Discrete Structures and Theory of Logic

Lecture-22

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Field

An algebraic structure (F, +, .), where F is a set and + and . are two binary operators defined on set F, is said to be field if it satisfies following properties:-

(1) (R, +) is an abelian group.

(2) (R' , .) is an abelian group, where R' = R-$\{0\}$.

(3) Distributive property must hold i.e. a.(b+c) = a.b + a.c and (b+c).a = b.a + c.a, $\forall$ a,b,c $\in$ F.

Field

Example: The ring of rational numbers (Q , $+$, .) is a field.

Solution: Since (Q , $+$, .) is a ring therefore we have to show only second property of field i.e. (Q', .) is an abelian.

Since (Q , $+$, .) is ring therefore (Q', .) is a semigroup. Now, we hae to find identity element and inverse.

Clearly 1 is an identity element.

Consider an element a $\in$ Q'. clearly the inverse of a is $1/a$. Therefore inverse property is also satisfied.

If a,b $\in$ Q' then a.b $=$ b.a, therefore commutative property is satisfied. Since all the properties of an abelian group is satisfied within Q'. Therefore, (Q', .) is an abelian group.

Therefore, (Q , $+$, .) is a field.

Example: (R, $+$, .) is a field.

Ring with zero divisors

If a and b are two non-zero elements of a ring R such that $a.b = 0$, then a and b are divisors of 0(or o divisors). In particular, a is a left divisor of 0 and b is right divisor of 0.

Example: The ring of integers do not have zero divisors. Because there exist no two non-zero integers such that their product is zero.

Ring homomorphism

Let (R, $+$, .) and (S, $\oplus, \odot$) be rings. A mapping f: R $\rightarrow$ S is called a ring homomorphism from (R, $+$, .) to (S, $\oplus, \odot$) if for any a,b $\in$ R,

f(a+b) $=$ f(a)$\oplus$ f(b) and f(a.b) $=$ f(a)$\odot$ f(b)

Boolean ring

A ring R is said to be boolean ring if $a^2 =$ a, $\forall$ a $\in$ R.

Ring

Example: Show that a Boolean ring is always commutative.

Solution: It is proved in the previous example.

Example: If (R, $+$, .) is a ring with unity, then show that, for all a $\in$ R,

(i) (-1).a $=$ -a

(ii) (-1).(-1) $=$ 1

Solution:

(i) a $+$ (-1).a $=$ 1.a $+$ (-1).a

$=$ (1$+$(-1)).a

$=$ 0.a

$=$ 0

$\Rightarrow$ -a $=$ (-1).a

(ii) (-1).(-1) $=$ -((-1).1) $=$ -(-(1.1)) $=$ -(-(1)) $=$ 1 (since $(a^{-1})^{-1}$ $=$ a)

Example: Explain Boolean ring with suitable example.
Solution: A ring R is said to be boolean ring if $a^2 =$ a, $\forall$ a $\in$ R.
Example of Boolean ring is $(Z_2, +_2, \times_2)$ because
$Z_2 = \{0,1\}$ and $0^2 = 0 \times_2 0 = 0$, $1^2 = 1 \times_2 1 = 1$.

Note: $(Z_n, +_n, \times_n)$ is a field iff n is prime number.

Example: Determine all values of x from the given field which satisfies the given equation:-

(i) x + 1 = -1 over Z_2, Z_3, Z_5 *and* Z_7

(ii) 2x + 1 = 2 over $Z_3,$ *and* Z_5

(iii) 5x + 1 = 2 over Z_5

Solution:

(i) Consider field Z_2. $Z_2 = \{0,1\}$. Now, we have to find which values of Z_2 satisfies following x + 1 = -1.

Here, -1 indicate the additive inverse of 1.Clearly, in this field, additive inverse of 1 is 1, therefore the given equation is modified as x + 1 = 1.

Clearly x = 0 satisfies this equation.

Ring

Consider field Z_3. $Z_3 = \{0,1,2\}$. In this field, additive inverse of 1 is 2, therefore the given equation is modified as x + 1 = 2.
Clearly x = 1 satisfies this equation.
Consider field Z_5. $Z_5 = \{0,1,2,3,4\}$. In this field, additive inverse of 1 is 4, therefore the given equation is modified as x + 1 = 4.
Clearly x = 3 satisfies this equation.
Consider field Z_7. $Z_7 = \{0,1,2,3,4,5,6\}$. In this field, additive inverse of 1 is 6, therefore the given equation is modified as x + 1 = 6.
Clearly x = 5 satisfies this equation.

(ii) Consider field Z_3. $Z_3 = \{0,1,2\}$. Now, we have to find which values of Z_3 satisfies following $2x + 1 = 2$.
Clearly $x = 2$ satisfies this equation.
Consider field Z_5. $Z_5 = \{0,1,2,3,4\}$. Now, we have to find which values of Z_5 satisfies following $2x + 1 = 2$.
Clearly $x = 3$ satisfies this equation.
(iii) Consider field Z_5. $Z_5 = \{0,1,2,3,4\}$. Now, we have to find which values of Z_5 satisfies following $5x + 1 = 2$.
Clearly there is no x in Z_5 which satisfies this equation.

Exercise

1. Show that $(Z_7, +_7, \times_7)$ is a commutative ring with identity.
2. We are given the ring ({a,b,c,d}, $+$, .), whose operations are given by the following table:-

$+$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

.	a	b	c	d
a	a	a	a	a
b	a	c	a	c
c	a	a	a	a
d	a	c	a	a

Is it commutative ring? Does it have an identity? what is the zero of this ring? Find the additive inverse pf each of its elements.

Exercise

1. Show that $(I, \oplus, \odot)$ is a commutative ring with identity, where the operations $\oplus$ and $\odot$ are defined, for any a,b $\in$ I as a$\oplus$b $=$ a$+$b-1 and a$\odot$b $=$ a$+$b-ab.
2. Prove that $(R, +, *)$ is a ring with zero divisors, where R is 2x2 matrix and $+$ and $*$ are usual addition and multiplication operations.