Discrete Structures and Theory of Logic

Lecture-45

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January 16, 2021

Solution of Non-Homogeneous Linear Recurrence Equation

In this case, we find homogeneous and particular solution both. The final solution will be addition of both. Here, f(n)$\neq$0.
The solution of non-homogeneous equation is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Method to find Particular Solution

The particular solution of a recurrence relation can be obtained by the method of inspection, since the particular solution depend on the form of f(n).

Recurrence Relation

We guess the solution according to following table:-

S. No.	f(n)	Guessing solution
1	b^n (If b is not a root of characteristic equation)	A b^n
2	Polynomial P(n) of degree m	$A_0+A_1n+A_2n^2+........+A_mn^m$
3	c^nP(n)(If c is not a root of characteristic equation and Polynomial P(n) of degree m)	$c^n(A_0 + A_1n + A_2n^2 + ... + A_mn^m)$
4	b^n (If b is a root of characteristic equation of multiplicity s)	A$n^s b^n$
5	c^nP(n) (If b is a root of characteristic equation of multiplicity t)	$n^t(A_0 + A_1n + A_2n^2 + ... + A_mn^m)b^n$

Example: Solve the recurrence relation

$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$ ....(1)

Solution: The homogeneous equation will be

$a_n + 5a_{n-1} + 6a_{n-2} = 0$

The characteristic equation will be

$\alpha^2 + 5\alpha + 6 = 0$

$\Rightarrow (\alpha + 2)(\alpha + 3) = 0$

$\Rightarrow \alpha =$ -2, -3

Therefore, the homogeneous solution of recurrence equation will be

$a_n^{(h)} = c_1(-2)^n + c_2(-3)^n$

For particular solution:

Here, f(n) $= 3n^2 - 2n + 1$

Clearly, f(n) is the polynomial equation of degree 2. Therefore using above table, we guess the following solution:-

$a_n = A_0 + A_1 n + A_2 n^2$.................(2)

Put the value of a_n in equation (1),

$(A_0 + A_1 n + A_2 n^2) + 5(A_0 + A_1(n-1) + A_2(n-1)^2) + 6(A_0 + A_1(n-2) + A_2(n-2)^2) = 3n^2 - 2n + 1$

$(A_0 + 5A_0 - 5A_1 + 5A_2 + 6A_0 - 12A_1 + 24A_2) + (A_1 + 5A_1 - 10A_2 + 6A_1 - 24A_2)n + (A_2 + 5A_2 + 6A_2)n^2 = 3n^2 - 2n + 1$

$(12A_0 - 17A_1 + 29A_2) + (12A_1 - 34A_2)n + 12A_2 n^2 = 3n^2 - 2n + 1$

Comparing the coefficients of power of n on both sides

$12A_0 - 17A_1 + 29A_2 = 1$ ....(3)

$12A_1 - 34A_2 =$ -2 ....(4)

$12A_2 = 3$..........(5)

After solving equations (3), (4) and (5), we get

$A_0 = 47/288$, $A_1 = 13/24$, $A_2 = 1/4$

Therefore, particular solution is

$a_n^{(p)} = (47/288) + (13/24)\text{n} + (1/4)n^2$

Therefore, the final solution of given recurrence relation will be the following:-

$$a_n = a_n^{(h)} + a_n^{(p)}$$
$$= c_1(-2)^n + c_2(-3)^n + (47/288) + (13/24)\text{n} + (1/4)n^2$$

Exercise:

Solve the following recurrence relations:-

1. $a_{n+2} - 5a_{n+1} + 6a_n = n^2$
2. $a_n - 6a_{n-1} + 8a_{n-2} = 3^n$
3. $a_n + 5a_{n-1} + 6a_{n-2} = 42(4)^n$
4. $a_n + a_{n-1} = 3n2^n$
5. $a_n - 2a_{n-1} = 32^n$
6. $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)2^n$

Example: Solve the recurrence relation

$a_n - 5a_{n-1} + 6a_{n-2} = 2^n + n$ ....(1)

Solution: The homogeneous solution will be

$a_n^{(h)} = c_1(2)^n + c_2(3)^n$

For particular solution:

Here, f(n) $= 2^n + n$

Therefore, we guess the solution as following:-

Let $a_n = A_0 n 2^n + (A_1 + A_2 n)$

Put this in equation (1), we get

$A_0 =$ -2, $A_1 = 7/4$, $A_2 = 1/2$

Therefore the solution will be

$a_n = c_1 2^n + c_2 3^n - 2n2^n + (7/4) + (1/2)n$

Generating Functions

The generating function of a sequence of numbers $a_0, a_1, a_2, \ldots\ldots\ldots\ldots, a_n, \ldots$ is defined as

$G(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots + a_nx^n + \ldots\ldots$

$\quad\quad = \sum_{n=0}^{\infty} a_nx^n$

Example: Find the generating functions for the following sequences

1. 1,1,1,1,1,.................
2. 1,2,3,4,...................
3. 0,1,2,3,4,....................
4. 1,a,a^2,a^3,..................

Solution:

1. The generating function of this sequence will be the following:-
   G(x) = 1+x+x^2+ x^3+ x^4+........................
   $= \frac{1}{(1-x)}$
2. The generating function of this sequence will be the following:-
   G(x) = 1+2x+3x^2+4x^3+........................
   xG(x)= x+2x^2+3x^3+........................
   Subtracting from above, we get
   (1-x)G(x) = 1+x+x^2+ x^3+ x^4+........................
   (1-x)G(x) = $\frac{1}{(1-x)}$
   Therefore, G(x) = $\frac{1}{(1-x)^2}$

3. The generating function of this sequence will be the following:-

G(x) = $0+x+2x^2+3x^3+4x^4+$.........................

$= x(1+2x+3x^2+4x^3+$.........................$)$

Therefore, G(x) = $\frac{x}{(1-x)^2}$

4. The generating function of this sequence will be the following:-

G(x) = $1+ax+a^2x^2+a^3x^3+a^4x^4+$.........................

$= 1+ax+(ax)^2+(ax)^3+(ax)^4+$.........................

$= \frac{1}{(1-ax)}$

Example: Find the generating functions for the following sequences

1. 0,0,1,1,1,.................
2. 1,1,0,1,1,1,1,..................
3. 1,0,-1,0,1,0,-1,0,1,.....................
4. 3,-3,3,-3,3,-3,..................

Example: Find the generating function of a sequence $< a_k >$ if $a_k = 2+3\text{k}$.

Solution: The generating function of a sequence whose general term is 2, is

G1(x) $= \frac{2}{(1-x)}$

The generating function of a sequence whose general term is 3k, is

G2(x) $= \frac{3x}{(1-x)^2}$

Hence the required generating function is

G(x) = G1(x)+ G1(x)= $\frac{2}{(1-x)}+\frac{3x}{(1-x)^2}$