Theory of Automata and Formal Language

Lecture-27

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Closure Properties of Context Free Languages

Theorem: Show that the family of context free languages is closed under union operation.

Proof: Let L_1 and L_2 be two context free languages generated by context free grammar $G_1 = (V_1, \Sigma_1, S_1, P_1)$ and $G_1 = (V_2, \Sigma_2, S_2, P_2)$ respectively.

Now, we construct the grammar G as the following;-

$G = (V, \Sigma, S, P)$

Where, V $= V_1 \cup V_2 \cup \{S\}$

$\Sigma = \Sigma_1 \cup \Sigma$

and P $= P_1 \cup P_2 \cup \{S \rightarrow S_1 | S_2\}$

Context Free Grammar

Now, we have to show that

$\mathrm{L(G)} = L(G_1) \cup L(G_2)$

Let $\mathrm{x} \in \mathrm{L(G)} \Leftrightarrow \mathrm{S} \overset{*}{\Rightarrow} \mathrm{x}$

$$\begin{aligned} &\Leftrightarrow \mathrm{S} \Rightarrow S_1 \overset{*}{\Rightarrow} \mathrm{x} \text{ or} \\ &\qquad \mathrm{S} \Rightarrow S_2 \overset{*}{\Rightarrow} \mathrm{x} \text{ or} \\ &\Leftrightarrow S_1 \overset{*}{\Rightarrow} \mathrm{x} \text{ or } S_2 \overset{*}{\Rightarrow} \mathrm{x} \\ &\Leftrightarrow \mathrm{x} \in L(G_1) \text{ or } \mathrm{x} \in L(G_2) \\ &\Leftrightarrow \mathrm{x} \in L(G_1) \cup L(G_2) \end{aligned}$$

Therefore, $\mathrm{L(G)} = L(G_1) \cup L(G_2)$

Clearly, G is a CFG for $L_1 \cup L_2$, therefore $L_1 \cup L_2$ is also a context free language.

Context Free Grammar

Theorem: Show that the family of context free languages is closed under concatenation operation.

Proof: Let L_1 and L_2 be two context free languages generated by context free grammar $G_1 = (V_1, \Sigma_1, S_1, P_1)$ and $G_1 = (V_2, \Sigma_2, S_2, P_2)$ respectively.

Now, we construct the grammar G as the following;-

$G = (V, \Sigma, S, P)$

Where, V = $V_1 \cup V_2 \cup \{S\}$

$\Sigma = \Sigma_1 \cup \Sigma$

and P = $P_1 \cup P_2 \cup \{S \rightarrow S_1.S_2\}$

Now, we have to show that

L(G) = $L(G_1).L(G_2)$

Let x $\in$ L(G) $\Leftrightarrow$ S $\stackrel{*}{\Rightarrow}$ x

$\Leftrightarrow$ S $\Rightarrow S_1.S_2 \stackrel{*}{\Rightarrow}$ x

$\Leftrightarrow S_1 \stackrel{*}{\Rightarrow} x_1$ and $S_2 \stackrel{*}{\Rightarrow} x_2$ (Let x= x_1x_2)

$\Leftrightarrow x_1.x_2 \in L(G_1).L(G_2)$

$\Leftrightarrow$ x$\in L(G_1).L(G_2)$

Therefore, L(G) = $L(G_1).L(G_2)$

Clearly, G is a CFG for $L_1.L_2$, therefore $L_1.L_2$ is also a context free language.

Theorem: Show that the family of context free languages is closed under kleene closure operation.

Proof: Let L is a context free languages and G is a context free grammar generating L.

$G = (V, \Sigma, S, P)$

Now, we construct a grammar G' as the following:-

$G' = (V', \Sigma, S', P')$

Where, V' = V$\cup$\{S'\}

P' = P$\cup\{S' \to SS'|\epsilon\}$

Now, we have to show that L(G') = (L(G))*.

Context Free Grammar

Let x $\in$ L(G') $\Leftrightarrow$ S' $\overset{*}{\Rightarrow}$ x

$\Leftrightarrow$ S' $\Rightarrow S.S.S.........S(n - times) \overset{*}{\Rightarrow}$ x

$\Leftrightarrow S \overset{*}{\Rightarrow} x_1, S \overset{*}{\Rightarrow} x_2 ....S \overset{*}{\Rightarrow} x_n$ (Let x= $x_1 x_2........x_n$)

$\Leftrightarrow\ x_1 \in L(G),\ x_2 \in L(G), ..., x_n \in L(G)$

$\Leftrightarrow\ x_1.x_2..........x_n \in (L(G))^n$

$\Leftrightarrow\ x \in (L(G))^*\ \ (L(G))^n \subseteq (L(G))^*$

Therefore, L(G') = (L(G))*.

Therefore, G' is a grammar generating the language L*. Hence L* is a context free language

Theorem: Show that the family of context free languages is not closed under intersection operation.

Proof: Consider the two context free languages:-

$L_1 = \{a^n b^n c^m \; ! \; n \geq 0, m \geq 0\}$

$L_2 = \{a^n b^m c^m \; ! \; n \geq 0, m \geq 0\}$

The intersection of these languages is

$L_1 \cap L_2 = \{a^n b^n c^n \; ! \; n \geq 0\}$

We know that this language is not context free.

Therefore, the family of context free languages is not closed under intersection operation.

Context Free Grammar

Theorem: Show that the family of context free languages is not closed under complement operation.

Proof: Consider the following formula

$$L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})} \quad ....(1)$$

Suppose the context free languages are closed under complement operation.

From (1), If L_1 and L_2 are context free languages, then $\overline{L_1}$ and $\overline{L_2}$ are also context free language.

Since $\overline{L_1}$ and $\overline{L_2}$ are context free, therefore $\overline{L_1} \cup \overline{L_2}$ will also be context free language. Therefore $\overline{(\overline{L_1} \cup \overline{L_2})}$ is also context free.

Since the R.H.S. of equation (1) is context free, therefore L.H.S. is also context free. But, by previous theorem $L_1 \cap L_2$ is not context free, therefore context free languages is not closed under complement operation.