

Discrete Structures and Theory of Logic

Lecture-13

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Theorem: Let $\langle L, \preceq \rangle$ be a lattice. For any $a, b, c \in L$,

$$(1) a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$$

$$(2) a \wedge (b \vee c) \succeq (a \wedge b) \vee (a \wedge c)$$

Proof:

(1) Using definition of least upper bound, $a \preceq (a \vee b)$(1)

Using definition of greatest lower bound, $b \wedge c \preceq b$ (2)

Using definition of least upper bound, $b \preceq (a \vee b)$(3)

From (2) and (3), $b \wedge c \preceq b \preceq (a \vee b)$

Therefore, $b \wedge c \preceq (a \vee b)$(4)

From (1) and (4), $(a \vee b)$ is the upper bound of a and $b \wedge c$, therefore

$\text{lub}\{a, b \wedge c\} \preceq (a \vee b)$ i.e.

$$a \vee (b \wedge c) \preceq (a \vee b) \text{(5)}$$

Similarly, using definition of least upper bound, $a \preceq (a \vee c)$(6)

Using definition of greatest lower bound, $b \wedge c \preceq c$ (7)

Using definition of least upper bound, $c \preceq (a \vee c)$(8)

From (7) and (8), $b \wedge c \preceq c \preceq (a \vee c)$

Therefore, $b \wedge c \preceq (a \vee c)$(9)

From (6) and (9), $(a \vee c)$ is the upper bound of a and $b \wedge c$, therefore

$$\text{lub}\{a, b \wedge c\} \preceq (a \vee c) \text{ i.e.}$$

$$a \vee (b \wedge c) \preceq (a \vee c) \text{(10)}$$

From (5) and (10), $a \vee (b \wedge c)$ is lower bound of $(a \vee b)$ and $(a \vee c)$, therefore

$$a \vee (b \wedge c) \preceq \text{glb}\{(a \vee b), (a \vee c)\} \text{ i.e.}$$

$$a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$$

Now, it is proved.

(2) Using definition of greatest lower bound, $(a \wedge b) \preceq a$ (1)

Using definition of least upper bound, $b \preceq (b \vee c)$(2)

Using definition of greatest lower bound, $(a \wedge b) \preceq b$(3)

From (2) and (3), $(a \wedge b) \preceq b \preceq (b \vee c)$

Therefore, $(a \wedge b) \preceq b \vee c$(4)

From (1) and (4), $(a \wedge b)$ is the lower bound of a and $b \vee c$, therefore

$(a \wedge b) \preceq \text{glb}\{a, b \vee c\}$ i.e.

$(a \wedge b) \preceq a \wedge (b \vee c)$ (5)

Similarly, using definition of greatest lower bound, $(a \wedge c) \preceq a$ (6)

Using definition of least upper bound, $c \preceq (b \vee c)$(7)

Using definition of greatest lower bound, $(a \wedge c) \preceq c$(8)

From (7) and (8), $(a \wedge c) \preceq c \preceq (b \vee c)$

Therefore, $(a \wedge c) \preceq (b \vee c)$(9)

From (6) and (9), $(a \wedge c)$ is the lower bound of a and $(b \vee c)$, therefore

$(a \wedge c) \preceq \text{glb}\{a, b \vee c\}$ i.e.

$(a \wedge c) \preceq a \wedge (b \vee c)$ (10)

From (5) and (10), $a \wedge (b \vee c)$ is upper bound of $(a \wedge b)$ and $(a \wedge c)$, therefore

$\text{lub}\{(a \wedge b), (a \wedge c)\} \preceq a \wedge (b \vee c)$ i.e.

$(a \wedge b) \vee (a \wedge c) \preceq a \wedge (b \vee c)$

Now, it is proved.

Theorem: Let $\langle L, \preceq \rangle$ be a lattice. For any $a, b, c \in L$,

$$a \preceq c \Leftrightarrow a \vee (b \wedge c) \preceq (a \vee b) \wedge c$$

Proof:

First part: In this part, we will prove that if $a \preceq c$ then $a \vee (b \wedge c) \preceq (a \vee b) \wedge c$

Suppose $a \preceq c$.

Using definition of least upper bound, $a \preceq (a \vee b)$(1)

Using definition of greatest lower bound, $(b \wedge c) \preceq b$ (2)

Using definition of least upper bound, $b \preceq (a \vee b)$ (3)

Therefore, from (2) and (3), $(b \wedge c) \preceq b \preceq (a \vee b)$

Therefore, $(b \wedge c) \preceq (a \vee b)$ (4)

From (1) and (4), $(a \vee b)$ is the upper bound of a and $b \wedge c$, therefore

$$\text{lub}\{a, b \wedge c\} \preceq (a \vee b) \text{ i.e.}$$

$$a \vee (b \wedge c) \preceq (a \vee b) \text{(5)}$$

Now, using definition of greatest lower bound, $(b \wedge c) \preceq c$ (6)

Since $a \preceq c$, therefore using (6), c is an upper bound of a and $(b \wedge c)$.

Therefore

$$a \vee (b \wedge c) \preceq c \text{(7)}$$

from (5) and (7), $a \vee (b \wedge c)$ is lower bound of $(a \vee b)$ and c . Therefore,

$$a \vee (b \wedge c) \preceq (a \vee b) \wedge c$$

Second part: In this part, we will prove that if $a \vee (b \wedge c) \preceq (a \vee b) \wedge c$ then $a \preceq c$.

$$\text{Now, } a \preceq a \vee (b \wedge c) \preceq (a \vee b) \wedge c \preceq c$$

Therefore, $a \preceq c$.

Since both parts are proved. Therefore, it is proved.

Exercise

1. Show that in a lattice if $a \preceq b \preceq c$, then
 $a \vee b = b \wedge c$ and $(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c)$
2. Show that in a lattice if $a \preceq b$ and $c \preceq d$, then
 $(a \wedge c) \preceq (b \wedge d)$
3. In a lattice, show that
 - (a) $(a \wedge b) \vee (c \wedge d) \preceq (a \vee c) \wedge (b \vee d)$
 - (b) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \preceq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$
4. Show that a lattice with three or fewer elements is a chain.