Discrete Structures and Theory of Logic

Unit-3

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Partial ordered set and Hasse diagram

1 Partial ordered relation and Partial ordered set

1.1 Partial ordered relation

Consider a relation R defined on set S. Relation R is said to be partial ordered relation if R satisfies following properties:-
(1) R is reflexive, i.e., xRx for every x $\in$ S.
(2) R is anti-symmetric, i.e., if xRy and yRx, then x = y.
(3) R is transitive, i.e., xRy and yRz, then xRz.

1.2 Partial ordered set (POSET)

Consider a relation R defined on set S. If R is a partial order relation, then the combination of set S and partial order relation R is said to be partial ordered set i.e. POSET. We denote POSET by $< S, \preceq >$, where $\preceq$ denotes partial ordered relation.

1.3 Totally ordered relation and set

Let $< S, \preceq >$ bea partially ordered set. If for eery a,b $\in$ S, we hae either a$\preceq$b or b$\preceq$a, then $\preceq$ is called a totally ordered relation defined on set P.
And the ordered pair $< S, \preceq >$ is called a totally ordered set.

1.4 Some examples

Example: Let R be the set of real numbers and relation $\preceq$ is less than or equal i.e. a$\preceq$b iff a$\leq$b. Is $< R, \preceq >$ a POSET?
Solution: $< R, \preceq >$ will be a POSET if $\preceq$ is partial ordered relation. $\preceq$ will be partial ordered relation if this relation satisfies reflexive, anti-symmetric and transitive.
Here, the relation is less than or equal.
Each real number will be related to itself because each real number is equal to itself. Therefore, this relation is reflexive.
a is less than or equal to b and b is less than or equal to a , this is only possible iff a=b. Therefore this relation is anti-symmetric.
Consider a$\preceq$b and b$\preceq$c. It imply that a$\leq$b and b$\leq$c. It imply that a$\leq$c. Therefore R is transitive.
Since all the three properties are satisfies, therefore, this relation is partial ordered relation.

Example: Let A = {1,2,3}. Show that ¡P(A),$\subseteq$¿ is POSET, P(A) is power set o A.

Solution: Here P(A) = {ϕ, {1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}} and relation is subset i.e. set B related set C iff B⊆C.
For reflexive: Since each set is subset of itself, therefore this relation is reflexive.
For anti-symmetric: Consider B and C are two elements of P(A) such that B⊆C and C⊆B. Clearly it will be only true when B=C, otherwise it will be false. Therefore this relation is anti-symmetric.
For transitive: Consider B, C and D are three elements of P(A) such that B⊆C, C⊆D. It will imply that B⊆D. Therefore this relation is transitive.
Since all the three properties are satisfies, therefore, this relation is partial ordered relation. And the ordered pair $< P(A), \subseteq >$ is POSET.

Example: Let D(n) is the set of all positive divisors of n. And relation $\preceq$ is defined as:- a $\preceq$ b iff a divides b. Is $< D(n), \preceq >$ POSET?
Solution: Two elements a and b of D(n) will be related iff a divides b.
For reflexive: Since each element divides to itself, therefore this relation is reflexive.
For anti-symmetric: Consider a and b are two elements of D(n) such that a$\preceq$b and b$\preceq$c. Clearly it will be only true when a=b, otherwise it will be false. Therefore this relation is anti-symmetric.
For transitive: Consider a, b and c are three elements of D(n) such that a$\preceq$b, b$\preceq$c. It will imply that a$\preceq$c. Therefore this relation is transitive.
Since all the three properties are satisfies, therefore, this relation is partial ordered relation. And the ordered pair $< D(n), \preceq >$ is POSET.

1.5 Cover, Successor, Predecessor

In a partially ordered set $< S, \preceq >$, an element b $\in$ S is said to be cover of element a $\in$ S if a $\preceq$ b and if there does not exist any element c $\in$ S such that a $\preceq$ c $\preceq$ b.
a is the immediate predecessor of b and b is the immediate successor of a.

2 Hasse diagram

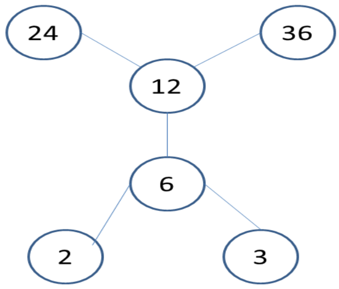
It is the graphical representation of POSET.

In this diagram, we make nodes corresponding to each elements in the POSET, that is the number of nodes is equal to the number of elements in the POSET. Edges will undirected. These edges will link elements a and b if b is cover of a such that a will be at lower side of b and b will be at upper side from a. In other words, two elements will be connected by an edge if one element is an immediate successor of another element.

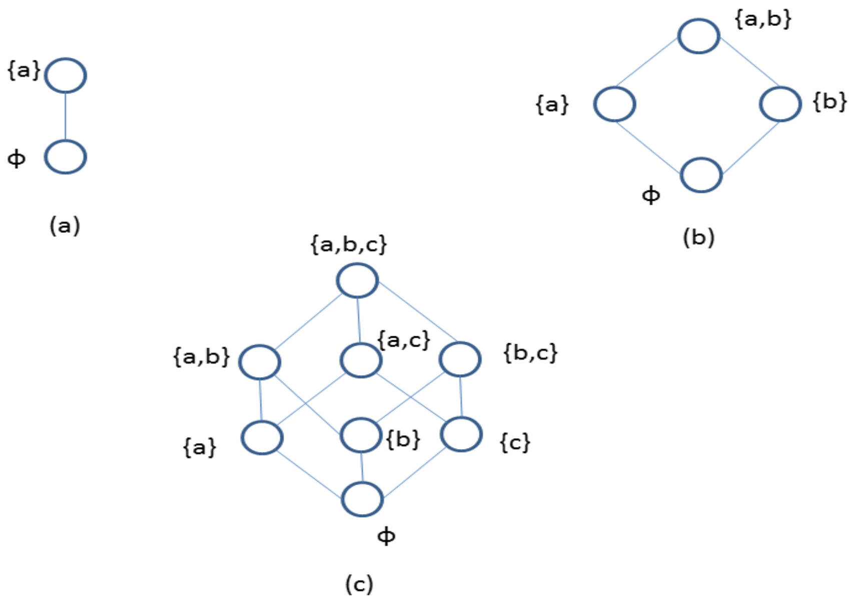
2.1 Some examples

Example: Let S = {2,3,6,12,24,36} and the relation $\preceq$ be such that a $\preceq$ b if a divides b. Draw the Hasse diagram of this POSET $< S, \preceq >$.
Solution:

Example: Let A be a given finite set and P(A) its power set. Let $\subseteq$ be the inclusion relation on the elements of P(A). Draw the Hasse diagram of $< P(A), \subseteq >$ for
(a) A = {a}, (b) A = {a,b}, (c) A = {a,b,c}



