_2}\hat{\mathcal{K}}[u]}{dx_2^{n_2}} + ... + b_0\hat{\mathcal{K}}[u] = 0 \\ &and \quad \frac{\partial^2\hat{\mathcal{K}}[u]}{\partial x_1 \partial x_2} = 0, \end{aligned} \tag{7.5}$$

where $a_i\, i \in \{0, 1, \ldots, n_1 - 1\}$ and $b_j\, j \in \{0, 1, \ldots, n_2 - 1\}$ are constants to be determined.

7.1 CRD equations with non-linearities

Consider the generalised CRD operator with different non-linearities in (2+1) dimension can be expressed as

$$\hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} B_i(u)\frac{\partial u}{\partial x_i} + C(u)$$

Quadratic non-linearity kind:

$$\begin{aligned} A_1(u) &= c_1u + c_0, \\ A_2(u) &= \beta_1u + \beta_0, \\ B_1(u) &= d_1u + d_0, \\ B_2(u) &= \lambda_1u + \lambda_0, \\ C(u) &= k_2u^2 + k_1u + k_0. \end{aligned} \tag{7.6}$$

Cubic non-linearity kind:

$$\begin{aligned} A_1(u) &= c_2u^2 + c_1u + c_0, \\ A_2(u) &= \beta_2u^2 + \beta_1u + \beta_0, \\ B_1(u) &= d_2u^2 + d_1u + d_0, \\ B_2(u) &= \lambda_2u^2 + \lambda_1u + \lambda_0, \\ C(u) &= k_3u^3 + k_2u^2 + k_1u + k_0. \end{aligned} \tag{7.7}$$

Other non-linearity:

$$\begin{aligned} A_1(u) &= c_3u^3 + c_2u^2 + c_1u + c_0, \\ A_2(u) &= \beta_3u^3 + \beta_2u^2 + \beta_1u + \beta_0 \\ B_1(u) &= d_3u^3 + d_2u^2 + d_1u + d_0, \\ B_2(u) &= \lambda_3u^3 + \lambda_2u^2 + \lambda_1u + \lambda_0, \\ C(u) &= k_4u^4 + k_3u^3 + k_2u^2 + k_1u + k_0 \end{aligned} \tag{7.8}$$

where $c_i, \beta_i, d_i, \lambda_i, k_i$ for $i = 1, 2$ are real arbitrary constants. Using the invariance conditions mentioned in (7.4) and (7.5) and $\hat{\mathcal{K}}[u]$ as above, we get different invariant subspaces for various values of n_1, n_2. We will specifically study the invariant subspaces of kind Type I and Type II of these operators. After direct symbolic computation, we get the subspaces as given in the table. We get exponential spaces, polynomial spaces, trigonometric spaces and their combination as invariant subspaces. To illustrate method, let us consider some specific values for n_1, n_2. Suppose both take value two then the invariance condition for Type I and Type II subspaces is given by

For kind *Type I* the invariance condition given in (7.4) reads as

$$\begin{aligned} & L_{x_1}[u] \equiv \frac{d^2u}{dx_1^2} + a_1\frac{du}{dx_1}a_0u = 0, L_{x_2}[u] \equiv \frac{d^2u}{dx_2^2} + b_1\frac{du}{dx_2} + b_0u = 0 \\ \Longrightarrow & L_{x_1}[\hat{\mathcal{K}}[u]] \equiv \frac{d^2\hat{\mathcal{K}}[u]}{dx_1^2} + a_1\frac{d\hat{\mathcal{K}}[u]}{dx_1} + a_0\hat{\mathcal{K}}[u] = 0, L_{x_2}[\hat{\mathcal{K}}[u]] \equiv \frac{d^2\hat{\mathcal{K}}[u]}{dx_2^2} + b_1\frac{d\hat{\mathcal{K}}[u]}{dx_2}b_0\hat{\mathcal{K}}[u] = 0 \end{aligned} \tag{7.9}$$

