Discrete Structures and Theory of Logic

Unit-1

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Set Theory

1 Set

1.1 Definition

A well-defined collection of distinct objects can be considered to be a set.

Elements of a set can be just about anything from real physical objects to abstract mathematical objects. An important feature of a set is that its elements are distinct or uniquely identifiable.

A set is typically expressed by curly braces, {} enclosing its elements. If A is a set and a is an element of it, then we write a $\in$ A. The fact that a is not an element of A is written as a $\notin$ A. For instance, if A is the set {1, 2, 4, 9}, then 1 $\in$ A; 4 $\in$ A; 2 $\in$ A and 9 $\in$ A. But 7 $\notin$ A; 10 $\notin$ A, the English word 'four' is not in A, etc.

1.2 Representation of sets

We can represent sets in two ways.
1. Tabular form or roster form: Listing the elements of a set inside a pair of braces { } is called the roster form.
2. Set builder form: In the set builder form, all the elements of the set, must possess a single property to become the member of that set.

1.3 Examples

1. Let X = {apple, tomato, orange}. Here, orange $\in$ X, but potato $\notin$ X.
2. X = $\{a_1, a_2, \ldots\ldots\ldots\ldots, a_{10}\}$. Then, $a_{100} \in$ X.
3. Observe that the sets {1, 2, 3} and {3, 1, 2} are equal.
4. Let X = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Then X is the set of first 10 natural numbers. Or equivalently, X is the set of integers between 0 and 11.
5. X = {x : x is a prime number}.
6. X = {x : $0 < x \leq 10$ and x is an even integer }

Clearly examples 1, 2, 3, and 4 are in roster form, but 5 and 6 are in set builder form.

2 Types of set

2.1 Finite and Infinite sets

A set is said to be finite if the number of elements in the set is finite otherwise it is said to be infinite.

For example, a set of days in a week, set of months in a year, and a set of integer lie between 1 and 100 are finite sets. But set of integers, set of real numbers, and set of stars in sky are infinite sets.

2.2 Null or Empty set

A set which does not contain any element, is said to be null set. It is denoted by ϕ.
Example: set A = {a ! a is an integer lie between 4 and 5}

2.3 Singleton set

A set is said to be singleton set if it contains only one element.

2.4 Universal set

A universal set is the set of all elements under consideration, denoted by capital U or sometimes capital E.
Example: If we consider the elements are integers, then universal set will be the set of integer numbers. Similarly, if the elements are days of a week, then the set of all days in a week will be the universal set.

2.5 Subset

Consider two sets A and B. Set B is said to be subset of A if all the elements of B belong into A. It is denoted by $\subseteq$ symbol. That is, B $\subseteq$ A.
Example: Consider three sets A, B and C such that A= { 2, 3, 5, 8}, B= { 3, 5}, C={ 2, 9, 5, 8}. Clearly B is a subset of A but B is not a subset of C. Similarly, neither A is a subset of A nor C is a subset of A.
Note (1) Every set A is a subset of itself i.e. A $\subseteq$ A.
(2) The null set ϕ is a subset of any set.
(3) If A $\subseteq$ B and B $\subseteq$ C then A $\subseteq$ C.

2.6 Superset

Consider two sets A and B. Set A is said to be superset of B if all the elements of B belong into A. It is denoted by $\supseteq$ symbol. That is, A $\supseteq$ B.
Example: Consider three sets A, B and C such that A= { 2, 3, 5, 8}, B= { 3, 5}, C={ 2, 9, 5, 8}. Clearly A is a superset of B but C is not a superset of B. Similarly, neither A is a superset of A nor C is a superset of A.

2.7 Proper and Improper subsets

A set B is said to proper subset of set A if B is a subset of A and not equal to A that is B $\subseteq$ A and A $\neq$ B. It is denoted by $\supset$. Therefore, we can represent proper subset as B $\subset$ A.
A set B is said to improper subset of set A if B is a subset of A and equal to A that is B $\subseteq$ A and A = B.
Example: Consider four sets A, B, C and D such that A= { 2, 3, 5, 8}, B= { 3, 5},

C={ 2, 3, 5}, D={ 2, 3, 5, 8}. Clearly, B and C are the proper subsets and D is an improper subset.

2.8 Equal set

Two sets are said to be equal if both contains same elements. That is, if A $\subseteq$ B and B $\subseteq$ A then A = B.

2.9 Power set

The power set of a set A is the set of all the subsets of set A. It is denoted by P(A) or 2^A.
Example: (1) Consider set A = { a, b, c}. Then the power set of A is, P(A) = {ϕ ,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}.
(2) The power set of null or empty set will be ϕ.
Note If set A has n elements then number of elements in the power set of A will be 2^n.

3 Operations defined on set

3.1 Union operation

For any two sets A and B, the union of A and B is the set of all the elements which are belongs into A or B or both. It is denoted by A$\cup$B. Mathematically, it is defined as following:-
A$\cup$B = { x ! x $\in$ A or x $\in$ B}
Example: Let A = { a, b, c } and B = { d, e, c }. Then union of A and B will be, A$\cup$B = { a, b, c, d, e }.

3.2 Intersection operation

For any two sets A and B, the intersection of A and B is the set of all the elements which are belong into both A and B . It is denoted by A$\cap$B. Mathematically, it is defined as following:-
A$\cap$B = { x ! x $\in$ A and x $\in$ B}
Example: Let A = { a, b, c } and B = { d, e, c }. Then intersection of A and B will be, A$\cap$B = { c }.

3.3 Set difference operation

For any two sets A and B, the set difference of A and B is the set of all the elements which are belong into A but not belong into B . It is denoted by A-B. Mathematically, it is defined as following:-
A-B = { x ! x $\in$ A and x $\notin$ B}
Example: Let A = { a, b, c } and B = { d, e, c }. Then set difference of A and B will be, A-B = {a, b}.

3.4 Complement operation

Let U is the universal set. For any set A, the complement of A is the set of all the elements U, which are not belong into A. It is denoted by A^c or A'. Mathematically, it is defined as following:-
A'= U-A = { x ! x $\in$ U and x $\notin$ A}
Example: Let A = { a, b, c } and U = { a, b, c, d, e, f, g }. Then complement of A will be, A' = { d, e, f, g}.

3.5 Symmetric difference operation

For any two sets A and B, the symmetric difference of A and B is denoted by A$\bigoplus$B. Mathematically, it is defined as following:-
A$\bigoplus$B = (A-B) $\cup$ (B-A)
Example: Let A = { a, b, c } and B = { d, e, c }. Then symmetric difference of A and B will be, A$\bigoplus$B = {a, b, d, e }.

3.6 Disjoint sets

Two sets A and B are said to be disjoint if there is no common elements between A and B. That is, A and B are disjoint iff A$\cap$B = ϕ.
Example: Let A = { a, b, c } and B = { d, e, f }. Here A and B are disjoint because A$\cap$B = ϕ.

3.7 Mutually disjoint sets

A collection of sets S={ $A_1, A_2, ..., A_n$} is said to be mutually disjoint if each pair of A_i and A_j in S are disjoint. That is, S is mutually disjoint if $A_i \cap A_j = \phi$, $\forall$ i,j = 1,2,.....,n. and i$\neq$j.
Example: Let A = { {1, 2}, { 3} }, B = { {1}, {2, 3}} and C = { {1, 2, 3}}. These sets A, B and C are mutually disjoint because A$\cap$B = ϕ, B$\cap$C = ϕ, and A$\cap$C = ϕ.

