

Discrete Structures and Theory of Logic

Lecture-20

Dr. Dharmendra Kumar

(Associate Professor)

United College of Engineering and Research, Prayagraj

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Boolean algebra

A lattice is said to be boolean algebra if it is both complemented and distributive. It is denoted by $\langle B, \wedge, \vee, ', 0, 1 \rangle$.

Some examples

Example: $\langle P(S), \cap, \cup, ', \phi, S \rangle$ is a boolean algebra. Because it is complemented and distributive both. In this lattice, 0 is ϕ and 1 is S. Each elements of P(S) has a complement.

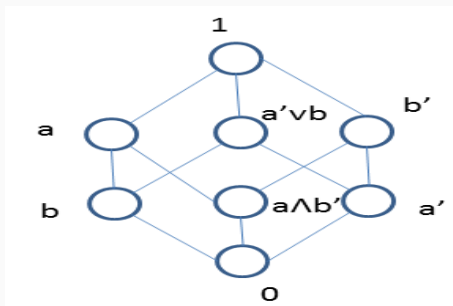
Example: $\langle D(30), gcd, lcm, ', 1, 30 \rangle$ is a boolean algebra.

Sub-boolean algebra

Let $\langle B, \wedge, \vee, ', 0, 1 \rangle$ be a boolean algebra and $S \subseteq B$. If S contains the elements 0 and 1 and is closed under the operations \wedge, \vee and $'$, then $\langle S, \wedge, \vee, ', 0, 1 \rangle$ is called a sub-boolean algebra.

Boolean Algebra

Example: Consider the following boolean algebra.



Let the subsets of this boolean algebra are:-

$$S_1 = \{a, a', 0, 1\}$$

$$S_2 = \{a' \vee b, a \wedge b', 0, 1\}$$

$$S_3$$

$$= \{a \wedge b', b', a, 1\}$$

$$S_4 = \{b', a \wedge b', a', 0\}$$

$$S_5 = \{a, b', 0, 1\}$$

Find out which are sub-boolean algebra.

Boolean Algebra

Solution: Consider the subset S_1 . Since S_1 is subset of B , therefore it satisfies distributive property. Now, 0 and 1 are also belongs into this subset. Complement of each element is also belong into S_1 . Since a and a' are complement of each other. we know 0 and 1 are complement of each other. Therefore, S_1 is sub-boolean algebra.

Consider the subset S_2 . Since S_1 is subset of B , therefore it satisfies distributive property. Now, 0 and 1 are also belongs into this subset. Clearly from diagram, complement of $a' \vee b = a \wedge b'$. complement of $0 = 1$. Therefore, S_2 is sub-boolean algebra.

Consider the subset S_3 . In this subset 0 is not belong, therefore this subset is not a boolean algebra.

Consider the subset S_4 . In this subset 1 is not belong, therefore this subset is not a boolean algebra.

Consider the subset S_5 . In this subset, complement of a and b' do not exists, therefore this subset is also not a boolean algebra.

Boolean expression

Definition: A Boolean expression always produces a Boolean value. A Boolean expression is composed of a combination of the Boolean constants (True or False), Boolean variables and logical connectives. Each Boolean expression represents a Boolean function.

A Boolean expression in n variables x_1, x_2, \dots, x_n is any finite string of symbols formed in the following manner:-

- (1) 0 and 1 are Boolean expressions.
- (2) x_1, x_2, \dots, x_n are Boolean expressions.
- (3) If α_1 and α_2 are Boolean expressions, then $\alpha_1 \wedge \alpha_2$ and $\alpha_1 \vee \alpha_2$ are also Boolean expressions.
- (4) If α is a Boolean expression then α' is also a Boolean expression.
- (5) All the expressions formed by step 1 to 4, are also Boolean expressions

Boolean Algebra

Example: x_1 , $x_1' \vee x_2$, $(x_2' \vee x_1)' \wedge (x_3 \vee x_1)$, and $(x_1' \vee x_1) \wedge x_2 \wedge x_3'$ are all Boolean expressions.