For kind *Type II* the invariance condition given in (7.5) reads as

$$
\begin{aligned}
&L_{x_1}[u] \equiv \frac{d^2u}{dx_1^2} + a_1\frac{du}{dx_1}a_0u = 0, L_{x_2}[u] \equiv \frac{d^2u}{dx_2^2} + b_1\frac{du}{dx_2} + b_0u = 0, \frac{\partial^2 u}{\partial x_1 \partial x_2} = 0\\
\Longrightarrow &L_{x_1}[\hat{\mathcal{K}}[u]] \equiv \frac{d^2\hat{\mathcal{K}}[u]}{dx_1^2} + a_1\frac{d\hat{\mathcal{K}}[u]}{dx_1} + a_0\hat{\mathcal{K}}[u] = 0, L_{x_2}[\hat{\mathcal{K}}[u]] \equiv \frac{d^2\hat{\mathcal{K}}[u]}{dx_2^2} + b_1\frac{d\hat{\mathcal{K}}[u]}{dx_2}b_0\hat{\mathcal{K}}[u] = 0,\\
&\frac{\partial^2\hat{\mathcal{K}}[u]}{\partial x_1\partial x_2} = 0
\end{aligned}
\tag{7.10}
$$

and the differential consequences with respect to x_1, x_2 which results in an overdetermined system of equations.The over determined system is obtained by substituting the differential operator on invariance condition mentioned above in both cases type I and type II.The substitution of general Convection-Reaction-Diffusion operator on the equations (7.9) and(7.10) yields the following equations.
Invariance condition for Type I subspaces

$$
\begin{aligned}
&2(A_2)_u u_{x_1x_2}^2 + [4(A_2)_{uu}u_{x_2} - 2(A_2)_u b_1 + 2(B_2)_u]u_{x_1x_2} + (A_1)_{uuu}u_{x_1}^4 + [-5(A_1)_{uu}a_1 + (B_1)_{uu}]u_{x_1}^3 +\\
&[u_{x_2}^2(A_2)_{uuu} + (-(A_2)_{uu}b_1 + (B_2)_{uu})u_{x_2} - 6(A_1)_{uu}a_0u + 4(A_1)_u a_1^2 - (A_2)_{uu}b_0u - 3(A_1)_u a_0\\
&-2(A_2)_u b_0 - 2(B_1)_u a_1 + C_{uu}]u_{x_1}^2 + [7(A_1)_u a_0a_1u - 3(B_1)_u a_0u]u_{x_1} + [-(A_2)_{uu}a_0u - (A_2)_u a_0]u_{x_2}^2 +\\
&+ [(A_2)_u a_0b_1u - (B_2)_u a_0u]u_{x_2} + 3(A_1)_u a_0^2u^2 + (A_2)_u b_0u^2a_0 - C_u a_0u + a_0C = 0\\
&\qquad\qquad\qquad\qquad and\\
&[-A_1a_1 + 2(A_1)_u u_{x_1} + B_1]u_{x_1x_2x_2} + 2(A_1)_u u_{x_1x_2}^2 + [(4(A_1)_u u_{x_2} + 2(A_1)_{u)}b_1)u_{x_1} +\\
&(-2(A_1)_u a_1 + 2(B_1)_u)u_{x_2} - A_1b_1a_1 + B_1b_1]u_{x_1x_2} + [(A_1)_{uuu}u_{x_2}^2 - (A_1)_{uu}b_0u + b_0(A_1)_u]u_{x_2}^2 +\\
&[(-(A_1)_{uu}a_1 + (B_1)_{uu})u_{x_2}^2 + (A_1)_u a_1b_0u - b_0A_1a_1 - (B_1)_u b_0u + b_0B_1]u_{x_1} + (A_2)_{uuu}u_{x_2}^4 +\\
&[-5(A_2)_{uu}b_1 + (B_2)_{uu}]u_{x_2}^3 + [-(A_1)_{uu}a_0u - 6(A_2)_{uu}b_0u + 4(A_2)_u b_1^2 - 2(B_2)_u b_1 - 2(A_1)_{u)}a_0\\
&-3(A_2)_u b_0 + C_{uu}]u_{x_2}^2 + [7(A_2)_u b_0b_1u - 3(B_2)_u b_0u]u_{x_2} + (A_1)_u a_0b_0u^2 + 3(A_2)_u b_0^2u^2 - C_u b_0u + b_0C = 0
\end{aligned}
\tag{7.11}
$$

Invariance condition for Type II subspaces

$$
\begin{aligned}
&(A_1)_{uuu}u_{x_1}^4 + [-5(A_1)_{uu}a_1 + (B_1)_{uu}]u_{x_1}^3 + [u_{x_2}^2(A_2)_{uuu} + (-(A_2)_{uu}b_1 + (B_2)_{uu})u_{x_2}\\
&- 6(A_1)_{uu}a_0u + 4(A_1)_u a_1^2 - (A_2)_{uu}b_0u - 3(A_1)_u a_0 - 2(A_2)_u b_0 - 2(B_1)_u a_1 + C_{uu}]u_{x_1}^2\\
&+[7(A_1)_u a_0a_1u - 3(B_1)_u a_0u]u_{x_1} + [-(A_2)_{uu}a_0u - (A_2)_u a_0]u_{x_2}^2 + [(A_2)_u a_0b_1u - (B_2)_u a_0u]u_{x_2}\\
&+ 3(A_1)_u a_0^2u^2 + (A_2)_u b_0u^2a_0 - C_u a_0u + a_0C = 0
\end{aligned}
$$

