Discrete Structures and Theory of Logic

Unit-5

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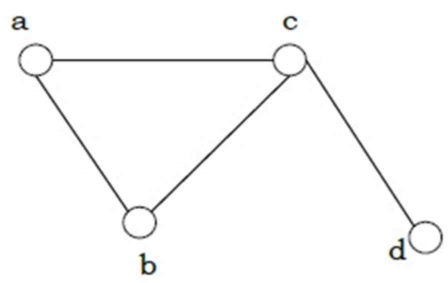
Graph Theory

1 Graph definition

A graph is an ordered pair G=(V,E) consisting of a nonempty set V of vertices and a set E of edges.

Example:



1.1 Order of a graph

The number of vertices in a graph is said to be the order of the graph.

1.2 Degree of a Vertex

The degree of a vertex v of a graph G (denoted by deg (v)) is the number of edges incident with the vertex v.

In-degree: The number of incoming edges at a vertex is said to be in-degree of that vertex.

Out-degree: The number of out going edges from a vertex is said to be out-degree of that vertex.

1.3 Even and Odd Vertex

If the degree of a vertex is even, the vertex is called an even vertex and if the degree of a vertex is odd, the vertex is called an odd vertex.

1.4 Degree of a Graph

The degree of a graph is the largest vertex degree of that graph.

1.5 Isolated Vertex

A vertex with degree zero is called an isolated vertex.

1.6 Pendant Vertex

A vertex with degree one is called a pendent vertex.

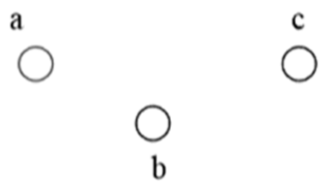
2 Types of Graph

There are different types of graphs.

2.1 Null Graph

A null graph is a graph in which there are no edges between its vertices. A null graph is also called empty graph.

Example:



2.2 Trivial Graph

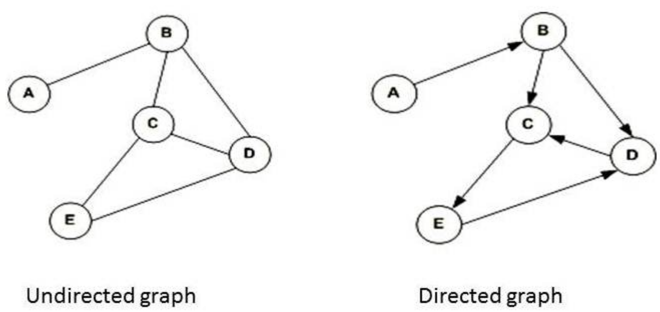
A trivial graph is the graph which has only one vertex.

2.3 Directed and Undirected Graph

An undirected graph is a graph whose edges are not directed.

A directed graph is a graph in which the edges are directed by arrows.

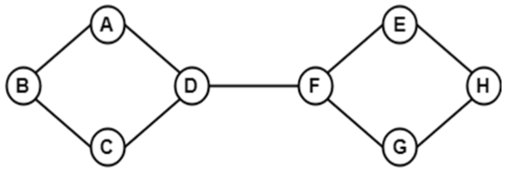
Directed graph is also known as digraphs.



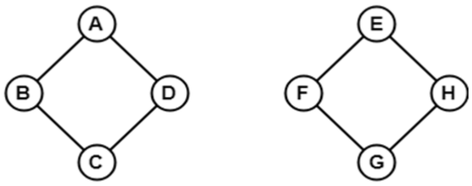
2.4 Connected and Disconnected Graph

A connected graph is a graph in which we can visit from any one vertex to any other vertex. In a connected graph, at least one edge or path exists between every pair of vertices.

A disconnected graph is a graph in which any path does not exist between every pair of vertices.



Connected graph

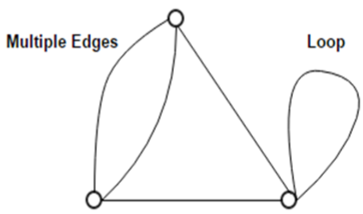


Disconnected graph

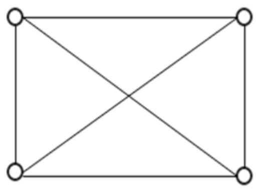
2.5 Simple graph

A simple graph is the undirected graph with no parallel edges and no loops.
A simple graph which has n vertices, the degree of every vertex is at most n -1.

Example:



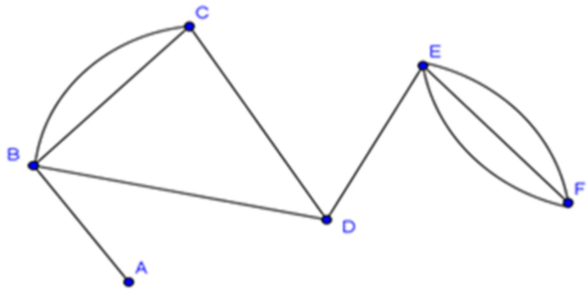
Not a Simple Graph



Simple Graph

2.6 Multi-graph

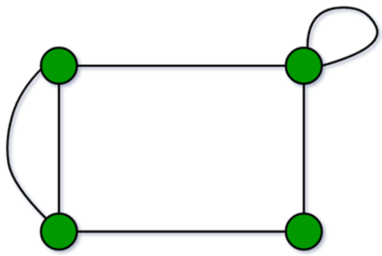
A graph having no self loops but having parallel edge(s) in it is called as a multi-graph.
Example:



2.7 Pseudo Graph

A graph in which loops and multiple edges are allowed is called psuedograph.

Example: Following graph is pseudo graph:-



Handshaking lemma

In any graph (simple) G, the sum of degree of all vertices is equal to 2e, where e is the number of edges that is
$\sum_{v \in V} deg_G(v)$= 2e.
Proof:
Since the degree of a vertex is the number of edges incident with that vertex, the sum of degree counts the number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, so each edge gets counted twice , once at each end. Thus the sum of degree is equal twice the number of edges.

