# Discrete Structures and Theory of Logic

## Lecture-17

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# Subgroup

**Theorem:** The intersection of any two subgroups of a group (G,o) is again a subgroup of (G,o).

**Proof:** Let ($H_1$,o) and ($H_2$,o) are the two subgroups of (G,o).

Let a $\in H_1 \cap H_2$ and b $\in H_1 \cap H_2$

$\Rightarrow$ (a $\in H_1$ and a $\in H_2$) and (b $\in H_1$ and b $\in H_2$)

$\Rightarrow$ (a $\in H_1$ and b $\in H_1$) and (a $\in H_2$ and b $\in H_2$)

$\Rightarrow$ a$ob^{-1} \in H_1$ and a$ob^{-1} \in H_2$ ( Since $H_1$ and $H_2$ are subgroups.)

$\Rightarrow$ a$ob^{-1} \in H_1 \cap H_2$

Therefore, $H_1 \cap H_2$ is also a subgroup.

# Subgroup

**Example:** The union of two subgroups is not necessarily a subgroup.

**Solution:** Let G be the additive group of integers.

$H_1 = \{$ ..............,-6,-4,-2,0,2,4,6,..........$\}$

and $H_2 = \{$ ............,-9,-6,-3,0,3,6,9,..........$\}$

are both subgroups of G. But $H_1 \cup H_2$ is not subgroup.

# Subgroup

## Coset

Let H be subgroup of G and let $a \in G$. Then set $\{aoh \; ! \;\; h \in H\}$ is called the left coset generated by a and H and is denoted by aH. And right coset is denoted by $Ha = \{hoa \; ! \;\; h \in H\}$.

## Index of a subgroup in a group

If H is a subgroup of a group G, then the number of distinct right(or left) cosets of H in G is called the index of H in G and it is denoted by [G:H].

**Example:** Let G be the additive group of integers i.e. G = { ..........-3,-2,-1,0,1,2,3,4,........}. Let H = { ..............,-9,-6,-3,0,3,6,9,..............} be the subgroup of G. Determine the index of H in G.

**Solution:** The index of H in G = The number of left cosets of H in G.

Now, we calculate all distinct left cosets.

0+H = H = { ..............,-9,-6,-3,0,3,6,9,..............}.

1+H = {.............,-8,-5,-2,1,4,7,10,..............}

2+H = {.............., -7,-4,-1,2,5,8,11,...................}

3+H = { ..............,-9,-6,-3,0,3,6,9,..............}.

4+H = {.............,-8,-5,-2,1,4,7,10,..............}

Clearly the number of distinct left cosets is 3. Therefore, the index of H in G = 3.

# Normal subgroup

A subgroup H of G is said to be normal subgroup of G if Ha = aH, $\forall$ a $\in$ G.

**Theorem:** A subgroup H of G is normal iff $g^{-1}hg \in$ H, $\forall$ h $\in$ H, g$\in$ G.

**Proof:**

**First part:** Let H be a normal subgroup of G.

Let h $\in$ H, $g \in$ G. Since H is normal subgroup, therefore $gH = Hg$.

Now,$hg \in Hg \Rightarrow$ hg $\in gH$ (since $gH = Hg$)

$$\Rightarrow g^{-1}hg \in (g^{-1}g)H$$
$$\Rightarrow g^{-1}hg \in \text{eH}$$
$$\Rightarrow g^{-1}hg \in \text{H}$$

Now, it is proved.

# Normal subgroup

**Second part:** Suppose $g^{-1}hg \in$ H, $\forall$ h $\in$ H, $g \in$ G ..................(1)

Now, we have to show that $gH = Hg$.

Let $hg \in Hg$. $\Rightarrow (gg^{-1})hg \in Hg$

$\Rightarrow g(g^{-1}hg) \in Hg$

$\Rightarrow gh' \in$ Hg (using (1))

$\Rightarrow gh' \in$ gH

$\Rightarrow g(g^{-1}hg) \in gH$

$\Rightarrow hg \in gH$

Therefore, $Hg \subseteq gH$ ......................(2)

# Normal subgroup

Now, let $gh \in gH$. $\Rightarrow (g^{-1})^{-1})hg^{-1}g \in gH$

$\Rightarrow h'g \in gH$ (using (1))

$\Rightarrow h'g \in Hg$

$\Rightarrow (g^{-1})^{-1})hg^{-1}g \in Hg$

$\Rightarrow gh \in Hg$

Therefore, $gH \subseteq Hg$ ......................(3)

From (2) and (3), $gH = Hg$, $\forall\ g \in$ G.

Therefore, H is normal subgroup of G. Now, it is proved.

# Normal subgroup

**Example:** If H is a subgroup of G such that $a^2 \in$ H for every a$\in$ G, then prove that H is a normal subgroup of G.

**Solution:** Let a $\in$ G. Then $a^2 \in$ H.
We know that if $a^{-1}ba \in$ H, then H is normal subgroup, for b $\in$ H.
Here, b = $a^2$, therefore, $a^{-1}ba = a^{-1}a^2a = a^2$ = b
Since, b $\in$ H, therefore $a^{-1}ba \in$ H. Hence, H is a normal subgroup.