Discrete Structures and Theory of Logic

Unit-4

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Mathematical Logic

1 Statement(Proposition)

All the declarative sentences to which it is possible to assign one and only one of the two possible truth values(True or False) are called statements.
There are two types of statements. (i) Primitive Statement (ii) Compound Statement

1.1 Primitive Statement

The statement which do not contain any of the connective is called primitive statement.

1.2 Compound Statement

The statement which contain more than one primitive statements is called compound statement.

Primitive statements are denoted by P, Q, R, S etc.

Example: Find out following sentences are statement or not.

1. Canada is a country.
2. Mumbai is the capital of India.
3. This statement is false.
4. $1+101 = 110$
5. Close the door.
6. Toronto is an old city.
7. Please go from here.

Solution:
Sentences (1), (2), (4), (6) are the statements.

2 Connective

Connectives are used to make compound sentences. Following are the main connectives:-

1. Negation($\neg$)

2. Conjunction($\wedge$)

3. Disjunction($\vee$)

4. Conditional($\rightarrow$)

5. Biconditional($\leftrightarrow$)

2.1 Negation

If P denotes a statement, then the negation of P is written as $\neg P$ and read as "not P". Truth table of negation is the following:-
Example: Consider the statement

P	$\neg P$
T	F
F	T

(1) P: London is a city.
Then $\neg P$ will be
London is not a city.

(2) Q: I went to my class yesterday.
$\neg Q$: I did not go to to my class yesterday.

2.2 Conjunction

The conjunction of two statements P and Q is denoted by P$\wedge$Q which is read as "P and Q". Truth table for this is the following:-
Example: Form the conjunction of the following statements

P	Q	P$\wedge$Q
T	T	T
T	F	F
F	T	F
F	F	F

P: It is raining today.
Q: There are 20 tables in this room.
Solution:
P$\wedge$Q: It is raining today and there are 20 tables in this room.

Example: Translate the following statement in to symbolic form
Jack and Jill went up the hill.
Solution:
P: Jack went up the hill.
Q: Jill went up the hill.
Symbolic form is P$\wedge$Q.

Example: Consider the following statements
(1) Roses are red and violets are blue.
(2) He opened the book and started to read.
(3) Jack and Jill are cousins.
Statement (1) can be written in the form of $\wedge$.
But (2) and (3) can not be written in the form of $\wedge$.

2.3 Disjunction

The disjunction of two statements P and Q is denoted by P$\vee$Q which is read as "P or Q". Truth table for this is the following:-
Example: Consider the following statements

P	Q	P$\vee$Q
T	T	T
T	F	T
F	T	T
F	F	F

(1) I shall watch the game on television or go to the game.
(2) There is something wrong with the bulb or with the wiring.
(3) Twenty or thirty animals were killed in the fire today.
Statement (1) and (2)can be written in the form of $\vee$.
But(3) can not be written in the form of $\vee$.

Example: Using the following statements
R: Mark is rich.
H: Mark is happy.
Write the following statements in the symbolic form.
(a) Mark is poor but happy.
(b) Mark is rich but unhappy.
(c) Mark is neither rich nor happy.
(d) Mark is poor or he is both rich and unhappy.
Solution:
(a) $\neg R \wedge H$
(b) $R \wedge \neg H$
(c) $\neg R \wedge \neg H$
(d) $\neg R \vee (R \wedge \neg H)$

2.4 Conditional

If P and Q are two statements, then conditional of P and Q is denoted by 'P$\rightarrow$Q'. It is read as "if P then Q".
Example: Write the following statements in the symbolic form.
If either Jerry takes Calculus or Ken takes Sociology, then Larry will take English.
Solution:

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

P: Jerry takes Calculus.
Q: Ken takes Sociology.
R: Larry takes English.
Then symbolic form of given statement is
$(P \vee Q) \rightarrow R$

Example: Write the following statements in the symbolic form.
The crop will be destroyed if there is a flood.
Solution:
P: There is a flood.
Q: The crop will be destroyed.
Then symbolic form of given statement is
$P \rightarrow Q$

Note: $P \rightarrow Q = \neg P \vee Q$ **Example:** Construct truth table for $(P \rightarrow Q) \wedge (Q \rightarrow P)$.

2.5 Biconditional

If P and Q are two statements, then biconditional of P and Q is denoted by 'P↔Q'. It is read as "P if and only if Q". **Note:** $P \leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$

P	Q	P↔Q
T	T	T
T	F	F
F	T	F
F	F	T

Example: Construct truth table for the following formula:-
$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$

2.6 Exercise

1. Show that the truth values of the following formulas are independent of their components.

   (a) $(P \wedge (P \rightarrow Q)) \rightarrow Q$
   (b) $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
   (c) $((P \rightarrow Q) \wedge (Q \rightarrow R) \rightarrow (P \rightarrow R)$
   (d) $(P \leftrightarrow Q) \leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

2. Construct truth table for the following formulas:-

(a) $(Q \wedge (P \to Q)) \to P$

(b) $\neg(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R))$

3 Well formed formulas

A well formed formulas are defined as following:

1. A statement variable standing alone is a well formed formula.
2. If A is a well formed formula, then $\neg A$ is also a well formed formula.
3. If A and B are well formed formulas, then (A$\wedge$ B), (A$\vee$ B), (A$\to$ B) and (A$\leftrightarrow$ B) are also well formed formulas.
4. A string of symbols containing the statement variables, connectives and parenthesis is a well formed formula iff it can be obtained by finitely many applications of the rules 1, 2 and 3.

