Discrete Structures and Theory of Logic

Lecture-23

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Exercise

1. Show that the truth values of the following formulas are independent of their components.
    1.1 $(P \wedge (P \rightarrow Q)) \rightarrow Q$
    1.2 $(P \rightarrow Q) \leftrightarrow (\neg P \vee Q)$
    1.3 $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$
    1.4 $(P \leftrightarrow Q) \leftrightarrow ((P \wedge Q) \vee (\neg P \wedge \neg Q))$
2. Construct truth table for the following formulas:-
    2.1 $(Q \wedge (P \rightarrow Q)) \rightarrow P$
    2.2 $\neg(P \vee (Q \wedge R)) \leftrightarrow ((P \vee Q) \wedge (P \vee R))$

Well formed formulas

A well formed formulas are defined as following:

1. A statement variable standing alone is a well formed formula.
2. If A is a well formed formula, then $\neg A$ is also a well formed formula.
3. If A and B are well formed formulas, then (A$\wedge$ B), (A$\vee$ B), (A$\rightarrow$ B) and (A$\leftrightarrow$ B) are also well formed formulas.
4. A string of symbols containing the statement variables, connectives and parenthesis is a well formed formula iff it can be obtained by finitely many applications of the rules 1, 2 and 3.

Example: Some well-formed formulas are
$\neg(P \wedge Q)$, $\neg(P \vee Q)$, $(P \rightarrow (P \vee Q))$ etc.
Example: Following are not well-formed formulas
$\neg P \wedge Q$, $(P \rightarrow Q$, $(P \rightarrow Q) \rightarrow (\wedge Q))$ and $(P \wedge Q) \rightarrow Q)$

Tautology and Contradiction

A statement formula which is always true, is called a tautology. A statement formula which is always false, is called a contradiction.

Example: $(P \vee (\neg P))$, $\neg(P \wedge \neg P)$ are tautology.

If a statement formula A has the truth value t for at least one combination of truth values assigned to $P_1, P_2, ..., P_n$, then A is said to be satisfiable.

Exercise:

From the formula given below, select those which are well-formed and indicate which ones are tautologies or contradictions.

1. $(P \to (P \lor Q))$
2. $((P \to (\neg P)) \to \neg P)$
3. $((\neg Q \land P) \land Q)$
4. $((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$
5. $((\neg P \to Q) \to (Q \to P))$
6. $((P \land Q) \leftrightarrow P)$

Equivalence formulas

Let A and B be the two statement formulas and let $P_1, P_2, ..., P_n$ denote all the ariables occurring in both A and B.
If the truth value of A is equal to the truth value of B for every possible sets of truth values assigned to $P_1, P_2, ..., P_n$, then A and B are said to be equivalent.
The equivalence of two formulas are denoted by $A \Leftrightarrow B$.

Example:

1. $\neg\neg P$ is equivalent to P.
2. $P \vee P$ is equivalent to P.
3. $(P \wedge \neg P) \vee Q$ is equivalent to Q.
4. $P \vee \neg P$ is equivalent to $q \vee \neg q$.

Note: $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

Equivalence formulas:

1. **Idempotent law**

(i) $P \vee P \Leftrightarrow P$

(ii) $P \wedge P \Leftrightarrow P$

2. **Associative law**

(i) $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$

(ii) $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$

3. **Commutative law**

(i) $P \vee Q \Leftrightarrow Q \vee P$

(ii)$P \wedge Q \Leftrightarrow Q \wedge P$

4. **Distributive law**

(i) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

(ii) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5. **Identities law**

(i) $P \vee F \Leftrightarrow P$,

(ii) $P \wedge T \Leftrightarrow P$

(iii) $P \vee T \Leftrightarrow T$,

(iv) $P \wedge F \Leftrightarrow F$

(v) $P \vee \neg P \Leftrightarrow T$,

(vi) $P \wedge \neg P \Leftrightarrow F$

6. **Absorption law**

(i) $P \vee (P \wedge Q) \Leftrightarrow P)$

(ii) $P \wedge (P \vee Q) \Leftrightarrow P$

7. **DeMorgan's law**

(i) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

(ii)$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Example: Show that

1. $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R$
2. $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$
3. $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$
4. $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$
5. $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$
   are a tautology.