Theorem: The number of vertices of odd degree in a graph G is always even.
Proof: We know that, the sum of the degrees of all ertices in a graph G is twice the number of edges in G i.e. $\sum_{v \in V} deg_G(v)$= 2e
$\sum_{v \in EVEN} deg_G(v) + \sum_{v \in ODD} deg_G(v)$ = 2e, where EVEN is the set of even degree vertices and ODD is the set odd degree vertices.
$\Rightarrow \sum_{v \in ODD} deg_G(v)$ = 2e - $\sum_{v \in EVEN} deg_G(v)$
= even number - even number = even number
$\Rightarrow \sum_{v \in ODD} deg_G(v)$ = even number
Hence the number of vertices of odd degree in a graph is even.

Note: In a graph G with n≥2, there are two vertices of equal degree.

Example: Is there a simple graph with degree sequence (1,3,3,3,5,6,6)?

Example: Is there a simple graph with seven vertices having degree sequence (1,3,3,4,5,6,6)?

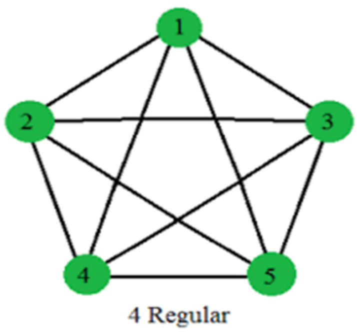
Example: Is there a simple graph with degree sequence (1,1,3,3,3,4,6,7)?

Example: Show that the maximum number of edges in a simple graph with n vertices is n(n-1)/2.

2.8 Regular Graph

A graph is called a regular if all the vertices of the graph have the same degree. If degree of each vertex is k, then the graph is called k-regular graph.

Example: Following graph is regular.



Regular graph

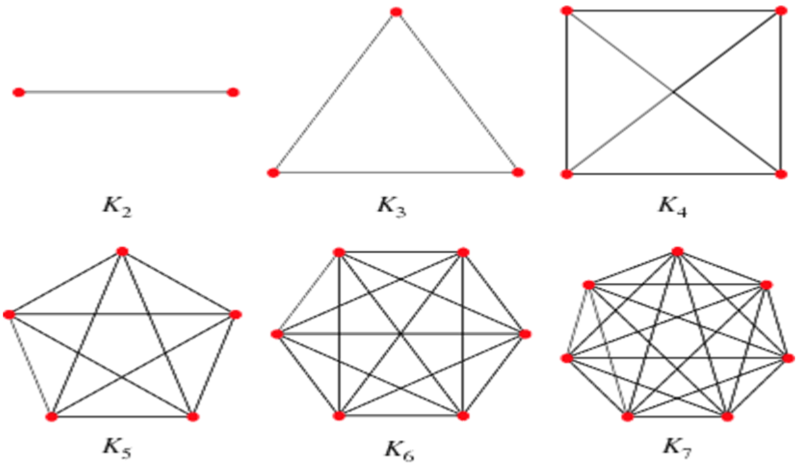
Example: Determine the number of edges in a graph with 6 vertices, 2 of degree 4 and 4 of degree 2. And also draw this graph.
Solution:
Let e is the number of edges in the graph. Since the sum of degree of all vertices is 2e, therefore
2*4+4*2 = 2e $\Rightarrow$ e = 8
Hence the number of edges in the graph is 8.

Example: Does there exists a 4-regular graph with 6 vertices? If so, construct a graph.
Solution:

2.9 Complete Graph

A graph G is said to be complete if every vertex in G is connected with every other vertex. A complete graph is denoted by K_n, where n is the number of vertex in G. Number of edges in complete graph K_n is exactly n(n-1)/2 edges.

Example: Draw the complete graph for n=2 to 7.
Solution:

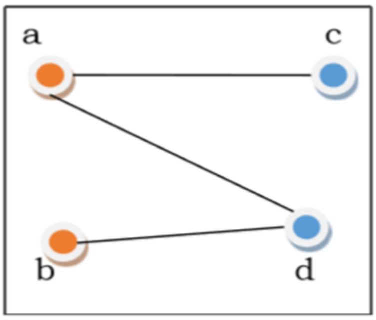


Complete graphs

2.10 Bipartite Graph

If the vertex-set of a graph G can be split into two disjoint sets, V_1 and V_2, in such a way that each edge in the graph joins a vertex in V_1 to a vertex in V_2, and there are no edges in G that connect two vertices in V_1 or two vertices in V_2, then the graph G is called a bipartite graph.

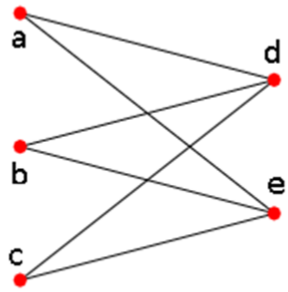
Example: Following graph is bipartite.



Bipartite graph

2.11 Complete Bipartite Graph

A complete bipartite graph is a bipartite graph in which each vertex in the first set is joined to every single vertex in the second set. The complete bipartite graph is denoted by $K_{m,n}$ where the graph G contains m vertices in the first set and n vertices in the second set.
Example: Draw the complete bipartite graph for m=3 and n=2.
Note: Number of edges in complete bipartite graph $K_{m,n}$ is m*n and number of vertices is m+n.