3.8 Cardinality of a set

The number of elements in a set is said to be cardinality of a set. It is denoted by || symbol. That is, if A is a set then cardinality of set A is denoted by $|A|$.
Note:
(1) The number of elements in a set A is also represented by n(A).
(2) n(A$\cup$B) = n(A) + n(B) - n(A$\cap$B)
(3) n(A$\cup B \cup C$) = n(A) + n(B) + n(C) - n(A$\cap$B) - n(B$\cap$C) - n(A$\cap$C) + n(A$\cap B \cap C$)

4 Some examples

Example: Show that A $\subseteq$ B $\Leftrightarrow$ A$\cap$B = A
Solution: In this question, we have to prove two parts.
First part: In this part, we have to show that if A $\subseteq$ B then A$\cap$B = A.

Suppose $A \subseteq B$.
Let $x \in A$. Since $A \subseteq B$ therefore $x \in B$. Clearly x is belong into both A and B. Therefore x also belongs into $A \cap B$. Therefore

$$A \subseteq A \cap B \text{ ....}(1)$$

Let $x \in A \cap B$. Therefore $x \in A$ and $x \in B$. Therefore we can say $x \in A$. Therefore

$$A \cap B \subseteq A \text{ ....}(2)$$

Using equations(1) and (2), $A \cap B = A$.

Second part: In this part we have to show that if $A \cap B = A$ then $A \subseteq B$.
Let $x \in A$. Since $A \cap B = A$ therefore $x \in A \cap B$. This imply that $x \in A$ and $x \in B$. Therefore we can say $x \in B$. Therefore

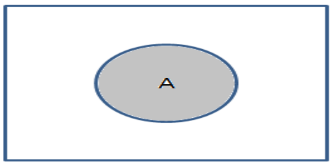
$$A \subseteq B.$$

Example: Show that
(a) A-B = A∩B'
(b) $A \subseteq B \Leftrightarrow B' \subseteq A'$
Solution:
(a) Let $x \in A\text{-}B \Rightarrow x \in A$ and $x \notin B$
$\Rightarrow x \in A$ and $x \in B'$
$\Rightarrow x \in A \cap B'$
Therefore, $A\text{-}B \subseteq A \cap B'$ ....(1)
Now, let $x \in A \cap B' \Rightarrow x \in A$ and $x \in B'$
$\Rightarrow x \in A$ and $x \notin B$
$\Rightarrow x \in A\text{-}B$
Therefore, $A \cap B' \subseteq A\text{-}B$ ....(2)
Using equations (1) and (2), $A\text{-}B = A \cap B'$
(b) First part: Suppose $A \subseteq B$.
Let let $x \in B' \Rightarrow x \notin B$
$\Rightarrow x \notin A$ (Since $A \subseteq B$)
$\Rightarrow x \in A'$
Therefore, $B' \subseteq A'$
Second part: Suppose $B' \subseteq A'$.
Let let $x \in A \Rightarrow x \notin A'$
$\Rightarrow x \notin B'$ (Since $B' \subseteq A'$)
$\Rightarrow x \in B$
Therefore, $A \subseteq B$
Using first and second parts, we can say that
$A \subseteq B \Leftrightarrow B' \subseteq A'$
Example: Show that for any two sets A and B,
$A\text{-}(A \cap B) = A\text{-}B$
Solution: Let $x \in A\text{-}(A \cap B) \Leftrightarrow x \in A$ and $x \notin A \cap B$
$\Leftrightarrow x \in A$ and $x \in (A \cap B)'$
$\Leftrightarrow x \in A$ and ($x \notin A$ or $x \notin B$)
$\Leftrightarrow$ ($x \in A$ and $x \notin A$) or ($x \in A$ and $x \notin B$)
$\Leftrightarrow$ FALSE or ($x \in A$ and $x \notin B$)
$\Leftrightarrow$ ($x \in A$ and $x \notin B$)
$\Leftrightarrow x \in A\text{-}B$

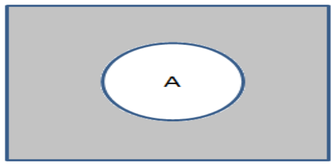
Therefore, $A-(A\cap B) = A-B$

5 Venn Diagram

Venn Diagram is a diagram representing mathematical or logical sets pictorially as circles or closed curves within an enclosing rectangle (the universal set), common elements of the sets being represented by intersections of the circles.
The universal set U is represented by a set of points in a rectangle and a subset A of U is represented by a circle or some other closed curve inside the rectangle.



A



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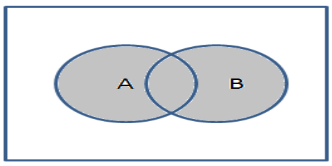

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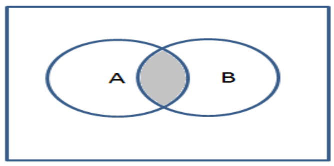
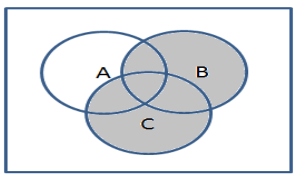
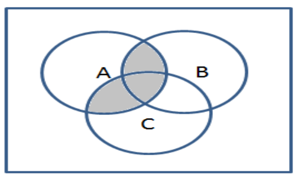


A∩B

Example: Using Venn diagram, show that
(a) $A\cap(B\cup C) = (A\cap B) \cup (A\cap C)$
(b) $A\cup(B\cap C) = (A\cup B) \cap (A\cup C)$
Solution:
(a)
LHS =



BUC

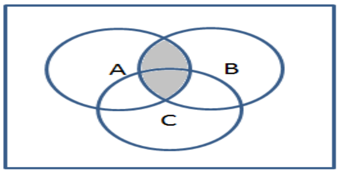


A ∩(BUC)

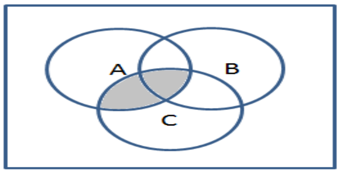
RHS =

6 Some examples

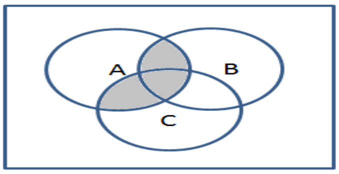
Example: In a group of 60 people, 40 speak Hindi, 20 speak both English and Hindi and all people speak at least one of the two languages. How many people speak only English and not Hindi? How many speak English?
Solution:
Total people = 60
Hindi speaking people = 40



A ∩B

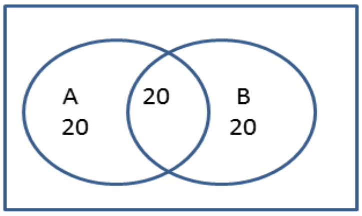


A ∩C



(A ∩B) U (A ∩C)

Both English and Hindi speaking = 20
Let A = The set of Hindi speaking people
and B = The set of English speaking people
Therefore, n(A) = 40, n(A∪B) = 60, and n(A∩B) = 20.
Number of people that speak only English and not Hindi is
n(B-A) = n(A∪B)-n(A) = 60-40 = 20
Number of people that speak English is
n(B) = n(A∪B)-n(A) + n(A∩B) = 60-40+20 = 40
By Venn diagram, we can also compute these values.