Example: Some well-formed formulas are
$\neg(P \wedge Q)$, $\neg(P \vee Q)$, $(P \to (P \vee Q))$ etc.

Example: Following are not well-formed formulas
$\neg P \wedge Q$, $(P \to Q$, $(P \to Q) \to (\wedge Q))$ and $(P \wedge Q) \to Q)$

4 Tautology and Contradiction

4.1 Tautology

A statement formula which is always true, is called a tautology.

4.2 Contradiction

A statement formula which is always false, is called a contradiction.

Example: $(P \vee (\neg P))$, $\neg(P \wedge \neg P)$ are tautology.

4.3 Satisfiable

If a statement formula A has the truth value t for at least one combination of truth values assigned to $P_1, P_2, ..., P_n$, then A is said to be satisfiable.

4.4 Exercise:

From the formula given below, select those which are well-formed and indicate which ones are tautologies or contradictions.

1. $(P \to (P \vee Q))$
2. $((P \to (\neg P)) \to \neg P)$
3. $((\neg Q \wedge P) \wedge Q)$
4. $((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$
5. $((\neg P \to Q) \to (Q \to P))$
6. $((P \wedge Q) \leftrightarrow P)$

5 Equivalence formulas

Let A and B be the two statement formulas and let $P_1, P_2, ..., P_n$ denote all the ariables occurring in both A and B.
If the truth value of A is equal to the truth value of B for every possible sets of truth values assigned to $P_1, P_2, ..., P_n$, then A and B are said to be equivalent.
The equivalence of two formulas are denoted by $A \Leftrightarrow B$.

Example:

1. $\neg\neg P$ is equivalent to P.
2. $P \vee P$ is equivalent to P.
3. $(P \wedge \neg P) \vee Q$ is equivalent to Q.
4. $P \vee \neg P$ is equivalent to $q \vee \neg q$.

Note: $(P \to Q) \Leftrightarrow (\neg P \vee Q)$

Equivalence formulas:

1. **Idempotent law**
   (i) $P \vee P \Leftrightarrow P$
   (ii) $P \wedge P \Leftrightarrow P$

2. **Associative law**
   (i) $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$
   (ii) $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$

3. **Commutative law**
   (i) $P \vee Q \Leftrightarrow Q \vee P$
   (ii)$P \wedge Q \Leftrightarrow Q \wedge P$

4. **Distributive law**
   (i) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$
   (ii) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5. **Identities law**
   (i) $P \vee F \Leftrightarrow P$, (ii) $P \wedge T \Leftrightarrow P$
   (iii) $P \vee T \Leftrightarrow T$, (iv) $P \wedge F \Leftrightarrow F$
   (v) $P \vee \neg P \Leftrightarrow T$, (vi) $P \wedge \neg P \Leftrightarrow F$

6. **Absorption law**
   (i) $P \vee (P \wedge Q) \Leftrightarrow P)$
   (ii) $P \wedge (P \vee Q) \Leftrightarrow P$

7. **DeMorgan's law**
   (i) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
   (ii)$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Example: Show that

1. $P \rightarrow (Q \rightarrow R) \Leftrightarrow P \rightarrow (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \rightarrow R$
2. $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$
3. $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$
4. $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$
5. $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ are a tautology.

6 Duality law

Two formulas P and Q are said to be dual of each others if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called dual of each other. If the formula P contains the special symbols T or F, then Q, its dual, is obtained by replacing T by F and F by T.

Example: Dual of $(P \wedge Q) \vee T$ is $(P \vee Q) \wedge F$.

7 Converse, Inverse and Contrapositive

For any statement formula $P \rightarrow Q$, the statement formula $Q \rightarrow P$ is called the converse, $\neg P \rightarrow \neg Q$ is called its inverse and $\neg Q \rightarrow \neg P$ is called its contrapositive.

Note: $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$ and $Q \rightarrow P \Leftrightarrow \neg P \rightarrow \neg Q$

8 Tautological Implication

A statement A is said to be tautologically imply a statement B iff $A \to B$ is tautology. We shall denote this by $A \Rightarrow B$, which is read as "A implies B".

$A \Rightarrow B$ guarantees that B has the truth value T whenever A has the truth value T.