Equivalent Boolean expressions: Two Boolean expressions α and β are said to be equivalent if one can be obtained from the other by a finite number of applications of the identities of a Boolean algebra.

Minterm: A Boolean expression of n variables in the following form is said to be minterm.

$$x_1^{\alpha_1} \wedge x_2^{\alpha_2} \wedge \dots \wedge x_n^{\alpha_n}$$

where α_i is either 0 or 1, x_i^0 stands for x_i' and x_i^1 stands for x_i .

Maxterm: A Boolean expression of n variables in the following form is said to be maxterm.

$$x_1^{\alpha_1} \vee x_2^{\alpha_2} \vee \dots \vee x_n^{\alpha_n}$$

where α_i is either 0 or 1, x_i^0 stands for x_i' and x_i^1 stands for x_i .

Boolean Algebra

Canonical sum of product form: A Boolean expression is said to be in canonical sum of product form if it is the join of only minterms.

For example, for three variables, Boolean expression $(x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3')$ is in canonical sum of product form.

Canonical product of sum form: A Boolean expression is said to be in canonical product of sum form if it is the meet of only maxterms.

For example, for three variables, Boolean expression $(x_1' \vee x_2' \vee x_3') \wedge (x_1' \vee x_2 \vee x_3')$ is in canonical product of sum form.

Boolean Algebra

Example: Obtain the values of the Boolean expression (1) $x_1 \wedge (x_1' \vee x_2)$ (2) $x_1 \wedge x_2$ and (3) $x_1 \vee (x_1 \wedge x_2)$ over the ordered pairs of the two elements Boolean algebra.

Solution: Let $B = \{0,1\}$. Consider $x_1 = 0$ and $x_2 = 1$.

$$\begin{aligned}(1) \quad x_1 \wedge (x_1' \vee x_2) &= (x_1 \wedge x_1') \vee (x_1 \wedge x_2) \\ &= 0 \vee (x_1 \wedge x_2) \\ &= x_1 \wedge x_2 \\ &= 0 \wedge 1 \text{ (putting the values of } x_1 = 0 \text{ and } x_2 = 1 \text{)} \\ &= 0.\end{aligned}$$

$$\begin{aligned}(2) \quad x_1 \wedge x_2 &= 0 \wedge 1 \text{ (putting the values of } x_1 = 0 \text{ and } x_2 = 1 \text{)} \\ &= 0.\end{aligned}$$

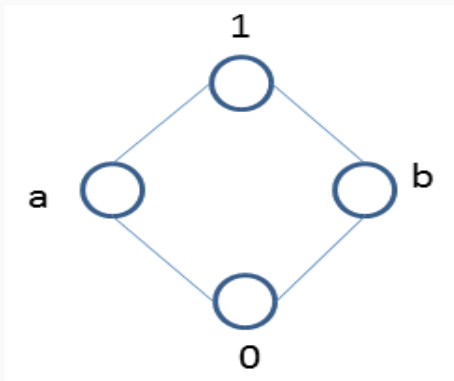
$$\begin{aligned}(3) \quad x_1 \vee (x_1 \wedge x_2) &= 0 \vee (0 \wedge 1) \\ &= 0 \vee 0 = 0.\end{aligned}$$

Boolean Algebra

Example: Find the value of following Boolean expression

$$x_1 \wedge x_2 \wedge [(x_1 \wedge x_4) \vee x_2' \vee (x_3 \wedge x_1')]$$

for $x_1 = a$, $x_2 = 1$, $x_3 = b$, $x_4 = 1$, where $1, a, b \in B$ and the Boolean algebra B is the following:-



Solution:

$$\begin{aligned}f(x_1, x_2, x_3) &= x_1 \wedge x_2 \wedge [(x_1 \wedge x_4) \vee x_2' \vee (x_3 \wedge x_1')] \\&= a \wedge 1 \wedge [(a \wedge 1) \vee 1' \vee (b \wedge a')] \\&= a \wedge [a \vee 0 \vee (b \wedge b)] \\&= a \wedge [a \vee b] \\&= a \wedge 1 \\&= a\end{aligned}$$