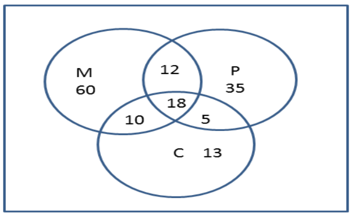


From above diagram,
Number of people that speak only English and not Hindi = 20
Number of people that speak English = 40

Example: A class has 175 students. In which, the number of students studying subjects are the following:-
Mathematics: 100, Physics: 70, Chemistry: 46; Mathematics and Physics: 30, Mathematics and Chemistry: 28, Physics and Chemistry: 23, Mathematics, Physics and Chemistry: 18. Find the following:-
(1) How many students are enrolled in Mathematics alone; Physics alone and Chemistry alone.
(2) The number of students who have not offered any of these subjects.
Solution:
Total students = 175
Let M, P, C denote the sets of students enrolled in Mathematics, Physics and Chemistry.
Therefore,
n(M) = 100, n(P) = 70, n(C) = 46, n(M∩P) = 30, n(P∩C) = 23,n(M∩C) = 28, n(M∩$P \cap C$) = 18.

Therefore, n(M$\cup P\cup C$) = n(M)+n(P)+n(C)-n(M$\cap$P)-n(P$\cap$C)-n(M$\cap$C)+n(M$\cap P\cap C$)
= 100+70+46-30-23-28+18
= 153

(1) Number of students enrolled in Mathematics alone is
n(M)-n(M$\cap$P)-n(M$\cap$C)+ n(M$\cap P\cap C$) = 100-30-28+18 = 60
Number of students enrolled in Physics alone is
n(P)-n(M$\cap$P)-n(P$\cap$C)+ n(M$\cap P\cap C$) = 70-30-23+18 = 35
Number of students enrolled in Chemistry alone is
n(C)-n(P$\cap$C)-n(M$\cap$C)+ n(M$\cap P\cap C$) = 46-23-28+18 = 13
(2) Number of students who have not offered any subjects is
Total students - n(M$\cup P\cup C$) = 175 - 153 = 22
By Venn diagram:



From above diagram,
Number of students enrolled in Mathematics alone = 60
Number of students enrolled in Physics alone = 35
Number of students enrolled in Chemistry alone = 13
Number of students who have not offered any subjects = total students - (60+35+13+12+10+5+18)
= 175 - 153 = 22

Example: A total of 1232 student have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages? AKTU(2018) **Solution:** Let S, F and R denotes the set of students have taken course Spanish, French and Russian. Therefore,
n(S) = 1232, n(F) = 879, n(R) = 114, n(S$\cap$F) = 103, n(S$\cap$R) = 23,n(F$\cap$R) = 14, n(S$\cup F\cup R$) = 2092
n(S$\cup F\cup R$) = n(S)+n(F)+n(R)-n(S$\cap$F)-n(S$\cap$R)-n(F$\cap$R)+n(S$\cap F\cap R$)
2092 = 1232+879+114-103-23-13+n(S$\cap F\cap R$)
n(S$\cap F\cap R$) = 2231- 2225 = 6

Example: Find the number of integers between 1 and 250 that are not divisible by any of the integers 2, 3, and 5.
Solution: Let A, B and C denotes the set integers between 1 and 250 that are divisible by 2, 3 and 5 respectively. Therefore,

$$n(A) = \left\lfloor \frac{250}{2} \right\rfloor = 125, \qquad n(B) = \left\lfloor \frac{250}{3} \right\rfloor = 83$$

$$n(C) = \left\lfloor \frac{250}{5} \right\rfloor = 50, \qquad n(A\cap B) = \left\lfloor \frac{250}{6} \right\rfloor = 41$$

$$n(A\cap C) = \left\lfloor \frac{250}{10} \right\rfloor = 25, \qquad n(B\cap C) = \left\lfloor \frac{250}{15} \right\rfloor = 16$$

$n(A \cap B \cap C) = \left\lfloor \frac{250}{30} \right\rfloor = 8$

Number of integers divisible by any of 2,3, and 5 is

= $n(A \cup B \cup C)$

= n(A)+n(B)+n(C)-n(A∩B)-n(B∩C)-n(A∩C)+$n(A \cap B \cap C)$

= 125+83+50-41-25-16+8 = 266-82 = 184

Therefore, number of integers not divisible by any of 2,3, and 5 is

= Total integers between 1 and 250 - $n(A \cup B \cup C)$

= 250-184 = 66

7 Equivalence laws in set theory

(1) Idempotent laws

(a) A∪A = A

(b) A∩A = A

(2) Associative laws

(a) A∪(B∪C) = (A∪B)∪C

(b) A∩(B∩C) = (A∩B)∩C

(3) Commutative laws

(a) A∪B = B∪A

(b) A∩B = B∪A

(4) Distributive laws

(a) A∪(B∩C) = (A∪B)∩(A∪C)

(b) A∩(B∪C) = (A∩B)∪(A∩C)

(5) Absorption laws

(a) A∪(A∩B) = A

(b) A∩(A∪B) = A

(6) Identity laws

(a) A∪ϕ = A (b) A∩ϕ = ϕ

(c) A∪U = U (d) A∩U = A

(7) Complement laws

(a) A∪A' = U (b) A∩A' = ϕ

(c) U' = ϕ (d) ϕ' = U

(8) DeMorgan's laws

(a) (A∪B)' = A'∩B'

(b) (A∩B)' = A'∪B'

8 Partition of a set

Let S be a given set and A = $\{A_1, A_2, A_3, ..., A_n\}$, where each A_i, for i = 1,2,3,......,n, is a subset of S.

A is called the partition of set S if it satisfies the following two conditions:-

(1) $\cup_{i=1}^{n} A_i$ = S

(2) $A_i \cap A_j = \phi$, ∀ i,j = 1,2,3,.......,n and i≠j.

Example: Consider set S = { a, b, c, d } and A = {{a, b }, { c, d }}.

In this example, A is the partition of S because

$\{a, b\} \cup \{c, d\} = S$ and $\{a, b\} \cap \{c, d\} = \phi$.

9 Multiset

A multiset is an unordered collection of elements, in which the multiplicity of an element may be one or more than one or zero. The multiplicity of an element is the number of times the element repeated in the multiset. In other words, we can say that an element can appear any number of times in a set.

Example:
A = {l, l, m, m, n, n, n, n}
B = {a, a, a, a, a, c}

9.1 Operations defined on Multisets

9.1.1 Union of Multisets

The Union of two multisets A and B is a multiset such that the multiplicity of an element is equal to the maximum of the multiplicity of an element in A and B and is denoted by A∪B.

Example:
Let A = {l, l, m, m, n, n, n, n}
B = {l, m, m, m, n},
Therefore, A∪B = {l, l, m, m, m, n, n, n, n}

9.1.2 Intersection of Multisets

The intersection of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the minimum of the multiplicity of an element in A and B and is denoted by A∩B.

Example:
Let A = {l, l, m, n, p, q, q, r}
B = {l, m, m, p, q, r, r, r, r}
Therefore, A∩B = {l, m, p, q, r}

9.1.3 Difference of Multisets

The difference of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the multiplicity of the element in A minus the multiplicity of the element in B if the difference is +ve, and is equal to 0 if the difference is 0 or negative

Example:
Let A = {l, m, m, m, n, n, n, p, p, p}
B = {l, m, m, m, n, r, r, r}
A - B = {n, n, p, p, p}

9.1.4 Sum of Multisets

The sum of two multisets A and B, is a multiset such that the multiplicity of an element is equal to the sum of the multiplicity of an element in A and B **Example:**
Let A = {l, m, n, p, r}

B = {l, l, m, n, n, n, p, r, r}
A + B = {l, l, l, m, m, n, n, n, n, p, p, r, r, r}

10 Countable and Uncountable sets

A set is said to be countable if:
(1) It is finite, or
(2) It has the same cardinality (size) as the set of natural numbers.
Equivalently, a set is countable if it has the same cardinality as some subset of the set of natural numbers. Otherwise, it is uncountable.
For example, the set of integers, the set of rational numbers or the set of algebraic numbers are countable set. An uncountable set is the set of real numbers.
For example, the set of real numbers between 0 and 1 is an uncountable set because no matter what, you'll always have at least one number that is not included in the set. This set does not have a one-to-one correspondence with the set of natural numbers.

