Discrete Structures and Theory of Logic

Lecture-44

Dharmendra Kumar

January 15, 2021

Recurrence Relation and Generating Function

Recurrence Relation

A recurrence relation for the sequence $< a_n >$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence.

Example: Consider the following recurrence relation:-

$a_n = a_{n-1} - a_{n-2}$

for n=2,3,4,5,......., with the conditions $a_0 = 3$, $a_1 = 5$.

Example: The sequence 1,1,2,3,5,8,.................., is defined by the recurrence relation

$a_n = a_{n-1} + a_{n-2}$

with initial conditions $a_0 = 1$, $a_1 = 1$.

Order of the Recurrence Relation

The order of a recurrence relation is the difference between the largest and the smallest subscript appearing in the the relation.

Example: Consider the following recurrence relation:-

$a_n = a_{n-1} + a_{n-2}$

The order of this relation $= 2$

Consider another recurrence relation:-

$a_{n+3} - a_{n+2} + a_{n+1} - a_n = 0$

The order of this relation $= 3$

Degree of the Recurrence Relation

The degree of a recurrence relation is the highest power of a_n occurring in that relation.

Example: Consider the following recurrence relation:-

${a_n}^3 + 3a_{n-1}^2 + 6a_{n-2}^2 + 4a_{n-3} = 0$

The degree of this relation $= 3$

Consider another recurrence relation:-

$a_{n+2}^2 + 4a_{n+1} + 5a_n = 0$

The degree of this relation $= 2$

Linear Recurrence Relation with Constant Coefficients

A recurrence relation of the form

$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots\ldots\ldots\ldots\ldots + c_k a_{n-k} = f(n)$

where c_i's are constants, is called a linear recurrence relation with constant coefficients of k^{th} order, provided c_0 and c_k, both are non-zero. f(n) is the function of the independent variable 'n' only.

Example: (i) $3a_n + 6a_{n-1} = 2^n$

is the first order linear recurrence relation with constant coefficients.

(ii) $2a_n + 5a_{n-2} = n^2 + n$

is the second order linear recurrence relation with constant coefficients.

Note: A recurrence relation is said to be linear if its degree is one.

Homogeneous Linear Recurrence Relation

A linear recurrence realtion is said to be homogeneous if f(n) = 0. If f(n)$\neq$ 0, then it is said to be non-homogeneous.

Solution of Linear Equation

The solution of linear equation consists of two parts (i) homogeneous solution (ii) particular solution. That is,
$a = a^h + a^p$

Solution of Homogeneous Linear Recurrence Equation

Example: Find the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$, and $a_1 = 7$.

Solution:

Let the solution be $a_n = \alpha^n$.

Put $a_n = \alpha^n$ in the given recurrence relation.

$\alpha^n = \alpha^{n-1} + 2\alpha^{n-2}$

$\Rightarrow \alpha^2 - \alpha - 2 = 0$

$\Rightarrow (\alpha - 2)(\alpha + 1) = 0$

$\Rightarrow \alpha = 2$, -1

Therefore, the solution of recurrence equation will be
$a_n = c_1 2^n + c_2(-1)^n$ .... (1)
Now, we find c_1 and c_2 by using initial values $a_0 = 2$, and $a_1 = 7$.
For n=0, equation (1) will be
$a_0 = c_1 2^0 + c_2(-1)^0$
therefore, $c_1 + c_2 = 2$ ....(2)
For n=1, equation (1) will be
$a_1 = c_1 2^1 + c_2(-1)^1$
therefore, $2c_1 - c_2 = 7$ ....(3)
After solving equations (2) and (3), we get $c_1 = 3$ and $c_2 =$ -1.
Putting c_1 and c_2 in equation (1), we get the final solution
$a_n = 3 * 2^n - (-1)^n$

Recurrence Relation

Example: Find the solution of the recurrence relation

$a_n - 6a_{n-1} + 9a_{n-2} = 0$

with $a_0 = 1$, and $a_1 = 6$.

Solution:

Let the solution be $a_n = \alpha^n$.

Put $a_n = \alpha^n$ in the given recurrence relation.

$\alpha^n - 6\alpha^{n-1} + 9\alpha^{n-2}{=}0$

$\Rightarrow \alpha^2 - 6\alpha + 9 = 0$

$\Rightarrow (\alpha - 3)^2 = 0$

$\Rightarrow \alpha = 3, 3$

Therefore, the solution of recurrence equation will be

$a_n = (c_1 + c_2 n)(3)^n$ .... (1)

Recurrence Relation

Now, we find c_1 and c_2 by using initial values $a_0 = 1$, and $a_1 = 6$.
For n=0, equation (1) will be
$a_0 = (c_1 + c_2.0)(3)^0$
therefore, $c_1 = 1$ ....(2)
For n=1, equation (1) will be
$a_1 = (c_1 + c_2.1)(3)^1$
therefore, $3c_1 + 3c_2 = 6$ ....(3)
After solving equations (2) and (3), we get $c_1 = 1$ and $c_2 = 1$.
Putting c_1 and c_2 in equation (1), we get the final solution
$a_n = (1 + n)(3)^n$

Example: Find the solution of the recurrence relation

$a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$

Solution:

Let the solution be $a_n = \alpha^n$.

Put $a_n = \alpha^n$ in the given recurrence relation.

$\alpha^n - 5\alpha^{n-1} + 8\alpha^{n-2} - 4\alpha^{n-3}=0$

$\Rightarrow \alpha^3 - 5\alpha^2 + 8\alpha - 4 = 0$

$\Rightarrow (\alpha - 1)(\alpha - 2)^2 = 0$

$\Rightarrow \alpha = 1, 2, 2$

Therefore, the solution of recurrence equation will be

$a_n = c_1(1)^n + (c_2 + c_3 n)(2)^n$

Example: Find the solution of the recurrence relation

$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 0$

Solution:

Let the solution be $a_n = \alpha^n$.

Put $a_n = \alpha^n$ in the given recurrence relation.

$\alpha^n + 6\alpha^{n-1} + 12\alpha^{n-2} + 8\alpha^{n-3}$=0

$\Rightarrow \alpha^3 + 6\alpha^2 + 12\alpha + 8 = 0$

$\Rightarrow (\alpha + 2)^3 = 0$

$\Rightarrow \alpha =$ -2, -2, -2

Therefore, the solution of recurrence equation will be

$a_n = (c_1 + c_2 n + c_3 n^2)(-2)^n$

Recurrence Relation

Example: Find the solution of the recurrence relation

$4a_n - 20a_{n-1} + 17a_{n-2} - 4a_{n-3} = 0$

Solution:

Let the solution be $a_n = \alpha^n$.

Put $a_n = \alpha^n$ in the given recurrence relation.

$4\alpha^n - 20\alpha^{n-1} + 17\alpha^{n-2} - 4\alpha^{n-3}=0$

$\Rightarrow 4\alpha^3 - 20\alpha^2 + 17\alpha - 4 = 0$

$\Rightarrow (\alpha - 4)(2\alpha - 1)^2 = 0$

$\Rightarrow \alpha = 4,\ 1/2,\ 1/2$

Therefore, the solution of recurrence equation will be

$a_n = c_1(4)^n + (c_2 + c_3 n)(1/2)^n$