Exercise

1. Show the following implications.
   (a) $(P \wedge Q) \Rightarrow (P \to Q)$
   (b) $P \Rightarrow (Q \to P)$
   (c) $(P \to (Q \to R)) \Rightarrow (P \to Q) \to (P \to R)$
2. Show the following equivalences.
   (a) $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to \neg Q)$
   (b) $P \to (Q \vee R) \Leftrightarrow (P \to Q) \vee (P \to R)$
   (c) $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$
   (d) $\neg(P \leftrightarrow Q) \Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$
3. Show the following implications without constructing truth tables.
   (a) $(P \to Q) \Rightarrow P \to (P \wedge Q)$
   (b) $(P \to Q) \to Q)) \Rightarrow (P \vee Q)$
   (c) $((P \vee \neg P) \to Q) \to ((P \vee \neg P) \to R) \Rightarrow (Q \to R)$
   (d) $(Q \to (P \wedge \neg P)) \to (R \to (P \wedge \neg P)) \Leftrightarrow (R \to Q)$

8.1 Formulas with distinct truth tables

A statement formula containing n variables must have as its truth table one of the 2^{2^n} possible truth table, each of them having 2^n rows.

9 Functionally complete set of connectives

A set of connectives by which every formula can be expressed in terms of an equivalent formula containing the connectives from this set, is called a functionally complete set of connectives.

Minimal functionally complete set: A functional complete set is said to be minimal functionally complete set if its proper subset is not functionally complete.

Example:

1. Are the sets $\{\wedge, \vee, \neg\}, \{\wedge, \neg\}, and\{\vee, \neg\}$ functionally complete?
2. Is the set $\{\wedge, \vee, \neg\}$ minimal functionally complete?

3. Are the sets $\{\wedge, \neg\}$, and $\{\vee, \neg\}$ minimal functionally complete?
4. Are the sets $\{\neg\}, \{\wedge\}, \{\vee\} and \{\wedge, \vee\}$ functionally complete?

Example: Write the formulas which are equivalent to the formulas given below and which contain the connectives $\wedge$ and $\neg$.

1. $\neg(p \leftrightarrow (Q \rightarrow (R \vee P))$
2. $((P \vee Q) \wedge R) \rightarrow (P \vee R)$

10 Some other connectives

10.1 NAND Connective

It is denoted by $\uparrow$.
$P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$
Truth table for this is the following

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

10.2 NOR Connective

It is denoted by $\downarrow$.
$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$
Truth table for this is the following **Example:** Express the connectives $\neg, \wedge and \vee$ in the

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

terms of $\uparrow$ only.
Solution:

1. $\neg P \Leftrightarrow \neg P \vee \neg P \Leftrightarrow \neg(P \wedge P) \Leftrightarrow P \uparrow P$
2. $P \wedge Q \Leftrightarrow \neg\neg(P \wedge Q) \Leftrightarrow \neg(P \uparrow Q) \Leftrightarrow (P \uparrow Q) \uparrow (P \uparrow Q)$
3. $P \vee Q \Leftrightarrow \neg\neg(P \vee Q) \Leftrightarrow \neg(\neg P \wedge \neg Q) \Leftrightarrow (\neg P) \uparrow (\neg Q) \Leftrightarrow (P \uparrow P) \uparrow (Q \uparrow Q)$

Example: Express the connectives $\neg, \wedge and \vee$ in the terms of $\downarrow$ only.
Solution:

1. $\neg P \Leftrightarrow \neg P \wedge \neg P \Leftrightarrow \neg(P \vee P) \Leftrightarrow P \downarrow P$

2. $P \vee Q \Leftrightarrow \neg\neg(P \vee Q) \Leftrightarrow \neg(P \downarrow Q) \Leftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$

3. $P \wedge Q \Leftrightarrow \neg\neg(P \wedge Q) \Leftrightarrow \neg(\neg P \vee \neg Q) \Leftrightarrow (\neg P) \downarrow (\neg Q) \Leftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q)$

Note: NAND or NOR is functionally complete.

10.3 Exercise

1. Express $P \to (\neg P \to Q)$ in terms of $\uparrow$ only. Express same formula in terms of $\downarrow$ only.

2. Express $P \uparrow Q$ in terms of $\downarrow$ only.

3. Show the following:-

   (a) $\neg(P \uparrow Q) \Leftrightarrow \neg P \downarrow \neg Q$

   (b) $\neg(P \downarrow Q) \Leftrightarrow \neg P \uparrow \neg Q$

4. Write a formula which is equivalent to the formula
   $P \wedge (Q \leftrightarrow R)$
   and contains the connective NAND($\uparrow$) only. Obtain an equivalent formula which contains the connective NOR($\downarrow$) only.

5. Show the following equivalences.

   (a) $A \to (P \vee C) \Leftrightarrow (A \wedge \neg P) \to C$

   (b) $(P \to C) \wedge (Q \to C) \Leftrightarrow (P \vee Q) \to C$

   (c) $((Q \wedge A) \to C) \wedge (A \to (P \vee C)) \Leftrightarrow (A \wedge (P \to Q)) \to C$

   (d) $((P \wedge Q \wedge A) \to C) \wedge (A \to (P \vee Q \vee C)) \Leftrightarrow (A \wedge (P \leftrightarrow Q)) \to C$

11 Normal Form

There are following types of normal form.

