Discrete Structures and Theory of Logic

Lecture-5

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Ordered pairs and Cartesian products

Ordered pair

An ordered pair consists of two objects in a given fixed order. An ordered pair is not a set of two elements. The ordering of two objects is important. We denote ordered pair by (x,y).

The equality of two ordered pairs is defined by $(x,y) = (u,v) \Leftrightarrow x=u$ and $y=v$.

Cartesian products

Let A and B be any two sets. The set of all ordered pairs such that the first member of the ordered pair is an element of A and the second member is an element of B, is called the Cartesian product of A and B. It is denoted by $A \times B$.

Mathematically, it is defined as

$$A \times B = \{ (x,y) \,!\, x \in A \text{ and } y \in B \}$$

Some examples

Example If A $=\{$a, b$\}$ and B $=\{$ 1, 2 ,3$\}$, then find A$\times$B, B$\times$A, A$\times$A, B$\times$B, and (A$\times$B)$\cap$(B$\times$A).

Solution:

A$\times$B $=\{$ (a,1),(a,2),(a,3),(b,1),(b,2),(b,3)$\}$

B$\times$A $=\{$ (1,a),(1,b),(2,a),2,b),(3,a),(3,b)$\}$

A$\times$A $=\{$ (a,a),(a,b),(b,a),(b,b)$\}$

B$\times$B $=\{$ (1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$\}$

(A$\times$B)$\cap$(B$\times$A) $=\phi$

Example If A $=\phi$ and B $=\{$ 1, 2 ,3$\}$, then what are A$\times$B, B$\times$A?

Solution: A$\times$B $=\phi$ and B$\times$A $=\phi$

Some examples(cont.)

Example Prove that

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution:

(a)
$$
\begin{aligned}
A \times (B \cup C) &= \{(x,y) \,!\, x \in A \text{ and } y \in (B \cup C) \} \\
&= \{(x,y) \,!\, x \in A \text{ and } (y \in B \text{ or } y \in C) \} \\
&= \{(x,y) \,!\, (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \} \\
&= \{(x,y) \,!\, (x,y) \in A \times B \text{ or } (x,y) \in A \times C \} \\
&= \{(x,y) \,!\, (x,y) \in (A \times B) \cup (A \times C) \} \\
&= (A \times B) \cup (A \times C)
\end{aligned}
$$

Therefore, $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Some examples(cont.)

(b) $A \times (B \cap C) = \{(x,y) \; ! \; x \in A \text{ and } y \in (B \cap C) \}$
$= \{(x,y) \; ! \; x \in A \text{ and } (y \in B \text{ and } y \in C) \}$
$= \{(x,y) \; ! \; (x \in A \text{ and } y \in B) \text{and} (x \in A \text{ and } y \in C) \}$
$= \{(x,y) \; ! \; (x,y) \in A \times B \text{ and } (x,y) \in A \times C \}$
$= \{(x,y) \; ! \; (x,y) \in (A \times B) \cap (A \times C) \}$
$= (A \times B) \cap (A \times C)$

Therefore, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Exercise

1. Determine the following:-
   1.1 $\phi \cap \{\phi\}$
   1.2 $\{\phi\} \cap \{\phi\}$
   1.3 $\{\phi, \{\phi\}\} - \phi$
2. Determine A×B×C, B^2, A^3, $B^2 \times A$, where A = {1}, B = {a, b} and C = {2, 3}.
3. Prove that
   3.1 (A∩B)∪(A∩B') = A
   3.2 A∩(A'∪B) = A∩B
4. Show that (A∩B)∪C = A∩(B∪C) iff C ⊆ A

Exercise(cont.)

1. Draw Venn diagram for the following:-
    1.1 B'
    1.2 $(A \cup B)'$
    1.3 B-A'
    1.4 $A' \cup B$
    1.5 $A' \cap B$
2. Show that
    2.1 (A-B)-C = (A-C)-(B-C)
    2.2 $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$
3. Let A and B be sets. Show that $A \times B \neq B \times A$. Under what condition $A \times B = B \times A$?

Relation

A relation is defined as the subset of Cartesian product. That is, if R is a relation defined from the set A to B, then

$$R \subseteq A \times B$$

Mathematically, $R = \{(x,y) \,!\, x \in A \text{ and } y \in B\}$

Element a related b by relation R if $(a,b) \in R$. It is denoted by aRb.

Examples

1. R = {(x,y) ! x is the father of y}
2. R = {(a,2), (b,2), (c,3)}
3. Let A = {a, b, c, d}. Some relations R defined on set A are:
    3.1 R = A×A
    3.2 R = {(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (b, c)}
    3.3 R = {(a, a), (b, b), (c, c)}
    3.4 R = {(a, a), (b, b), (a, b), (b, a), (c, d)}
    3.5 R = {(a, a), (a, c), (c, a), (a, b), (b, a), (c, c), (b, b)}

Note: Consider set A and B with m and n number of elements respectively. The number of relations which can be defined from set A to B will be 2^{mn}.

Domain and Range of a relation

Domain of a relation

Domain of a relation R is the set of first element of all the ordered pairs belong into the relation R. Mathematically, it is defined as

$$\text{Domain}(R) = \{\ a\ !\ \exists\ b, \text{ such that } (a,b) \in R\}$$

Range of a relation

Range of a relation R is the set of second element of all the ordered pairs belong into the relation R. Mathematically, it is defined as

$$\text{Range}(R) = \{\ b\ !\ \exists\ a, \text{ such that } (a,b) \in R\}$$

Types of relation

Universal relation

A relation R defined from A to B is said to be universal relation if it contains all the ordered pairs defined from set A to B. That is, if R $=$ A$\times$B, then R is said to be universal set.

Void relation

If R does not contain any ordered pair, then it is said to be void or empty relation. That is, if R $= \phi$ then R is said to be empty relation.

Identity relation

A relation R defined on set A is said to be identity relation if

R $= \{$(a,a) ! for all a $\in$ A$\}$

Inverse relation

A relation R' is said to be inverse relation of R defined on set A if

R' $= \{$(a,b) ! (b,a) $\in$ R$\}$