Discrete Structures and Theory of Logic

Lecture-19

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Inverse function

Let f: X$\rightarrow$Y is a function. If f is a bijective function then we can define the inverse function of f. It is denoted by f^{-1}. It is defined as f^{-1}: Y$\rightarrow$ X. If f(a) = b then f^{-1}(b) = a.

Example: Let X = {1,2,3} and Y = {p,q,r}. f: X$\rightarrow$ Y be given by

f = {(1,p), (2,q), (3,q)}

Is inverse of this function possible?

Solution: Inverse of this function is not possible because this function is not bijective. This function is not bijective because f is not onto function.

Inverse function

Example: Let R be the set of real numbers and let f: R$\rightarrow$ R be given by

f = $\{$(x, x^2) ! x $\in$ R$\}$

Is inverse of this function possible?

Solution: Inverse of this function is not possible because this function is not bijective. This function is not bijective because f is not onto function.

Example: Let R be the set of real numbers and let f: R$\rightarrow$ R be given by

f = $\{$(x, x+2) ! x $\in$ R$\}$

Is inverse of this function possible?

Solution: Inverse of this function is possible because this function is bijective.

Identity function and Invertible function

Identity function

A function I_X: X$\rightarrow$ X is called an identity function if I_X(x) $=$ x, $\forall$ x $\in$ X.

Invertible function

A function f is said to be invertible function if there exists an inverse function of this function.

Note:

(1) If f: X$\rightarrow$Y is invertible then $f^{-1}$of $= I_X$ and fo$f^{-1} = I_Y$

(2) Let f: X$\rightarrow$ Y and g: Y$\rightarrow$ X are two functions. The function g is equal to f^{-1} only if gof $= I_X$ and fog $= I_Y$.

Some examples

Example: Show that the functions f(x) = x^3 and g(x) = $x^{1/3}$, for x $\in$ R are inverses of one another.

Solution: These functions will be inverse of each other if gof = I = fog.

gof(x) = g(f(x)) = g(x^3) = $(x^3)^{1/3}$ = $x^{3/3}$ = x = I(x).

fog(x) = f(g(x)) = f($x^{1/3}$) = $(x^{1/3})^3$ = $x^{3/3}$ = x = I(x).

Therefore these functions are inverses of each others.

Some examples

Example: Let F_X be the set of all bijective functions from X to X, where X = $\{1,2,3\}$. Find all the elements of F_X and also find the inverse of each element.

Solution: Since the number of elements in set X is 3, therefore the number of bijective functions will be $3! = 6$. These functions are:-

$f_1 = \{(1,1),(2,2),(3,3)\}$ $\quad$ $f_2 = \{(1,1),(2,3),(3,2)\}$

$f_3 = \{(1,2),(2,1),(3,3)\}$ $\quad$ $f_4 = \{(1,2),(2,3),(3,1)\}$

$f_5 = \{(1,3),(2,2),(3,1)\}$ $\quad$ $f_6 = \{(1,3),(2,1),(3,2)\}$

Inverse of these functions is determined by interchanging values in each ordered pairs of corresponding functions.

Note: If a set X has n elements, then the number of bijective functions from X to X is n!.

Exercise

1. Let f: R $\rightarrow$R and g: R $\rightarrow$R are two functions such that f(x) = $x^2 - 2$ and g(x) = x+4, where R is the set real numbers. Find fog and gof. State wheteher these functions are injective, surjective and bijective.
2. If f: X $\rightarrow$Y and g: Y$\rightarrow$ Z and both f and g are onto, show that gof is also onto. Is gof one-one if both g and f are one-one?
3. Let f: R $\rightarrow$R be given by f(x) = x^2-2. Find f^{-1}.

Exercise(cont.)

How many functions are there from X to Y for the sets given below? Find also the number of functions which are one-one, onto and bijective.

1. $X = \{1,2,3\}$, $Y = \{1,2,3\}$
2. $X = \{1,2,3,4\}$, $Y = \{1,2,3\}$
3. $X = \{1,2,3\}$, $Y = \{1,2,3,4\}$
4. $X = \{1,2,3,4,5\}$, $Y = \{1,2,3\}$
5. $X = \{1,2,3\}$, $Y = \{1,2,3,4,5\}$

Number of Onto Functions

If a set A has m elements and set B has n elements, $m \geq n$ then the number of onto functions from A to B
$= n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots\ldots\ldots - {}^nC_{n-1}(1)^m$

Number of One-One Functions

If set A has m elements and set B has n elements, $n \geq m$, then the number of one-one functions from A to B
$= {}^nP_m$

Exercise(cont.)

1. Let X = $\{1,2,3,4\}$. Define a function f: X $\rightarrow$X such that f$\neq I_X$ an is one-one. Find f^2, f^3, f^{-1} and fof^{-1}. Can you find another one-one function g: X $\rightarrow$X such that g $\neq I_X$ but gog = I_X?
2. Let f: X$\rightarrow$Y and X=Y=R, the set of real numbers. Find f^{-1} if
   2.1 f(x) = x^2
   2.2 f(x) = $\frac{(2x-1)}{5}$
3. Let f and g be the functions from the set of integers to the set of integers defined by f (x) = 2x + 3 and g(x) = 3x + 2. What is the composition of f and g? What is the composition of g and f?