Discrete Structures and Theory of Logic

Lecture-7

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Example: If R and S are both reflexive then show that R$\cup$S and R$\cap$S are also reflexive.

Solution: Since R and s are reflexive, therefore (a,a) $\in$ R and (a,a) $\in$ S, $\forall$ a. Since (a,a) $\in$ R and (a,a) $\in$ S, $\forall$ a, therefore (a,a) $\in$ R$\cup$S and (a,a) $\in$ R$\cap$S, $\forall$ a. Therefore, R$\cup$S and R$\cap$S are also reflexive.

Example: If R and S are both reflexive, symmetric, and transitive then show that R$\cup$S and R$\cap$S are also reflexive, symmetric, and transitive.

Some examples(cont.)

Solution:

For reflexive: Since R and s are reflexive, therefore (a,a) $\in$ R and (a,a) $\in$ S, $\forall$ a. Since (a,a) $\in$ R and (a,a) $\in$ S, $\forall$ a, therefore (a,a) $\in$ R$\cap$S, $\forall$ a. Therefore, R$\cap$S is also reflexive.

For symmetric: Since R is symmetric, therefore if (a,b) $\in$ R then (b,a) $\in$ R. Similarly, since S is symmetric, therefore if (a,b) $\in$ S then (b,a) $\in$ S.

Let (a,b) $\in$ R$\cap$S. It imply that (a,b) $\in$ R and (a,b) $\in$ S. Since R and s are symmetric therefore (b,a) $\in$ R and (b,a) $\in$ S. It imply that (b,a) $\in$ R$\cap$S. Therefore R$\cap$S is symmetric.

For transitive: Let (a,b) and (b,c) $\in$ R$\cap$S. Therefore

$\Rightarrow$ (a,b) and (b,c) $\in$ R and (a,b) and (b,c) $\in$ S

$\Rightarrow$ (a,c) $\in$ R and (a,c) $\in$ S (Since R and S are transitive)

$\Rightarrow$ (a,c) $\in$ R$\cap$S.

Therefore, R$\cap$S is also transitive.

Equivalence relation

Definition

A relation R defined on set A is said to be an equivalence relation if it satisfies reflexive, symmetric, and transitive properties.

Example: Let A = $\{1,2,3,4\}$ and R = $\{(1,1),(1,4),(4,1),(4,4),(2,2),(2,3),$
Is this relation an equivalence relation?

Solution: Since (1,1),(2,2),(3,3), and (4,4) are belongs into R, therefore R is reflexive.

Clearly in R if (a,b) $\in$ R then (b,a) $\in$ R. Here both (1,4) and (4,1) $\in$ R and both (3,2) and (2,3) $\in$ R. Therefore R is symmetric.

Clearly in R if (a,b) $\in$ R and (b,c) $\in$ R then (a,c) $\in$ R. Here, for pair (1,4) and (4,1), its transitive pair (1,1) and (4,4) are also belong into R. Similarly, or pair (2,3) and (3,2), its transitive pair (2,2) and (3,3) are also belong into R. Therefore R is transitive.

Clearly, R satisfies all the three properties. Therefore, R is an equivalence relation.

Some examples(cont.)

Example: Let A $= \{1,2,3,4,5,6\}$ and R $= \{(a,b) \,!\, (a\text{-}b)$ is divisible by 3 $\}$. Show that R is an equivalence relation.

Solution: In this example R will be

R $= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(1,4),(4,1),(2,5),(5,2),(3,6),(6,3)\}$

Clearly R satisfies reflexive, symmetric and transitive, therefore R is an equivalence relation.

Example: Let X $= \{a,b,c,d,e\}$ and let C $= \{\{a,b\},\{c\},\{d,e\}\}$. Show that the partition C defines an equivalence relation on X.

Solution: The relation defined by partition C will be the following

R $= \{(a,a),(b,b),(a,b),(b,a),(c,c),(d,d),(e,e),(d,e),(e,d)\}$

Clearly relation R is an equivalence relation because R satisfies all the three properties.

Some examples(cont.)

Example: Let S be the set of lines on a plane. Define a relation R on set S as following:- aRb if line a is parallel to line b, $\forall$ a,b $\in$ S. Is relation R an equivalence relation.

Solution: Since each line in a plane is parallel to itself, therefore R satisfies reflexive property.

We know that if line a is parallel to line b then line b is also parallel to line a. Therefore R satisfies symmetric property.

We know that if line a is parallel to line b and line b is parallel to line c, then line a is also parallel to line c. Therefore R satisfies transitive property.

Since R satisfies all the three properties, therefore R is an equivalence relation.

Exercise

1. Let R denote a relation on the set of ordered pairs of positive integers such that
   (x,y)R(u,v) iff xv = yu.
   Show that R is an equivalence relation.
2. Given a set S = $\{$1,2, 3, 4,5$\}$. Find the equivalence relation defined on S which generates the partition $\{\ \overline{1,2}, \overline{3}, \overline{4,5}\ \}$.
3. Prove that the relation "congruence modulo m" defined as
   $$\cong = \{\ (a,b)\ !\ (a\text{-}b) \text{ is divisible by m } \}$$
   over the set of positive integers is an equivalence relation.
   Show that if a $\cong$ b and c $\cong$ d, then (a+c) $\cong$ (b+d).

Exercise(cont.)

1. Let R_1 be a relation defined on R, the set of real numbers, such that $R_1 = \{$(x,y) ! $|x - y| < 1$ $\}$. Is R_1 an equivalence relation ? Justify. AKTU(2019)
2. Let R be a binary relation on the set of all positive integers such that:

   R = {(a,b) ! a-b is an odd positive integers}

   Is R reflexive ? Symmetric? Transitive?