11 Exercise

1. Give another description of the following sets and indicate those which are infinite sets.
   (a) { x ! x is an integer and $5 \leq x \leq 12$}
   (b) { 2, 4, 8, ... }
   (c) All the countries of the world.
2. Given S = { 2, a, { 3 }, 4 } and R = { { a }, 3, 4, 1 }, indicate whether the following are true or false.
   (a) $\{a\} \in S$
   (b) $\{a\} \in R$
   (c) $\{ a, 4, \{3\}\} \subseteq S$
   (d) $\{\{a\}, 1, 3, 4 \} \subset R$
   (e) $R = S$
   (f) $\{a\} \subseteq S$
   (g) $\{a\} \subseteq R$
   (h) $\phi \subset R$
   (i) $\phi \subseteq \{\{a\}\} \subseteq R \subseteq U$
   (j) $\{\phi\} \subseteq S$
   (k) $\{\phi\} \in R$
   (l) $\{\phi\} \subseteq \{\{3\}, 4\}$
3. Show that
   $(R \subseteq S) \wedge (S \subset Q) \Rightarrow R \subset Q$
   Is it correct to replace $R \subset Q$ by $R \subseteq Q$? Explain your answer.

4. Determine the power sets of the followings:-
   (a) { a, { b } }
   (b) { 1, ϕ }
   (c) { 1, 2, 3, 4 }
5. What is the power set of the empty set? What is the power set of the set $\{\phi\}$?
6. Find the numbers between 1 to 500 that are not divisible by any of the integers 2 or 3 or 5 or 7. AKTU(2019)
7. Determine whether each of these statements is true or false.
   (a) $0 \in \phi$
   (b) $\phi \in 0$
   (c) $0 \subset \phi$
   (d) $\phi \subset 0$
   (e) $0 \in 0$
   (f) $0 \subset 0$
   (g) $\phi \subseteq \phi$

12 Ordered pairs and Cartesian products

12.1 Ordered pair

An ordered pair consists of two objects in a given fixed order. An ordered pair is not a set of two elements. The ordering of two objects is important. We denote ordered pair by (x,y).

The equality of two ordered pairs is defined by (x,y) = (u,v) $\Leftrightarrow$ x=u and y=v.

12.2 Cartesian products

Let A and B be any two sets. The set of all ordered pairs such that the first member of the ordered pair is an element of A and the second member is an element of B, is called the Cartesian product of A and B. It is denoted by A×B.

Mathematically, it is defined as

A×B = { (x,y) ! x $\in$ A and y $\in$ B }

Example If A = {a, b} and B = { 1, 2 ,3}, then find A×B, B×A, A×A, B×B, and (A×B)∩(B×A).

Solution:

A×B = { (a,1),(a,2),(a,3),(b,1),(b,2),(b,3)}
B×A = { (1,a),(1,b),(2,a),2,b),(3,a),(3,b)}
A×A = { (a,a),(a,b),(b,a),(b,b)}
B×B = { (1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)}
(A×B)∩(B×A) = ϕ

Example If A = ϕ and B = { 1, 2 ,3}, then what are A×B, B×A?

Solution: A×B = ϕ and B×A = ϕ

Example Prove that
(a) A×(B∪C) = (A×B)∪(A×C)
(b) A×(B∩C) = (A×B)∩(A×C)
Solution:
(a) A×(B∪C) = {(x,y) ! x ∈ A and y ∈ (B∪C) }
= {(x,y) ! x ∈ A and (y ∈ B or y ∈ C) }
= {(x,y) ! (x ∈ A and y ∈ B) or (x ∈ A and y ∈ C) }
= {(x,y) ! (x,y) ∈ A×B or (x,y) ∈ A×C }
= {(x,y) ! (x,y) ∈ (A×B)∪(A×C) }
= (A×B)∪(A×C)
Therefore, A×(B∪C) = (A×B)∪(A×C).
(b) A×(B∩C) = {(x,y) ! x ∈ A and y ∈ (B∩C) }
= {(x,y) ! x ∈ A and (y ∈ B and y ∈ C) }
= {(x,y) ! (x ∈ A and y ∈ B) and (x ∈ A and y ∈ C) }
= {(x,y) ! (x,y) ∈ A×B and (x,y) ∈ A×C }
= {(x,y) ! (x,y) ∈ (A×B)∩(A×C) }
= (A×B)∩(A×C)
Therefore, A×(B∩C) = (A×B)∩(A×C).

13 Exercise

1. Determine the following:-
   (a) $\phi \cap \{\phi\}$
   (b) $\{\phi\} \cap \{\phi\}$
   (c) $\{\phi, \{\phi\}\} - \phi$
2. Determine A×B×C, B^2, A^3, $B^2 \times A$, where A = {1}, B = {a, b} and C = {2, 3}.
3. Prove that
   (a) (A∩B)∪(A∩B') = A
   (b) A∩(A'∪B) = A∩B
4. Show that (A∩B)∪C = A∩(B∪C) iff C ⊆ A
5. Draw Venn diagram for the following:-
   (a) B'
   (b) (A∪B)'
   (c) B-A'
   (d) A'∪B
   (e) A'∩B
6. Show that
   (a) (A-B)-C = (A-C)-(B-C)

(b) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$

7. Let A and B be sets. Show that $A \times B \neq B \times A$. Under what condition $A \times B = B \times A$?

Relation

14 Relation

14.1 Definition

A relation is defined as the subset of Cartesian product. That is, if R is a relation defined from the set A to B, then

$R \subseteq A \times B$

Mathematically, R = $\{(x,y)$! $x \in A$ and $y \in B\}$

Element a related b by relation R if (a,b)$\in$ R. It is denoted by aRb.

14.2 Examples

1. R = $\{(x,y)$! x is the father of y$\}$
2. R = $\{(a,2), (b,2), (c,3)\}$
3. Let A = $\{a, b, c, d\}$. Some relations R defined on set A are:
   (a) R = A$\times$A
   (b) R = $\{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (b, c)\}$
   (c) R = $\{(a, a), (b, b), (c, c)\}$
   (d) R = $\{(a, a), (b, b), (a, b), (b, a), (c, d)\}$
   (e) R = $\{(a, a), (a, c), (c, a), (a, b), (b, a), (c, c), (b, b)\}$

Note: Consider set A and B with m and n number of elements respectively. The number of relations which can be defined from set A to B will be 2^{mn}.

14.3 Domain of a relation

Domain of a relation R is the set of first element of all the ordered pairs belong into the relation R. Mathematically, it is defined as

Domain(R) = $\{$ a ! $\exists$ b, such that (a,b) $\in$ R$\}$

14.4 Range of a relation

Range of a relation R is the set of second element of all the ordered pairs belong into the relation R. Mathematically, it is defined as

Range(R) = $\{$ b ! $\exists$ a, such that (a,b) $\in$ R$\}$

15 Types of relation

15.1 Universal relation

A relation R defined from A to B is said to be universal relation if it contains all the ordered pairs defined from set A to B. That is, if R = A×B, then R is said to be universal set.

15.2 Void relation

If R does not contain any ordered pair, then it is said to be void or empty relation. That is, if R = ϕ then R is said to be empty relation.

