

Discrete Structures and Theory of Logic

Lecture-29

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Free and Bound variables

Given a formula containing a part of the form $(\forall x)P(x)$ or $(\exists x)P(x)$, such a part is called an x -bound part of the formula. Any occurrence of x in an x -bound part of a formula is called a bound occurrence of x , while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence.

Example:

- (i) $(\forall x)P(x, y)$
- (ii) $(\forall x)(A(x) \rightarrow R(x))$
- (iii) $(\forall x)(A(x) \rightarrow (\exists y)R(x, y))$
- (iv) $(\exists x)(A(x) \wedge R(x))$
- (v) $(\exists x)A(x) \wedge R(x)$

Example:

Let $P(x)$: x is a person.

$F(x,y)$: x is the father of y .

$M(x,y)$: x is the mother of y .

Write the predicate form of the following statement

" x is the father of the mother of y ."

Solution: Let z as the mother of y . Therefore, statement will be

$$(\exists z)(P(z) \wedge F(x, z) \wedge M(z, y))$$

Universe of discourse

The domain of a variable is known as the universe of discourse.

Example: If the discussion refers to human beings only, then the universe of discourse is the class of human beings.

Example: Consider the predicate

$Q(x)$: x is less than 5.

and the statements $(\forall x)Q(x)$ and $(\exists x)Q(x)$. If the universe of discourse is given by the following sets, then find the truth value of statements $(\forall x)Q(x)$ and $(\exists x)Q(x)$.

(1) $\{-1, 0, 1, 2, 4\}$

(2) $\{3, -2, 7, 8, -5\}$

(3) $\{15, 20, 24\}$

Solution:

$(\forall x)Q(x)$ is true for (1) and false for (2) and (3).

$(\exists x)Q(x)$ is true for (1) and (2) and false for (3).

Inference theory of the predicate calculus

Some equivalences

$$(i) \neg((\forall x)A(x)) \Leftrightarrow (\exists x)\neg A(x)$$

$$(ii) \neg((\exists x)A(x)) \Leftrightarrow (\forall x)\neg A(x)$$

$$(iii) A(x) \rightarrow B(x) \Leftrightarrow \neg A(x) \vee B(x)$$

Some rules

(1) Universal Specification rule (US rule)

$$(\forall x)A(x) \Rightarrow A(x)$$

(2) Universal Generalization rule (UG rule)

$$A(x) \Rightarrow (\forall x)A(x)$$

(3) Existential Specification rule (ES rule)

$$(\exists x)A(x) \Rightarrow A(y)$$

(4) Existential Generalization rule (EG rule)

$$A(y) \Rightarrow (\exists x)A(x)$$

Some implications and equivalences

$$(1) (\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$$

$$(2) (\forall x)(A(x) \wedge B(x)) \Leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$$

$$(3) \neg(\exists x)A(x) \Leftrightarrow (\forall x)\neg A(x)$$

$$(4) \neg(\forall x)A(x) \Leftrightarrow (\exists x)\neg A(x)$$

$$(5) (\forall x)A(x) \vee (\forall x)B(x) \Rightarrow (\forall x)(A(x) \vee B(x))$$

$$(6) (\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$$

Example: Prove the following:-

$$(1) (\exists x)(A(x) \rightarrow B(x)) \Leftrightarrow (\forall x)A(x) \rightarrow (\exists x)B(x)$$

$$(2) (\exists x)A(x) \rightarrow (\forall x)B(x) \Leftrightarrow (\forall x)(A(x) \rightarrow B(x))$$

Proof:

$$(1) (\exists x)(A(x) \rightarrow B(x))$$

$$\Leftrightarrow (\exists x)(\neg A(x) \vee B(x))$$

$$\Leftrightarrow (\exists x)\neg A(x) \vee (\exists x)B(x)$$

$$\Leftrightarrow \neg(\forall x)A(x) \vee (\exists x)B(x)$$

$$\Leftrightarrow (\forall x)A(x) \rightarrow (\exists x)B(x)$$

$$(2) (\exists x)A(x) \rightarrow (\forall x)B(x)$$

$$\Leftrightarrow \neg(\exists x)A(x) \vee (\forall x)B(x)$$

$$\Leftrightarrow (\forall x)\neg A(x) \vee (\forall x)B(x)$$

$$\Leftrightarrow (\forall x)(\neg A(x) \vee B(x))$$

$$\Leftrightarrow (\forall x)(A(x) \rightarrow B(x))$$

Example: Show that $(\forall x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$.

Solution:

(1) $(\forall x)(H(x) \rightarrow M(x))$, By rule P

(2) $H(s) \rightarrow M(s)$, By rule US and (1)

(3) $H(s)$, By rule P

(4) $M(s)$, By rule T, (2), (3) and modus ponens