Complete bipartite graph $K_{3,2}$

2.12 Isomorphism of graph

Suppose G=(V,E) and G'=(V',E') are two graphs. A function f: V→ V' is called a graph isomorphism if

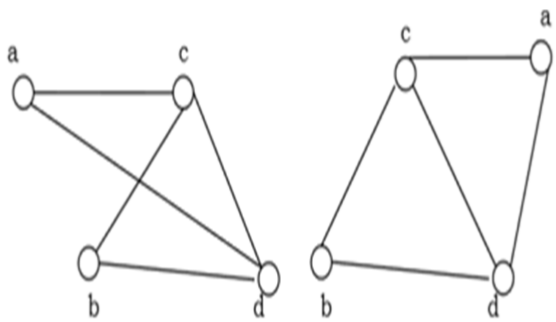
1. f is bijective.
2. For all a,b ∈ V, (a,b)∈ E iff (f(a),f(b))∈ E'.

If such function exists then graph G and G' are said to be isomorphic to each other.

2.12.1 Conditions for Graph Isomorphism

1. Both graphs G and G' must have the same number of vertices.
2. Both graphs G and G' must have the same number of edges.
3. Degree sequence of both graphs are same.

Example: Is the following graphs isomorphism?
Solution:



Example: Is the following graphs isomorphism?
Solution:
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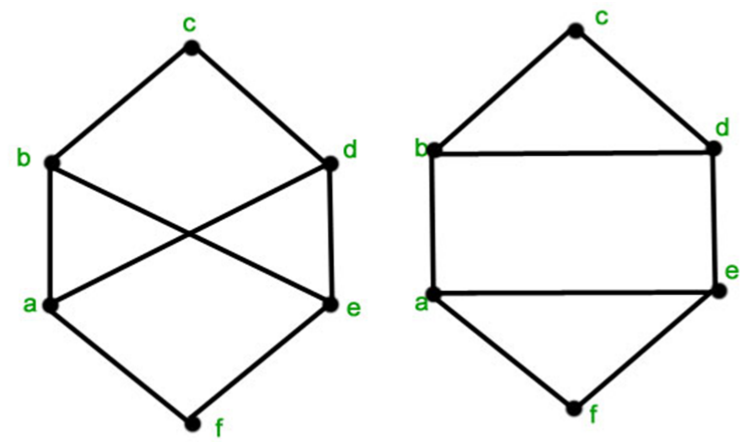
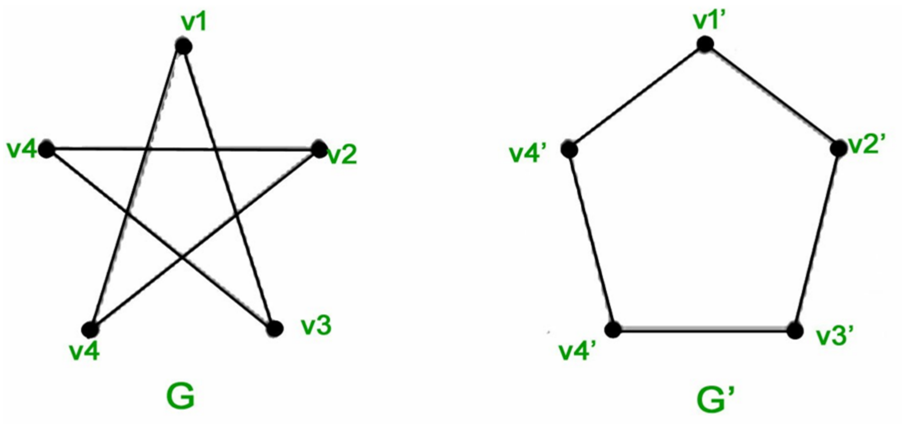




2.13 Homomorphism of graph

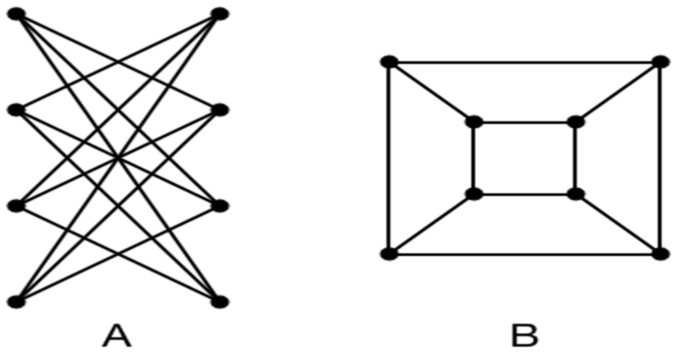
Suppose G=(V,E) and G'=(V',E') are two graphs. A function f: V→ V' is called a graph homomorphism if for all a,b ∈ V, if (a,b)∈ E then (f(a),f(b))∈ E'.
If such function exists then graph G and G' are said to be homomorphic to each other.

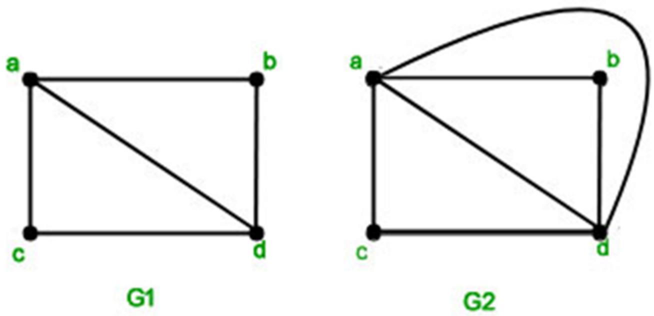
2.14 Euler Graphs

A graph G is called Euler graph if it contains an Euler cycle.
Euler cycle: An Euler cycle is a cycle which contains every edges of the graph and no edge is repeated.
Euler path: A path is called an Euler path if it contains every edges of the graph and no edge is repeated.

Example: Which graphs shown below an Euler?
Solution:
Example: Which graphs shown below an Euler?
Solution:

Note: A graph is an Euler iff every vertex has an even degree.