15.3 Identity relation

A relation R defined on set A is said to be identity relation if
R = {(a,a) ! for all a $\in$ A}

15.4 Inverse relation

A relation R' is said to be inverse relation of R defined on set A if
R' = {(a,b) ! (b,a) $\in$ R}

16 Properties of binary relations in a set

There are following properties of which can be defined on a set. Consider the set is A.

16.1 Reflexive property

A binary relation R defined on set A is said to be satisfies reflexive property if every element of set A is related to itself. That is, aRa , $\forall$ a $\in$ A. That is, (a,a) $\in$ R, $\forall$ a $\in$ A.

16.2 Symmetric property

A binary relation R defined on set A is said to be satisfies symmetric property if (a,b) $\in$ R then (b,a) $\in$ R, $\forall$ a,b $\in$ A. That is, if aRb then bRa, $\forall$ a,b $\in$ A.

16.3 Transitive property

A binary relation R defined on set A is said to be satisfies transitive property if (a,b) $\in$ R and (b,c) $\in$ R then (a,c) $\in$ R, $\forall$ a,b,c $\in$ A.

16.4 Irreflexive property

A binary relation R defined on set A is said to be satisfies irreflexive property if no element of set A is related to itself. That is, (a,a) $\notin$ R, $\forall$ a $\in$ A.

16.5 Anti-symmetric property

A binary relation R defined on set A is said to be satisfies anti-symmetric property if (a,b) $\in$ R and (b,a) $\in$ R then a=b, $\forall$ a,b $\in$ A.

16.6 Asymmetric property

A binary relation R defined on set A is said to be satisfies asymmetric property if (a,b) $\in$ R then (b,a) $\notin$ R, $\forall$ a,b $\in$ A.

Note: A relation which satisfies reflexive property is said to be reflexive relation. A relation which satisfies symmetric property is said to be symmetric relation. A relation which satisfies transitive property is said to be transitive relation. A relation which satisfies irreflexive property is said to be irreflexive relation. A relation which satisfies anti-symmetric property is said to be anti-symmetric relation. A relation which satisfies asymmetric property is said to be asymmetric relation.

Example: Consider the following relations defined on set A = {1,2,3,4}. Find out which of these satisfies which of the above properties i.e. reflexive, symmetric, transitive, irreflexive, anti-symmetric, and asymmetric.

1. {(2,2),(2,3),(2,4),(3,2),(3,3),(3,4)}
2. {(1,1),(2,2),(2,1),(1,2),(3,3),(4,4)}
3. {(2,4),(4,2)}
4. {(1,2),(2,3),(3,4)}
5. {(1,1),(2,2),(3,3),(4,4)}
6. {(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)}

Solution:

1. Transitive.
2. Reflexive, symmetric, transitive.
3. Symmetric, irreflexive.
4. Irreflexive, anti-symmetric, asymmetric.
5. Reflexive, symmetric, transitive, anti-symmetric.
6. Irreflexive.

Example: Give an example of a relation which is satisfies corresponding properties.

1. Neither reflexive nor irreflexive.
2. Both symmetric and anti-symmetric.
3. Reflexive, transitive but not symmetric.

4. Symmetric, transitive but not reflexive.

5. Reflexive, symmetric but not transitive.

6. Reflexive, transitive but neither symmetric nor anti-symmetric.

Example: Which of the following relations are transitive?
R_1 = {(1,1)}, R_2 = {(1,2),(2,2)}, R_3 = {(1,2),(2,3),(1,3),(2,1)}

Example: Given S = {1,2,3,4}, and a relation R on S defined by
R = {(1,2),(4,3),(2,2),(2,1),(3,1)}
Show that R is not transitive. Find a relation $R_1 \supseteq R$ such that R_1 is transitive. Can you find another relation $R_2 \supseteq R$ which is also transitive?

Example: Given S = {1,2,3,.........,10}, and a relation R on S defined by
R = { (a,b) ! a+b = 10}
Which of the properties of a relation satisfy R?

Example: If R and S are both reflexive then show that R∪S and R∩S are also reflexive.
Solution: Since R and s are reflexive, therefore (a,a) ∈ R and (a,a) ∈ S, ∀ a. Since (a,a) ∈ R and (a,a) ∈ S, ∀ a, therefore (a,a) ∈ R∪S and (a,a) ∈ R∩S, ∀ a. Therefore, R∪S and R∩S are also reflexive.

Example: If R and S are both reflexive, symmetric, and transitive then show that and R∩S is also reflexive, symmetric, and transitive.
Solution:
For reflexive Since R and s are reflexive, therefore (a,a) ∈ R and (a,a) ∈ S, ∀ a. Since (a,a) ∈ R and (a,a) ∈ S, ∀ a, therefore (a,a) ∈ R∩S, ∀ a. Therefore, R∩S is also reflexive.
For symmetric Since R is symmetric, therefore if (a,b) ∈ R then (b,a) ∈ R. Similarly, since S is symmetric, therefore if (a,b) ∈ S then (b,a) ∈ S.
Let (a,b) ∈ R∩S. It imply that (a,b) ∈ R and (a,b) ∈ S. Since R and s are symmetric therefore (b,a) ∈ R and (b,a) ∈ S. It imply that (b,a) ∈ R∩S. Therefore R∩S is symmetric.
For transitive
Let (a,b) and (b,c) ∈ R∩S. ⇒ (a,b) and (b,c) ∈ R and (a,b) and (b,c) ∈ S
⇒ (a,c) ∈ R and (a,c) ∈ S (Since R and S are transitive)
⇒ (a,c) ∈ R∩S.
Therefore, R∩S is also transitive.

Example: Is if R and S are both reflexive, symmetric, and transitive then R∪S is reflexive, symmetric, and transitive?

17 Equivalence relation

17.1 Definition

A relation R defined on set A is said to be an equivalence relation if it satisfies reflexive, symmetric, and transitive properties.

17.2 Some examples

Example: Let A = {1,2,3,4} and R = {(1,1),(1,4),(4,1),(4,4),(2,2),(2,3),(3,2),(3,3)}. Is this relation an equivalence relation?
Solution: Since (1,1),(2,2),(3,3), and (4,4) are belongs into R, therefore R is reflexive.
Clearly in R if (a,b) $\in$ R then (b,a) $\in$ R. Here both (1,4) and (4,1) $\in$ R and both (3,2) and (2,3) $\in$ R. Therefore R is symmetric.
Clearly in R if (a,b) $\in$ R and (b,c) $\in$ R then (a,c) $\in$ R. Here, for pair (1,4) and (4,1), its transitive pair (1,1) and (4,4) are also belong into R. Similarly, or pair (2,3) and (3,2), its transitive pair (2,2) and (3,3) are also belong into R. Therefore R is transitive.

Example: Let A = {1,2,3,4,5,6} and R = {(a,b) ! (a-b) is divisible by 3 }. Show that R is an equivalence relation.
Solution: In this example R will be

R = {(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,4),(4,1),(2,5),(5,2),(3,6),(6,3)}

Clearly R satisfies reflexive, symmetric and transitive, therefore R is an equivalence relation.

Example: Let S be the set of lines on a plane. Define a relation R on set S as following:- aRb if line a is parallel to line b, $\forall$ a,b $\in$ S. Is relation R an equivalence relation.
Solution: Since each line in a plane is parallel to itself, therefore R satisfies reflexive property.
We know that if line a is parallel to line b then line b is also parallel to line a. Therefore R satisfies symmetric property.
We know that if line a is parallel to line b and line b is parallel to line c, then line a is also parallel to line c. Therefore R satisfies transitive property.
Since R satisfies all the three properties, therefore R is an equivalence relation.

