Design and Analysis of Algorithms

Lecture-3

Dharmendra Kumar (Associate Professor)
Department of Computer Science and Engineering
United College of Engineering and Research,
Prayagraj

Asymptotic Notations

- The notations we use to describe the asymptotic running time of an algorithm are defined in terms of functions whose domains are the set of natural numbers N={ 0, 1, 2, …………}.
- We will use asymptotic notations to describe the running times of algorithms.
- Following notations are used to define the running time of algorithms.
  1. Θ–notation
  2. O-notation
  3. Ω-notation
  4. o-notation
  5. ω-notation

Asymptotic Notations

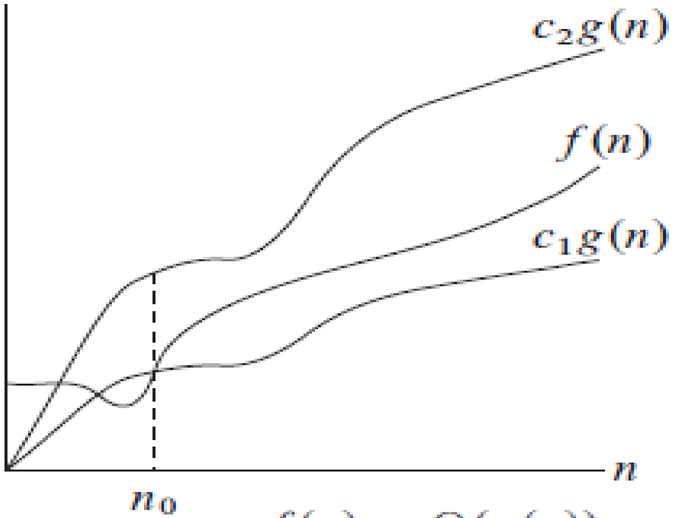
Θ–notation (Theta notation)

- For a given function g(n), it is denoted by Θ(g(n)).
- It is defined as following:-

$\Theta(g(n)) = \{ f(n) \,!\, \exists$ positive constants c_1, c_2 and n_0 such that

$0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n), \forall\, n \geq n_0 \}$

- This notation is said to be tight bound.
- If $f(n) \in \Theta(g(n))$ then $f(n) = \Theta(g(n))$



$f(n) = \Theta(g(n))$

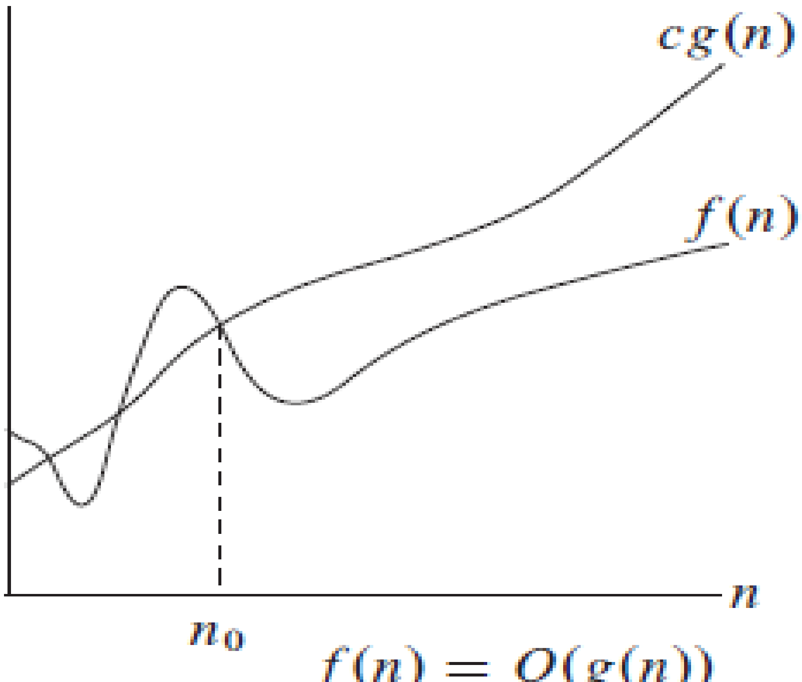
Asymptotic Notations

O–notation (Big-oh notation)

- For a given function g(n), it is denoted by O(g(n)).
- It is defined as following:-

$O(g(n)) = \{ f(n) \mid \exists$ positive constants c and n_0 such that

$0 \leq f(n) \leq cg(n), \forall\, n \geq n_0 \}$



- This notation is said to be upper bound.
- If $f(n) \in O(g(n))$ then $f(n) = O(g(n))$
- If $f(n) = \Theta(g(n))$ then $f(n) = O(g(n))$