and

$$[(A_1)_{uuu}u_{x_2}^2-(A_1)_{uu}b_0u+b_0(A_1)_u]u_{x_2}^2+[(-(A_1)_{uu}a_1+(B_1)_{uu})u_{x_2}^2+(A_1)_ua_1b_0u-b_0A_1a_1-(B_1)_ub_0u$$
$$+b_0B_1]u_{x_1}+(A_2)_{uuu}u_{x_2}^4+[-5(A_2)_{uu}b_1+(B_2)_{uu}]u_{x_2}^3+[-(A_1)_{uu}a_0u-6(A_2)_{uu}b_0u+4(A_2)_ub_1^2$$
$$-2(B_2)_ub_1-2(A_1)_{u)}a_0-3(A_2)_ub_0+C_{uu}]u_{x_2}^2+[7(A_2)_ub_0b_1u-3(B_2)_ub_0u]u_{x_2}+(A_1)_ua_0b_0u^2$$
$$+3(A_2)_ub_0^2u^2-C_ub_0u+b_0C=0 \tag{7.12}$$

But in practical, it is not always easy to reduce and solve the above mentioned invariance condition to an over determined system of partial differential equations. For the sake of convenience, here we illustrate the overdetermined equations obtained for case of $n_1, n_2 = 2$ having cubic non linearity on $\hat{\mathcal{K}}[u]$ mentioned in (7.7) yields the corresponding operator as given below

$$\begin{aligned}\hat{\mathcal{K}}[u] \equiv & \frac{\partial}{\partial x_1}\left[(c_2u^2+c_1u+c_0)\left(\frac{\partial u}{\partial x_1}\right)\right]+\frac{\partial}{\partial x_2}\left[(\beta_2u^2+\beta_1u+\beta_0)\left(\frac{\partial u}{\partial x_2}\right)\right]\\ & +(d_2u^2+d_1u+d_0)\left(\frac{\partial u}{\partial x_1}\right)+(\lambda_2u^2+\lambda_1u+\lambda_0)\left(\frac{\partial u}{\partial x_2}\right)\\ & +k_3u^3+k_2u^2+k_1u+k_0\end{aligned} \tag{7.13}$$

where $c_i, \beta_i, d_i, \lambda_i, k_i$for $i \in \{1, 2\}$ are real constants. For illustrative convenience we here only present the over determined equation so obtained by the symbolic computation. The invariance condition on the operator for Type II subspaces yields the following over determined system

$$8a_1^2c_2-18a_0c_2-4a_1d_2-6b_0\beta_2+6k_3=0, 4a_1^2c_1-3a_0c_1-2a_1d_1-2b_0\beta_1+2k_2=0,$$
$$8b_1^2\beta_2-6a_0c_2-18b_0\beta_2-4b_1\lambda_2+6k_3=0, 4b_1^2\beta_1-2a_0c_1-3b_0\beta_1-2b_1\lambda_1+2k_2=0, a_0\beta_1=0,$$
$$2(a_1^2c_2+b_1^2\beta_2-a_1d_2-b_1\lambda_2+3k_3)-14(a_0c_2+b_0\beta_2)=0, 6a_1c_2-2d_2=0,$$
$$a_1^2c_1+b_1^2\beta_1-4a_0c_1-a_1d_1-4b_0\beta_1-b_1\lambda_1+2k_2=0, a_0\beta_2=0, b_0c_2=0, b_0c_1=0,$$
$$2a_0b_1\beta_2-2a_0\lambda_2=0, a_0k_0=0, 6b_1\beta_2-2\lambda_2=0, a_0b_1\beta_1-a_0\lambda_1=0, 2d_2-2a_1c_2=0,$$
$$b_0b_1\beta_0-b_0\lambda_0=0, 2a_0a_1c_1-2a_0d_1=0, a_0a_1c_0-a_0d_0=0, 2a_1b_0c_2-2b_0d_2=0, a_1b_0c_1-b_0d_1=0,$$
$$2\lambda_2-10b_1\beta_2=0, 2d_2-10a_1c_2=0, 2\lambda_2-2b_1\beta_2=0, 7a_0a_1c_1-3a_0d_1=0, 7b_0b_1\beta_1-3b_0\lambda_1=0,$$
$$14b_0b_1\beta_2-6b_0\lambda_2=0, 14a_0a_1c_2-6a_0d_2=0, 3a_0a_1c_2-3a_0d_2=0, 2b_0b_1\beta_1-2b_0\lambda_1=0, b_0k_0=0,$$
$$3b_0b_1\beta_2-3b_0\lambda_2=0, 3a_0^2c_1+a_0b_0\beta_1-a_0k_2=0, 2a_0b_0c_2+6b_0^2\beta_2-2b_0k_3=0, a_0b_0c_1+3b_0^2\beta_1-b_0k_2=0,$$
$$6a_0^2c_2+2a_0b_0\beta_2-2a_0k_3=0 \tag{7.14}$$