2.15 Hamiltonian Graphs

A graph G is called Hamiltonian graph if it contains an Hamiltonian cycle.
Hamiltonian cycle: An Hamiltonian cycle is a cycle which contains every vertex of the graph and no vettex is repeated.
Hamiltonian path: A path is called an Hamiltonian path if it contains every vertex of the graph and no vertex is repeated.

Example: Which graphs shown below a Hamiltonian?
Solution:
Note: A simple connected graph G of order n≥3 vertices is Hamiltonian if deg(v)≥ n/2 for every v in G.

Note: Let G be a simple graph with n vertices and m edges where m is atleast 3. If $m \geq \frac{(n-1)(n-2)}{2} + 2$, then G is Hamiltonian.

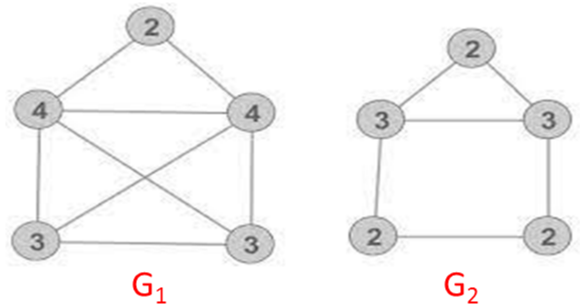
2.16 Planar Graph

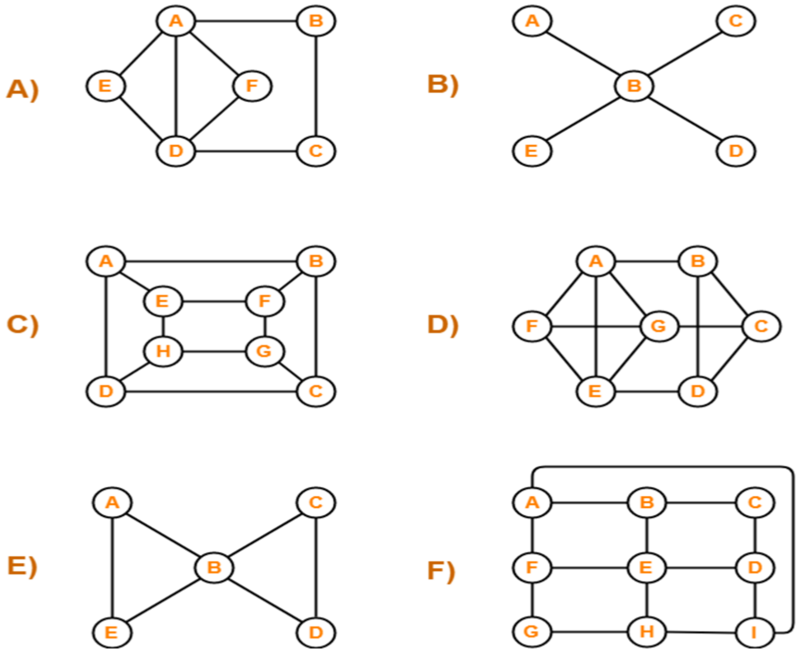
A graph is said to be planar if it can be drawn on a plane without crossing their edges.

Example: Are the following graphs planar?

Example: Are the following graphs planar?

Note: The complete graph of five vertices is not planar.





2.16.1 Properties of Planar Graphs:

1. If a connected planar graph G has e edges and r regions, then r $\geq$ (2/3)e.

2. If a connected planar graph G has e edges and v vertices, then 3v-e$\geq$6.

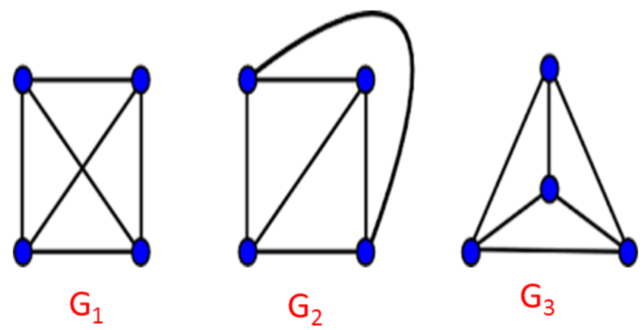
3. A complete graph K_n is a planar if and only if n<5.

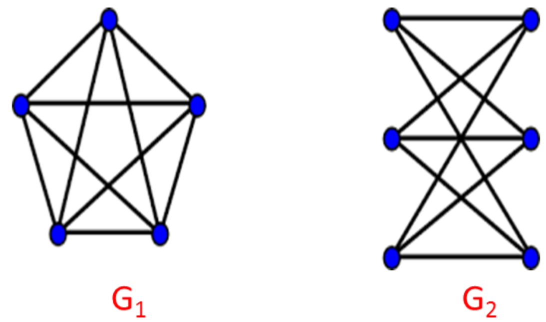
4. A complete bipartite graph K_{mn} is planar if and only if m < 3 or n <3.

Example: Prove that complete graph K_4 is planar.
Solution: The complete graph K_4 contains 4 vertices and 6 edges.
We know that for a connected planar graph 3v-e$\geq$6.Hence for K_4, we have 3x4-6 = 6 which satisfies the property. Thus K_4 is a planar graph. Hence Proved.

2.16.2 Euler's Formula

Let G be a connected planar graph and let n, e and r denote respectively the number of vertices, edges and region in a plane representation of G, then n-e+r = 2.





3 Matrix representation of Graphs

There are two principal ways to represent a graph G with the matrix, i.e., adjacency matrix and incidence matrix representation.