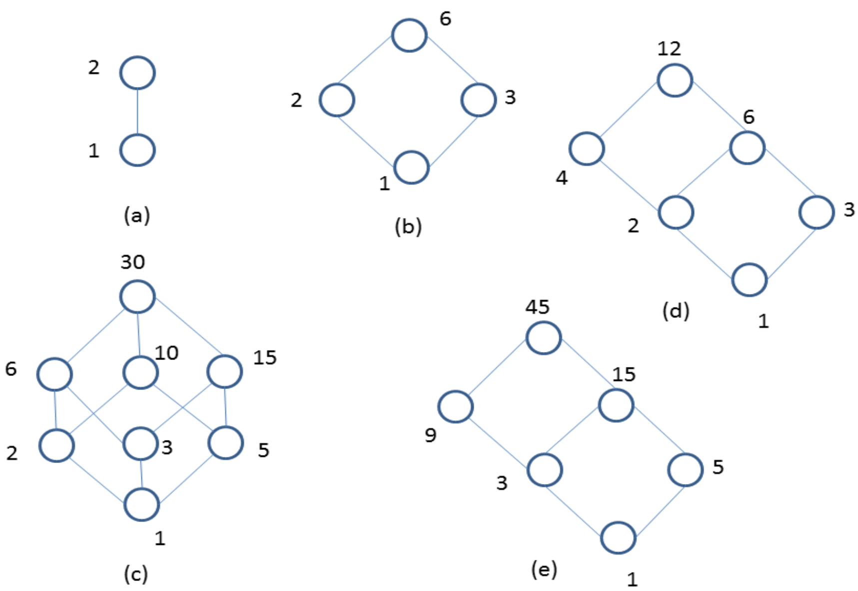
Solution:



Example: Let A be the set of factors of a particular positive integer m and let $\preceq$ be the relation divides, i.e.
$\preceq$ = {(a,b) ! a divides b}
Draw Hasse diagram for (a) m = 2 (b) m = 6 (c) m = 30 (d) m = 12 (e) m = 45
Solution: Since A be the set of factors of a particular positive integer m, therefore A will be for each case:-
(a) A = {1,2} (b) A = {1,2,3,6} (c) A = {1,2,3,5,6,10,15,30}
(d) A = {1,2,3,4,6,12} (e) A = {1,3,5,9,15,45}

2.2 Least and Greatest element

An element a $\in$ S is said to be the least element of the POSET $< S, \preceq >$, if a$\preceq$b, $\forall$ b $\in$ S.
An element b $\in$ S is said to be the greatest element of the POSET $< S, \preceq >$, if a$\preceq$b, $\forall$ a $\in$ S.



2.3 Minimal and Maximal element

An element a $\in$ S is said to be the minimal element of the POSET $< S, \preceq >$, if there is no element b $\in$ S such that b$\preceq$a.
An element b $\in$ S is said to be the maximal element of the POSET $< S, \preceq >$, if there is no element a $\in$ S such that b$\preceq$a.

2.4 Upper bound and Lower bound

Let $< S, \preceq >$ be a POSET and let A$\subseteq$S.
An element u$\in$S is said to be upper bound of set A if a$\preceq$u, $\forall$ a$\in$ A.
An element l$\in$S is said to be lower bound of set A if l$\preceq$a, $\forall$ a$\in$ A.

2.5 Least upper and Greatest lower bound

An upper bound u of the set A is said to be least upper bound of set A if u$\preceq$u', $\forall$ upper bound u' of A.
An upper bound l of the set A is said to be greatest lower bound of set A if l'$\preceq$l, $\forall$ lower bound l' of A.

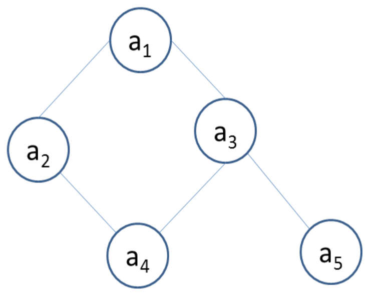
2.6 Well ordered set

A POSET is said to be well-ordered set if for every non-empty subset of it has a least element.

2.7 Some examples

Example: Consider the following Hasse diagram:-

Find the least and greatest element of this POSET if they exist. Also find minimal and maximal elements. Find the upper and lower bounds of $\{a_2, a_3, a_4\}$, $\{a_3, a_4, a_5\}$,



$\{a_1, a_2, a_3\}$. Also indicate the least upper bound and greatest lower bound of these subsets if they exists.

Solution:

Least element = does not exists because no element in this POSET is related to all the elements.

Greatest element = a_1 because all the elements of the POSET is related to a_1

Minimal elements = a_4, a_5

Maximal elements = a_1

Consider the set $\{a_2, a_3, a_4\}$.

lower bounds = a_4, because a_4 is related to all the elements of the set $\{a_2, a_3, a_4\}$

upper bounds = a_1, because all the elements of the set $\{a_2, a_3, a_4\}$ is related to a_1.

greatest lower bound = a_4

least upper bound = a_1

Consider the set $\{a_3, a_4, a_5\}$.

lower bounds = does not exists because no element is related to all the elements of the set $\{a_3, a_4, a_4\}$

upper bounds = a_1, a_3, because all the elements of the set $\{a_3, a_4, a_5\}$ is related to a_1, a_3.

greatest lower bound = does not exists

least upper bound = a_3

Consider the set $\{a_1, a_2, a_3\}$.

lower bounds = a_4, because a_4 is related to all the elements of the set $\{a_1, a_2, a_3\}$

upper bounds = a_1, because all the elements of the set $\{a_1, a_2, a_3\}$ is related to a_1.

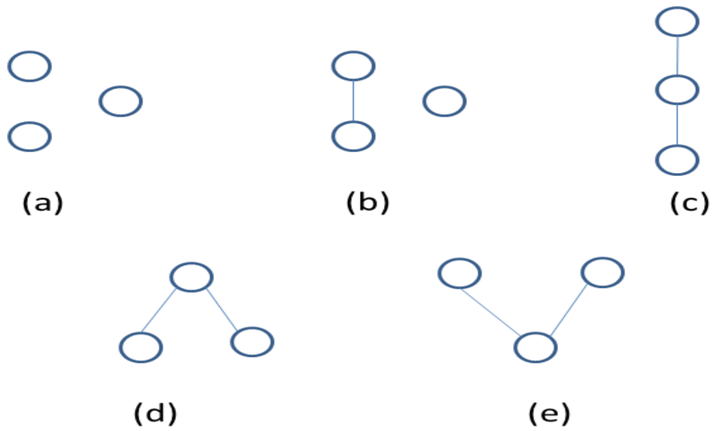
greatest lower bound = a_4

least upper bound = a_1

Example: Show that there are only five distinct Hasse diagrams for partial ordered sets that contain three elements.

Solution: All the distinct Hasse diagrams corresponding to three elements are the followings:-

Clearly no other Hasse diagrams can be drawn for three elements. If we make any other diagram for three elements then it will be equivalent to any one of the above. Therefore, Hasse diagram corresponding to three elements will be five.



