

Discrete Structures and Theory of Logic

Lecture-12

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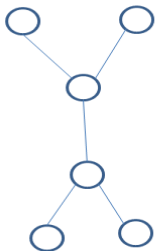
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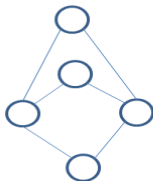
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Exercise

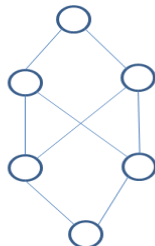
1. Find out the following POSETs are lattices or not.



(a)



(b)



(c)

2. Draw the diagram of lattices $\langle S_n, D \rangle$ for $n = 4, 6, 10, 12, 15, 45, 60, 75$ and 210 . For what values of n , do you expect $\langle S_n, D \rangle$ to be a chain?

Exercise

3. Let R be the set of real numbers in $[0,1]$ and \preceq be the usual operation of "less than or equal" on R . Show that $\langle R, \preceq \rangle$ is a lattice. What are the operations of meet and join on this lattice?

4. Let the sets $S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7$ be given by

$$S_0 = \{a,b,c,d,e,f\} \quad S_1 = \{a,b,c,d,e\} \quad S_2 = \{a,b,c,e,f\}$$

$$S_3 = \{a,b,c,e\} \quad S_4 = \{a,b,c\} \quad S_5 = \{a,b\}$$

$$S_6 = \{a,c\} \quad S_7 = \{a\}$$

Draw the diagram of $\langle L, \subseteq \rangle$, where $L = \{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$.

Principle of Duality

Any statement about lattices involving the operations \wedge and \vee remains true if \wedge is replaced by \vee and \vee is replaced by \wedge .

The operations \wedge and \vee are said to be dual of each other. For example $a \wedge b$ is the dual of $a \vee b$.

Properties of lattices

(1) Idempotent law

$$a \wedge a = a ,$$

$$a \vee a = a$$

(2) Commutative law

$$a \wedge b = b \wedge a ,$$

$$a \vee b = b \vee a$$

(3) Associative law

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c ,$$

$$a \vee (b \vee c) = (a \vee b) \vee c$$

(4) Absorption law

$$a \wedge (a \vee b) = a ,$$

$$a \vee (a \wedge b) = a$$

Lattice

Theorem: Let $\langle L, \preceq \rangle$ be a lattice. For any $a, b \in L$,
$$a \preceq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$$

Proof: In this theorem, we have to prove many parts.

First part: In this part, we will prove $a \preceq b \Leftrightarrow a \wedge b = a$.

Suppose $a \preceq b$. Since $a \preceq a$, therefore $a = \text{lower bound(l.b.)}$ of a and b .

Since a is lower bound therefore, $a \preceq \text{greatest lower bound(g.l.b.)}$ of a and b .

hence $a \preceq a \wedge b \dots\dots\dots(1)$

By the definition of glb, $a \wedge b \preceq a \dots\dots\dots(2)$

from (1) and (2), $a \wedge b = a$.

Conversely, suppose $a \wedge b = a$.

By the definition of glb,

$a \wedge b \preceq b$

Since $a \wedge b = a$, therefore $a \preceq b$.

Second part: In this part, we will prove $a \preceq b \Leftrightarrow a \vee b = b$.

Suppose $a \preceq b$. Since $b \preceq b$, therefore $b = \text{upper bound(u.b.)}$ of a and b .

Since b is an upper bound therefore, least upper bound(l.u.b.) of a and $b \preceq b$.

hence $a \vee b \preceq b$ (1)

By the definition of lub, $b \preceq a \vee b$ (2)

from (1) and (2),

$a \vee b = b$.

Conversely, suppose $a \vee b = b$.

By the definition of lub,

$a \preceq a \vee b$

Since $a \vee b = b$, therefore $a \preceq b$.

Third part: In this part, we will prove $a \wedge b = a \Leftrightarrow a \vee b = b$.

Suppose $a \wedge b = a$.

Now, $a \vee b = (a \wedge b) \vee b = b$, by absorption law.

Suppose $a \vee b = b$.

Now, $a \wedge b = a \wedge (a \vee b) = a$, by absorption law.