Example: Let X = {a,b,c,d,e} and let C = {{a,b},{c},{d,e}}. Show that the partition C defines an equivalence relation on X.
Solution: The relation defined by partition C will be the following
R = {(a,a),(b,b),(a,b),(b,a),(c,c),(d,d),(e,e),(d,e),(e,d)}
Clearly relation R is an equivalence relation because R satisfies all the three properties.

17.3 Exercise

1. Let R denote a relation on the set of ordered pairs of positive integers such that (x,y)R(u,v) iff xv = yu.
   Show that R is an equivalence relation.

2. Given a set S = {1,2, 3, 4,5}. Find the equivalence relation defined on S which generates the partition { $\overline{1,2},\overline{3},\overline{4,5}$ }.

3. Prove that the relation "congruence modulo m" defined as

   $\cong$ = { (a,b) ! (a-b) is divisible by m }

   over the set of positive integers is an equivalence relation. Show that if a $\cong$ b and c $\cong$ d, then (a+c) $\cong$ (b+d).

4. Let R_1 be a relation defined on R, the set of real numbers, such that $R_1 = \{(x,y) ! |x - y| < 1 \}$. Is R_1 an equivalence relation ? Justify. AKTU(2019)

5. Let R be a binary relation on the set of all positive integers such that:
   R = {(a,b) ! a-b is an odd positive integers}
   Is R reflexive ? Symmetric? Transitive?

18 Equivalence class

Let R is an equivalence relation defined on set S. For any a ∈ S, the equivalence class of a is the set of all the elements of set S which are related from a. It is denoted by [a]. Mathematically it is defined as [a] = { b ∈ S ! aRb i.e. (a,b) ∈ R}.

Example: Let Z be the set of integers and let R be the relation called " Congruence modulo 3 ". Determine the equivalence classes generated by the elements of Z. That is, R = { (a,b) ! a,b ∈ Z and (a-b) is divisible by 3 }.
Solution: The equivalence classes for this relation are the followings:-
[0] = {.........,-12,-9,-6,-3,0,3,6,9,12,...........}
[1] = {.........,-11,-8,-5,-2,1,4,7,10,13,...........}
[2] = {.........,-10,-7,-4,-1,2,5,8,11,14,...........}

19 Matrix and Graph representation of the relations

19.1 Matrix representation

Let A = $\{a_1, a_2, ..., a_m\}$, B = $\{b_1, b_2, ..., b_n\}$, and R be a relation from A to B. Then the relation matrix corresponding to relation R will be m×n order matrix. Let this matrix is M. Then

$$m_{ij} = 1 \quad \text{if } (a_i, b_j) \in R$$
$$= 0 \quad \text{if } (a_i, b_j) \notin R$$

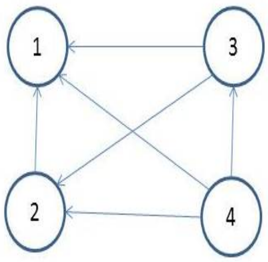
where m_{ij} is the element of matrix in i^th row and in j^th column.
Example: Consider a relation R = $\{(a_1, b_1), (a_2, b_1), (a_3, b_2), (a_2, b_2)\}$, and A = $\{a_1, a_2, a_3\}$, B = $\{b_1, b_2\}$. Find the relation matrix for R.
Solution:

	b_1	b_2
a_1	1	0
a_2	1	1
a_3	0	1

19.2 Graph representation

Let R be a relation defined in a set A = $\{a_1, a_2, ..., a_m\}$. The nodes in the graph corresponds to the elements in set a. Therefore, the number of nodes in the graph will be equal to number of elements in the set A. This graph will be directed graph. If $(a_i, a_j) \in$

R, then the directed edge will be from a_i to a_j in the graph.
Example: Let A = {1,2,3,4} and R = {(a,b) ! a > b}. Draw the graph of R and also give its matrix.
Solution:



	1	2	3	4
1	0	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	1	1	0

20 Composition of binary relations

Let R be a relation from A to B and S be a relation from B to C. Then a relation RoS is called composition of relation R and S. It is defined as:-

RoS = {(a,c) ! a $\in$ A and c $\in$ C and $\exists$ b $\in$ B such that (a,b) $\in$ R and (b,c) $\in$ S }

Example: Let R = {(1,2),(3,4),(2,2)} and S = {(4,2),(2,5),(3,1),(1,3)}. Find RoS, SoR and Ro(SoR).
Solution:
RoS = {(1,5),(3,2),(2,5)}
SoR = {(4,2),(3,2),(1,4)}
Ro(SoR) = {(3,2)}

Example: Let R and S be two relations on a set of positive integers I such that R = {(a, 2a) ! a $\in$ I } and S = {(a, 7a) ! a $\in$ I }. Find RoS, RoR, RoRoR and RoSoR.
Solution:
RoS = {(a,14a) ! a $\in$ I }
RoR = {(a,4a) ! a $\in$ I }
RoRoR = {(a,8a) ! a $\in$ I }
RoSoR = {(a,28a) ! a $\in$ I }

21 Closure of a relation

Consider R be relation defined on a set S.

21.1 Reflexive closure

The reflexive closure of a relation R is the smallest reflexive relation that contains R as a subset. It is denoted by r(R). Mathematically, it is defined as :-
r(R) = R$\cup I_S$
Where I_S is the identity relation defined on set S.

21.2 Symmetric closure

The symmetric closure of a relation R is the smallest symmetric relation that contains R as a subset. It is denoted by s(R). Mathematically, it is defined as :-
s(R) = R$\cup R^{-1}$
Where R^{-1} is the inverse relation of R.

21.3 Transitive closure

The transitive closure of a relation R is the smallest transitive relation that contains R as a subset. It is denoted by t(R). **Example:** Let S= {1,2,3,4}. Consider the following relation defined on the set S:-
R = { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}
Find reflexive, symmetric and transitive closure of R.
Solution:
Reflexive closure r(R) = R $\cup$ I_S
= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}$\cup$ { (1,1),(2,2),(3,3),(4,4)}
= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2),(3,3),(4,4)}
Symmetric closure s(R) = R $\cup R^{-1}$
={ (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}$\cup${ (1,1),(2,2),(2,1),(1,3),(3,1),(2,4)}
= { (1,1),(2,2),(1,2),(2,1),(1,3),(3,1),(4,2),(2,4)}
Transitive closure t(R) = R $\cup$ The set of ordered pairs to satisfy the transitive property
= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}$\cup$ { (3,3),(3,2)}
= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2),(3,3),(3,2)}

Example: How many reflexive relations are defined on the set with n elements? AKTU(2019)
Solution: According to reflexive property, each reflexive relation contains all the pairs like (a,a), where a belongs into the set. Total number of ordered pairs defined in the set with n elements is n^2. The number of ordered pairs like (a,a) will be n. Therefore, the remaining elements like (a,b) and a$\neq$b will be n^2 - n. Since the relation is a subset of set of ordered pairs, therefore total number of reflexive relations will be $2^{(n^2-n)}$.

Example: How many symmetric relations are defined on the set with n elements? AKTU(2019)
Solution: Consider the set is S with n elements. Relation is defined on the set S. The total number of relations defined on set S will be n^2, because relation is the subset of S$\times$S.
Now, if relation satisfies the symmetric property, then (a,b) and (b,a) belongs into the relation together. Therefore, the set whose all the subsets are reflexive relation contains $\frac{(n^2-n)}{2} + n = \frac{(n^2+n)}{2}$. Here, n is the number of ordered pairs like (a,a).
Therefore the total number of symmetric relations = $2^{\frac{(n^2+n)}{2}}$.