2.8 Exercise

1. Draw the Hasse diagram of the following sets under the partial ordering relation "divides" and indicate those which are totally ordered.
   (a) {2,6,24}
   (b) {3,5,15}
   (c) {1,2,3,6,12}
   (d) {2,4,8,16}
   (e) {3,9,27,54}
2. Give an example of a set A such that $< P(A), \subseteq >$ is a totally ordered set.
3. Give a relation which is both partial ordering and an equivalence on a set.
4. Draw all the Hasse diagrams corresponding to four elements.

Lattice

3 Lattice

3.1 Definition

A POSET $< L, \subseteq >$ is said to be lattice if for every pair of elements a,b $\in$ L, its greatest lower bound and least upper bound exists.
Greatest lower bound is denoted by a$\wedge$b and least upper bound is denoted by a$\vee$b.

3.2 Some examples

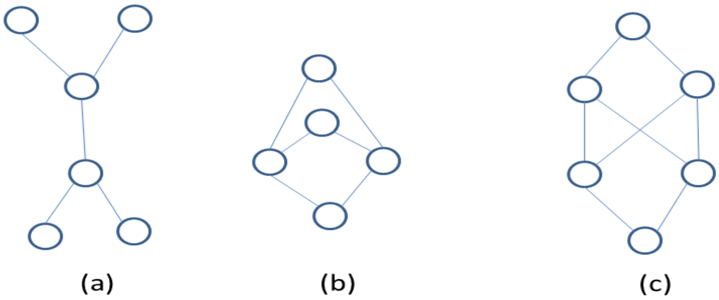
Example: Is POSET $< P(A), \subseteq >$ a lattice, where A = {1,2,3}?
Solution: In this POSET, greatest lower bound of two elements of P(A) is equivalent to intersection of those elements and least upper bound of two elements of P(A) is equivalent to union of those elements.
Clearly, intersection and union of any two elements of P(A) always exists in P(A). Therefore this POSET is lattice.

Example: Let I_+ be the set of all positive integers and D denote the relation of division in I_+ such that for any a,b $\in I_+$, aDb iff a divides b. Is (I_+,D) a lattice?
Solution: In this example, first we have to check this set is POSET or Not. After this, we have to check for lattice.
Clearly, this set is POSET because it satisfies all the three properties.
Clearly, for each pair of integers, its least upper bound and greatest lower bound exists because the operation is division and set is set of all positive integers. For example, consider two elements 4 and 6. Its lub = 12 and glb = 2. Both 2 and 12 belongs into I_+. Similarly for elements 3 and 4, lub = 12 and glb = 1. Here 1 and 12 both belong into I_+. Therefore this set is lattice.

Example: Let n be a positive integer and S_n be the set of all positive divisors of n. And D is a division relation. Is (S_6,D), (S_8,D), (S_{24},D), and(S_{30},D) lattices?
Solution: All these are lattices.

3.3 Exercise

1. Find out the following POSETs are lattices or not.



2. Draw the diagram of lattices $< S_n, D >$ for n = 4, 6, 10, 12, 15, 45, 60, 75 and 210. For what values of n, do you expect $< S_n, D >$ to be a chain?

3. Let R be the set of real numbers in [0,1] and $\preceq$ be the usual operation of "less than or equal" on R. Show that $< R, \preceq >$ is a lattice. What are the operations of meet and join on this lattice?

4. Let the sets $S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7$ be given by
S_0 = {a,b,c,d,e,f} S_1 = {a,b,c,d,e} S_2 = {a,b,c,e,f} S_3 = {a,b,c,e}
S_4 = {a,b,c} S_5 = {a,b} S_6 = {a,c} S_7 = {a}
Draw the diagram of $< L, \subseteq >$, where L = $\{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$.

3.4 Principle of Duality

Any statement about lattices involving the operations $\wedge$ and $\vee$ remains true if $\wedge$ is replaced by $\vee$ and $\vee$ is replaced by $\wedge$.

The operations $\wedge$ and $\vee$ are said to be dual of each other. For example $a\wedge b$ is the dual of $a\vee b$.

3.5 Properties of lattices

(1) Idempotent law

$a\wedge a = a$, $\quad$ $a\vee a = a$

(2) Commutative law

$a\wedge b = b\wedge a$, $\quad$ $a\vee b = b\vee a$

(3) Associative law

$a\wedge(b\wedge c) = (a\wedge b)\wedge c$, $\quad$ $a\vee(b\vee c) = (a\vee b)\vee c$

(4) Absorption law

$a\wedge(a\vee b) = a$, $\quad$ $a\vee(a\wedge b) = a$

Theorem: Let $< L, \preceq >$ be a lattice. For any a,b $\in$ L,

$a \preceq b \Leftrightarrow a\wedge b = a \Leftrightarrow a\vee b = b$

Proof: In this theorem,we have to prove many parts.
First part: In this part, we will prove $a \preceq b \Leftrightarrow a\wedge b = a$.
Suppose $a \preceq b$. Since $a \preceq a$, therefore a = lower bound(l.b.) of a and b.
Since a is lower bound therefore, a$\preceq$ greatest lower bound(g.l.b.) of a and b.
hence $a \preceq a\wedge b$ ....(1)
By the definition of glb, $a\wedge b \preceq a$ ....(2)
from (1) and (2),
$a\wedge b = a$.
Conversely, suppose $a\wedge b = a$.
By the definition of glb,
$a\wedge b \preceq b$
Since $a\wedge b = a$, therefore $a \preceq b$.

Second part: In this part, we will prove $a \preceq b \Leftrightarrow a\vee b = b$.
Suppose $a \preceq b$. Since $b \preceq b$, therefore b = upper bound(u.b.) of a and b.
Since b is an upper bound therefore, least upper bound(l.u.b.) of a and b $\preceq$ b.
hence $a\vee b \preceq b$ ....(1)
By the definition of lub, $b \preceq a\vee b$ ....(2)
from (1) and (2),
$a\vee b = b$.
Conversely, suppose $a\vee b = b$.
By the definition of lub,
$a \preceq a\vee b$
Since $a\vee b = b$, therefore $a \preceq b$.

Third part: In this part, we will prove $a\wedge b = a \Leftrightarrow a\vee b = b$.
Suppose $a \wedge b = a$.
Now, $a\vee b = (a\wedge b)\vee b = b$, by absorption law.
Suppose $a \vee b = b$.
Now, $a\wedge b = a\wedge(a\vee b) = a$, by absorption law.

