Discrete Structures and Theory of Logic

Lecture-41

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Unit-5

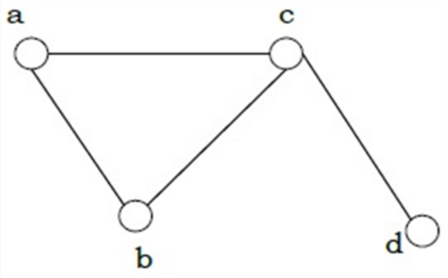
Graph Theory and Combinatorics

Graph Theory

Graph definition

A graph is an ordered pair G=(V,E) consisting of a nonempty set V of vertices and a set E of edges.

Example:



Order of a graph

The number of vertices in a graph is said to be the order of the graph.

Degree of a Vertex

The degree of a vertex v of a graph G (denoted by deg (v)) is the number of edges incident with the vertex v.

In-degree: The number of incoming edges at a vertex is said to be in-degree of that vertex.

Out-degree: The number of out going edges from a vertex is said to be out-degree of that vertex.

Even and Odd Vertex If the degree of a vertex is even, the vertex is called an even vertex and if the degree of a vertex is odd, the vertex is called an odd vertex.

Degree of a Graph

The degree of a graph is the largest vertex degree of that graph.

Isolated Vertex A vertex with degree zero is called an isolated vertex.

Pendant Vertex A vertex with degree one is called a pendent vertex.

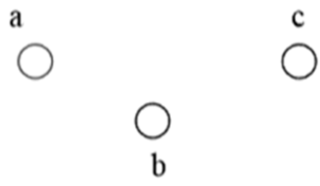
Types of Graph

There are different types of graphs.

Null Graph

A null graph is a graph in which there are no edges between its vertices. A null graph is also called empty graph.

Example:



Trivial Graph

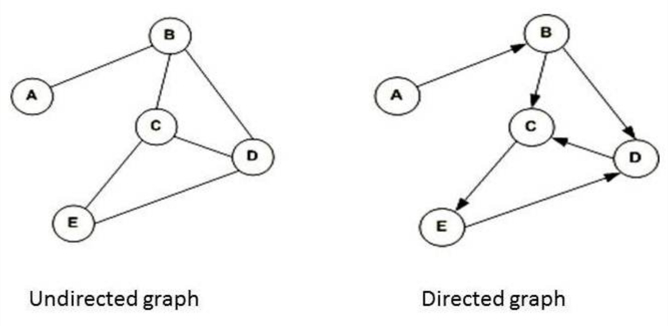
A trivial graph is the graph which has only one vertex.

Directed and Undirected Graph

An undirected graph is a graph whose edges are not directed.

A directed graph is a graph in which the edges are directed by arrows.

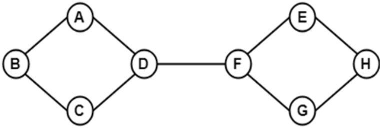
Directed graph is also known as digraphs.



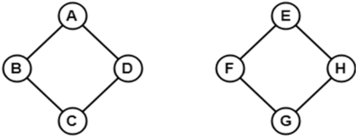
Connected and Disconnected Graph

A connected graph is a graph in which we can visit from any one vertex to any other vertex. In a connected graph, at least one edge or path exists between every pair of vertices.

A disconnected graph is a graph in which any path does not exist between every pair of vertices.



Connected graph



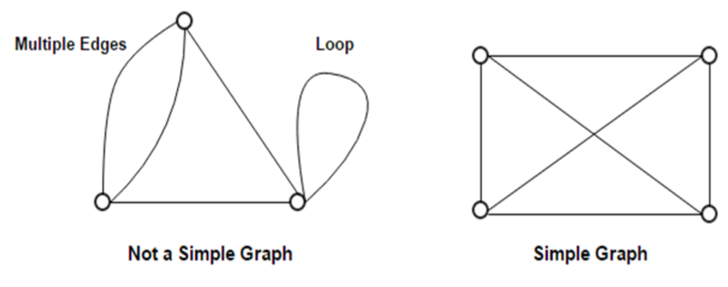
Disconnected graph

Simple graph

A simple graph is the undirected graph with no parallel edges and no loops.

A simple graph which has n vertices, the degree of every vertex is at most n-1.

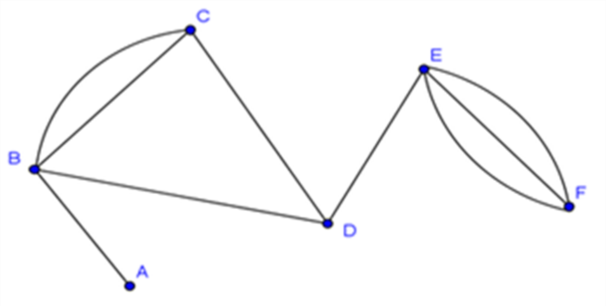
Example:



Multi-graph

A graph having no self loops but having parallel edge(s) in it is called as a multi-graph.

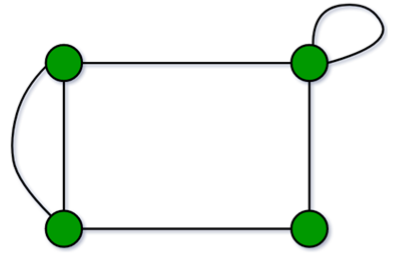
Example:



Pseudo Graph

A graph in which loops and multiple edges are allowed is called psuedograph.

Example: Following graph is pseudo graph:-



Handshaking lemma

In any graph (simple) G, the sum of degree of all vertices is equal to 2e, where e is the number of edges that is
$\sum_{v \in V} deg_G(v)$= 2e.

Proof:

Since the degree of a vertex is the number of edges incident with that vertex, the sum of degree counts the number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, so each edge gets counted twice , once at each end. Thus the sum of degree is equal twice the number of edges.

Theorem: The number of vertices of odd degree in a graph G is always even.

Proof: We know that, the sum of the degrees of all vertices in a graph G is twice the number of edges in G i.e. $\sum_{v \in V} deg_G(v)$= 2e

$\sum_{v \in EVEN} deg_G(v) + \sum_{v \in ODD} deg_G(v) =$ 2e, where EVEN is the set of even degree vertices and ODD is the set odd degree vertices.

$\Rightarrow \sum_{v \in ODD} deg_G(v) =$ 2e - $\sum_{v \in EVEN} deg_G(v)$

$=$ even number - even number $=$ even number

$\Rightarrow \sum_{v \in ODD} deg_G(v) =$ even number

Hence the number of vertices of odd degree in a graph is even.

Note: In a graph G with n$\geq$2, there are two vertices of equal degree.

Example: Is there a simple graph with degree sequence (1,3,3,3,5,6,6)?

Example: Is there a simple graph with seven vertices having degree sequence (1,3,3,4,5,6,6)?

Example: Is there a simple graph with degree sequence (1,1,3,3,3,4,6,7)?

Example: Show that the maximum number of edges in a simple graph with n vertices is n(n-1)/2.