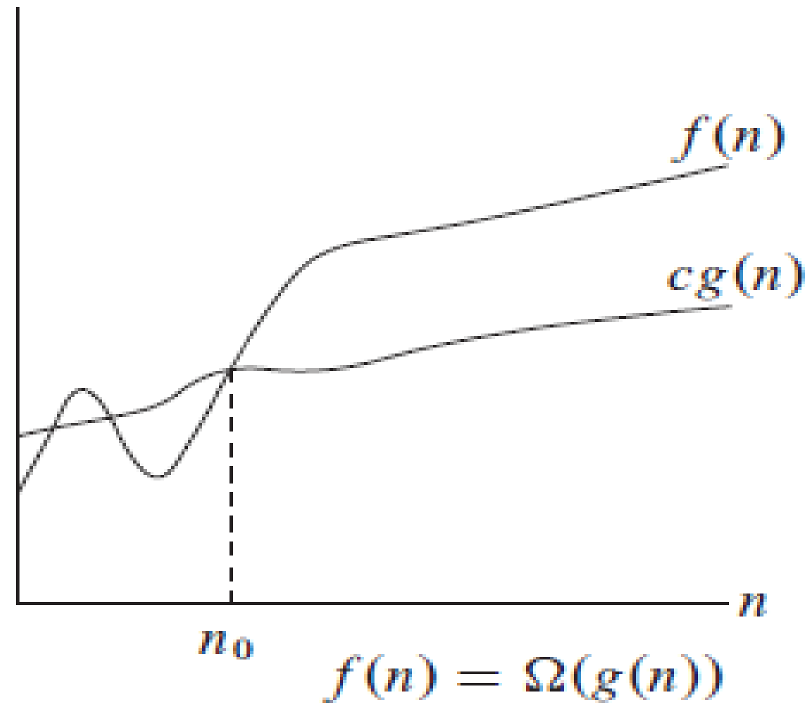
Asymptotic Notations

Ω–notation (Big-omega notation)

- For a given function g(n), it is denoted by $\Omega(g(n))$.
- It is defined as following:-

$\Omega(g(n)) = \{ f(n) \,!\, \exists$ positive constants c and n_0 such that

$0 \le cg(n) \le f(n), \forall\, n \ge n_0 \}$



- This notation is said to be lower bound.
- If $f(n) \in \Omega(g(n))$ then $f(n) = \Omega(g(n))$
- If $f(n) = \Theta(g(n))$ then $f(n) = \Omega(g(n))$
- $f(n) = \Theta(g(n))$ iff $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$

Asymptotic Notations

o–notation (little-oh notation)

- The asymptotic upper bound provided by O-notation may or may not be asymptotically tight.
- o-notation denotes an upper bound that is not asymptotically tight.
- For a given function g(n), it is denoted by o(g(n)).
- It is defined as following:-

o(g(n)) = { f(n) ! for any positive constants c, there exists a constant n_o such that

$0 \le f(n) < cg(n), \forall\, n \ge n_o$ }

Asymptotic Notations

ω–notation (little-omega notation)

- The asymptotic lower bound provided by **Ω**-notation may or may not be asymptotically tight.
- **ω**-notation denotes an upper bound that is not asymptotically tight.
- For a given function g(n), it is denoted by **ω**(g(n)).
- It is defined as following:-

$\boldsymbol{\omega}(g(n)) = \{ f(n) \,!$ for any positive constants c, there exists a constant n_0 such that

$0 \leq cg(n) < f(n), \forall\, n \geq n_0 \}$

Asymptotic Notations

Example: Show that $(1/2)n^2 - 3n = \theta(n^2)$.

Solution: Using definition of θ-notation,

$$c_1 g(n) \leq f(n) \leq c_2 g(n), \quad \forall\, n \geq n_0$$

In this question, $f(n) = (1/2)n^2 - 3n$ and $g(n) = n^2$, therefore

$$c_1\, n^2 \leq (1/2)n^2 - 3n \leq c_2\, n^2, \quad \forall\, n \geq n_0$$

We divide above by n^2, we get

$$c_1 \leq (1/2) - (3/n) \leq c_2\,, \quad \forall\, n \geq n_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Now, we have to find c_1, c_2 and n_0, such that equation (1) is satisfied.

Consider, left part of (1), $c_1 \leq (1/2) - (3/n)$ .... (2)

The value of c_1 will be positive value less than or equal to the minimum value of $(1/2) - (3/n)$. Minimum value of $(1/2) - (3/n) = 1/14$. Therefore, $c_1 = 1/14$. This value of c_1 will satisfy equation (2) for $n \geq 7$.

Here, $c_1 = 1/14$ and $n \geq 7$ which satisfy (2).

Asymptotic Notations

Consider, right part of (1), $(1/2) - (3/n) \leq c_2$, ... (3)

The value of c_2 will be positive value greater than or equal to the maximum value of $(1/2) - (3/n)$. Maximum value of $(1/2) - (3/n) = 1/2$. Therefore, $c_2 = 1/2$. This value of c_2 will satisfy equation (3) for $n \geq 1$.

Here, $c_2 = 1/2$ and $n \geq 1$ which satisfy (3).

Therefore, for $c_1 = 1/14$, $c_2 = 1/2$ and $n_0 = 7$, equation (1) is satisfied.

Hence by using definition of θ-notation,

$$(1/2)n^2 - 3n = \theta(n^2).$$

It is proved.

Asymptotic Notations

Example: Show that $2n+5 = O(n^2)$.