Theorem: Let $< L, \preceq >$ be a lattice. For any a,b,c $\in$ L,

(1) a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$ (a$\vee$c)
(2) a$\wedge$(b$\vee$c) $\succeq$ (a$\wedge$b)$\vee$ (a$\wedge$c)

Proof:

(1) Using definition of least upper bound, a $\preceq$ (a$\vee$b)..........(1)
Using definition of greatest lower bound, b$\wedge$c $\preceq$ b ....(2)
Using definition of least upper bound, b $\preceq$ (a$\vee$b)..........(3)
From (2) and (3), b$\wedge$c $\preceq$ b $\preceq$ (a$\vee$b)
Therefore, b$\wedge$c $\preceq$ (a$\vee$b).....................(4)
From (1) and (4), (a$\vee$b) is the upper bound of a and b$\wedge$c, therefore
lub{a , b$\wedge$c} $\preceq$ (a$\vee$b) i.e.
a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b) ....(5)
Similarly, using definition of least upper bound, a $\preceq$ (a$\vee$c)..........(6)
Using definition of greatest lower bound, b$\wedge$c $\preceq$ c ....(7)
Using definition of least upper bound, c $\preceq$ (a$\vee$c)..........(8)
From (7) and (8), b$\wedge$c $\preceq$ c $\preceq$ (a$\vee$c)
Therefore, b$\wedge$c $\preceq$ (a$\vee$c).....................(9)
From (6) and (9), (a$\vee$c) is the upper bound of a and b$\wedge$c, therefore
lub{a , b$\wedge$c} $\preceq$ (a$\vee$c) i.e.
a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$c) ....(10)
From (5) and (10), a$\vee$(b$\wedge$c) is lower bound of (a$\vee$b) and (a$\vee$c), therefore
a$\vee$(b$\wedge$c) $\preceq$ glb{(a$\vee$b), (a$\vee$c)} i.e.
a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$(a$\vee$c)
Now, it is proved.

(2) Using definition of greatest lower bound, (a$\wedge$b)$\preceq$ a ....(1)
Using definition of least upper bound, b $\preceq$ (b$\vee$c)..........(2)
Using definition of greatest lower bound, (a$\wedge$b)$\preceq$ b..........(3)
From (2) and (3), (a$\wedge$b)$\preceq$ b $\preceq$ (b$\vee$c)
Therefore, (a$\wedge$b) $\preceq$ b$\vee$c.....................(4)
From (1) and (4), (a$\wedge$b) is the lower bound of a and b$\vee$c, therefore
(a$\wedge$b)$\preceq$ glb{a , b$\vee$c} i.e.
(a$\wedge$b)$\preceq$ a$\wedge$(b$\vee$c) ....(5)
Similarly, using definition of greatest lower bound, (a$\wedge$c)$\preceq$ a ....(6)
Using definition of least upper bound, c $\preceq$ (b$\vee$c)..........(7)
Using definition of greatest lower bound, (a$\wedge$c)$\preceq$ c..........(8)
From (7) and (8), (a$\wedge$c)$\preceq$ c $\preceq$ (b$\vee$c)
Therefore, (a$\wedge$c) $\preceq$ (b$\vee$c).....................(9)
From (6) and (9), (a$\wedge$c) is the lower bound of a and (b$\vee$c), therefore
(a$\wedge$c)$\preceq$ glb{a , b$\vee$c} i.e.
(a$\wedge$c)$\preceq$ a$\wedge$(b$\vee$c) ....(10)
From (5) and (10), a$\wedge$(b$\vee$c) is upper bound of (a$\wedge$b) and (a$\wedge$c), therefore
lub{(a$\wedge$b), (a$\wedge$c)} $\preceq$ a$\wedge$(b$\vee$c) i.e.
(a$\wedge$b)$\vee$(a$\wedge$c)} $\preceq$ a$\wedge$(b$\vee$c)
Now, it is proved.

Theorem: Let $< L, \preceq >$ be a lattice. For any a,b,c $\in$ L,
a $\preceq$ c $\Leftrightarrow$ a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$c

Proof:

First part: In this part, we will prove that if a $\preceq$ c then a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$c
Suppose a $\preceq$ c.
Using definition of least upper bound, a $\preceq$ (a$\vee$b)..........(1)
Using definition of greatest lower bound, (b$\wedge$c) $\preceq$ b ....(2)
Using definition of least upper bound, b $\preceq$ (a$\vee$b) ....(3)
Therefore, from (2) and (3), (b$\wedge$c) $\preceq$ b $\preceq$ (a$\vee$b)
Therefore, (b$\wedge$c) $\preceq$ (a$\vee$b) ....(4)
From (1) and (4), (a$\vee$b) is the upper bound of a and b$\wedge$c, therefore
lub{a , b$\wedge$c} $\preceq$ (a$\vee$b) i.e.
a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b) ....(5)
Now, using definition of greatest lower bound, (b$\wedge$c) $\preceq$ c ....(6)
Since a $\preceq$ c, therefore using (6), c is an upper bound of a and (b$\wedge$c). Therefore
a$\vee$(b$\wedge$c) $\preceq$ c ....(7)
from (5) and (7), a$\vee$(b$\wedge$c) is lower bound of (a$\vee$b) and c. Therefore,
a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$c
Second part: In this part, we will prove that if a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$c then a $\preceq$ c.
Now, a $\preceq$ a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$c $\preceq$ c
Therefore, a $\preceq$ c.
Since both parts are proved. Therefore, it is proved.

3.6 Exercise

1. Show that in a lattice if a $\preceq$ b $\preceq$ c, then
   a$\vee$b = b$\wedge$c and (a$\wedge$b)$\vee$(b$\wedge$c) = b = (a$\vee$b)$\wedge$(a$\vee$c)

2. Show that in a lattice if a $\preceq$ b and c $\preceq$ d, then
   (a$\wedge$c) $\preceq$ (b$\wedge$d)

3. In a lattice, show that

   (a) (a$\wedge$b)$\vee$ (c$\wedge$d) $\preceq$ (a$\vee$c)$\wedge$ (b$\vee$d)

   (b) (a$\wedge$b)$\vee$ (b$\wedge$c) $\vee$ (c$\wedge$a) $\preceq$ (a$\vee$b)$\wedge$ (b$\vee$c) $\wedge$ (c$\vee$a)

4. Show that a lattice with three or fewer elements is a chain.

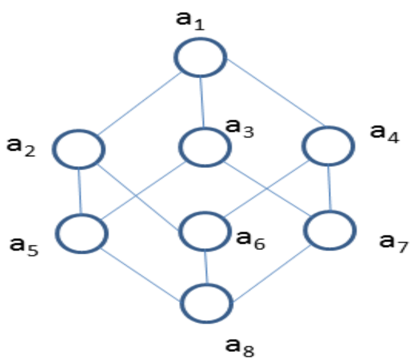
3.7 Lattices as algebraic system

A lattice is an algebraic system $< L, \wedge, \vee >$ with two binary operations $\wedge$ and $\vee$ on L which are satisfy commutative, associative, absorption and idempotent properties.

3.8 Sublattice

Let $< L, \wedge, \vee >$ be a lattice and let S $\subseteq$ L be a subset of L. Then $< S, \wedge, \vee >$ is said to be sublattice of $< L, \wedge, \vee >$ iff $< S, \wedge, \vee >$ is also a lattice.

Example: Consider the following lattice L = $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$.
Let $S_1 = \{a_1, a_2, a_4, a_6\}$, $S_2 = \{a_3, a_5, a_7, a_8\}$, and $S_3 = \{a_1, a_2, a_4, a_8\}$.
Find out $< S_1, \prec >, < S_2, \prec >, and < S_3, \prec >$ sublattices or not.