11.1 Disjunctive normal form

A statement formula is said to be in disjunctive normal form if it is the disjunction of conjunction.
Example: $(P \wedge Q) \vee (\neg Q \wedge R)$

11.2 Conjunctive normal form

A statement formula is said to be in conjunctive normal form if it is the conjunction of disjunction.
Example: $(P \vee Q) \wedge (\neg Q \vee R)$

11.3 Principal disjunctive normal form

A statement formula is said to be in principal disjunctive normal form if it is the disjunction of minterms only.
Example: $(P \wedge Q) \vee (\neg P \wedge Q)$

11.4 Principal conjunctive normal form

A statement formula is said to be in principal conjunctive normal form if it is the conjunction of maxterms only.
Example: $(P \vee Q) \wedge (\neg P \vee Q)$

11.5 Exercise

1. Obtain disjunctive normal form of the followings:-
   (a) $P \wedge (P \rightarrow Q)$
   (b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

   Also find the conjunctive normal form of above formulas.

2. Obtain the principal disjunctive normal form of the followings:-
   (a) $\neg P \vee Q$
   (b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$
   (c) $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

3. Obtain the principal conjunctive normal form of the followings:-
   (a) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$
   (b) $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$
   (c) $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

12 Theory of inference for statement calculus

Let A and B be two statement formulas. We say that "B logically follows from A" or "B is a alid conclusion of the premise A" iff $A \rightarrow B$ is a tautology. That is, $A \Rightarrow B$.

A conclusion C follows from a set of premises $\{H_1, H_2, ..., H_m\}$ iff
$H_1 \wedge H_2 \wedge .... \wedge H_m \Rightarrow C$

12.1 Exercise

1. Determine whether the conclusion C follows logically from the premises H_1 and H_2.

(a) $H_1 : P \rightarrow Q$	$H_2 : P$	$C : Q$
(b) $H_1 : P \rightarrow Q$	$H_2 : \neg P$	$C : Q$
(c) $H_1 : P \rightarrow Q$	$H_2 : \neg(P \wedge Q)$	$C : \neg P$
(d) $H_1 : \neg P$	$H_2 : P \leftrightarrow Q$	$C : \neg(P \wedge Q)$
(e) $H_1 : P \rightarrow Q$	$H_2 : Q$	$C : P$

2. Show that the conclusion C follows from the premises $H_1, H_2, ...$ in the following cases:-

(a) $H_1 : P \to Q$	$C : P \to (P \wedge Q)$		
(b) $H_1 : \neg P \vee Q$	$H_2 : \neg(Q \wedge \neg R)$	$H_3 : \neg R$	$C : \neg P$
(c) $H_1 : \neg P$	$H_2 : P \vee Q$	$C : Q$	
(d) $H_1 : \neg Q$	$H_2 : P \to Q$	$C : \neg P$	
(e) $H_1 : P \to Q$	$H_2 : Q \to R$	$C : P \to R$	
(f) $H_1 : R$	$H_2 : P \vee \neg P$	$C : R$	

3. Determine whether the conclusion C is valid in the following, when $H_1, H_2, ...$ are the premises.

(a) $H_1 : P \to Q$	$H_2 : \neg Q$	$C : P$	
(b) $H_1 : P \vee Q$	$H_2 : P \to R$	$H_3 : Q \to R$	$C : R$
(c) $H_1 : P \to (Q \to R)$	$H_2 : P \wedge Q$	$C : R$	
(d) $H_1 : P \to (Q \to R)$	$H_2 : R$	$C : P$	
(e) $H_1 : \neg P$	$H_2 : P \vee Q$	$C : P \wedge Q$	

4. Without constructing a truth table, show that $A \wedge E$ is not a valid consequence of
   $A \leftrightarrow B, B \leftrightarrow (C \wedge D), C \leftrightarrow (A \vee E)\ and\ A \vee E$
   Also show that $A \vee C$ is not a valid consequence of
   $A \leftrightarrow (B \to C), B \leftrightarrow (\neg A \vee \neg C), C \leftrightarrow (A \vee \neg B)\ and\ B$

12.2 Implication rules

1. **Simplification**
   $P \wedge Q \Rightarrow P,\ P \wedge Q \Rightarrow Q$

2. **Addition**
   $P \Rightarrow P \vee Q,\ Q \Rightarrow P \vee Q$

3. **Disjunctive Syllogism**
   $\neg P, P \vee Q \Rightarrow Q$

4. **Modus Ponens**
   $P, P \to Q \Rightarrow Q$

5. **Modus Tollens**
   $\neg Q, P \to Q \Rightarrow \neg P$

6. **Hypothetical Syllogism**
   $P \to Q, Q \to R \Rightarrow P \to R$

7. **Dilemma**
   $P \vee Q, P \to R, Q \to R \Rightarrow R$

8. **Conjunction**
   $P, Q \Rightarrow P \wedge Q$

12.3 Rules of Inference

Rule P: A premise may be introduced at any point in the derivation.
Rule T: A formula S may be introduced in a derivation if S is tautology implied by any one or more of the preceding formulas in the derivation.

