

Discrete Structures and Theory of Logic

Lecture-7

Dr. Dharmendra Kumar

(Associate Professor)

United College of Engineering and Research, Prayagraj

September 17, 2023

Some examples

Example: If R and S are both reflexive then show that $R \cup S$ and $R \cap S$ are also reflexive.

Solution: Since R and S are reflexive, therefore $(a,a) \in R$ and $(a,a) \in S, \forall a$. Since $(a,a) \in R$ and $(a,a) \in S, \forall a$, therefore $(a,a) \in R \cup S$ and $(a,a) \in R \cap S, \forall a$. Therefore, $R \cup S$ and $R \cap S$ are also reflexive.

Example: If R and S are both reflexive, symmetric, and transitive then show that $R \cap S$ is also reflexive, symmetric, and transitive.

Some examples(cont.)

Solution:

For reflexive: Since R and s are reflexive, therefore $(a,a) \in R$ and $(a,a) \in S, \forall a$. Since $(a,a) \in R$ and $(a,a) \in S, \forall a$, therefore $(a,a) \in R \cap S, \forall a$. Therefore, $R \cap S$ is also reflexive.

For symmetric: Since R is symmetric, therefore if $(a,b) \in R$ then $(b,a) \in R$. Similarly, since S is symmetric, therefore if $(a,b) \in S$ then $(b,a) \in S$.

Let $(a,b) \in R \cap S$. It imply that $(a,b) \in R$ and $(a,b) \in S$. Since R and s are symmetric therefore $(b,a) \in R$ and $(b,a) \in S$. It imply that $(b,a) \in R \cap S$. Therefore $R \cap S$ is symmetric.

For transitive: Let (a,b) and $(b,c) \in R \cap S$. Therefore

$\Rightarrow (a,b)$ and $(b,c) \in R$ and (a,b) and $(b,c) \in S$

$\Rightarrow (a,c) \in R$ and $(a,c) \in S$ (Since R and S are transitive)

$\Rightarrow (a,c) \in R \cap S$.

Therefore, $R \cap S$ is also transitive.

Equivalence relation

Definition

A relation R defined on set A is said to be an equivalence relation if it satisfies reflexive, symmetric, and transitive properties.

Example: Let $A = \{1,2,3,4\}$ and $R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (3,2), (3,3)\}$. Is this relation an equivalence relation?

Solution: Since $(1,1), (2,2), (3,3)$, and $(4,4)$ are belongs into R , therefore R is reflexive.

Clearly in R if $(a,b) \in R$ then $(b,a) \in R$. Here both $(1,4)$ and $(4,1) \in R$ and both $(3,2)$ and $(2,3) \in R$. Therefore R is symmetric.

Clearly in R if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$. Here, for pair $(1,4)$ and $(4,1)$, its transitive pair $(1,1)$ and $(4,4)$ are also belong into R . Similarly, or pair $(2,3)$ and $(3,2)$, its transitive pair $(2,2)$ and $(3,3)$ are also belong into R . Therefore R is transitive.

Clearly, R satisfies all the three properties. Therefore, R is an equivalence relation.

Some examples(cont.)

Example: Let $A = \{1,2,3,4,5,6\}$ and $R = \{(a,b) \mid (a-b) \text{ is divisible by } 3\}$. Show that R is an equivalence relation.

Solution: In this example R will be

$$R = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (1,4), (4,1), (2,5), (5,2), (3,6), (6,3)\}$$

Clearly R satisfies reflexive, symmetric and transitive, therefore R is an equivalence relation.

Example: Let $X = \{a,b,c,d,e\}$ and let $C = \{\{a,b\}, \{c\}, \{d,e\}\}$. Show that the partition C defines an equivalence relation on X .

Solution: The relation defined by partition C will be the following
 $R = \{(a,a), (b,b), (a,b), (b,a), (c,c), (d,d), (e,e), (d,e), (e,d)\}$

Clearly relation R is an equivalence relation because R satisfies all the three properties.

Some examples(cont.)

Example: Let S be the set of lines on a plane. Define a relation R on set S as following:- aRb if line a is parallel to line b , $\forall a, b \in S$. Is relation R an equivalence relation.

Solution: Since each line in a plane is parallel to itself, therefore R satisfies reflexive property.

We know that if line a is parallel to line b then line b is also parallel to line a . Therefore R satisfies symmetric property.

We know that if line a is parallel to line b and line b is parallel to line c , then line a is also parallel to line c . Therefore R satisfies transitive property.

Since R satisfies all the three properties, therefore R is an equivalence relation.

Exercise

1. Let R denote a relation on the set of ordered pairs of positive integers such that
 $(x,y)R(u,v)$ iff $xv = yu$.
Show that R is an equivalence relation.
2. Given a set $S = \{1,2, 3, 4,5\}$. Find the equivalence relation defined on S which generates the partition $\{ \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5} \}$.
3. Prove that the relation "congruence modulo m " defined as
$$\cong = \{ (a,b) \mid (a-b) \text{ is divisible by } m \}$$
over the set of positive integers is an equivalence relation.
Show that if $a \cong b$ and $c \cong d$, then $(a+c) \cong (b+d)$.

Exercise(cont.)

1. Let R_1 be a relation defined on \mathbb{R} , the set of real numbers, such that $R_1 = \{(x,y) \mid |x - y| < 1\}$. Is R_1 an equivalence relation ? Justify. AKTU(2019)
2. Let R be a binary relation on the set of all positive integers such that:

$$R = \{(a,b) \mid a-b \text{ is an odd positive integer}\}$$

Is R reflexive ? Symmetric? Transitive?