Invariant subspaces admitted by non-linear Convection Reaction Diffusion operator after solving this equation is tabulated as in Table 1,in which we obtained exponential spaces, polynomial spaces and exponential and polynomial spaces along with their operators.Proceeding similarly we obtained invariant subspaces of type I and II for different values of n_1, n_2.

Table 1: Invariant subspaces and operators obtained from (7.14)

Cases	$\hat{\mathcal{K}}[u]=\sum_{i=1}^{2}(A_i(u)u_{x_i})_{x_i}+\sum_{i=1}^{2}B_i(u)u_{x_i}+C(u)$ $A_i(u),\ B_i(u),\ \&\ C(u), i\in\{1,2\}$	Invaraint Subspaces
1.	$\begin{cases} A_1(u)=\frac{-b_1^2\beta_1}{a_1^2}u+c_0 \\ A_2(u)=\beta_1 u+\beta_0 \\ B_1(u)=\frac{-4b_1^2\beta_1+b_1\lambda_1}{a_1}u+d_0 \\ B_2(u)=\lambda_1 u+\lambda_0 \\ C(u)=(-2b_1^2\beta_1+b_1\lambda_1)u^2+k_1u+k_0 \end{cases}$	$\mathcal{Y}_3=Span\{1,e^{-a_1x_1},e^{-b_1x_2}\}$
2.	$\begin{cases} A_1(u)=c_2u^2+c_1u+c_0 \\ A_2(u)=\beta_2u^2+\beta_1u+\beta_0 \\ B_1(u)=d_1u+d_0 \\ B_2(u)=\lambda_1u+\lambda_0 \\ C(u)=k_1u+k_0 \end{cases}$	$\mathcal{Y}_3=Span\{1,x_1,x_2\}$
3.	$\begin{cases} A_1(u)=c_0 \\ A_2(u)=\beta_0 \\ B_1(u)=d_0 \\ B_2(u)=\lambda_0 \\ C(u)=k_1u+k_0 \end{cases}$	$\left\{\ \mathcal{Y}_4=Span\{1,e^{-a_1x_1},e^{-b_1x_2},e^{-a_1x_1-b_1x_2}\}\right.$

8 Exact solutions

Here in this section we would like to present exact solution of some C-R-D equations in (2+1) dimension using the obtained invariant subspaces.

Example 4. *Let us consider the nonlinear generalised CRD equation*

$$\hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} B_i(u)\frac{\partial u}{\partial x_i} + C(u),$$

where $A_i(u)$, $B_i(u)$ and $C(u)$ represents the diffusion, convection and reaction phenomenon respectively.

From the above table we are going to find the exact solution corresponding to the invariant subspaces.

CASE 1:

Let us assume for the values of $n_1 = 1$ and $n_2 = 2$. For illustrative convenience we here only present the over determined equation so obtained by the symbolic computation. The invariance condition on the operator for Type I subspaces yields the following system of algebraic equations

$$\begin{aligned}
&a_0k_0 = 0, b_0k_0 = 0,\\
&-a_0\beta_1 = 0, -4a_0\beta_2 = 0,\\
&-10b_1\beta_2 + 2\lambda_2 = 0, a_0b_1\beta_1 - a_0\lambda_1 = 0,\\
&2a_0b_1\beta_2 - 2a_0\lambda_2 = 0, 7b_0b_1\beta_1 - 3b_0\lambda_1 = 0,\\
&14b_0b_1\beta_2 - 6b_0\lambda_2 = 0, -6a_0^2b_0c_2 + 2a_0b_0d_2 + 6b_0^2\beta_2 = 0,\\
&-6a_0^3c_2 + 2a_0^2d_2 + 2a_0b_0\beta_2 = 0, -2a_0^2b_0c_1 + a_0b_0d_1 + 3b_0^2\beta_1 - b_0k_2 = 0,\\
&-2a_0^3c_1 + a_0^2d_1 + a_0b_0\beta_1 - a_0k_2 = 0, 18a_0^2c_2 + 8b_1^2\beta_2 - 6a_0d_2 - 18b_0\beta_2 - 4b_1\lambda_2 = 0,\\
&4a_0^2c_1 + 4b_1^2\beta_1 - 2a_0d_1 - 3b_0\beta_1 - 2b_1\lambda_1 + 2k_2 = 0
\end{aligned} \tag{8.1}$$