Example: Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P.
Solution:

(1) $P \to Q$	By rule P
(2) $Q \to R$	By rule P
(3) $P \to R$	By rule T, (1), (2) and hypothetical syllogism
(4) P	By rule P
(5) R	By rule T, (3), (4) and modus ponens

Example: Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$.
Solution:

(1) $(C \vee D) \to \neg H$	By rule P
(2) $\neg H \to (A \wedge \neg B)$	By rule P
(3) $C \vee D \to (A \wedge \neg B)$	By rule T, (1), (2) and hypothetical syllogism
(4) $(A \wedge \neg B) \to (R \vee S)$	By rule P
(5) $C \vee D \to (R \wedge S)$	By rule T, (3), (4) and hypothetical syllogism
(6) $C \vee D$	By rule P
(7) $R \vee S$	By rule T, (5), (6) and modus ponens

Example: Show that $R \vee S$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$.
Solution:

(1) $P \vee Q$	By rule P
(2) $\neg P \to Q$	By rule T, (1) and $(P \to Q \Leftrightarrow \neg P \vee Q)$
(3) $Q \to S$	By rule P
(4) $\neg P \to S$	By rule T, (2), (3) and hypothetical syllogism
(5) $\neg S \to P$	By rule T, (4) and $(\neg P \to S \Leftrightarrow \neg S \to P)$
(6) $P \to R$	By rule P
(7) $\neg S \to R$	By rule T, (5), (6) and hypothetical syllogism
(8) $R \vee S$	By rule T, (7)and $(P \to Q \Leftrightarrow \neg P \vee Q)$

Example: Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.
Solution:

(1) $P \to M$	By rule P
(2) $\neg M$	By rule P
(3) $\neg P$	By rule T, (1), (2) and modus tollens
(4) $P \vee Q$	By rule P
(5) Q	By rule T, (3), (4) and disjunctive syllogism
(6) $Q \to R$	By rule P
(7) R	By rule T, (3), (4) and modus ponens
(8) $R \wedge (P \vee Q)$	By rule T, (4), (7) and conjunction

Example: Show that $R \to S$ can be derived from the premises $\neg R \vee P$, $P \to (Q \to S)$, and Q.
Solution:
(1) $P \to (Q \to S)$
(2) $\neg R \vee P$
(3) $R \to P$
(4) $R \to (Q \to S)$
(5) $\neg R \vee \neg Q \vee S$
(6) $Q \to (\neg R \vee S)$
(4) Q
(5) $\neg R \vee S$
(9) $R \to S$

Example: "If there was a ball game, then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore there was no ball game." Show that these statements constitute a valid argument.
Solution:
Let
P: There was a ball game.
Q: Traveling was difficult.
R: They arrived on time.
Therefore sentences in symbolic form will be:
Premises: $P \to Q$, $R \to \neg Q$, R and Conclusion: $\neg P$
(1) $R \to \neg Q$
(2) R
(3) $\neg Q$
(4) $P \to Q$
(5)$\neg P$

Rule CP: If we can derive S from r and a set of premises, then we can derive $R \to S$ from the set of premises alone.

Example: If A works hard, then either B or C will enjoy themselves. If B enjoy himself, then A will not work hard. If D enjoys himself, then C will not. Therefore, if A works hard, then D will not enjoy himself.
Solution:
Let
A: A works hard.
B: B will enjoy himself.
C: C will enjoy himself.
D: D will enjoy himself.
Therefore sentences in symbolic form will be:
Premises: $A \to (B \vee C)$, $B \to \neg A$, $D \to \neg C$ and Conclusion: $A \to \neg D$
(1) A
(2) $A \to B \vee C$
(3) $B \vee C$ (4) $\neg C \to B$
(5) $D \to \neg C$

(6) $D \to B$
(7) $B \to \neg A$
(8) $D \to \neg A$
(9) $A \to \neg D$
(10)$\neg D$

12.4 Consistency of premises and Indirect method of proof

Consistency of premises

- A set of premises $H_1, H_2, ..., H_m$ is said to be consistent if their conjunction has the truth value T for some assignment of the truth values of the atomic variables appearing in $H_1, H_2, ..., H_m$.
- If for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, ..., H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, ..., H_m$ are called inconsistent.
- Alternatively, a set of formulas $H_1, H_2, ..., H_m$ is inconsistent if their conjunction implies a contradiction, that is,
  $H_1 \wedge H_2 \wedge .... \wedge H_m \Rightarrow R \wedge \neg R$
  Where R is any formula.

Indirect method of proof

- In order to show that a conclusion C follows logically from the premises $H_1, H_2, ..., H_m$, we assume that C is false and $\neg C$ as an additional premises.
- If the new set of premises is inconsistent, then the assumption that $\neg C$ is true does not hold simultaneously with $H_1, H_2, ..., H_m$ being true. Therefore, C is true whenever $H_1, H_2, ..., H_m$ is true. Thus, C follows logically from the premises $H_1, H_2, ..., H_m$.

