

Discrete Structures and Theory of Logic

Lecture-35

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Subgroup

Theorem: The intersection of any two subgroups of a group (G,o) is again a subgroup of (G,o) .

Proof: Let (H_1,o) and (H_2,o) are the two subgroups of (G,o) .

Let $a \in H_1 \cap H_2$ and $b \in H_1 \cap H_2$

$$\Rightarrow (a \in H_1 \text{ and } a \in H_2) \text{ and } (b \in H_1 \text{ and } b \in H_2)$$

$$\Rightarrow (a \in H_1 \text{ and } b \in H_1) \text{ and } (a \in H_2 \text{ and } b \in H_2)$$

$\Rightarrow aob^{-1} \in H_1$ and $aob^{-1} \in H_2$ (Since H_1 and H_2 are subgroups.)

$$\Rightarrow aob^{-1} \in H_1 \cap H_2$$

Therefore, $H_1 \cap H_2$ is also a subgroup.

Example: The union of two subgroups is not necessarily a subgroup.

Solution: Let G be the additive group of integers.

$$H_1 = \{ \dots, -6, -4, -2, 0, 2, 4, 6, \dots \}$$

$$\text{and } H_2 = \{ \dots, -9, -6, -3, 0, 3, 6, 9, \dots \}$$

are both subgroups of G . But $H_1 \cup H_2$ is not subgroup.

Coset

Let H be subgroup of G and let $a \in G$. Then set $\{ah \mid h \in H\}$ is called the left coset generated by a and H and is denoted by aH . And right coset is denoted by $Ha = \{ha \mid h \in H\}$.

Index of a subgroup in a group

If H is a subgroup of a group G , then the number of distinct right(or left) cosets of H in G is called the index of H in G and it is denoted by $[G:H]$.

Subgroup

Example: Let G be the additive group of integers i.e. $G = \{ \dots -3, -2, -1, 0, 1, 2, 3, 4, \dots \}$. Let $H = \{ \dots, -9, -6, -3, 0, 3, 6, 9, \dots \}$ be the subgroup of G . Determine the index of H in G .

Solution: The index of H in G = The number of left cosets of H in G .

Now, we calculate all distinct left cosets.

$$0+H = H = \{ \dots, -9, -6, -3, 0, 3, 6, 9, \dots \}.$$

$$1+H = \{ \dots, -8, -5, -2, 1, 4, 7, 10, \dots \}$$

$$2+H = \{ \dots, -7, -4, -1, 2, 5, 8, 11, \dots \}$$

$$3+H = \{ \dots, -9, -6, -3, 0, 3, 6, 9, \dots \}.$$

$$4+H = \{ \dots, -8, -5, -2, 1, 4, 7, 10, \dots \}$$

Clearly the number of distinct left cosets is 3. Therefore, the index of H in $G = 3$.

Normal subgroup

A subgroup H of G is said to be normal subgroup of G if $Ha = aH$, $\forall a \in G$.

Theorem: A subgroup H of G is normal iff $g^{-1}hg \in H$, $\forall h \in H$, $g \in G$.

Proof:

First part: Let H be a normal subgroup of G .

Let $h \in H$, $g \in G$. Since H is normal subgroup, therefore $gH = Hg$.

Now, $hg \in Hg \Rightarrow hg \in gH$ (since $gH = Hg$)

$$\Rightarrow g^{-1}hg \in (g^{-1}g)H$$

$$\Rightarrow g^{-1}hg \in eH$$

$$\Rightarrow g^{-1}hg \in H$$

Now, it is proved.

Normal subgroup

Second part: Suppose $g^{-1}hg \in H, \forall h \in H, g \in G$ (1)

Now, we have to show that $gH = Hg$.

$$\begin{aligned}\text{Let } hg \in Hg. &\Rightarrow (gg^{-1})hg \in Hg \\ &\Rightarrow g(g^{-1}hg) \in Hg \\ &\Rightarrow gh' \in Hg \text{ (using (1))} \\ &\Rightarrow gh' \in gH \\ &\Rightarrow g(g^{-1}hg) \in gH \\ &\Rightarrow hg \in gH\end{aligned}$$

Therefore, $Hg \subseteq gH$ (2)

Normal subgroup

Now, let $gh \in gH$. $\Rightarrow (g^{-1})^{-1})hg^{-1}g \in gH$
 $\Rightarrow h'g \in gH$ (using (1))
 $\Rightarrow h'g \in Hg$
 $\Rightarrow (g^{-1})^{-1})hg^{-1}g \in Hg$
 $\Rightarrow gh \in Hg$

Therefore, $gH \subseteq Hg$ (3)

From (2) and (3), $gH = Hg, \forall g \in G$.

Therefore, H is normal subgroup of G . Now, it is proved.

Normal subgroup

Example: If H is a subgroup of G such that $a^2 \in H$ for every $a \in G$, then prove that H is a normal subgroup of G .

Solution: Let $a \in G$. Then $a^2 \in H$.

We know that if $a^{-1}ba \in H$, then H is normal subgroup, for $b \in H$.

Here, $b = a^2$, therefore, $a^{-1}ba = a^{-1}a^2a = a^2 = b$

Since, $b \in H$, therefore $a^{-1}ba \in H$. Hence, H is a normal subgroup.