Discrete Structures and Theory of Logic

Lecture-31

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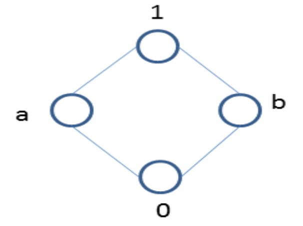
Boolean Algebra

Note: Given a Boolean expression $\alpha(x_1, x_2, ..., x_n)$ and a Boolean algebra $< B, \wedge, \vee, ', 0, 1 >$, we can obtain the values of the Boolean expression for every n-tuple of B^n. Let us now consider a function $f_{\alpha,B} : B^n \to B$ such that for any n-tuple $< a_1, a_2, ..., a_n > \in B^n$, the value of $f_{\alpha,B}$ is equal to the value of the Boolean expression $\alpha(x_1, x_2, ..., x_n)$, that is,

$f_{\alpha,B}(a_1, a_2, ..., a_n) = \alpha(x_1, x_2, ..., x_n)$

for all $(a_1, a_2, ..., a_n) \in B^n$. We shall call $f_{\alpha,B}$ the function associated with the Boolean expression $\alpha(x_1, x_2, ..., x_n)$.

Example: Find the value of the function $f_{\alpha,B} : B^3 \rightarrow B$ for $x_1 =$ a, $x_2 = 1$,and $x_3 =$ b, where a,b,1 are the elements of the Boolean algebra is shown in the following figure:-



and $\alpha(x_1, x_2, ..., x_n)$ is the expression whose binary valuation is given in the following table:-

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x_1	x_2	x_3	$\alpha(x_1, x_2, x_3)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

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Solution: From the table,

$$f_{\alpha,B}(x_1, x_2, x_3) = (x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3)$$
$$\vee (x_1 \wedge x_2' \wedge x_3)$$

$$\begin{aligned}
\alpha(a, 1, b) &= (a' \wedge 1' \wedge b') \vee (a' \wedge 1 \wedge b') \vee (a' \wedge 1 \wedge b) \vee (a \wedge 1' \wedge b) \\
&= (b \wedge 0 \wedge a) \vee (b \wedge 1 \wedge a) \vee (b \wedge 1 \wedge b) \vee (a \wedge 0 \wedge b) \\
&= 0 \vee (b \wedge a) \vee b \vee 0 \\
&= (b \wedge a) \vee b \\
&= 0 \vee \mathrm{b} \\
&= \mathrm{b}
\end{aligned}$$

Boolean function

Let $< B, \wedge, \vee, ', 0, 1 >$ ba Boolean algebra. A function f: $B^n \rightarrow$ B which is associated with a Boolean expression in n-variables is called a Boolean function.

Note: For a two elements Boolean algebra, the number of functions from B^n to B is 2^{2^n}. Here, every function from B^n to B is Boolean function.

Symmetric Boolean expression

A Boolean expression in n variables is called symmetric if interchanging any two variables results in an equivalent expression.

Example: Following expressions are symmetric.

(a) $(x_1 \wedge x_2') \vee (x_1' \wedge x_2)$

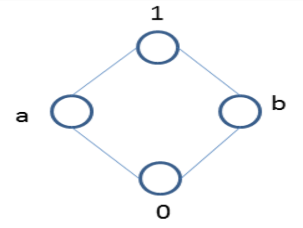
(b) $(x_1 \wedge x_2 \wedge x_3') \vee (x_1 \wedge x_2' \wedge x_3) \vee (x_1' \wedge x_2 \wedge x_3)$

Exercise

1. Find the canonical sum of product form of the following Boolean expressions:-
   1.1 $x_1 \vee x_2$
   1.2 $x_1 \vee (x_2 \wedge x_3')$
   1.3 $(x_1 \vee x_2)' \vee (x_1' \wedge x_3)$
2. Show that
   2.1 $(a \wedge (b' \vee c))' \wedge (b' \vee (a \wedge c')')' = (a \wedge b \wedge c')$
   2.2 $a' \wedge ((b' \vee c)' \vee (b \wedge c)) \vee ((a \vee b')' \wedge c) = a' \wedge b$

Exercise

3. Given an expression $\alpha(x_1, x_2, x_3)$ defined to be Σ 0,3,5,7, determine the value of $\alpha(a, b, 1)$, where a,b,1 $\in$ B and $< B, \wedge, \vee, ', 0, 1 >$ is the following Boolean algebra.



Exercise

4. Obtain simplified Boolean expressions which are equivalent to these expressions:-

(a) $m_0 + m_7$

(b) $m_0 + m_1 + m_2 + m_3$

(c) $m_5 + m_7 + m_9 + m_1 1 + m_1 3$

Where m_j are the minterms in the variables x_1, x_2, x_3, and x_4.

Minimization of Boolean function or expression

We shall minimize the Boolean function or expression using Karnaugh map.

Example: Minimize the following function using K-map.

f(a,b,c) $= \Sigma(0, 1, 4, 6)$

Solution:

a \ bc	00	01	11	10
0	1	1		
1	1			1

K-map

The minimized function will be, f(a,b,c) $= (a' \wedge b') \vee (a \wedge c')$.

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Example: Minimize the following function using K-map.

f(a,b,c,d) $= \Sigma(0, 5, 7, 8, 12, 14)$

Solution:

ab \ cd	00	01	11	10
00	1			
01		1	1	
11	1			1
10	1			

K-map

The minimized function will be, f(a,b,c,d) $= (a' \wedge b \wedge d') \vee (b' \wedge c' \wedge d') \vee (a \wedge b \wedge d')$.

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Example: Minimize the following function using K-map.

f(a,b,c,d) $= \Sigma(0, 1, 2, 3, 13, 15)$

Solution:

ab \ cd	00	01	11	10
00	1	1	1	1
01				
11		1	1	
10				

K-map

The minimized function will be, f(a,b,c,d) $= (a' \wedge b') \vee (a \wedge b \wedge d)$.