Example: Show that the following premises are inconsistent.
(1) If Jack misses many classes through illness, then he fails high school.
(2) If Jack fails high school, then he is uneducated.
(3) If Jack reads a lot of books, then he is not uneducated.
(4) Jack misses many classes through illness and reads a lot of books.
Solution:
E: Jack misses many classes.
S: Jack fails high school.
A: Jack reads a lot of books.
H: Jack is uneducated.
The premises are $E \to S$, $S \to H$, $A \to \neg H$, and $E \wedge A$.
(1) $E \to S$
(2) $S \to H$
(3) $E \to H$ (4) $A \to \neg H$
(5) $H \to \neg A$
(6) $E \to \neg A$

(7) $\neg E \vee \neg A$
(8) $\neg(E \wedge A)$
(9) $E \wedge A$
(10)$\neg(E \wedge A) \wedge (E \wedge A)$ Contradiction

12.5 Exercise

1. Show the validity of the following arguments, for which the premises are given on the left and the conclusion on the right.
   (a) $\neg(P \wedge \neg Q)$, $\neg Q \vee R$, $\neg R$ C: $\neg P$
   (b) $(A \to B) \wedge (A \to C)$, $\neg(B \wedge C)$, $D \vee A$ C: D
   (c) $\neg J \to (M \vee N)$, $(H \vee G) \to \neg J$, $H \vee G$ C: $M \vee N$
   (d) $(P \to Q)$, $(\neg Q \vee R) \wedge \neg R$, $\neg(\neg P \wedge S)$ C: $\neg S$
2. Derive the following using rule CP if necessary.
   (a) $\neg P \vee Q$, $\neg Q \vee R$, $R \to S \Rightarrow P \to S$
   (b) P, $P \to (Q \to (R \wedge S))$, $\Rightarrow Q \to S$
   (c) $(P \vee Q) \to R \Rightarrow (P \wedge Q) \to R$
   (d) $P \to (Q \to R)$, $Q \to (R \to S) \Rightarrow P \to (Q \to S)$
3. Show that the following sets of premises are inconsistent.
   (a) $P \to Q$, $P \to R$, $Q \to \neg R$, P
   (b) $A \to (B \to C)$, $D \to (B \wedge \neg C)$, $A \wedge D$
4. Show the following (use indirect method if needed).
   (a) $R \to \neg Q$, $R \vee S$, $S \to \neg Q$, $P \to Q \Rightarrow \neg P$
   (b) $S \to \neg Q$, $R \vee S$, $\neg R$, $\neg R \leftrightarrow Q \Rightarrow \neg P$
   (c) $\neg(P \to Q) \to \neg(R \vee S)$, $((Q \to P) \vee \neg R)$, $R \Rightarrow P \leftrightarrow Q$

13 Predicate Calculus

13.1 Predicate

Consider the statements.
John is a bachelor.
Smith is a bachelor.
The part "is a bachelor" is called a predicate.
Now, we denote the predicate as following:- B: is a bachelor.
Therefore the statement in predicate form will be:-
B(John), B(Smith).

Example: Write the following statements in predicate form.
(a) Jack is taller than Jill.
(b) Canada is to the north of the United States.
Solution:
(a) Let P: is taller than
Therefore, statement in predicate form will be
P(Jack, Jill)
(b)Let Q: is to the north of the
Therefore, statement in predicate form will be
Q(Canada, United States)

13.2 The statement function, variables and quantifiers

A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable. Such a statement function becomes a statement when the variable is replaced by the name of any object.
Let M(x) : x is a man.
H(x): x is a mortal.
The compound statement functions are
$M(x) \wedge H(x)$, $M(x) \rightarrow H(x)$, $\neg H(x)$, $M(x) \vee \neg H(x)$ etc.

Example: Consider the following statements
(a) All men are mortal.
(b) Every apple is red.
(c) Any integer is either positive or negative.
Write these statements in predicate form.
Solution:
(a) Let M(x) : x is a man.
H(x): x is a mortal.
Therefore, the predicate form of the statement will be
$(\forall x)(M(x) \rightarrow H(x))$
(b) Let A(x) : x is an apple.
R(x): x is a red.
Therefore, the predicate form of the statement will be
$(\forall x)(A(x) \rightarrow R(x))$
(c) Let I(x) : x is an integer.
P(x): x is either positive or negative integer.
Therefore, the predicate form of the statement will be
$(\forall x)(I(x) \rightarrow P(x))$

The symbol $(\forall x)$ or (x) is said to be universal quantifier. It is used in the statement which contains for all, every and for any.

Example: Find the predicate form of the following statement.
For any x and y, if x is taller than y, then y is not taller than x.
Solution:
$(\forall x)(\forall y)(G(x, y) \rightarrow \neg G(y, x))$

Example: Consider the following statements
(a)There exists a man.
(b)Some men are clever.
(c) Some real numbers are rational.
Write these statements in predicate form.
Solution:
Let M(x): x is a man.
C(x): x is clever.
R(x): x is real number.
Q(x): x is rational number.
(a) $(\exists x)M(x)$
(b) $(\exists x)(M(x) \wedge C(x))$
(c) $(\exists x)(R(x) \wedge Q(x))$

The symbol $(\exists x)$ is said to be existential quantifier. It is used in the statement which contains some or there exists.