After solving these algebraic equations, we obtain the invariant subspace as:
$W_1 = Span\{e^{-a_0x_1}\}$ and $W_2 = Span\{1, e^{-b_1x_2}\}$

For type I, the invariant subspace becomes

$$W_2 = Span\{e^{-b_1x_2 - a_0x_1}, e^{-a_0x_1}\}$$

Also, $A_1(u) = \frac{1}{3}\frac{d_2}{a_0}u^2 + c_1 u + c_0$,
$A_2(u) = \beta_0$,
$B_1(u) = d_2 u^2 + d_1 u + d_0$,
$B_2(u) = \lambda_0$,
and $C(u) = (-2a_0^2 c_1 + a_0 d_1)u^2 + k_1 u$.
Substituting and simplifying we get reduced equation as

$$\begin{aligned} \frac{dp_1(t)}{dt} &= c_0 p_1(t) a_0^2 + \beta_0 p_1(t) b_1^2 - d_0 p_1(t) a_0 - \lambda_0 p_1(t) b_1 + k_1 p_1(t), \\ \frac{dp_2(t)}{dt} &= k_1 p_2(t) + c_0 p_2(t) a_0^2 - d_0 p_2(t) a_0. \end{aligned}$$

Solving these reduced equation we get the required exact solution as

$$u(x_1, x_2, t) = C_1 e^{(a_0^2 c_0 + b_1^2 \beta_0 - a_0 d_0 - b_1 \lambda_0 + k_1)t - b_1 x_2 - a_0 x_1} + C_2 e^{(a_0^2 c_0 - a_0 d_0 + k_1)t - a_0 x_1},$$

where C_1 and C_2 are arbitrary constants.

CASE 2:

Now, let us consider for $n_1 = 2$ and $n_2 = 2$. The system of algebraic equations for type 1 is mentioned above (7.14), for which we got the invariant subspace as

$$W_4 = Span\{1, e^{-a_1 x_1}, e^{-b_1 x_2}, e^{-a_1 x_1 - b_1 x_2}\},$$

for which ,$A_1(u) = c_0$,
$A_2(u) = \beta_0$,
$B_1(u) = d_0$,
$B_2(u) = \lambda_0$, and
$C(u) = k_1 u + k_0$.
As proceeding the above case we get the reduced equations as follows:

$$\begin{gathered} \frac{dp_1(t)}{dt} = k_1 p_1(t) + k_0, \\ \frac{dp_2(t)}{dt} = c_0 p_2(t) a_1^2 - d_0 p_2(t) a_1 + k_1 p_2(t), \\ \frac{dp_3(t)}{dt} = \beta_0 p_3(t) b_1^2 - \lambda_0 p_3(t) b_1 + k_1 p_3(t), \\ \frac{dp_4(t)}{dt} = c_0 p_4(t) a_1^2 + \beta_0 p_4(t) b_1^2 - d_0 p_4(t) a_1 - \lambda_0 p_4(t) b_1 + k_1 p_4(t). \end{gathered}$$

Therefore, the exact solution corresponding to the invariant subspace
$W_4 = Span\{1, e^{-a_1 x_1}, e^{-b_1 x_2}, e^{-a_1 x_1 - b_1 x_2}\}$, is given by:

$$u(x_1,x_2,t) = \frac{-k_0}{k_1} + e^{k_1 t}C_1 + C_2 e^{(a_1^2 c_0 - a_1 d_0 + k_1)t - a_1 x_1} + C_3 e^{(b_1^2 \beta_0 - b_1 \lambda_0 + k_1)t - b_1 x_2} + C_4 e^{(a_1^2 c_0 + b_1^2 \beta_0 - a_1 d_0 - b_1 \lambda_0 + k_1)t - a_1 x_1 - b_1 x_2}$$

9 Special Kinds of Convection Reaction Diffusion Equation on (2+1) dimension and the invariant subspaces

Special type of CRD equation comprises of convection-diffusion equation, reaction-diffusion equation and diffusion equation which shows their importance in all fields of science and engineering. Here we present the general form of these equations and classified the invariant subspaces admitted by the corresponding operators as similar to the case of CRD equation on (2+1) dimension. The functions $A_i(u)$ denotes the process of diffusion, $B_i(u)$ denotes convection for $i \in \{1, 2\}$ and $C(u)$ denotes reaction of a system under consideration with space variables x_1, x_2 and time-variable t.

