# Discrete Structures and Theory of Logic Lecture-33

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A group (G,o) is said to be a cyclic group if there exists an element  $a \in G$  such that every element of G can be written as some power of G, that is G for some integer G is said to be the generator of G.

**Example:** Show that the set of integers with respect to + operation is cyclic group.

**Solution:** A group will be cyclic if there exists a generator in the group.

Consider an element 1 of this group.

$$(1)^1 = 1$$

$$(1)^2 = 1 + 1 = 2$$

$$(1)^3 = 1 + 1 + 1 = 3$$

$$(1)^4 = 1 + 1 + 1 + 1 = 4$$

Clearly 1, 2, 3, 4 are expressed in the power of 1. Similarly, we can expressed all the positive integers in the power of 1.

Now, 
$$(1)^{-1} = -1$$
  
 $(1)^{-2} = (1^{-1})^2 = (-1)^2$ 

$$(1)^{-2} = (1^{-1})^2 = (-1)^2 = -1 + (-1) = -2$$

$$(1)^{-3} = (1^{-1})^3 = (-1)^3 = -1 + (-1) + (-1) = -3$$

$$(1)^{-4} = (1^{-1})^4 = (-1)^4 = -1 + (-1) + (-1) + (-1) = -4$$

Clearly -1, -2, -3, -4 are expressed in the power of 1. Similarly, we can expressed all the negative integers in the power of 1.

Now, 
$$(1)^0 = 0$$

Clearly, all the integers are expressed in the powers of 1. Therefore, 1 is the generator of this group. Since generator exists, therefore the group is cyclic.

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**Example:** Is  $(G,+_6)$  a cyclic group, where  $G = \{0,1,2,3,4,5\}$ .

**Solution:** We have to find generator in this group.

Consider an element 1 of this group.

Now, 
$$(1)^1 = 1$$

$$(1)^2 = 1 + 61 = 2$$

$$(1)^3 = 1 +_6 1 +_6 1 = 3$$

$$(1)^4 = 1 + 61 + 61 + 61 = 4$$

$$(1)^5 = 1 +_6 1 +_6 1 +_6 1 = 5$$

$$(1)^6 = 1 +_6 1 +_6 1 +_6 1 +_6 1 = 0$$

Clearly all the elements of G are expressed in the power of 1, therefore 1 is a generator of G. Since generator exists, therefore the group is cyclic.

**Example:** Is the multiplicative group  $\{1, \omega, \omega^2\}$ , a cyclic group?

**Solution:** Consider an element  $\omega$  of G.

Now, 
$$(\omega)^1 = \omega$$
  
 $(\omega)^2 = \omega^2$   
 $(\omega)^3 = \omega^3 = 1$ 

Clearly all the elements of G are expressed in the power of  $\omega$ , therefore  $\omega$  is a generator of G. Since generator exists, therefore the group is cyclic.

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**Example:** Show that every cyclic group is an abelian group.

**Solution:** Consider (G,o) is a cyclic group. Since (G,o) is cyclic, therefore generator exists. Let it's generator is a.

Consider two elements b,c  $\in$  G. It can be expressed in the power of a. Let b =  $a^i$  and c =  $a^j$ .

Now, boc = 
$$a^i \circ a^j$$
  
=  $a^{i+j}$ 

 $=a^{j+i}$  (since set of integers with respect to addition operation is an abelian)

$$= a^j o a^i$$
  
 $= cob$ 

That is, boc = cob

Therefore group (G,o) is an abelian. Now, we can say, every cyclic group is an abelian group.

**Example:** Show that if a is a generator of a cyclic group G, then  $a^{-1}$  is also a generator of G.

**Solution:** Since a is a generator of G, therefore each elements of G can be epressed in the power of a.

Consider any element  $b \in G$  such that  $b = a^i$ . If we can epressed b in the power of  $a^{-1}$ , then  $a^{-1}$  will be also generator of G.

Now,  $b = a^i = (a^{-1})^{-i}$ . Clearly b is expressed in the power of  $a^{-1}$ , therefore  $a^{-1}$  is also generator of G.

