Discrete Structures and Theory of Logic

Lecture-46

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Solution of linear recurrence relation using generating function

Example: Solve the linear recurrence relation

$a_n - 3a_{n-1} + 2a_{n-2} = 0$, n ≥ 2

using the method of generating function with the initial conditions $a_0 = 2$, and $a_1 = 3$.

Solution:

$a_n - 3a_{n-1} + 2a_{n-2} = 0$

Multiply both sides by x^n and taking summation, we get

$\sum_{n=2}^{\infty} a_n x^n - 3\sum_{n=2}^{\infty} a_{n-1}x^n + 2\sum_{n=2}^{\infty} a_{n-2}x^n = 0$

Since G(x) $= \sum_{n=0}^{\infty} a_n x^n$, therefore

(G(x)-a_0-a_1x)- 3x(G(x)-a_0)+2x^2G(x) $= 0$

$\Rightarrow$ G(x)(1-3x+2x^2)-a_0-a_1x+3a_0x $= 0$

Recurrence Relation

Put the value of $a_0 = 2$ and $a_1 = 3$,

$$G(x) = \frac{2+3x-6x}{(1-3x+2x^2)} = \frac{2-3x}{(1-3x+2x^2)}$$

$$= \frac{2-3x}{(1-x)(1-2x)}$$

$$= \frac{1}{(1-x)} + \frac{1}{(1-2x)}$$

Therefore, the solution of recurrence relation will be
$a_n = (1)^n + (2)^n$
This is the final answer.

Recurrence Relation

Example: Solve the linear recurrence relation

$a_n - 2a_{n-1} - 3a_{n-2} = 0$, n ≥ 2

using the method of generating function with the initial conditions $a_0 = 3$, and $a_1 = 1$.

Solution:

$a_n - 2a_{n-1} - 3a_{n-2} = 0$

Multiply both sides by x^n and taking summation, we get

$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n - 3\sum_{n=2}^{\infty} a_{n-2} x^n = 0$

Since G(x) $= \sum_{n=0}^{\infty} a_n x^n$, therefore

(G(x)-a_0-a_1x)- 2x(G(x)-a_0)- 3x^2G(x) $= 0$

$\Rightarrow$ G(x)(1-2x-3x^2)-a_0-a_1x+2a_0x $= 0$

Recurrence Relation

Put the value of $a_0 = 3$ and $a_1 = 1$,

$$G(x) = \frac{3+x-6x}{(1-2x-3x^2)} = \frac{3-5x}{(1-2x-3x^2)}$$

$$= \frac{3-5x}{(1-3x)(1+x)}$$

$$= \frac{2}{(1+x)} + \frac{1}{(1-3x)}$$

Therefore, the solution of recurrence relation will be
$a_n = 2(-1)^n + (3)^n$
This is the final answer.

Example: Solve the linear recurrence relation

$a_n - 2a_{n-1} + a_{n-2} = 2^n$, n ≥ 2

using the method of generating function with the initial conditions $a_0 = 2$, and $a_1 = 1$.

Solution:

$a_n - 2a_{n-1} + a_{n-2} = 2^n$ Multiply both sides by x^n and taking summation, we get

$\sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1}x^n + \sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2} \infty 2^n x^n$

Since G(x) $= \sum_{n=0}^{\infty} a_n x^n$, therefore

(G(x)-a_0-a_1x)- 2x(G(x)-a_0)+ x^2G(x) $= \frac{4x^2}{(1-2x)}$

G(x)(1-2x+x^2)-a_0-a_1x+2a_0x $= \frac{4x^2}{(1-2x)}$

Recurrence Relation

Put the value of $a_0 = 2$ and $a_1 = 1$,

G(x)(1-2x+x^2) $=$ 2+x-4x $+ \frac{4x^2}{(1-2x)}$

G(x) $= \frac{2-7x+10x^2}{(1-2x+x^2)(1-2x)}$

G(x) $= \frac{2-7x+10x^2}{(1-x)^2(1-2x)}$

G(x) $= \frac{3}{(1-x)}$ - $\frac{5}{(1-x)^2} + \frac{4}{(1-2x)}$

Therefore, the solution of recurrence relation will be
$a_n = 3(1)^n$ -5(n+1) $+ 4(2)^n$

This is the final answer.

Recurrence Relation

Example: Solve the linear recurrence relation

$$a_{n+2} - 2a_{n+1} + a_n = 2^n, \text{ n} \geq 2$$

using the method of generating function with the initial conditions $a_0 = 2$, and $a_1 = 1$.

Solution:

$$a_{n+2} - 2a_{n+1} + a_n = 2^n$$

Multiply both sides by x^n and taking summation, we get

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=0}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_nx^n = \sum_{n=0} \infty 2^n x^n$$

Since G(x) $= \sum_{n=0}^{\infty} a_nx^n$, therefore

$$\frac{(G(x)-a_0-a_1x)}{x^2} \text{- } 2\ \frac{(G(x)-a_0)}{x} + \text{G(x)} = \sum_{n=0} \infty 2^n x^n$$

$$\Rightarrow \text{G(x)(1-2x+}x^2\text{)-}a_0\text{-}a_1\text{x+2}a_0\text{x} = \frac{x^2}{(1-2x)}$$

Recurrence Relation

Put the value of $a_0 = 2$ and $a_1 = 1$,

G(x)(1-2x+x^2) = 2+x-4x + $\frac{4x^2}{(1-2x)}$

$$G(x) = \frac{2-7x+7x^2}{(1-2x+x^2)(1-2x)}$$

$$= \frac{2-7x+7x^2}{(1-x)^2(1-2x)}$$

G(x) = $\frac{3}{(1-x)}$ - $\frac{2}{(1-x)^2}$ + $\frac{1}{(1-2x)}$

Therefore, the solution of recurrence relation will be

$a_n = 3(1)^n$ -2(n+1) + $(2)^n$

= 1 - 2n + 2^n

This is the final answer.

AKTU Examination Questions

1. Obtain the generating function for the sequence 4, 4, 4, 4, 4, 4, 4.
2. Solve the following recurrence equation using generating function

   G(K) -7G(K-1) + 10G(K-2) = 8K + 6
3. Solve the recurrence relation by the method of generating function

   $a_n - 7a_{n-1} + 10a_{n-2} = 0$, n $\geq$ 2, Given $a_0 = 3$ and $a_1 = 3$.
4. Find the recurrence relation from $y_n = \mathrm{A}2^n + \mathrm{B}(-3)^n$.
5. Solve the recurrence relation

   $y_{n+2} - 5y_{n+1} + 6y_n = 5^n$ subject to the condition $y_0 = 0$, $y_1 = 2$.
6. Solve the recurrence relation using generating function:

   $a_n - 7a_{n-1} + 10a_{n-2} = 0$ with $a_0 = 3$, and $a_1 = 3$.
7. Solve the recurrence relation

   $a_{r+2} - 5a_{r+1} + 6a_r = (r+1)^2$