Discrete Structures and Theory of Logic

Lecture-23

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Partial ordered relation

Consider a relation R defined on set S. Relation R is said to be partial ordered relation if R satisfies following properties:-

(1) R is reflexive, i.e., xRx for every $x \in S$.

(2) R is anti-symmetric, i.e., if xRy and yRx, then $x = y$.

(3) R is transitive, i.e., xRy and yRz, then xRz.

Partial ordered set (POSET)

Consider a relation R defined on set S. If R is a partial order relation, then the combination of set S and partial order relation R is said to be partial ordered set i.e. POSET. We denote POSET by $< S, \preceq >$, where $\preceq$ denotes partial ordered relation.

Totally ordered relation and set

Let $< S, \preceq >$ bea partially ordered set. If for eery a,b $\in$ S, we hae either a$\preceq$b or b$\preceq$a, then $\preceq$ is called a totally ordered relation defined on set P.
And the ordered pair $< S, \preceq >$ is called a totally ordered set.

POSET

Example: Let R be the set of real numbers and relation $\preceq$ is less than or equal i.e. a$\preceq$b iff a$\leq$b. Is $< R, \preceq >$ a POSET?

Solution: $< R, \preceq >$ will be a POSET if $\preceq$ is partial ordered relation. $\preceq$ will be partial ordered relation if this relation satisfies reflexive, anti-symmetric and transitive.

Here, the relation is less than or equal.

Each real number will be related to itself because each real number is equal to itself. Therefore, this relation is reflexive.

a is less than or equal to b and b is less than or equal to a , this is only possible iff a=b. Therefore this relation is anti-symmetric.

Consider a$\preceq$b and b$\preceq$c. It imply that a$\leq$b and b$\leq$c. It imply that a$\leq$c. Therefore R is transitive.

Since all the three properties are satisfies, therefore, this relation is partial ordered relation.

POSET

Example: Let A = $\{1,2,3\}$. Show that $< P(A), \subseteq >$ is POSET, P(A) is power set o A.

Solution: Here P(A) = $\{\phi, \{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$ and relation is subset i.e. set B related set C iff B$\subseteq$C.

For reflexive: Since each set is subset of itself, therefore this relation is reflexive.

For anti-symmetric: Consider B and C are two elements of P(A) such that B$\subseteq$C and C$\subseteq$B. Clearly it will be only true when B=C, otherwise it will be false. Therefore this relation is anti-symmetric.

For transitive: Consider B, C and D are three elements of P(A) such that B$\subseteq$C, C$\subseteq$D. It will imply that B$\subseteq$D. Therefore this relation is transitive.

Since all the three properties are satisfies, therefore, this relation is partial ordered relation. And the ordered pair $< P(A), \subseteq >$ is POSET.

POSET

Example: Let D(n) is the set of all positive divisors of n. And relation $\preceq$ is defined as:- a $\preceq$ b iff a divides b. Is $< D(n), \preceq >$ POSET?

Solution: Two elements a,b $\in$ D(n) will be related iff a divides b.

For reflexive: Since each element divides to itself, therefore this relation is reflexive.

For anti-symmetric: Consider a and b are two elements of D(n) such that a$\preceq$b and b$\preceq$c. Clearly it will be only true when a=b, otherwise it will be false. Therefore this relation is anti-symmetric.

For transitive: Consider a, b and c are three elements of D(n) such that a$\preceq$b, b$\preceq$c. It will imply that a$\preceq$c. Therefore this relation is transitive.

Since all the three properties are satisfies, therefore, this relation is partial ordered relation. And the ordered pair $< D(n), \preceq >$ is POSET.

Cover, Successor, Predecessor

In a partially ordered set $< S, \preceq >$, an element b $\in$ S is said to be cover of element a $\in$ S if a $\preceq$ b and if there does not exist any element c $\in$ S such that a $\preceq$ c $\preceq$ b.
a is the immediate predecessor of b and b is the immediate successor of a.

Hasse diagram

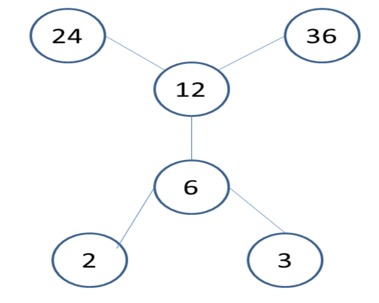
It is the graphical representation of POSET.

In this diagram, we make nodes corresponding to each elements in the POSET, that is the number of nodes is equal to the number of elements in the POSET. Edges will undirected. These edges will link elements a and b if b is cover of a such that a will be at lower side of b and b will be at upper side from a. In other words, two elements will be connected by an edge if one element is an immediate successor of another element.

Hasse diagram

Example: Let S $= \{2,3,6,12,24,36\}$ and the relation $\preceq$ be such that a $\preceq$ b if a divides b. Draw the Hasse diagram of this POSET $< S, \preceq >$.

Solution:

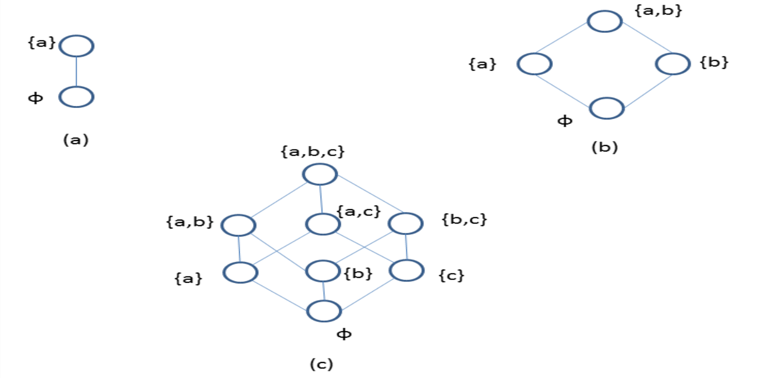


Hasse diagram

Example: Let A be a given finite set and P(A) its power set. Let $\subseteq$ be the inclusion relation on the elements of P(A). Draw the Hasse diagram of $< P(A), \subseteq >$ for

(a) A = {a}, (b) A = {a,b}, (c) A = {a,b,c}

Solution:



Hasse diagram

Example: Let A be the set of factors of a particular positive integer m and let $\preceq$ be the relation divides, i.e.
$\preceq = \{(a,b) ! \text{ a divides b}\}$
Draw Hasse diagram for (a) m = 2 (b) m = 6 (c) m = 30 (d) m = 12 (e) m = 45
Solution: Since A be the set of factors of a particular positive integer m, therefore A will be for each case:-
(a) A = $\{1,2\}$
(b) A = $\{1,2,3,6\}$
(c) A = $\{1,2,3,5,6,10,15,30\}$
(d) A = $\{1,2,3,4,6,12\}$
(e) A = $\{1,3,5,9,15,45\}$

Hasse diagram

Hasse diagrams are:-