Example: How many anti-symmetric relations are defined on the set with n elements?
Solution: Consider the set is S with n elements. Relation is defined on the set S. The total number of relations defined on set S will be n^2, because relation is the subset of S$\times$S.
The total number of ordered pairs related to itself = n. Clearly, all the subsets of these

ordered pairs are anti-symmetric. Therefore, the total anti-symmetric relations defined on these ordered pairs = 2^n.
The remaining ordered pairs which are not related to itself = $n^2 - n$
Since both (a,b) and (b,a) can not belong into any anti-symmetric relations, Therefore, we consider only orederd pair = $\frac{(n^2-n)}{2}$.
Therefore, there are three possibilities for ordered pairs (a,b) and (b,a).
First possibility: (a,b) and (b,a) both not belong.
Second possibility: (a,b) belong but (b,a) not belong.
Third possibility: (a,b) not belong but (b,a) belong.
Therefore, total number of anti-symmetric relations for these types of ordered pairs = $3^{\frac{(n^2-n)}{2}}$.
Therefore, total number of anti-symmetric relations for the set S = $2^n * 3^{\frac{(n^2-n)}{2}}$.

22 Exercise

1. Is the "divides" relation on the set of positive integers transitive? What is the reflexive and symmetric closure of the relation R = $\{$(a, b) — a > b$\}$ on the set of positive integers? AKTU(2019)

Function

23 Function

23.1 Definition

Let X and Y are any two sets. A relation f from X to Y is called a function if for every x∈X, there is a unique element y∈Y such that (x,y) ∈ f. It is denoted by f: X$rightarrow$Y.

23.2 Some examples

example: Let X = {1,2,3,4} and Y = {x,y,w,z} and f = {(1,x), (2,y),(3,w),(4,x)}. Is f a function?
Solution: Clearly in function f, each element of set X has an image in set Y and that image has an unique. Therefore, f is a function.

example: Let X = Y = R. Also let, f = { (x, x^2) ! x∈R } and g = { (x^2, x) ! x∈R }. Find out f and g is functions or not.
Solution: Here R is a set of real numbers. Clearly for f, each real number has a unique square because square of 2 is 4, 3 is 9, 4 is 16 etc. Therefore, f is a function.
For relation g, element 4 has two images 2 and -2. Similarly, 9 has two images 3 and -3. Therefore, g is not a function.

23.3 Domain, Range, and Co-domain

Consider a function f : X$rightarrow$Y.
Domain of a function f is X. Co-domain of function f is Y. And range of f will be the set of second elements of all the ordered pairs in f i.e. range ∈ Y.

24 Types of function

24.1 Onto function (Surjective function)

A function f: X→Y is said to be onto function if every element of Y is the image of some element of X. That is, if range(f) = Y, then f is onto.

24.2 Into function

A function f: X→Y is said to be into function iff there exists at least one element in Y which is not the image of any element in X. That is, range(f) ⊂ Y.

24.3 One-one function (Injective function)

A function f: X→Y is said to be one-one function if for all elements x_1, x_2 in X such that f(x_1) = f(x_2) then $x_1 = x_2$.

24.4 Many-one function

A function f: X→Y is said to be many-one function iff two or more elements of X have same image in Y.

24.5 Bijective function)

A function f: X→Y is said to be bijective function if f is both one-one and onto.

24.6 Exercise

1. Let N be the set of natural numbers including zero. Determine which of the following functions are one-one, onto and bijective.

   (a) f: N→N, f(j) = j^2+2

   (b) f: N→N, f(j) = j mod 3

   (c) f: N→N, f(j) = 1 , if j is odd
   = 0 , if j is even

   (d) f: N→{0,1}, f(j) = 0 , if j is odd
   = 1 , if j is even

2. Let I be the set of integers, I_+ the set of positive integers, and I_p = {0,1,2,3,...................,(p-1)}. Determine which of the following functions are one-one, onto and bijective.

   (a) f: I→I, f(j) = (j-1)/2 , if j is odd
   = j/2 , if j is even

   (b) f: $I_+ \to I_+$, f(x) = greatest integer $\leq \sqrt{(x)}$

   (c) $I_7 \to I_7$, f(x) = 3x mod 7

   (d) $I_4 \to I_4$, f(x) = 3x mod 4

3. List all possible functions from X = {a,b,c} to Y = {0,1} and indicate in each case whether the function is one-one, onto and bijective.

4. Show that the functions f and g which both are from N×N to N given by f(x,y) = x+y and g(x,y) = xy are onto but not one-one.

24.7 Composition of functions

Let f: X→Y and g: Y→Z be two functions. Then composition of f and g is denoted by gof. It is defined as gof : X →Z.
(gof)(x) = g(f(x))
Note: gof ≠ fog.

Example: Let X ={1,2,3}, Y = {p,q}, and Z = {a,b}. Also let f is a function from X to Y such that f = {(1,p), (2,p), (3,q)} and g is a function from Y to Z such that g = {(p,b), (q,b)}. Find gof.
Solution: gof = { (1,b), (2,b), (3,b)}

Example: Let X ={1,2,3}, and f, g, h and s be functions from X to X given by
f = {(1,2),(2,3),(3,1)}, g = {(1,2), (2,1), (3,3)}, h = {(1,1), (2,2), (3,1)}, and s = {(1,1), (2,2),(3,3)}
Find fog, gof, fohog, sog, gos, sos and fos.
Solution:
fog = {((1,3),(2,2),(3,1)}
gof = {((1,1),(2,3),(3,2)}
fohog = {((1,1),(2,2),(3,2)}
Similarly, we can calculate others.

Example: Let f(x) = x+2, g(x) = x-2, and h(x) = 3x, ∀ x∈ R, where R is the set of real numbers. Find gof, fog, fof, hog and fohog.
Solution:
gof(x) = g(f(x)) = g(x+2) = x+2-2 = x
fog(x) = f(g(x)) = f(x-2) = x-2+2 = x
fohog(x) = f(h(g(x))) = f(h(x-2)) = f(3(x-2)) = 3(x-2)+2 = 3x-4
Similarly, we can calculate others.

Example: Let f: R→R be given by f(x) = -x^2 and g: $R_+ \rightarrow R_+$ be given by g(x) = $\sqrt(x)$, where R_+ is the set of positive real numbers and R is the set of real numbers. Find fog. Is gof defined?
Solution:
fog(x) = f(g(x)) = f($\sqrt(x)$) = -($\sqrt(x)$)^2 = -x
gof can not be defined because square root of negative real number can not be a real number.

24.8 Inverse function

Let f: X→Y is a function. If f is a bijective function then we can define the inverse function of f. It is denoted by f^{-1}. It is defined as f^{-1}: Y→ X. If f(a) = b then f^{-1}(b) = a.

Example: Let X = {1,2,3} and Y = {p,q,r}. f: X→ Y be given by
f = {(1,p), (2,q), (3,q)}
Is inverse of this function possible?
Solution: Inverse of this function is not possible because this function is not bijective. This function is not bijective because f is not onto function.
Example: Let R be the set of real numbers and let f: R→ R be given by
f = {(x, x^2) ! x ∈ R}
Is inverse of this function possible?
Solution: Inverse of this function is not possible because this function is not bijective. This function is not bijective because f is not onto function.

Example: Let R be the set of real numbers and let f: R→ R be given by
f = {(x, x+2) ! x ∈ R}
Is inverse of this function possible?
Solution: Inverse of this function is possible because this function is bijective.

24.9 Identity function

A function I_X: X→ X is called an identity function if I_X(x) = x, ∀ x ∈ X.

24.10 Invertible function

A function f is said to be invertible function if there exists an inverse function of this function.

