

# **Discrete Structures and Theory of Logic**

## **Lecture-41**

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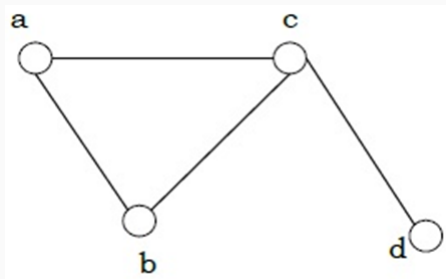
## Graph Theory

## Graph definition

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A graph is an ordered pair  $G=(V,E)$  consisting of a nonempty set  $V$  of vertices and a set  $E$  of edges.

**Example:**



## Order of a graph

The number of vertices in a graph is said to be the order of the graph.

## Degree of a Vertex

The degree of a vertex  $v$  of a graph  $G$  (denoted by  $\deg(v)$ ) is the number of edges incident with the vertex  $v$ .

**In-degree:** The number of incoming edges at a vertex is said to be in-degree of that vertex.

**Out-degree:** The number of out going edges from a vertex is said to be out-degree of that vertex.

**Even and Odd Vertex** If the degree of a vertex is even, the vertex is called an even vertex and if the degree of a vertex is odd, the vertex is called an odd vertex.

## Degree of a Graph

The degree of a graph is the largest vertex degree of that graph.

**Isolated Vertex** A vertex with degree zero is called an isolated vertex.

**Pendant Vertex** A vertex with degree one is called a pendent vertex.

## Types of Graph

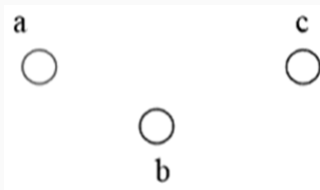
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There are different types of graphs.

### Null Graph

A null graph is a graph in which there are no edges between its vertices. A null graph is also called empty graph.

**Example:**



## Trivial Graph

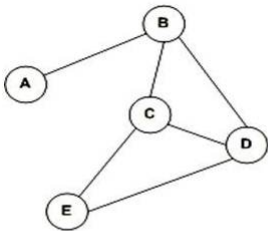
A trivial graph is the graph which has only one vertex.

## Directed and Undirected Graph

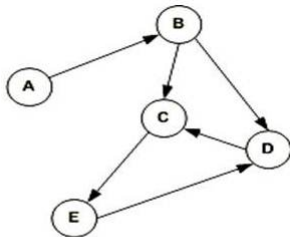
An undirected graph is a graph whose edges are not directed.

A directed graph is a graph in which the edges are directed by arrows.

Directed graph is also known as digraphs.



Undirected graph

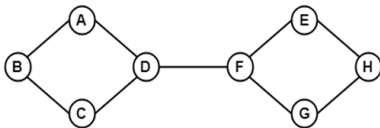


Directed graph

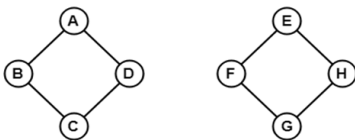
## Connected and Disconnected Graph

A connected graph is a graph in which we can visit from any one vertex to any other vertex. In a connected graph, at least one edge or path exists between every pair of vertices.

A disconnected graph is a graph in which any path does not exist between every pair of vertices.



Connected graph



Disconnected graph

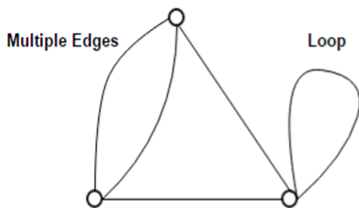


## Simple graph

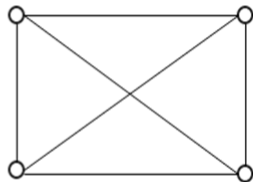
A simple graph is the undirected graph with no parallel edges and no loops.

A simple graph which has  $n$  vertices, the degree of every vertex is at most  $n-1$ .

**Example:**



**Not a Simple Graph**

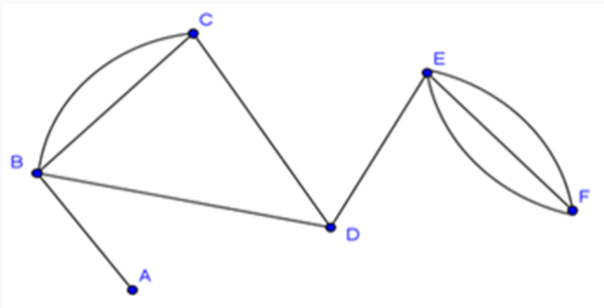


**Simple Graph**

## Multi-graph

A graph having no self loops but having parallel edge(s) in it is called as a multi-graph.

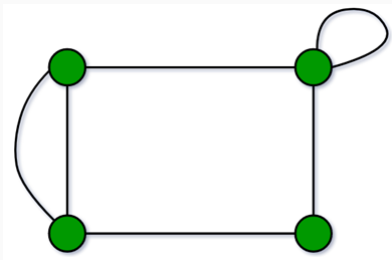
**Example:**



## Pseudo Graph

A graph in which loops and multiple edges are allowed is called psuedograph.

**Example:** Following graph is pseudo graph:-



## Handshaking lemma

In any graph (simple)  $G$ , the sum of degree of all vertices is equal to  $2e$ , where  $e$  is the number of edges that is

$$\sum_{v \in V} \deg_G(v) = 2e.$$

### **Proof:**

Since the degree of a vertex is the number of edges incident with that vertex, the sum of degree counts the number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices, so each edge gets counted twice, once at each end. Thus the sum of degree is equal twice the number of edges.

**Theorem:** The number of vertices of odd degree in a graph  $G$  is always even.

**Proof:** We know that, the sum of the degrees of all vertices in a graph  $G$  is twice the number of edges in  $G$  i.e.  $\sum_{v \in V} \deg_G(v) = 2e$   
 $\sum_{v \in \text{EVEN}} \deg_G(v) + \sum_{v \in \text{ODD}} \deg_G(v) = 2e$ , where EVEN is the set of even degree vertices and ODD is the set odd degree vertices.

$$\Rightarrow \sum_{v \in \text{ODD}} \deg_G(v) = 2e - \sum_{v \in \text{EVEN}} \deg_G(v)$$
$$= \text{even number} - \text{even number} = \text{even number}$$

$$\Rightarrow \sum_{v \in \text{ODD}} \deg_G(v) = \text{even number}$$

Hence the number of vertices of odd degree in a graph is even.

**Note:** In a graph  $G$  with  $n \geq 2$ , there are two vertices of equal degree.

**Example:** Is there a simple graph with degree sequence  $(1,3,3,3,5,6,6)$ ?

**Example:** Is there a simple graph with seven vertices having degree sequence  $(1,3,3,4,5,6,6)$ ?

**Example:** Is there a simple graph with degree sequence  $(1,1,3,3,3,4,6,7)$ ?

**Example:** Show that the maximum number of edges in a simple graph with  $n$  vertices is  $n(n-1)/2$ .