- **Convection-diffusion equation on (2+1) dimension**

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} B_i(u)\frac{\partial u}{\partial x_i} \tag{9.1}$$

- **Reaction-diffusion equation on (2+1) dimension**

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + C(u). \tag{9.2}$$

- **Diffusion equation on (2+1) dimension**

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) \tag{9.3}$$

Now, we will classify invariant subspaces determined by the generalised (2+1) dimensional time-fractional non-linear Convection-Diffusion equation as in (9.1), Reaction-Diffusion equation as in (9.5) and Diffusion equation as in (9.8) under quadratic non linearity(7.6),cubic non linearity(7.7) and other non linearity(7.8).Using the similar method we followed for CRD equation on (2+1) dimension,we find the invariance condition for the operators and using that we obtain an over determined system of equation from which we generates the invariant subspaces for these operators. Let us see an example of each special case.

Example 5. ***Convection-diffusion equation on (2+1) dimension***

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{2} B_i(u)\frac{\partial u}{\partial x_i}. \tag{9.4}$$

Let us consider for $n_1 = 1$ and $n_2 = 2$. The system of algebraic equations for type I is mentioned above (8.1), for which we got the invariant subspace as

$$W_2 = Span\{e^{-b_1x_2-a_0x_1}, e^{-a_0x_1}\}$$

is invariant under the differential operator $\hat{\mathcal{K}}$ if $A_1(u) = \frac{1}{3}\frac{d_2}{a_0}u^2 + \frac{1}{2}\frac{d_1}{a_0}u + c_0$,
$A_2(u) = \beta_0$,
$B_1(u) = d_2u^2 + d_1u + d_0$,
$B_2(u) = \lambda_0$.
Substituting and simplifying we get reduced equation as

$$\frac{dp_1(t)}{dt} = c_0p_1(t)a_0^2 - d_0p_1(t)a_0,$$

$$\frac{dp_2(t)}{dt} = c_0p_2(t)a_0^2 + \beta_0p_2(t)b_1^2 - d_0p_2(t)a_0 - \lambda_0p_2(t)b_1.$$

Solving these reduced equation we get the required exact solution as:

$$u(x_1, x_2, t) = C_1e^{a_0(a_0c_0-d_0)t-a_0x_1} + C_2e^{(a_0^2c_0+b_1^2\beta_0-a_0d_0-b_1\lambda_0)t-b_1x_2-a_0x_1},$$

where C_1 and C_2 are arbitrary constants.

Example 6. ***Reaction-diffusion equation on (2+1) dimension***

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right) + C(u). \tag{9.5}$$

Let us consider for $n_1 = 3$ and $n_2 = 2$. The system of algebraic equations for type II is given as