4 Types of morphism in lattice

4.1 Lattice Homomorphism

Let $< L, \otimes, \oplus >$ and $< S, \wedge, \vee >$ be two lattices. A mapping f: L $\rightarrow$ S is called lattice homomorphism from the lattice $< L, \otimes, \oplus >$ to $< S, \wedge, \vee >$ if for any (a,b) $\in$ L,

f(a$\otimes$b) = f(a)$\wedge$f(b) and f(a$\oplus$b) = f(a)$\vee$f(b)

4.2 Lattice Isomorphism

A homomorphism f: L $\rightarrow$ S is said to be isomorphism if f is bijective.
If there exists isomorphism between two lattices, then the lattices are called isomorphic.

4.3 Lattice Endomorphism

A homomorphism is said to be endomorphism if both lattices are same.

4.4 Lattice Automorphism

An isomorphism is said to be autoomorphism if both lattices are same.

4.5 Order-preserving

Let $< P, \preceq >$ and $< Q, \preceq' >$ be two POSETs. A mapping f: P$\rightarrow$Q is said to be order-preserving relatie to the ordering $\prec$ in P and $\prec$' in Q iff for any a,b $\in$ P such that a $\prec$ b, f(a) $\prec$' f(b).
Note: If f is homomorphism, then f is order-preserving.

4.6 Order-isomorphic

Two POSETs $< P, \preceq >$ and $< Q, \preceq' >$ are called order-isomorphic if there exists a mapping f: P$\rightarrow$Q which is bijective and if both f and f^1 are order-preserving.
Note: It may happen that a mapping f: P$\rightarrow$Q is bijective and order-preserving, but that f^1 is not order-preserving.

4.7 Direct product or Cartesian product

Let $< L, \otimes, \oplus >$ and $< S, \wedge, \vee >$ be two lattices. The algebraic system $< L \times S, *, + >$ in which the binary operations + and * on L×S are such that for any (a_1, b_1) and (a_2, b_2) in L×S

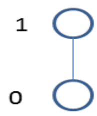
$(a_1, b_1) * (a_2, b_2) = (a_1 \otimes a_2, b_1 \wedge b_2)$

$(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$

is called the direct product of the lattices $< L, \otimes, \oplus >$ and $< S, \wedge, \vee >$.

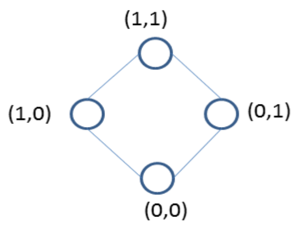
Note: L^2 = L×L and L^3 = L×L×L

Example: Let L = {0,1} and the lattice $< L, \prec>$ is

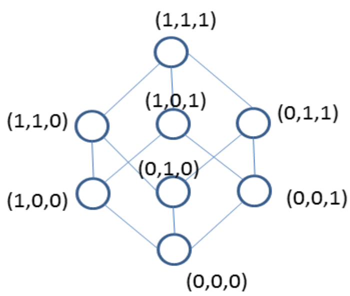


Find the lattices $< L^2, \prec_2>$ and $< L^3, \prec_3>$.

Solution: Lattices $< L^2, \prec_2>$ and $< L^3, \prec_3>$ are drawn as following:-



<L², ≤₂>



<L³, ≤₃>

Note: The partial ordering relation $\preceq^n$ on L^n can be defined for any a,b $\in L^n$, where a = $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$, as

a$\prec_n$b $\Leftrightarrow a_i \preceq b_i$, $\forall$ i.

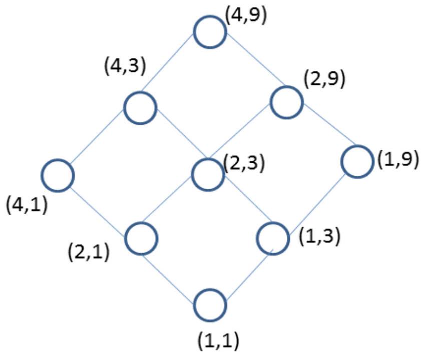
Where $\preceq$ means the relation of "less than or equal to" on {0,1}.

Example: Consider the chains of divisors of 4 and 9, that is L_1 = {1,2,4} and L_2 = {1,3,9}, and the partial order relation of "division" on L_1 and L_2. Draw the Hasse diagram for $L_1 \times L_2$.

Solution: Hasse diagram for this lattice can be drawn as following:-

This diagram is same as the lattice of divisors of 36, where (a, b) is replaced by the product ab.

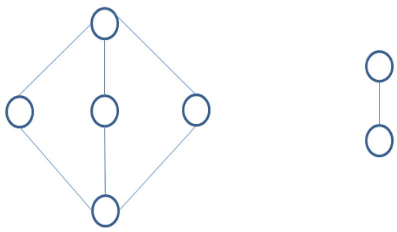
Example: Let S be any set containing n elements and P(S) be its power set. Then the lattice $< P(S), \cap, \cup >$ or $< P(S), \subseteq>$ is isomorphic to the lattice $< L^n, \prec_n>$.



4.8 Exercise

1. Find all the sublattices of the lattice $< D(n), / >$, for n = 12.

2. Draw the diagram of a lattice which is the direct product of the five element lattice and a two element lattice.



(a) (b)

5 Types of lattice

5.1 Complete lattice

A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

5.2 Bounded lattice

Bounds: The least and greatest elements of a lattice, if they exists, are called the bounds of the lattice and are denoted by 0 and 1 respectively.
Definition: A lattice which has both least and greatest elements i.e. 0 and 1, is called a bounded lattice.
Note: The bounds 0 and 1 of a lattice satisfy the following identities:-
For any a $\in$ L, a$\wedge$0 = 0, a$\wedge$1 = a
a$\vee$0 = a, a$\vee$1 = 1.

5.3 Complemented lattice

In a bounded lattice, an element b $\in$ L is said to be complement of an element a $\in$ L if a$\wedge$b = 0 and a$\vee$b = 1.

A lattice $< L, \wedge, \vee, 0, 1 >$ is said to be a complemented lattice if every element of L has at least one complement.

5.4 Some examples

Example: Is the lattice $< P(\{a, b, c\}), \subseteq>$ a complemented?
Solution: This lattice will be complemented if eery element has complement in this lattice.
P({a,b,c}) = {ϕ,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}
In this lattice, least element i.e. 0 = ϕ and greatest element i.e. 1 = {a,b,c}
The complement of ϕ will be {a,b,c}, because ϕ∧{a,b,c} = ϕ and ϕ∨{a,b,c} = {a,b,c}.
Similarly, the complement of {a,b,c} will be ϕ.
Similarly, {a}' = {b,c}, {b}' = {a,c}, {c}' = {a,b}.
{a,b}' = {c}, {a,c}' = {b}, and {b,c}' = {a}
Clearly each element has a complement, therefore this lattice is complemented.

