Discrete Structures and Theory of Logic

Lecture-26

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Lattice

Theorem: Let $< L, \preceq >$ be a lattice. For any a,b,c $\in$ L,

(1) $a \vee (b \wedge c) \preceq (a \vee b) \wedge (a \vee c)$

(2) $a \wedge (b \vee c) \succeq (a \wedge b) \vee (a \wedge c)$

Proof:

(1) Using definition of least upper bound, $a \preceq (a \vee b)$..........(1)

Using definition of greatest lower bound, $b \wedge c \preceq b$ ....(2)

Using definition of least upper bound, $b \preceq (a \vee b)$..........(3)

From (2) and (3), $b \wedge c \preceq b \preceq (a \vee b)$

Therefore, $b \wedge c \preceq (a \vee b)$.....................(4)

From (1) and (4), $(a \vee b)$ is the upper bound of a and $b \wedge c$, therefore

$lub\{a , b \wedge c\} \preceq (a \vee b)$ i.e.

$a \vee (b \wedge c) \preceq (a \vee b)$ ....(5)

Lattice

Similarly, using definition of least upper bound, a $\preceq$ (a$\vee$c)..........(6)
Using definition of greatest lower bound, b$\wedge$c $\preceq$ c ....(7)
Using definition of least upper bound, c $\preceq$ (a$\vee$c)..........(8)
From (7) and (8), b$\wedge$c $\preceq$ c $\preceq$ (a$\vee$c)
Therefore, b$\wedge$c $\preceq$ (a$\vee$c).....................(9)
From (6) and (9), (a$\vee$c) is the upper bound of a and b$\wedge$c, therefore

lub{a , b$\wedge$c} $\preceq$ (a$\vee$c) i.e.
a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$c) ....(10)

From (5) and (10), a$\vee$(b$\wedge$c) is lower bound of (a$\vee$b) and (a$\vee$c), therefore

a$\vee$(b$\wedge$c) $\preceq$ glb{(a$\vee$b), (a$\vee$c)} i.e.
a$\vee$(b$\wedge$c) $\preceq$ (a$\vee$b)$\wedge$(a$\vee$c)
Now, it is proved.

(2) Using definition of greatest lower bound, $(a \wedge b) \preceq a$ ....(1)
Using definition of least upper bound, $b \preceq (b \vee c)$..........(2)
Using definition of greatest lower bound, $(a \wedge b) \preceq b$..........(3)
From (2) and (3), $(a \wedge b) \preceq b \preceq (b \vee c)$
Therefore, $(a \wedge b) \preceq b \vee c$.....................(4)
From (1) and (4), $(a \wedge b)$ is the lower bound of a and $b \vee c$, therefore

$(a \wedge b) \preceq \text{glb}\{a , b \vee c\}$ i.e.

$(a \wedge b) \preceq a \wedge (b \vee c)$ ....(5)

Lattice

Similarly, using definition of greatest lower bound, $(a \wedge c) \preceq a$ ....(6)

Using definition of least upper bound, $c \preceq (b \vee c)$..........(7)

Using definition of greatest lower bound, $(a \wedge c) \preceq c$..........(8)

From (7) and (8), $(a \wedge c) \preceq c \preceq (b \vee c)$

Therefore, $(a \wedge c) \preceq (b \vee c)$.....................(9)

From (6) and (9), $(a \wedge c)$ is the lower bound of a and $(b \vee c)$, therefore

$(a \wedge c) \preceq \text{glb}\{a\ ,\ b \vee c\}$ i.e.

$(a \wedge c) \preceq a \wedge (b \vee c)$ ....(10)

From (5) and (10), $a \wedge (b \vee c)$ is upper bound of $(a \wedge b)$ and $(a \wedge c)$, therefore

$\text{lub}\{(a \wedge b), (a \wedge c)\} \preceq a \wedge (b \vee c)$ i.e.

$(a \wedge b) \vee (a \wedge c)\} \preceq a \wedge (b \vee c)$

Now, it is proved.

Theorem: Let $< L, \preceq >$ be a lattice. For any a,b,c $\in$ L,

$$a \preceq c \Leftrightarrow a \vee (b \wedge c) \preceq (a \vee b) \wedge c$$

Proof:

First part: In this part, we will prove that if $a \preceq c$ then $a \vee (b \wedge c) \preceq (a \vee b) \wedge c$

Suppose $a \preceq c$.

Using definition of least upper bound, $a \preceq (a \vee b)$..........(1)

Using definition of greatest lower bound, $(b \wedge c) \preceq b$ ....(2)

Using definition of least upper bound, $b \preceq (a \vee b)$ ....(3)

Therefore, from (2) and (3), $(b \wedge c) \preceq b \preceq (a \vee b)$

Therefore, $(b \wedge c) \preceq (a \vee b)$ ....(4)

From (1) and (4), $(a \vee b)$ is the upper bound of a and $b \wedge c$, therefore

$\text{lub}\{a , b \wedge c\} \preceq (a \vee b)$ i.e.

$a \vee (b \wedge c) \preceq (a \vee b)$ ....(5)

Now, using definition of greatest lower bound, $(b \wedge c) \preceq c$ ....(6)
Since $a \preceq c$, therefore using (6), c is an upper bound of a and $(b \wedge c)$.
Therefore
$a \vee (b \wedge c) \preceq c$ ....(7)
from (5) and (7), $a \vee (b \wedge c)$ is lower bound of $(a \vee b)$ and c. Therefore,
$a \vee (b \wedge c) \preceq (a \vee b) \wedge c$

Second part: In this part, we will prove that if $a \vee (b \wedge c) \preceq (a \vee b) \wedge c$ then $a \preceq c$.
Now, $a \preceq a \vee (b \wedge c) \preceq (a \vee b) \wedge c \preceq c$
Therefore, $a \preceq c$.
Since both parts are proved. Therefore, it is proved.

Exercise

1. Show that in a lattice if $a \preceq b \preceq c$, then
   $a \vee b = b \wedge c$ and $(a \wedge b) \vee (b \wedge c) = b = (a \vee b) \wedge (a \vee c)$
2. Show that in a lattice if $a \preceq b$ and $c \preceq d$, then
   $(a \wedge c) \preceq (b \wedge d)$
3. In a lattice, show that
   (a) $(a \wedge b) \vee (c \wedge d) \preceq (a \vee c) \wedge (b \vee d)$
   (b) $(a \wedge b) \vee (b \wedge c) \vee (c \wedge a) \preceq (a \vee b) \wedge (b \vee c) \wedge (c \vee a)$
4. Show that a lattice with three or fewer elements is a chain.