$$
\begin{aligned}
&d_0=0, d_1=0, d_2=0, d_3=0, \lambda_0=0, \lambda_1=0,\\
&\lambda_2=0, \lambda_3=0, a_0k_0=0, b_0k_0=0,\\
&2c_2=0, 6c_2=0, 30c_2=0, 6c_3=0, 18c_3=0, 60c_3=0, 90c_3=0, 2d_2=0, 6d_3=0,\\
&6\beta_3=0, 18\beta_3=0, -a_0c_0=0, -a_0\beta_1=0, -4a_0\beta_2=0, -9a_0\beta_3=0,\\
&6a_2\beta_3=0, -b_0d_1=0, -2b_0d_2=0, -3b_0d_3=0, -10c_1a_0=0, -2c_1a_0=0,\\
&-c_1b_0=0, -20c_2a_0=0, -3c_2a_0=0, -4c_2b_0=0, -2c_2b_0=0, -30c_3a_0=0, -4c_3a_0=0,\\
&-9c_3b_0=0, -3c_3b_0=0, 9\beta_3b_0^2=0, 3\beta_3b_0a_0=0, -7a_2c_1+3d_1=0,\\
&-a_2c_1+d_1=0, -14a_2c_2+6d_2=0, -8a_2c_2+12d_2=0, -2a_2c_2+2d_2=0, -24a_2c_3+36d_3=0,\\
&-21a_2c_3+9d_3=0, -3a_2c_3+3d_3=0, 6a_2c_3+6d_3=0, 3b_0^2\beta_1-b_0k_2=0,\\
&6b_0^2\beta_2-2b_0k_3=0, -10b_1\beta_2+2\lambda_2=0, -6b_1\beta_2+2\lambda_2=0,\\
&-6b_1\beta_2+6\lambda_2=0, -30b_1\beta_3+6\lambda_3=0, -18b_1\beta_3+6\lambda_3=0, -18b_1\beta_3+18\lambda_3=0,\\
&-6b_1\beta_3+6\lambda_3=0, a_0a_2c_1-4a_0d_1=0, 2a_0a_2c_2-8a_0d_2=0,\\
&3a_0a_2c_3-12a_0d_3=0, a_0b_0\beta_1-a_0k_2=0, 2a_0b_0\beta_2-2a_0k_3=0, a_0b_1\beta_1-a_0\lambda_1=0,\\
&2a_0b_1\beta_2-2a_0\lambda_2=0, 3a_0b_1\beta_3-3a_0\lambda_3=0, -2a_2b_1\beta_2+2a_2\lambda_2=0,\\
&-6a_2b_1\beta_3+6a_2\lambda_3=0, b_0b_1\beta_0-b_0\lambda_0=0, 2b_0b_1\beta_1-2b_0\lambda_1=0, 7b_0b_1\beta_1-3b_0\lambda_1=0,\\
&3b_0b_1\beta_2-3b_0\lambda_2=0, 14b_0b_1\beta_2-6b_0\lambda_2=0, 4b_0b_1\beta_3-4b_0\lambda_3=0,\\
&21b_0b_1\beta_3-9b_0\lambda_3=0, 12b_1^2\beta_3-45b_0\beta_3-6b_1\lambda_3=0,\\
&-54a_1c_3+6a_2d_3-24b_0\beta_3=0, 4b_1^2\beta_1-3b_0\beta_1-2b_1\lambda_1+2k_2=0, 8b_1^2\beta_2-18b_0\beta_2-4b_1\lambda_2+6k_3=0,\\
&3a_2^2c_3-36a_1c_3-3a_2d_3-36b_0\beta_3=0, 3b_1^2\beta_3-3a_1c_3-30b_0\beta_3-3b_1\lambda_3=0,\\
&3a_1a_2c_3-12a_2b_0\beta_3-72a_0c_3-9a_1d_3=0, -18a_1c_2+2a_2d_2-6b_0\beta_2+6k_3=0,\\
&a_2^2c_1-12a_1c_1-a_2d_1-6b_0\beta_1+6k_2=0, 2a_2^2c_2-24a_1c_2-2a_2d_2-18b_0\beta_2+18k_3=0, b_1^2\beta_1\\
&-a_1c_1-4b_0\beta_1-b_1\lambda_1+2k_2=0, 2b_1^2\beta_2-2a_1c_2-14b_0\beta_2-2b_1\lambda_2+6k_3=0,\\
&a_1a_2c_1-2a_2b_0\beta_1-4a_0c_1-3a_1d_1+2a_2k_2=0,\\
&2a_1a_2c_2-6a_2b_0\beta_2-28a_0c_2-6a_1d_2+6a_2k_3=0.
\end{aligned}
\tag{9.6}
$$

After solving these algebraic equations, we obtain the invariant subspace as

$$W_4=Span\{e^{\frac{1}{2}a_2+\frac{1}{2}\sqrt{(a_2^2-4a_1)}x_1}, e^{\frac{1}{2}a_2-\frac{1}{2}\sqrt{(a_2^2-4a_1)}x_1}, e^{\frac{1}{2}b_1+\frac{1}{2}\sqrt{(b_1^2-4b_0)}x_2}, e^{\frac{1}{2}b_1-\frac{1}{2}\sqrt{(b_1^2-4b_0)}x_2}\},$$

which is invariant under the differential operator if $A_1(u)=c_0$,
$A_2(u)=0$,

$C(u) = k_1 u$.