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**Example:** How many generators are there of the cyclic group G of order 8?

**Solution:** Since the group is cyclic, therefore there exists generator in this group. Let a is a generator.

Therefore,  $G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e\}.$ 

Now, consider an element  $a^2$ .

$$(a^{2})^{1} = a^{2}$$
  
 $(a^{2})^{2} = a^{4}$   
 $(a^{2})^{3} = a^{6}$   
 $(a^{2})^{4} = a^{8} = e$   
 $(a^{2})^{5} = a^{1}0 = a^{2}$ 

Clearly elements  $a^1$ ,  $a^3$ ,  $a^5$ ,  $a^7$  are not expressed in the power of  $a^2$ . Therefore  $a^2$  is not generator.

Now, consider an element  $a^3$ .

$$(a^{3})^{1} = a^{3}$$

$$(a^{3})^{2} = a^{6}$$

$$(a^{3})^{3} = a^{9} = a$$

$$(a^{3})^{4} = a^{1}2 = a^{4}$$

$$(a^{3})^{5} = a^{1}5 = a^{7}$$

$$(a^{3})^{6} = a^{1}8 = a^{2}$$

$$(a^{3})^{7} = a^{2}1 = a^{5}$$

$$(a^{3})^{8} = a^{2}4 = a^{8} = e$$

Clearly all the elements of G are expressed in the power of  $a^3$ , therefore  $a^3$  is a generator of G.

Now, consider an element  $a^4$ .

$$(a^4)^1 = a^4$$
  
 $(a^4)^2 = a^8 = e$   
 $(a^4)^3 = a^1 = a^4$ 

Clearly elements  $a^1, a^2, a^3, a^5, a^6, a^7$  are not expressed in the power of  $a^4$ . Therefore  $a^4$  is not a generator.

Now, consider an element  $a^5$ .

$$(a^{5})^{1} = a^{5}$$

$$(a^{5})^{2} = a^{1}0 = a^{2}$$

$$(a^{5})^{3} = a^{1}5 = a^{7}$$

$$(a^{5})^{4} = a^{2}0 = a^{4}$$

$$(a^{5})^{5} = a^{2}5 = a$$

$$(a^{5})^{6} = a^{3}0 = a^{6}$$

$$(a^{5})^{7} = a^{3}5 = a^{3}$$

$$(a^{5})^{8} = a^{4}0 = a^{8} = e^{3}$$

Clearly all the elements of G are expressed in the power of  $a^5$ , therefore  $a^5$  is a generator of G.

Similarly, we can show that  $a^7$  is a generator and  $a^6$  is not generator. Therefore the generators of this group are a,  $a^3$ ,  $a^5$ ,  $a^7$ . Total number of generators is 4.

**Example:** Show that the group ( $\{1,2,3,4,5,6\}, \times_7$ ) is cyclic.

How many generators of this group?

#### **Solution:**

Consider the element 3 of this group.

$$(3)^1 = 3$$

$$(3)^2 = 3 \times_7 3 = 2$$

$$(3)^3 = 3 \times_7 3 \times_7 3 = 6$$

$$(3)^4 = 3 \times_7 3 \times_7 3 \times_7 3 = 4$$

$$(3)^5 = 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 = 5$$

$$(3)^6 = 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 \times_7 3 = 1$$

Clearly all the elements of G are expressed in the power of 3, therefore 3 is a generator of G.

Since generator exists, therefore the group is cyclic.

Another generator will be 5. Because,

$$(5)^1 = 5$$

$$(5)^2 = 5 \times_7 5 = 4$$

$$(5)^3 = 5 \times_7 5 \times_7 5 = 6$$

$$(5)^4 = 5 \times_7 5 \times_7 5 \times_7 5 = 2$$

$$(5)^5 = 5 \times_7 5 \times_7 5 \times_7 5 \times_7 5 = 3$$

$$(5)^6 = 5 \times_7 5 \times_7 5 \times_7 5 \times_7 5 \times_7 5 = 1$$

No other elements will be generator. Therefore number of generators is 2 i.e. 3 and 5.