3.1 Adjacency Matrix Representation

If an undirected Graph G consists of n vertices, then the adjacency matrix of a graph is an n x n matrix A = $[a_{ij}]$ and defined by
a_{ij} = 1, if there exists an edge between vertex v_i and v_j
= 0, otherwise

3.2 Incidence Matrix Representation

If an undirected Graph G consists of n vertices and m edges, then the incidence matrix is an n x m matrix C = $[c_{ij}]$ and defined by
c_{ij} = 1, if the vertex v_i incident by edge e_j
= 0, otherwise

Note: The number of ones in an incidence matrix of the undirected graph (without loops) is equal to the sum of the degrees of all the vertices in a graph.

4 Graph Coloring

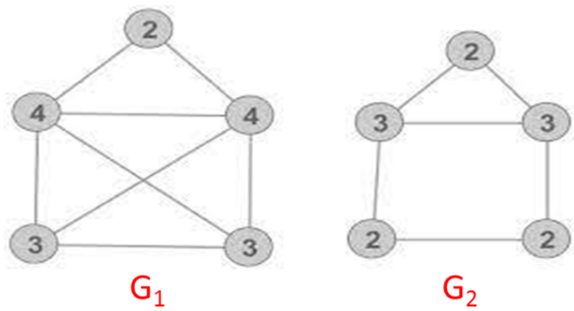
4.1 Vertex Coloring

Vertex coloring is an assignment of colors to the vertices of a graph 'G' such that no two adjacent vertices have the same color.

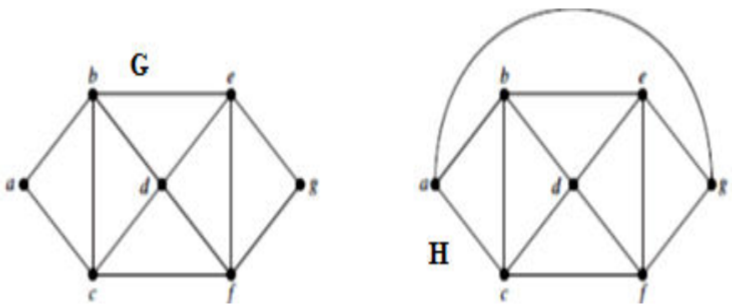
4.2 Chromatic Number

The minimum number of colors required for vertex coloring of graph 'G' is called as the chromatic number of G, denoted by X(G).

Example: Find the chromatic number of the following graphs



Example: Find the chromatic number of the following graphs



Note: $\chi(G) = 1$, if and only if 'G' is a null graph. If 'G' is not a null graph, then $\chi(G) \geq 2$.

Note: A graph 'G' is said to be n-coverable if there is a vertex coloring that uses at most n colors, i.e., X(G) $\geq$ n.

Note: The chromatic number of K_n is n.

4.3 Region Coloring

Region coloring is an assignment of colors to the regions of a planar graph such that no two adjacent regions have the same color. Two regions are said to be adjacent if they have a common edge.

Recurrence Relation and Generating Function

5 Recurrence Relation

A recurrence relation for the sequence $< a_n >$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

Example: Consider the following recurrence relation:-
$a_n = a_{n-1} - a_{n-2}$
for n=2,3,4,5,......., with the conditions $a_0 = 3$, $a_1 = 5$.

Example: The sequence 1,1,2,3,5,8,..................., is defined by the recurrence relation
$a_n = a_{n-1} + a_{n-2}$
with initial conditions $a_0 = 1$, $a_1 = 1$.

5.1 Order of the Recurrence Relation

The order of a recurrence relation is the difference between the largest and the smallest subscript appearing in the the relation.

Example: Consider the following recurrence relation:-
$a_n = a_{n-1} + a_{n-2}$
The order of this relation = 2

Consider another recurrence relation:-
$a_{n+3} - a_{n+2} + a_{n+1} - a_n = 0$
The order of this relation = 3

5.2 Degree of the Recurrence Relation

The degree of a recurrence relation is the highest power of a_n occurring in that relation.

Example: Consider the following recurrence relation:-
${a_n}^3 + 3a_{n-1}^2 + 6a_{n-2}^2 + 4a_{n-3} = 0$
The degree of this relation = 3

Consider another recurrence relation:-
$a_{n+2}^2 + 4a_{n+1} + 5a_n = 0$
The degree of this relation = 2

5.3 Linear Recurrence Relation with Constant Coefficients

A recurrence relation of the form
$c_0a_n + c_1a_{n-1} + c_2a_{n-2} + \ldots\ldots\ldots\ldots\ldots + c_ka_{n-k} = f(n)$
where c_i's are constants, is called a linear recurrence relation with constant coefficients of k^{th} order, provided c_0 and c_k, both are non-zero. f(n) is the function of the independent variable 'n' only.

Example: (i) $3a_n + 6a_{n-1} = 2^n$
is the first order linear recurrence relation with constant coefficients.

(ii) $2a_n + 5a_{n-2} = n^2 + n$
is the second order linear recurrence relation with constant coefficients.

Note: A recurrence relation is said to be linear if its degree is one.

5.4 Homogeneous Linear Recurrence Relation

A linear recurrence realtion is said to be homogeneous if f(n) = 0.
If f(n)$\neq$ 0, then it is said to be non-homogeneous.

5.5 Solution of Linear Equation

The solution of linear equation consists of two parts (i) homogeneous solution (ii) particular solution. That is,
$a = a^h + a^p$

5.6 Solution of Homogeneous Linear Recurrence Equation

Example: Find the solution of the recurrence relation
$a_n = a_{n-1} + 2a_{n-2}$
with $a_0 = 2$, and $a_1 = 7$.
Solution:
Let the solution be $a_n = \alpha^n$.
Put $a_n = \alpha^n$ in the given recurrence relation.
$\alpha^n = \alpha^{n-1} + 2\alpha^{n-2}$
$\Rightarrow \alpha^2 - \alpha - 2 = 0$
$\Rightarrow (\alpha - 2)(\alpha + 1) = 0$
$\Rightarrow \alpha = 2$, -1
Therefore, the solution of recurrence equation will be
$a_n = c_1 2^n + c_2(-1)^n$ .... (1)
Now, we find c_1 and c_2 by using initial values $a_0 = 2$, and $a_1 = 7$.