Substituting and simplifying we get reduced equation as:

$$\frac{dp_1(t)}{dt} = \frac{1}{4}c_0 p_1(t) a_2^2 - \frac{1}{2}c_0 p_1(t) a_2 \sqrt{a_2^2 - 4a_1} + \frac{1}{4}c_0 p_1(t)(a_2^2 - 4a_1) + k_1 p_1(t),$$
$$\frac{dp_2(t)}{dt} = \frac{1}{4}c_0 p_2(t) a_2^2 + \frac{1}{2}c_0 p_2(t) a_2 \sqrt{a_2^2 - 4a_1} + \frac{1}{4}c_0 p_2(t)(a_2^2 - 4a_1) + k_1 p_2(t),$$
$$\frac{dp_3(t)}{dt} = k_1 p_3(t),$$
$$\frac{dp_4(t)}{dt} = k_1 p_4(t).$$

Solving these reduced equation we get the required exact solution as:

$$\begin{aligned} u(x_1, x_2, t) =& C_1 e^{(-(\frac{1}{2}(c_0 a_2\sqrt{a_2^2-4a_1} - c_0 a_2^2 + 2a_1 c_0 - 2k_1))t)\frac{1}{2}a_2 + \frac{1}{2}\sqrt{(a_2^2-4a_1)}x_1} + \\ & C_2 e^{(\frac{1}{2}(c_0 a_2\sqrt{a_2^2-4a_1} + c_0 a_2^2 - 2a_1 c_0 + 2k_1))t\frac{1}{2}a_2 - \frac{1}{2}\sqrt{(a_2^2-4a_1)}x_1} + \\ & C_3 e^{k_1 t \frac{1}{2}b_1 + \frac{1}{2}\sqrt{(b_1^2-4b_0)}x_2} + \\ & C_4 e^{k_1 t \frac{1}{2}b_1 - \frac{1}{2}\sqrt{(b_1^2-4b_0)}x_2}, \end{aligned} \tag{9.7}$$

where C_1 ,C_2, C_3 and C_4 are arbitrary constants.

Example 7. ***diffusion equation on (2+1) dimension***

$$\frac{\partial u}{\partial t} = \hat{\mathcal{K}}[u] \equiv \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(A_i(u)\frac{\partial u}{\partial x_i}\right). \tag{9.8}$$

Let us consider for $n_1 = 1$ and $n_2 = 1$. The system of algebraic equations for type I as

$$\begin{aligned} & -6a_0^3 c_2 - 6a_0 b_0^2 \beta_2 + 2a_0^2 d_2 + 2a_0 b_0 \lambda_2 = 0, -2a_0^3 c_1 - 2a_0 b_0^2 \beta_1 + a_0^2 d_1 + a_0 b_0 \lambda_1 - a_0 k_2 = 0, \\ & a_0 k_0 = 0, -a_0^2 c_2 b_0 - 6a_0^2 c_2 b_0 + 2a_0 d_2 b_0 - 6b_0^3 \beta_2 + 2b_0^2 \lambda_2 + a_0^2 c_2 b_0 = 0, \\ & -a_0^2 c_1 b_0 - 2a_0^2 c_1 b_0 + a_0 d_1 b_0 + b_0^2 \lambda_1 - k_2 b_0 + a_0^2 c_1 b_0 - 2b_0^3 \beta_1 = 0, \\ & b_0 k_0 = 0, d_2 = 0, d_1 = 0, d_0 = 0, \lambda_2 = 0, \lambda_1 = 0, \lambda_0 = 0, k_2 = 0, k_1 = 0, k_0 = 0 \end{aligned} \tag{9.9}$$

After solving these algebraic equations, we obtain the invariant subspace as:

$$W_1 = Span\{e^{-a_0 x_1 - b_0 x_2}\},$$

which is invariant under the differential operator if , $A_1(u) = \frac{-b_0^2\beta_2}{a_0^2}u^2 + c_0$, $A_2(u) = \beta_2 u^2 + \beta_0$.

Substituting and simplifying we get reduced equation as:

$$\frac{dp_1(t)}{dt} = p_1(t) a_0^2 c_0 + p_1(t) b_0^2 \beta_0.$$

Solving these reduced equation we get the required exact solution as

$$u(x_1, x_2, t) = C_1 e^{(a_0^2 c_0 + b_0^2 \beta_0)t - a_0 x_1 - b_0 x_2}, \tag{9.10}$$

where C_1 is an arbitrary constant.

10 Conclusion:

This study was shown a detailed investigation for finding the exact solutions of scalar and coupled nonlinear partial differential equations (PDEs) using the invariant subspace method. Moreover, we were explained how to extend the invariant subspace method of (1+1)-dimensional PDEs to (2+1)-dimensional PDEs. More precisely, this systematic study mainly was investigated how to find exact solutions for the two-dimensional convection-reaction-diffusion equation using the obtained invariant subspaces. The applicability and effectiveness of this method have been illustrated through the various types of nonlinear PDEs. The obtained exact solutions can be expressed in terms of trigonometric, polynomial and exponential functions. Hence this investigation shows that the invariant subspace method is a very effective and powerful mathematical tool for deriving the exact solution of various types of scalar and coupled system of non-linear PDEs.

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