Example: Is the lattice $< D(30), / >$ a complemented?
Solution: Here, D(30) = {1,2,3,5,6,10,15,30}
Two elements a and b will be complement of each other iff a∧b = 0 and a∨b = 1.
In this eample, 0(least element) = 1 and 1(greatest element) = 30.
Since 2∧15 = 1 and 2∨15 = 30, therefore 2 and 15 are complement of each other.
Since 3∧10 = 1 and 3∨10 = 30, therefore 3 and 10 are complement of each other.
Since 5∧6 = 1 and 5∨6 = 30, therefore 5 and 6 are complement of each other.
Since 1∧30 = 1 and 1∨30 = 30, therefore 1 and 30 are complement of each other.
Clearly each element has a complement, therefore this lattice is complemented.

Example: Is the lattice $< D(12), / >$ a complete?
Solution: Here, D(12) = {1,2,3,4,6,12}
Since this lattice is finite, therefore every subset of this set has a least upper bound and greatest lower bound. Clearly, consider the set {2,3,4}. The least upper bound of this set is 12 because each elements of this set divides 12 and no other elements in this. The greatest lower bound will be 1 because 1 diides to each elements of this set. Similarly, we can check for any subset of the given lattice.
Therefore this lattice is complete.

5.5 Distributive lattice

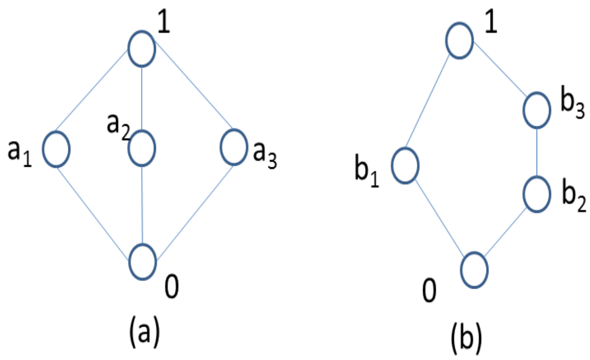
A lattice $< L, \wedge, \vee >$ is called a distributive lattice if for any a,b,c ∈ L,
$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

5.6 Modular lattice

A lattice $< L, \wedge, \vee >$ is called a modular lattice if for any a,b,c ∈ L,
a⪯c $\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$.

5.7 Some examples

Example: Check the following lattices to be modular or distributive.



Solution:
(a) For modular lattice:
Consider three elements a,b,c belongs into the lattice such that a$\preceq$c.
Let a = a_1, b = a_2, and c = 1.
Therefore, $a \vee (b \wedge c) = a_1 \vee (a_2 \wedge 1) = a_1 \vee a_2 = 1$
and $(a \vee b) \wedge c = (a_1 \vee a_2) \wedge 1 = 1 \wedge 1 = 1$
Therefore, a$\preceq$c $\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$ for a = a_1, b = a_2, and c = 1.
Similarly, we can show that a$\preceq$c $\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$ for any a,b,c belongs into lattice such that a$\preceq$c. Therefore, this lattice is modular lattice.
For distributive lattice:
Consider three elements a,b,c belongs into the lattice.
Let a = a_1, b = a_2, and c = a_3.
Therefore, $a \wedge (b \vee c) = a_1 \wedge (a_2 \vee a_3) = a_1 \wedge 1 = a_1$
and $(a \wedge b) \vee (a \wedge c) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3) = 0 \vee 0 = 0$
Clearly, $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$ for a = a_1, b = a_2, and c = a_3. Therefore this lattice is not distributive.

Example: Show that every chain is a distributive lattice.
Solution: Consider any three elements a,b,c of a chain. There will be six different relations exist between these elements.
Case-1: ($a \preceq b \preceq c$):
In this case, $a \wedge (b \vee c) = a \wedge c =$ a
and $(a \wedge b) \vee (a \wedge c) = a \vee a =$ a
Therefore, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
Similarly, $a \vee (b \wedge c) = a \vee b =$ b
and $(a \vee b) \wedge (a \vee c) = b \wedge c =$ b
Therefore, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
Clearly, both properties of distributive lattice are satisfied for this case.

Case-2: ($a \preceq c \preceq b$):
In this case, $a \wedge (b \vee c) = a \wedge b =$ a
and $(a \wedge b) \vee (a \wedge c) = a \vee a =$ a
Therefore, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Similarly, $a \vee (b \wedge c) = a \vee c =$ c
and $(a \vee b) \wedge (a \vee c) = b \wedge c =$ c
Therefore, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
Clearly, both properties of distributive lattice are satisfied for this case.

Similarly, we can show for other four cases that properties of distributive lattice are satisfied. Other four case are (3) $(b \preceq a \preceq c)$ (4) $(b \preceq c \preceq a)$ (5) $(c \preceq a \preceq b)$ (6) $(c \preceq b \preceq a)$.
Since, in all the cases, properties of distributive lattice are satisfied, therefore a chain is distributive lattice.

Example: Let $< L, \wedge, \vee >$ be a distributive lattice. For any a,b,c $\in$ L,

a$\wedge$b = a$\wedge$c and a$\vee$b = a$\vee$c $\Rightarrow$ b = c.

Solution:
LHS = b = $b \wedge (b \vee a)$ (using absorption law)
= $b \wedge (a \vee b)$ (using commutative law)
= $b \wedge (a \vee c)$ (using given equality a$\vee$b = a$\vee$c)
= $(b \wedge a) \vee (b \wedge c)$ (using distributive law)
= $(a \wedge b) \vee (b \wedge c)$ (using commutative law)
= $(a \wedge c) \vee (b \wedge c)$ (using given equality a$\wedge$b = a$\wedge$c)
= $(a \wedge b) \vee c)$ (using distributive law)
= $(a \wedge c) \vee c)$ (using given equality a$\wedge$b = a$\wedge$c)
= c (using absorption law)
= RHS
Therefore, b = c

Example: In a distributive lattice, every element has a unique complement.
Solution: Consider an element a belong into given lattice L. Suppose b and c are two complements of a in L. Therefore,
a$\wedge$b = 0, a$\vee$b = 1 , and a$\wedge$c = 0, a$\vee$c = 1. ....(1)
Now, we have to show that b and c will be equal.
b = b$\wedge$1 (using identity law)
= b$\wedge$(a$\vee$c) (using equation (1))
= $(b \wedge a) \vee (b \wedge c)$ using distributive law)
= 0$\vee(b \wedge c)$ (using equation (1))
= (a$\wedge$c)$\vee(b \wedge c)$ (using equation (1))
= $(a \wedge b) \vee c$ (using distributive law)
= 0$\vee c$ (using equation (1))
= c (using identity law)
Therefore, b = c. That is, we can say, every element has a unique complement.

5.8 Exercise

1. Find the complements of every elements of the lattice $< D(n), / >$ for n = 75.

2. Show that in a lattice with two or more elements, no element is its own complement.

3. Show that a chain of three or more elements is not complemented.

4. Which of the two lattices $< D(n), / >$ for n = 30 and n = 45 are complemented? Are these lattices are distributive?

5. Show that De-Morgan's law given by
   (a∧b)' = a'∨b' and (a∨b)' = a'∧b'
   hold in complemented and distributive lattice.