For n=0, equation (1) will be
$a_0 = c_1 2^0 + c_2(-1)^0$
therefore, $c_1 + c_2 = 2$ ....(2)
For n=1, equation (1) will be
$a_1 = c_1 2^1 + c_2(-1)^1$
therefore, $2c_1 - c_2 = 7$ ....(3)
After solving equations (2) and (3), we get $c_1 = 3$ and $c_2 =$ -1.
Putting c_1 and c_2 in equation (1), we get the final solution
$a_n = 32^n - (-1)^n$

Example: Find the solution of the recurrence relation
$a_n - 6a_{n-1} + 9a_{n-2} = 0$
with $a_0 = 1$, and $a_1 = 6$.
Solution:
Let the solution be $a_n = \alpha^n$.
Put $a_n = \alpha^n$ in the given recurrence relation.
$\alpha^n - 6\alpha^{n-1} + 9\alpha^{n-2}$=0
$\Rightarrow \alpha^2 - 6\alpha + 9 = 0$
$\Rightarrow (\alpha - 3)^2 = 0$
$\Rightarrow \alpha = 3, 3$
Therefore, the solution of recurrence equation will be
$a_n = (c_1 + c_2 n)(3)^n$ .... (1)
Now, we find c_1 and c_2 by using initial values $a_0 = 1$, and $a_1 = 6$.
For n=0, equation (1) will be
$a_0 = (c_1 + c_2.0)(3)^0$
therefore, $c_1 = 1$ ....(2)
For n=1, equation (1) will be
$a_1 = (c_1 + c_2.1)(3)^1$
therefore, $3c_1 + 3c_2 = 6$ ....(3)
After solving equations (2) and (3), we get $c_1 = 1$ and $c_2 = 1$.
Putting c_1 and c_2 in equation (1), we get the final solution
$a_n = (1 + n)(3)^n$

Example: Find the solution of the recurrence relation
$a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$
Solution:
Let the solution be $a_n = \alpha^n$.
Put $a_n = \alpha^n$ in the given recurrence relation.
$\alpha^n - 5\alpha^{n-1} + 8\alpha^{n-2} - 4\alpha^{n-3}$=0
$\Rightarrow \alpha^3 - 5\alpha^2 + 8\alpha - 4 = 0$
$\Rightarrow (\alpha - 1)(\alpha - 2)^2 = 0$
$\Rightarrow \alpha = 1, 2, 2$
Therefore, the solution of recurrence equation will be
$a_n = c_1(1)^n + (c_2 + c_3 n)(2)^n$

Example: Find the solution of the recurrence relation
$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$
Solution:

Let the solution be $a_n = \alpha^n$.
Put $a_n = \alpha^n$ in the given recurrence relation.
$\alpha^n + 6\alpha^{n-1} + 12\alpha^{n-2} + 8\alpha^{n-3}$=0
$\Rightarrow \alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$
$\Rightarrow (\alpha + 2)^3 = 0$
$\Rightarrow \alpha$ = -2, -2, -2
Therefore, the solution of recurrence equation will be
$a_n = (c_1 + c_2 n + c_3 n^2)(-2)^n$

Example: Find the solution of the recurrence relation
$4a_n - 20a_{n-1} + 17a_{n-2} - 4a_{n-3} = 0$
Solution:
Let the solution be $a_n = \alpha^n$.
Put $a_n = \alpha^n$ in the given recurrence relation.
$4\alpha^n - 20\alpha^{n-1} + 17\alpha^{n-2} - 4\alpha^{n-3}$=0
$\Rightarrow 4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$
$\Rightarrow (\alpha - 4)(2\alpha - 1)^2 = 0$
$\Rightarrow \alpha$ = 4, 1/2, 1/2
Therefore, the solution of recurrence equation will be
$a_n = c_1(4)^n + (c_2 + c_3 n)(1/2)^n$

5.7 Solution of Non-Homogeneous Linear Recurrence Equation

In this case, we find homogeneous and particular solution both. The final solution will be addition of both.
Here, f(n)≠0.

5.7.1 Method to find Particular Solution

The particular solution of a recurrence relation can be obtained by the method of inspection, since the particular solution depend on the form of f(n).
We guess the solution according to following table:-
The solution of non-homogeneous equation is

S. No.	f(n)	Guessing solution
1	b^n (If b is not a root of characteristic equation)	A b^n
2	Polynomial P(n) of degree m	$A_0 + A_1 n + A_2 n^2 + ... + A_m n^m$
3	c^nP(n)(If c is not a root of characteristic equation and Polynomial P(n) of degree m)	$c^n(A_0 + A_1 n + A_2 n^2 + ... + A_m n^m)$
4	b^n (If b is a root of characteristic equation of multiplicity s)	A$n^s b^n$
5	c^nP(n) (If b is a root of characteristic equation of multiplicity t)	$n^t(A_0 + A_1 n + A_2 n^2 + ... + A_m n^m)b^n$

$a_n = a_n^{(h)} + a_n^{(p)}$

Example: Solve the recurrence relation
$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$ ....(1)
Solution: The homogeneous equation will be
$a_n + 5a_{n-1} + 6a_{n-2} = 0$
The characteristic equation will be
$\alpha^2 + 5\alpha + 6 = 0$
$\Rightarrow (\alpha + 2)(\alpha + 3) = 0$
$\Rightarrow \alpha$ = -2, -3
Therefore, the homogeneous solution of recurrence equation will be
$a_n^{(h)} = c_1(-2)^n + c_2(-3)^n$