13.3 Free and Bound variables

Given a formula containing a part of the form (∀ x)P(x) or (∃ x)P(x), such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of a formula is called a bound occurrence of x , while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence.

Example:
(i) $(\forall x)P(x, y)$
(ii) $(\forall x)(A(x) \rightarrow R(x))$
(iii) $(\forall x)(A(x) \rightarrow (\exists y)R(x, y))$
(iv) $(\exists x)(A(x) \wedge R(x))$
(v) $(\exists x)A(x) \wedge R(x)$
Example: Let P(x): x is a person.
F(x,y): x is the father of y.
M(x,y): x is the mother of y.
Write the predicate form of the following statement
" x is the father of the mother of y."
Solution: Let z as the mother of y. Therefore, statement will be
$(\exists z)(P(z) \wedge F(x, z) \wedge M(z, y))$

13.4 Universe of discourse

The domain of a variable is known as the universe of discourse.
Example: If the discussion refers to human beings only, then the universe of discourse is the class of human beings.

Example: Consider the predicate

Q(x): x is less than 5.
and the statements $(\forall x)Q(x)$ and $(\exists x)Q(x)$. If the universe of discourse is given by the following sets, then find the truth value of statements $(\forall x)Q(x)$ and $(\exists x)Q(x)$.
(1) {-1, 0, 1, 2, 4}
(2) {3, -2, 7, 8, -5}
(3) {15, 20, 24}
Solution: $(\forall x)Q(x)$ is true for (1) and false for (2) and (3).
$(\exists x)Q(x)$ is true for (1) and (2) and false for (3).

14 Inference theory of the predicate calculus

14.1 Some equivalences

(i) $\neg((\forall x)A(x)) \Leftrightarrow (\exists x)\neg A(x))$
(ii) $\neg((\exists x)A(x)) \Leftrightarrow (\forall x)\neg A(x)$
(iii) $A(x) \rightarrow B(x) \Leftrightarrow \neg A(x) \vee B(x)$

14.2 Some rules

(1) Universal Specification rule (US rule)

$(\forall x)A(x) \Rightarrow A(x)$

(2) Universal Generalization rule (UG rule)

$A(x) \Rightarrow (\forall x)A(x)$

(3) Existential Specification rule (ES rule)

$(\exists x)A(x) \Rightarrow A(y)$

(4) Existential Generalization rule (EG rule)

$A(y) \Rightarrow (\exists x)A(x)$

14.3 Some implications and equivalences

(1) $(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$
(2) $(\forall x)(A(x) \wedge B(x)) \Leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$
(3) $\neg(\exists x)A(x) \Leftrightarrow (\forall x)\neg A(x)$
(4) $\neg(\forall x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
(5) $(\forall x)A(x) \vee (\forall x)B(x) \Rightarrow (\forall x)(A(x) \vee B(x))$
(6) $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$

Note: If a formula does not depend upon the variable x, then $\forall x)A \Leftrightarrow A$ and $\exists x)A \Leftrightarrow A$

Example: Prove the following:-
(1) $(\exists x)(A(x) \rightarrow B(x)) \Leftrightarrow (\forall x)A(x) \rightarrow (\exists x)B(x)$
(2) $(\exists x)A(x) \rightarrow (\forall x)B(x)) \Leftrightarrow (\forall x)(A(x) \rightarrow B(x))$
Proof:
(1) $(\exists x)(A(x) \rightarrow B(x))$

$\Leftrightarrow (\exists x)(\neg A(x) \vee B(x))$
$\Leftrightarrow (\exists x)\neg A(x) \vee (\exists x)B(x)$
$\Leftrightarrow \neg(\forall x)A(x) \vee (\exists x)B(x)$
$\Leftrightarrow (\forall x)A(x) \rightarrow (\exists x)B(x)$

(2) $(\exists x)A(x) \rightarrow (\forall x)B(x)$
$\Leftrightarrow \neg(\exists x)A(x) \vee (\forall x)B(x)$
$\Leftrightarrow (\forall x)\neg A(x) \vee (\forall x)B(x)$
$\Leftrightarrow (\forall x)(\neg A(x) \vee B(x))$
$\Leftrightarrow (\forall x)(A(x) \rightarrow B(x))$

Example: Show that $(\forall x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$.
Solution:
(1) $(\forall x)(H(x) \rightarrow M(x))$, By rule P
(2) $H(s) \rightarrow M(s)$, By rule US and (1)
(3) H(s) , By rule P
(4) M(s) , By rule T, (2), (3) and modus ponens

Example: Show that $(\forall x)(P(x) \rightarrow Q(x)) \wedge (\forall x)(Q(x) \rightarrow R(x)) \Rightarrow (\forall x)(P(x) \rightarrow R(x))$.
Solution:
(1) $(\forall x)(P(x) \rightarrow Q(x))$, By rule P
(2) $P(y) \rightarrow Q(y)$, By rule US and (1)
(3) $(\forall x)(Q(x) \rightarrow R(x))$, By rule P
(4) $Q(y) \rightarrow R(y)$, By rule US and (3)
(5) $P(y) \rightarrow R(y)$, By rule T, (2), (3) and hypothetical syllogism
(6) $(\forall x)(P(x) \rightarrow R(x))$, By rule UG and (5)