Solution: Using definition of O-notation,

$$f(n) \leq cg(n), \forall n \geq n_0$$

In this question, $f(n) = 2n+5$ and $g(n) = n^2$, therefore

$$2n+5 \leq c n^2 \quad \forall n \geq n_0$$

We divide above by n^2, we get

$$(2/n)+(5/n^2) \leq c, \quad \forall n \geq n_0 \quad \ldots\ldots\ldots(1)$$

Now, we have to find c and n_0, such that equation (1) is satisfied.

Asymptotic Notations

The value of c will be positive value greater than or equal to the maximum value of $(2/n)+(5/n^2)$.

Maximum value of $(2/n)+(5/n^2) = 7$.

Therefore, $c = 7$.

Clearly equation (1) is satisfied for $c = 7$ and $n \geq 1$.

Hence by using definition of O-notation ,

$$2n+5 = O(n^2).$$

It is proved.

Asymptotic Notations

Example: Show that $2n^2+5n+6 = \mathbf{\Omega}(n)$.

Solution: Using definition of $\mathbf{\Omega}$ -notation,

$$cg(n) \leq f(n) \text{ , } \forall\ n \geq n_0$$

In this question, $f(n) = 2n^2+5n+6$ and $g(n) = n$, therefore

$$cn \leq 2n^2+5n+6 \text{ , } \quad \forall\ n \geq n_0$$

We divide above by n, we get

$$c \leq 2n + 5 + (6/n) \text{ , } \ \forall\ n \geq n_0 \quad \ldots\ldots\ldots(1)$$

Now, we have to find c and n_0, such that equation (1) is always satisfied.

Asymptotic Notations

The value of c will be positive value less than or equal to the minimum value of $2n + 5 + (6/n)$.

Minimum value of $2n + 5 + (6/n) = 12$.

Therefore, $c = 12$.

Clearly equation (1) is satisfied for $c = 12$ and $n \geq 2$.

Hence by using definition of $\mathbf{\Omega}$ -notation ,

$$2n^2+5n+6 = \mathbf{\Omega}(n).$$

It is proved.

Asymptotic Notations

Example: Show that $2n^2 = o(n^3)$.

Solution: Using definition of o-notation,

$$f(n) < cg(n) \;, \forall\, n \geq n_0$$

Here, $f(n) = 2n^2$, and $g(n) = n^3$. Therefore,

$$2n^2 < cn^3 , \qquad \forall\, n \geq n_0$$

We divide above by n^3, we get

$$(2/n) < c , \qquad \forall\, n \geq n_0 \;\ldots\ldots\ldots\ldots(1)$$

for $c = 1$, there will be $n_0 = 3$, which satisfy (1).

for $c = 0.5$, there will be $n_0 = 7$, which satisfy (1).

Therefore, for every c, there exists n_0 which satisfy (1).

Hence $2n^2 = o(n^3)$.

Asymptotic Notations

Example: Show that $2n^2 \neq o(n^2)$.

Solution: Using definition of o-notation,

$$f(n) < cg(n) \;, \forall\, n \geq n_0$$

Here, $f(n) = 2n^2$, and $g(n) = n^2$. Therefore,

$$2n^2 < cn^2 , \quad \forall\, n \geq n_0$$

We divide above by n^2, we get

$$2 < c , \quad \forall\, n \geq n_0 \;\ldots\ldots\ldots\ldots(1)$$

Clearly for $c = 1$, inequality (1) does not satisfy.

Therefore, for every c, there does not exist n_0 which satisfy (1). Hence $2n^2 \neq o(n^2)$.

Asymptotic Notations

Example: Show that $2n^2 = \boldsymbol{\omega}(n)$.

Solution: Using definition of $\boldsymbol{\omega}$ -notation,

$$cg(n) < f(n) \ , \ \forall \ n \geq n_0$$

Here, $f(n) = 2n^2$, and $g(n) = n$. Therefore,

$$cn < 2n^2 \ , \qquad \forall \ n \geq n_0$$

We divide above by n, we get

$$c < 2n \ , \qquad \forall \ n \geq n_0 \ \ldots\ldots\ldots\ldots(1)$$

for $c = 1$, there will be $n_0 = 1$, which satisfy (1).

for $c = 10$, there will be $n_0 = 6$, which satisfy (1).

Therefore, for every c, there exists n_0 which satisfy (1).

Hence $2n^2 = \boldsymbol{\omega}(n)$.

Asymptotic Notations

Example: Show that $2n^2 \neq \omega(n^2)$.

Solution: Using definition of ω-notation,

$$cg(n) < f(n)\ ,\ \forall\ n \geq n_0$$

Here, $f(n) = 2n^2$, and $g(n) = n^2$. Therefore,

$$cn^2 < 2n^2\ , \qquad \forall\ n \geq n_0$$

We divide above by n^2, we get

$$c < 2\ , \qquad \forall\ n \geq n_0 \ \ldots\ldots\ldots\ldots(1)$$

Clearly for $c = 3$, there does not exists n_0, which satisfy (1).

Therefore, for every c, there does not exist n_0 which satisfy (1). Hence $2n^2 \neq \omega(n^2)$.