Discrete Structures and Theory of Logic

Lecture-39

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Free and Bound variables

Given a formula containing a part of the form ($\forall$ x)P(x) or ($\exists$ x)P(x), such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of a formula is called a bound occurrence of x , while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence.

Example:

(i) $(\forall x)P(x, y)$

(ii) $(\forall x)(A(x) \to R(x))$

(iii) $(\forall x)(A(x) \to (\exists y)R(x, y))$

(iv) $(\exists x)(A(x) \wedge R(x))$

(v) $(\exists x)A(x) \wedge R(x)$

Example:

Let P(x): x is a person.

F(x,y): x is the father of y.

M(x,y): x is the mother of y.

Write the predicate form of the following statement

" x is the father of the mother of y."

Solution: Let z as the mother of y. Therefore, statement will be

$(\exists z)(P(z) \wedge F(x, z) \wedge M(z, y))$

Universe of discourse

The domain of a variable is known as the universe of discourse.
Example: If the discussion refers to human beings only, then the universe of discourse is the class of human beings.
Example: Consider the predicate
Q(x): x is less than 5.
and the statements $(\forall x)Q(x)$ and $(\exists x)Q(x)$. If the universe of discourse is given by the following sets, then find the truth value of statements $(\forall x)Q(x)$ and $(\exists x)Q(x)$.
(1) {-1, 0, 1, 2, 4}
(2) {3, -2, 7, 8, -5}
(3) {15, 20, 24}

Solution:

$(\forall x)Q(x)$ is true for (1) and false for (2) and (3).

$(\exists x)Q(x)$ is true for (1) and (2) and false for (3).

Inference theory of the predicate calculus

Some equivalences

(i) $\neg((\forall x)A(x)) \Leftrightarrow (\exists x)\neg A(x))$

(ii) $\neg((\exists x)A(x)) \Leftrightarrow (\forall x)\neg A(x)$

(iii) $A(x) \rightarrow B(x) \Leftrightarrow \neg A(x) \vee B(x)$

Some rules

(1) Universal Specification rule (US rule)

$(\forall x)A(x) \Rightarrow A(x)$

(2) Universal Generalization rule (UG rule)

$A(x) \Rightarrow (\forall x)A(x)$

(3) Existential Specification rule (ES rule)

$(\exists x)A(x) \Rightarrow A(y)$

(4) Existential Generalization rule (EG rule)

$A(y) \Rightarrow (\exists x)A(x)$

Some implications and equivalences

(1) $(\exists x)(A(x) \vee B(x)) \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

(2) $(\forall x)(A(x) \wedge B(x)) \Leftrightarrow (\forall x)A(x) \wedge (\forall x)B(x)$

(3) $\neg(\exists x)A(x) \Leftrightarrow (\forall x)\neg A(x)$

(4) $\neg(\forall x)A(x) \Leftrightarrow (\exists x)\neg A(x)$

(5) $(\forall x)A(x) \vee (\forall x)B(x) \Rightarrow (\forall x)(A(x) \vee B(x))$

(6) $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$

Example: Prove the following:-

(1) $(\exists x)(A(x) \to B(x)) \Leftrightarrow (\forall x)A(x) \to (\exists x)B(x)$

(2) $(\exists x)A(x) \to (\forall x)B(x)) \Leftrightarrow (\forall x)(A(x) \to B(x))$

Proof:

(1) $(\exists x)(A(x) \to B(x))$

$\Leftrightarrow (\exists x)(\neg A(x) \vee B(x))$

$\Leftrightarrow (\exists x)\neg A(x) \vee (\exists x)B(x)$

$\Leftrightarrow \neg(\forall x)A(x) \vee (\exists x)B(x)$

$\Leftrightarrow (\forall x)A(x) \to (\exists x)B(x)$

(2) $(\exists x)A(x) \to (\forall x)B(x)$

$\Leftrightarrow \neg(\exists x)A(x) \vee (\forall x)B(x)$

$\Leftrightarrow (\forall x)\neg A(x) \vee (\forall x)B(x)$

$\Leftrightarrow (\forall x)(\neg A(x) \vee B(x))$

$\Leftrightarrow (\forall x)(A(x) \to B(x))$

Example: Show that $(\forall x)(H(x) \rightarrow M(x)) \wedge H(s) \Rightarrow M(s)$.

Solution:

(1) $(\forall x)(H(x) \rightarrow M(x))$, By rule P

(2) $H(s) \rightarrow M(s)$, By rule US and (1)

(3) H(s) , By rule P

(4) M(s) , By rule T, (2), (3) and modus ponens