Note: (1) If f: X→Y is invertible then $f^{-1}$of = I_X and fof^{-1} = I_Y
(2) Let f: X→ Y and g: Y→ X are two functions. The function g is equal to f^{-1} only if gof = I_X and fog = I_Y.

Example: Show that the functions f(x) = x^3 and g(x) = $x^{1/3}$, for x ∈ R are inverses of one another.
Solution: These functions will be inverse of each other if gof = I = fog.
gof(x) = g(f(x)) = g(x^3) = $(x^3)^{1/3}$ = $x^{3/3}$ = x = I(x).
fog(x) = f(g(x)) = f($x^{1/3}$) = $(x^{1/3})^3$ = $x^{3/3}$ = x = I(x).
Therefore these functions are inverses of each others.

Example: Let F_X be the set of all bijective functions from X to X, where X = {1,2,3}. Find all the elements of F_X and also find the inverse of each element.
Solution: Since the number of elements in set X is 3, therefore the number of bijective functions will be 3! = 6. These functions are:-
f_1 = {(1,1),(2,2),(3,3)} f_2 = {(1,1),(2,3),(3,2)}
f_3 = {(1,2),(2,1),(3,3)} f_4 = {(1,2),(2,3),(3,1)}
f_5 = {(1,3),(2,2),(3,1)} f_6 = {(1,3),(2,1),(3,2)}
Inverse of these functions is determined by interchanging values in each ordered pairs of corresponding functions.

Note: If a set X has n elements, then the number of bijective functions from X to X is n!.

24.11 Exercise

1. Let f: R →R and g: R →R are two functions such that f(x) = x^2 – 2 and g(x) = x+4, where R is the set real numbers. Find fog and gof. State wheteher these functions are injective, surjective and bijective.

2. If f: X →Y and g: Y→ Z and both f and g are onto, show that gof is also onto. Is gof one-one if both g and f are one-one?

3. Let f: R $\rightarrow$R be given by f(x) = x^2-2. Find f^{-1}.
4. How many functions are there from X to Y for the sets given below? Find also the number of functions which are one-one, onto and bijective.
   (a) X = {1,2,3}, Y = {1,2,3}
   (b) X = {1,2,3,4}, Y = {1,2,3}
   (c) X = {1,2,3}, Y = {1,2,3,4}
   (d) X = {1,2,3,4,5}, Y = {1,2,3}
   (e) X = {1,2,3}, Y = {1,2,3,4,5}
5. Let X = {1,2,3,4}. Define a function f: X $\rightarrow$X such that f$\neq I_X$ an is one-one. Find f^2, f^3, f^{-1} and fof^{-1}. Can you find another one-one function g: X $\rightarrow$X such that g $\neq I_X$ but gog = I_X?
6. Let f: X$\rightarrow$Y and X=Y=R, the set of real numbers. Find f^{-1} if
   (a) f(x) = x^2
   (b) f(x) = $\frac{(2x-1)}{5}$
7. Let f and g be the functions from the set of integers to the set of integers defined by f (x) = 2x + 3 and g(x) = 3x + 2. What is the composition of f and g? What is the composition of g and f?

25 Peano's Axioms and Principle of Mathematical Induction

25.1 Peano's Axioms

These axioms are
(1) 0 $\in$ N (where 0 = ϕ)
(2) If n $\in$ N, then $n^+ \in$ N, where n^+ = n$\cup${n}
(3) If a subset S $\subseteq$ N possesses the properties
    (a) 0 $\in$ S, and
    (b) If n $\in$ S, then $n^+ \in$ S
Then S = N.

25.2 Principle of Mathematical Induction

25.2.1 Definition

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.
The technique involves two steps to prove a statement, as stated below:-
Step 1(Base step): It proves that a statement is true for the initial value.
Step 2(Inductive step): It proves that if the statement is true for the number n, then it is also true for the number n+1.

25.2.2 Some examples

Example: Show that n $< 2^n$, by principle of induction method.
Solution:
Base step: For n = 0.

$0 < 2^0 \Leftrightarrow 0 < 1$

This is true. Therefore, the given statement is true for n = 0.

Now, for n=1.

$1 < 2^1 \Leftrightarrow 1 < 2$

This is true. Therefore, the given statement is also true for n = 1.

Therefore, the statement is true for base step.
Inductive Step: Now suppose the statement is true for n=k. We shall prove it for n=k+1.
Since statement is true for n=k, therefore k $< 2^k$ ....(1)
For n = k+1.

$$\begin{aligned} k+1 &< 2^k + 1 && \text{Using equation (1)} \\ &< 2^k + 2^k \\ &= 2.2^k \\ &= 2^{k+1} \end{aligned}$$

Therefore, k+1 $< 2^{k+1}$.
Therefore, statement is also true for inductive step.
Hence the given statement is proved.

Example: Show that 2^n < n!, $\forall$ n $\geq$ 4 by principle of induction method.
Solution:
Base step: For n = 4.

$2^0 < 4! \Leftrightarrow 16 < 24$

This is true. Therefore, the given statement is true for n = 4.

Now, for n=5.

$2^5 < 5! \Leftrightarrow 32 < 120$

This is true. Therefore, the given statement is also true for n = 5.

Therefore, the statement is true for base step.
Inductive Step: Now suppose the statement is true for n=k. We shall prove it for n=k+1.
Since statement is true for n=k, therefore 2^k <k! ....(1)
For n = k+1.

$$\begin{aligned} 2^{k+1} &= 2.2^k \\ &< 2.k! && \text{Using equation (1)} \\ &< (k+1).k! \\ &= (k+1)! \end{aligned}$$

Therefore 2^{k+1} < (k+1)!
Therefore, statement is also true for inductive step.
Hence the given statement is proved.

Example: Show that n^3 + 2n is divisible by 3, by principle of induction method.
Solution:
Base step: For n = 1.
n^3 + 2n = 1^3 + 2x1 = 1+2 = 3

Clearly n^3 + 2n is divisible by 3, therefore it is true for n = 1.
For n = 2.
n^3 + 2n = 2^3 + 2x2 = 8+4 = 12
Clearly n^3 + 2n is divisible by 3. Therefore it is also true for n = 2.
Therefore, the statement is true for base step.
Inductive Step: Now suppose the statement is true for n = k. We shall prove it for n = k+1.
Since statement is true for n = k, therefore k^3 + 2k is divisible by 3. It can be written as k^3 + 2k = 3m.........................(1)
For n = k+1.

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2(k+1)\\ &= (k^3 + 2k) + 3(k^2 + k + 1)\\ &= 3m + 3(k^2 + k + 1) \qquad \text{Using equation (1)}\\ &= 3(m + (k^2 + k + 1))\end{aligned}$$

Clearly it is divisible by 3. Therefore it is also true for n = (k+1).
Therefore, statement is also true for inductive step.
Hence the given statement is proved.

25.2.3 Exercise

1. Show that S(n) = 1+2+3+.........................+n = $\frac{n(n+1)}{2}$
2. Prove that $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots$.......................+ $\frac{1}{n.(n+1)} = \frac{n}{(n+1)}$
3. Show that 2+2^2+2^3+..................................+ 2^n = 2^{n+1} - 2
4. Show that 3^n - 1 is a multiple of 2, for n = 1, 2, 3,................
5. Show that 1+3+5+.........+(2n - 1) = n^2, for n = 1,2, 3, ...
6. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} +$ ....+ $\frac{1}{\sqrt{n}} > \sqrt{n}$, for n $\geq$ 2 using principle of mathematical induction. AKTU(2019)
7. Prove by using mathematical induction that
7+77+777+...........+777..........7 = $\frac{7}{81}$[10^{n+1} - 9n - 10] $\forall$ n $\in$ N. AKTU(2019)