6. Show that in a complemented, distributive lattice,
   a⪯b ⇔ a∧b' = 0 ⇔ a'∨b = 1 ⇔ b'⪯a'

7. Show that every distributive lattice is modular, but not conversely.

Boolean algebra diagram

6 Boolean algebra

6.1 Definition

A lattice is said to be boolean algebra if it is both complemented and distributive. It is denoted by $< B, \wedge, \vee,', 0, 1 >$.

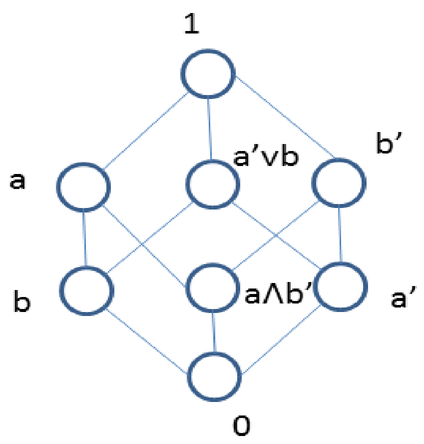
6.2 Some examples

Example: $< P(S), \cap, \cup,', \phi, S >$ is a boolean algebra. Because it is complemented and distributive both. In this lattice, 0 is ϕ and 1 is S. Each elements of P(S) has a complement.

Example: $< D(30), gcd, lcm,', 1, 30 >$ is a boolean algebra.

6.3 Sub-boolean algebra

Let $< B, \wedge, \vee,', 0, 1 >$ be a boolean algebra and S $\subseteq$ B. If S contains the elements 0 and 1 and is closed under the operations $\wedge, \vee$ and ', then $< S, \wedge, \vee,', 0, 1 >$ is called a sub-boolean algebra.
Example: Consider the following boolean algebra.



Let the subsets of this boolean algebra are:-
S_1 = {a,a',0,1} S_2 = {a'∨b ,a∧b',0,1} S_3 = {a∧b' ,b',a,1}
S_4 = {b' ,a∧b',a',0} S_5 = {a,b',0,1}
Find out which are sub-boolean algebra.

Solution: Consider the subset S_1. Since S_1 is subset of B, therefore it satisfies distributive property. Now, 0 and 1 are also belongs into this subset. Complement of each element is also belong into S_1. Since a and a' are complement of each other. we know 0 and 1 are complement of each other. Therefore, S_1 is sub-boolean algebra.
Consider the subset S_2. Since S_1 is subset of B, therefore it satisfies distributive property. Now, 0 and 1 are also belongs into this subset.
Clearly from diagram, complement of a'∨b = a∧b'. complement of 0 = 1. Therefore, S_2 is sub-boolean algebra.
Consider the subset S_3. In this subset 0 is not belong, therefore this subset is not a boolean algebra.
Consider the subset S_4. In this subset 1 is not belong, therefore this subset is not a boolean algebra.
Consider the subset S_5. In this subset, complement of a and b' do not exists, therefore this subset is also not a boolean algebra.

7 Boolean expression

7.1 Boolean expression

Definition: A Boolean expression always produces a Boolean value. A Boolean expression is composed of a combination of the Boolean constants (True or False), Boolean variables and logical connectives. Each Boolean expression represents a Boolean function.
A Boolean expression in n variables $x_1, x_2, ..., x_n$ is any finite string of symbols formed in the following manner:-
(1) 0 and 1 are Boolean expressions.
(2) $x_1, x_2, ..., x_n$ are Boolean expressions.
(3) If α_1 and α_2 are Boolean expressions, then $\alpha_1 \wedge \alpha_2$ and $\alpha_1 \vee \alpha_2$ are also Boolean expressions.
(4) If α is a Boolean expression then α' is also a Boolean expression.
(5) All the expressions formed by step 1 to 4, are also Boolean expressions.
Example: x_1, $x_1' \vee x_2$, $(x_2' \vee x_1)' \wedge (x_3 \vee x_1)$, and $(x_1' \vee x_1) \wedge x_2 \wedge x_3'$ are all Boolean expressions.

Equivalent Boolean expressions: Two Boolean expressions α and β are said to be equivalent if one can be obtained from the other by a finite number of applications of the identities of a Boolean algebra.

Minterm: A Boolean expression of n variables in the following form is said to be minterm.
$x_1^{\alpha_1} \wedge x_2^{\alpha_2} \wedge .... \wedge x_n^{\alpha_n}$
where α_i is either 0 or 1, x_i^0 stands for x_i' and x_i^1 stands for x_i.
Maxterm: A Boolean expression of n variables in the following form is said to be maxterm.
$x_1^{\alpha_1} \vee x_2^{\alpha_2} \vee .... \vee x_n^{\alpha_n}$
where α_i is either 0 or 1, x_i^0 stands for x_i' and x_i^1 stands for x_i.
Canonical sum of product form: A Boolean expression is said to be in canonical sum of product form if it is the join of only minterms.
For example, for three variables, Boolean expression $(x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3')$ is in

canonical sum of product form.

Canonical product of sum form: A Boolean expression is said to be in canonical product of sum form if it is the meet of only maxterms.

For example, for three variables, Boolean expression $(x_1' \vee x_2' \vee x_3') \wedge (x_1' \vee x_2 \vee x_3')$ is in canonical product of sum form.

Example: Obtain the values of the Boolean expression (1) $x_1 \wedge (x_1' \vee x_2)$ (2) $x_1 \wedge x_2$ and (3) $x_1 \vee (x_1 \wedge x_2)$

over the ordered pairs of the two elements Boolean algebra.

Solution: Let B = {0,1}. Consider $x_1 = 0$ and $x_2 = 1$.

(1) $x_1 \wedge (x_1' \vee x_2) = (x_1 \wedge x_1') \vee (x_1 \wedge x_2)$

$= 0 \vee (x_1 \wedge x_2)$

$= x_1 \wedge x_2$

$= 0 \wedge 1$ (putting the values of $x_1 = 0$ and $x_2 = 1$)

$= 0$.

(2) $x_1 \wedge x_2 = 0 \wedge 1$ (putting the values of $x_1 = 0$ and $x_2 = 1$)

$= 0$.

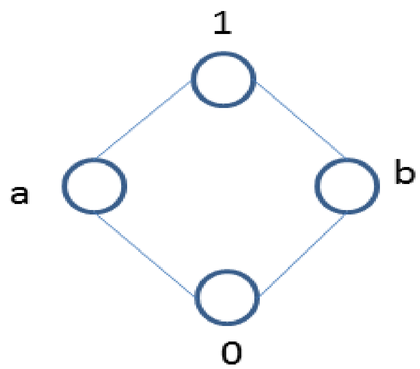
(3) $x_1 \vee (x_1 \wedge x_2) = 0 \vee (0 \wedge 1)$

$= 0 \vee 0 = 0$.