Example: Show that $(\exists x)M(x)$ follows logically from the premises.
$(\forall x)(H(x) \rightarrow M(x))$ and $(\exists x)H(x)$
Solution:
(1) $(\forall x)(H(x) \rightarrow M(x))$, By rule P
(2) $H(y) \rightarrow M(y)$, By rule US and (1)
(3) $(\exists x)H(x)$, By rule P
(4) H(y) , By rule ES and (3)
(5) M(y) , By rule T, (2), (4) and modus ponens
(6) $(\exists x)M(x)$, By rule EG and (5)

Example: Show that $(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$
Solution:
(1) $(\exists x)(P(x) \wedge Q(x))$, By rule P
(2) $P(y) \wedge Q(y)$, By rule ES
(3) P(y)
(4) Q(y)
(5) $(\exists x)P(x)$, By rule EG
(6) $(\exists x)Q(x)$, By rule EG
(7) $(\exists x)P(x) \wedge (\exists x)Q(x)$

Example: Show that from
(a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (\forall y)(M(y) \rightarrow W(y))$

(b) $(\exists y)(M(y) \wedge \neg W(y))$
the conclusion $(\forall x)(F(x) \to \neg S(x))$ follows.
Solution:
(1) $(\exists y)(M(y) \wedge \neg W(y))$, By rule P
(2) $(M(z) \wedge \neg W(z))$, By rule ES and (1)
(3) $\neg(M(z) \to W(z))$, By rule T and (2)
(4) $(\exists y)\neg(M(y) \to W(y))$, By rule EG
(5) $\neg(\forall y)\neg(M(y) \to W(y))$
(6) $(\exists x)(F(x) \wedge S(x)) \to (\forall y)(M(y) \to W(y))$
(7) $\neg(\exists x)(F(x) \wedge S(x))$, By rule T, (5), (6) and modus ponens
(8) $(\forall x)\neg(F(x) \wedge S(x))$
(9) $\neg(F(x) \wedge S(x))$, By rule US and (8)
(10) $(F(x) \to \neg S(x))$
(11) $(\forall x)(F(x) \to \neg S(x))$, By rule UG and (10)

Example: Show that $(\forall x)(P(x) \vee Q(x)) \Rightarrow (\forall x)P(x) \vee (\exists x)Q(x))$
Solution:
We shall use the indirect method of proof by assuming $\neg((\forall x)P(x) \vee (\exists x)Q(x))$ as an additional premise.
(1) $\neg((\forall x)P(x) \vee (\exists x)Q(x))$, By rule P(assumed)
(2) $\neg(\forall x)P(x) \wedge \neg(\exists x)Q(x))$, By rule T
(3) $\neg(\forall x)P(x)$
(4) $(\exists x)\neg P(x))$
(5) $\neg(\exists x)Q(x))$
(6) $(\forall x)\neg Q(x)$
(7) $\neg P(y)$, By rule ES and (4)
(8) $\neg Q(y)$, By rule US and (6)
(9) $\neg P(y) \wedge \neg Q(y)$
(10) $\neg(P(y) \vee Q(y))$
(11) $(\forall x)(P(x) \vee Q(x))$, By rule P
(12) $(P(y) \vee Q(y))$, By rule US and (11)
(13) $\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$, By rule T and (10),(12)
This is contradiction. Therefore the given statement is proved.

15 AKTU Examination Questions

1. Verify that the given propositions are tautology or not.
   (a) $p \vee \neg(p \wedge q)$
   (b) $\neg p \wedge q$
2. Write the contra positive of the implication: "if it is Sunday then it is a holiday".
3. Show that the propositions $p \to q$ and $\neg p \vee q$ are logically equivalent
4. Show that $((P \vee Q) \wedge \neg(\neg Q \vee \neg R)) \vee (\neg P \vee \neg Q) \vee (\neg P \vee \neg Q)$ is a tautology by using equivalences.

5. Obtain the principle disjunctive and conjunctive normal forms of the formula $(P \to R) \wedge (Q \leftrightarrow P)$.

6. Explain various Rules of Inference for Propositional Logic.

7. Prove the validity of the following argument "if the races are fixed so the casinos are crooked, then the tourist trade will decline. If the tourist trade decreases, then the police will be happy. The police force is never happy. Therefore, the races are not fixed."

8. Prove that $(P \vee Q) \to (P \wedge Q)$ is logically equivalent to $P \leftrightarrow Q$.

9. Express this statement using quantifiers:
"Every student in this class has taken some course in every department in the school of mathematical sciences".

10. Construct the truth table for the following statements:

    (a) $(P \to \neg Q) \to \neg P$

    (b) $P \leftrightarrow (\neg P \vee \neg Q)$