For particular solution:
Here, f(n) = $3n^2 - 2n + 1$
Clearly, f(n) is the polynomial equation of degree 2. Therefore using above table, we guess the following solution:-
$a_n = A_0 + A_1 n + A_2 n^2$.................(2)
Put the value of a_n in equation (1),
$(A_0 + A_1 n + A_2 n^2) + 5(A_0 + A_1(n-1) + A_2(n-1)^2) + 6(A_0 + A_1(n-2) + A_2(n-2)^2)$
$= 3n^2 - 2n + 1$
$(A_0 + 5A_0 - 5A_1 + 5A_2 + 6A_0 - 12A_1 + 24A_2) + (A_1 + 5A_1 - 10A_2 + 6A_1 - 24A_2)n + (A_2 + 5A_2 + 6A_2)n^2 = 3n^2 - 2n + 1$
$(12A_0 - 17A_1 + 29A_2) + (12A_1 - 34A_2)n + 12A_2 n^2 = 3n^2 - 2n + 1$
Comparing the coefficients of power of n on both sides
$12A_0 - 17A_1 + 29A_2 = 1$ ....(3)
$12A_1 - 34A_2$ = -2 ....(4)
$12A_2$ = 3..........(5)
After solving equations (3), (4) and (5), we get
A_0 = 47/288, A_1 = 13/24 , A_2 = 1/4
Therefore, particular solution is
$a_n^{(p)}$ = (47/288) + (13/24)n + (1/4)n^2
Therefore, the final solution of given recurrence relation will be the following:-
$a_n = a_n^{(h)} + a_n^{(p)}$
$= c_1(-2)^n + c_2(-3)^n$ + (47/288) + (13/24)n + (1/4)n^2

5.8 Exercise:

Solve the following recurrence relations:-

1. $a_{n+2} - 5a_{n+1} + 6a_n = n^2$
2. $a_n - 6a_{n-1} + 8a_{n-2} = 3^n$
3. $a_n + 5a_{n-1} + 6a_{n-2} = 42(4)^n$
4. $a_n + a_{n-1} = 3n2^n$
5. $a_n - 2a_{n-1} = 32^n$

6. $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$

Example: Solve the recurrence relation
$a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n$ ....(1)
Solution: The homogeneous solution will be
$a_n^{(h)} = c_1(2)^n + c_2(3)^n$

For particular solution:
Here, f(n) = $2^n + n$
Therefore, we guess the solution as following:-
Let $a_n = A_0 n 2^n + (A_1 + A_2 n)$
Put this in equation (1), we get
A_0 = -2, A_1 = 7/4, A_2 = 1/2
Therefore the solution will be
$a_n = c_1 2^n + c_2 3^n - 2n2^n + (7/4) + (1/2)n$

6 Generating Functions

The generating function of a sequence of numbers $a_0, a_1, a_2, ..., a_n, ...$ is defined as
G(x) = $a_0 + a_1 x + a_2 x^2 + ... + a_n x^n + ...$
$= \sum_{n=0}^{\infty} a_n x^n$

Example: Find the generating functions for the following sequences

1. 1,1,1,1,1,..................
2. 1,2,3,4,....................
3. 0,1,2,3,4,......................
4. 1,a,a^2,a^3,...................

Solution:

1. The generating function of this sequence will be the following:-
   G(x) = 1+x+x^2+ x^3+ x^4+..........................
   $= \frac{1}{(1-x)}$
2. The generating function of this sequence will be the following:-
   G(x) = 1+2x+3x^2+4x^3+..........................
   xG(x)= x+2x^2+3x^3+..........................
   Subtracting from above, we get
   (1-x)G(x) = 1+x+x^2+ x^3+ x^4+..........................
   (1-x)G(x) = $\frac{1}{(1-x)}$
   Therefore, G(x) = $\frac{1}{(1-x)^2}$
3. The generating function of this sequence will be the following:-
   G(x) = 0+x+2x^2+ 3x^3+ 4x^4+..........................
   = x(1+2x+3x^2+4x^3+..........................)
   Therefore, G(x) = $\frac{x}{(1-x)^2}$

4. The generating function of this sequence will be the following:-
   G(x) $= 1+\text{ax}+a^2x^2+ a^3x^3+ a^4x^4+$..........................
   $= 1+\text{ax}+(ax)^2+ (ax)^3+ (ax)^4+$..........................
   $= \frac{1}{(1-ax)}$

Example: Find the generating functions for the following sequences

1. 0,0,1,1,1,..................
2. 1,1,0,1,1,1,1,....................
3. 1,0,-1,0,1,0,-1,0,1,......................
4. 3,-3,3,-3,3,-3,...................

Solution:

Example: Find the generating function of a sequence $< a_k >$ if $a_k = 2+3\text{k}$.
Solution: The generating function of a sequence whose general term is 2, is
G1(x) $= \frac{2}{(1-x)}$

The generating function of a sequence whose general term is 3k, is

G2(x) $= \frac{3x}{(1-x)^2}$

Hence the required generating function is

G(x) = G1(x)+ G1(x)= $\frac{2}{(1-x)}+\frac{3x}{(1-x)^2}$

6.1 Solution of linear recurrence relation using generating function

Example: Solve the linear recurrence relation
$a_n - 3a_{n-1} + 2a_{n-2} = 0$, n ≥ 2
using the method of generating function with the initial conditions $a_0 = 2$, and $a_1 = 3$.
Solution:
$a_n - 3a_{n-1} + 2a_{n-2} = 0$

Multiply both sides by x^n and taking summation, we get

$\sum_{n=2}^{\infty} a_nx^n - 3\sum_{n=2}^{\infty} a_{n-1}x^n + 2\sum_{n=2}^{\infty} a_{n-2}x^n = 0$

Since G(x) $= \sum_{n=0}^{\infty} a_nx^n$, therefore

(G(x)-a_0-a_1x)- 3x(G(x)-a_0)+2x^2G(x) $= 0$

$\Rightarrow$ G(x)(1-3x+2x^2)-a_0-a_1x+3a_0x $= 0$

Put the value of $a_0 = 2$ and $a_1 = 3$,

G(x) $= \frac{2+3x-6x}{(1-3x+2x^2)} = \frac{2-3x}{(1-3x+2x^2)}$

$= \frac{2-3x}{(1-x)(1-2x)}$

$= \frac{1}{(1-x)} + \frac{1}{(1-2x)}$

Therefore, the solution of recurrence relation will be

$a_n = (1)^n + (2)^n$

This is the final answer.