Example: Find the value of following Boolean expression

$$x_1 \wedge x_2 \wedge [(x_1 \wedge x_4) \vee x_2' \vee (x_3 \wedge x_1')]$$

for x_1 = a, x_2 = 1, x_3 = b, x_4 = 1, where 1,a,b $\in$ B and the Boolean algebra B is the following:-



Solution:

$f(x_1, x_2, x_3) = x_1 \wedge x_2 \wedge [(x_1 \wedge x_4) \vee x_2' \vee (x_3 \wedge x_1')]$

$= a \wedge 1 \wedge [(a \wedge 1) \vee 1' \vee (b \wedge a')]$

$= a \wedge [a \vee 0 \vee (b \wedge b)]$

$= a \wedge [a \vee b]$

$= a \wedge 1$

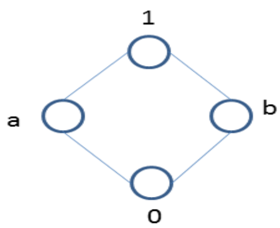
= a

Note: Given a Boolean expression $\alpha(x_1, x_2, ..., x_n)$ and a Boolean algebra $< B, \wedge, \vee, ', 0, 1 >$, we can obtain the values of the Boolean expression for every n-tuple of B^n. Let us now consider a function $f_{\alpha,B} : B^n \to B$ such that for any n-tuple $< a_1, a_2, ..., a_n > \in B^n$, the value of $f_{\alpha,B}$ is equal to the value of the Boolean expression $\alpha(x_1, x_2, ..., x_n)$, that is,

x_1	x_2	x_3	$\alpha(x_1, x_2, x_3)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

$f_{\alpha,B}(a_1, a_2, ..., a_n) = \alpha(x_1, x_2, ..., x_n)$
for all $(a_1, a_2, ..., a_n) \in B^n$. We shall call $f_{\alpha,B}$ the function associated with the Boolean expression $\alpha(x_1, x_2, ..., x_n)$.

Example: Find the value of the function $f_{\alpha,B} : B^3 \to B$ for x_1 = a, x_2 = 1,and x_3 = b, where a,b,1 are the elements of the Boolean algebra is shown in the following figure:-



and $\alpha(x_1, x_2, ..., x_n)$ is the expression whose binary valuation is given in the following table:-

Solution: From the table,

$f_{\alpha,B}(x_1, x_2, x_3) = (x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_2' \wedge x_3)$

$\alpha(a, 1, b) = (a' \wedge 1' \wedge b') \vee (a' \wedge 1 \wedge b') \vee (a' \wedge 1 \wedge b) \vee (a \wedge 1' \wedge b)$

$= (b \wedge 0 \wedge a) \vee (b \wedge 1 \wedge a) \vee (b \wedge 1 \wedge b) \vee (a \wedge 0 \wedge b)$

$= 0 \vee (b \wedge a) \vee b \vee 0$

$= (b \wedge a) \vee b$

= 0∨b

= b

7.2 Boolean function

Let $< B, \wedge, \vee,', 0, 1 >$ ba Boolean algebra. A function f: $B^n \to$ B which is associated with a Boolean expression in n-variables is called a Boolean function.

Note: For a two elements Boolean algebra, the number of functions from B^n to B is 2^{2^n}. Here, every function from B^n to B is Boolean function.

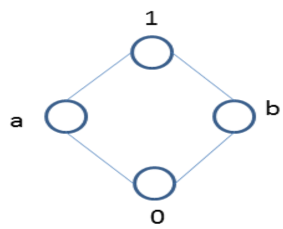
7.3 Symmetric Boolean expression

A Boolean expression in n variables is called symmetric if interchanging any two variables results in an equivalent expression.

Example: Following expressions are symmetric.
(a) $(x_1 \wedge x_2') \vee (x_1' \wedge x_2)$
(b) $(x_1 \wedge x_2 \wedge x_3') \vee (x_1 \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3)$

7.4 Exercise

1. Find the canonical sum of product form of the following Boolean expressions:-
   (a) $x_1 \vee x_2$
   (b) $x_1 \vee (x_2 \wedge x_3')$
   (c) $(x_1 \vee x_2)' \vee (x_1' \wedge x_3)$
2. Show that
   (a) $(a \wedge (b' \vee c))' \wedge (b' \vee (a \wedge c')')' = (a \wedge b \wedge c')$
   (b) $a' \wedge ((b' \vee c)' \vee (b \wedge c)) \vee ((a \vee b')' \wedge c) = a' \wedge b$
3. Given an expression $\alpha(x_1, x_2, x_3)$ defined to be Σ 0,3,5,7, determine the value of $\alpha(a, b, 1)$, where a,b,1 $\in$ B and $< B, \wedge, \vee, ', 0, 1 >$ is the following Boolean algebra.



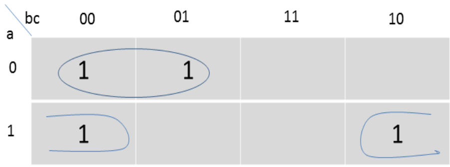
4. Obtain simplified Boolean expressions which are equivalent to these expressions:-
   (a) $m_0 + m_7$
   (b) $m_0 + m_1 + m_2 + m_3$
   (c) $m_5 + m_7 + m_9 + m_1 1 + m_1 3$
   Where m_j are the minterms in the variables x_1, x_2, x_3, and x_4.

7.5 Minimization of Boolean function or expression

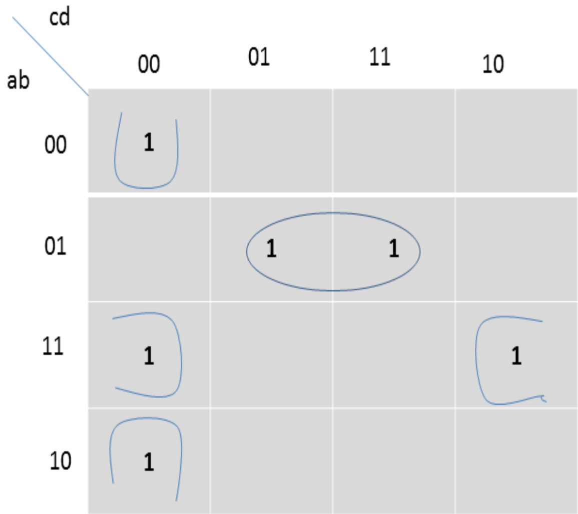
We shall minimize the Boolean function or expression using Karnaugh map.
Example: Minimize the following function using K-map.
f(a,b,c) = $\Sigma(0, 1, 4, 6)$
Solution:

The minimized function will be, f(a,b,c) = $(a' \wedge b') \vee (a \wedge c')$.

Example: Minimize the following function using K-map.
f(a,b,c,d) = $\Sigma(0, 5, 7, 8, 12, 14)$
Solution:



K-map



K-map

The minimized function will be, f(a,b,c,d) $= (a' \wedge b \wedge d') \vee (b' \wedge c' \wedge d') \vee (a \wedge b \wedge d')$.

Example: Minimize the following function using K-map.
f(a,b,c,d) $= \Sigma(0, 1, 2, 3, 13, 15)$
Solution:
The minimized function will be, f(a,b,c,d) $= (a' \wedge b') \vee (a \wedge b \wedge d)$.

ab \ cd	00	01	11	10
00	1	1	1	1
01				
11		1	1	
10				

K-map