Example: Solve the linear recurrence relation
$a_n - 2a_{n-1} - 3a_{n-2} = 0$, n ≥ 2
using the method of generating function with the initial conditions $a_0 = 3$, and $a_1 = 1$.
Solution:
$a_n - 2a_{n-1} - 3a_{n-2} = 0$

Multiply both sides by x^n and taking summation, we get

$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1}x^n - 3\sum_{n=2}^{\infty} a_{n-2}x^n = 0$

Since G(x) $= \sum_{n=0}^{\infty} a_n x^n$, therefore

(G(x)-a_0-a_1x)- 2x(G(x)-a_0)- $3x^2$G(x) $= 0$

$\Rightarrow$ G(x)(1-2x-$3x^2$)-a_0-a_1x+2a_0x $= 0$

Put the value of $a_0 = 3$ and $a_1 = 1$,

G(x) $= \frac{3+x-6x}{(1-2x-3x^2)} = \frac{3-5x}{(1-2x-3x^2)}$

$= \frac{3-5x}{(1-3x)(1+x)}$

$= \frac{2}{(1+x)} + \frac{1}{(1-3x)}$

Therefore, the solution of recurrence relation will be

$a_n = 2(-1)^n + (3)^n$

This is the final answer.

Example: Solve the linear recurrence relation
$a_n - 2a_{n-1} + a_{n-2} = 2^n$, n ≥ 2
using the method of generating function with the initial conditions $a_0 = 2$, and $a_1 = 1$.
Solution:

$a_n - 2a_{n-1} + a_{n-2} = 2^n$ Multiply both sides by x^n and taking summation, we get

$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2} \infty 2^n x^n$

Since G(x) $= \sum_{n=0}^{\infty} a_n x^n$, therefore

(G(x)-a_0-a_1x)- 2x(G(x)-a_0)+ x^2G(x) $= \frac{4x^2}{(1-2x)}$

G(x)(1-2x+x^2)-a_0-a_1x+2a_0x $= \frac{4x^2}{(1-2x)}$

Put the value of $a_0 = 2$ and $a_1 = 1$,

G(x)(1-2x+x^2) = 2+x-4x + $\frac{4x^2}{(1-2x)}$

G(x) $= \frac{2-7x+10x^2}{(1-2x+x^2)(1-2x)}$

G(x) $= \frac{2-7x+10x^2}{(1-x)^2(1-2x)}$

G(x) $= \frac{3}{(1-x)}$ - $\frac{5}{(1-x)^2}$ + $\frac{4}{(1-2x)}$

Therefore, the solution of recurrence relation will be

$a_n = 3(1)^n$ -5(n+1) + 4$(2)^n$

This is the final answer.

Example: Solve the linear recurrence relation
$a_{n+2} - 2a_{n+1} + a_n = 2^n$, n $\geq$ 2
using the method of generating function with the initial conditions $a_0 = 2$, and $a_1 = 1$.
Solution:
$a_{n+2} - 2a_{n+1} + a_n = 2^n$

Multiply both sides by x^n and taking summation, we get

$\sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0} \infty 2^n x^n$

Since G(x) $= \sum_{n=0}^{\infty} a_n x^n$, therefore

$\frac{(G(x)-a_0-a_1x)}{x^2}$- 2 $\frac{(G(x)-a_0)}{x}$+ G(x) $= \sum_{n=0} \infty 2^n x^n$

$\Rightarrow$ G(x)(1-2x+x^2)-a_0-a_1x+2a_0x $= \frac{x^2}{(1-2x)}$

Put the value of $a_0 = 2$ and $a_1 = 1$,

G(x)(1-2x+x^2) = 2+x-4x + $\frac{4x^2}{(1-2x)}$

G(x) $= \frac{2-7x+7x^2}{(1-2x+x^2)(1-2x)}$

$= \frac{2-7x+7x^2}{(1-x)^2(1-2x)}$

G(x) = $\frac{3}{(1-x)}$ - $\frac{2}{(1-x)^2}$ + $\frac{1}{(1-2x)}$

Therefore, the solution of recurrence relation will be

$a_n = 3(1)^n$ -2(n+1) + $(2)^n$
= 1 - 2n + 2^n
This is the final answer.

7 AKTU Examination Question

1. Obtain the generating function for the sequence 4, 4, 4, 4, 4, 4, 4.
2. Solve the following recurrence equation using generating function
   G(K) -7G(K-1) + 10G(K-2) = 8K + 6
3. Solve the recurrence relation by the method of generating function
   $a_n - 7a_{n-1} + 10a_{n-2} = 0$, n $\geq$ 2, Given $a_0 = 3$ and $a_1 = 3$.
4. Find the recurrence relation from y_n = A2^n + B$(-3)^n$.
5. Solve the recurrence relation
   $y_{n+2} - 5y_{n+1} + 6y_n = 5^n$ subject to the condition $y_0 = 0$, $y_1 = 2$.
6. Solve the recurrence relation using generating function:
   $a_n - 7a_{n-1} + 10a_{n-2} = 0$ with $a_0 = 3$, and $a_1 = 3$.
7. Solve the recurrence relation
   $a_{r+2} - 5a_{r+1} + 6a_r = (r+1)^2$