Discrete Structures and Theory of Logic

Lecture-34

Dharmendra Kumar

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Duality law

Two formulas P and Q are said to be dual of each others if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called dual of each other. If the formula P contains the special symbols T or F, then Q, its dual, is obtained by replacing T by F and F by T.

Example: Dual of $(P \wedge Q) \vee T$ is $(P \vee Q) \wedge F$.

Converse, Inverse and Contrapositive

For any statement formula $P \to Q$, the statement formula $Q \to P$ is called the converse, $\neg P \to \neg Q$ is called its inverse and $\neg Q \to \neg P$ is called its contrapositive.

Note: $P \to Q \Leftrightarrow \neg Q \to \neg P$ and $Q \to P \Leftrightarrow \neg P \to \neg Q$

Tautological Implication

A statement A is said to be tautologically imply a statement B iff $A \rightarrow B$ is tautology. We shall denote this by $A \Rightarrow B$, which is read as "A implies B".

$A \Rightarrow B$ guarantees that B has the truth value T whenever A has the truth value T.

Exercise

1. Show the following implications.

(a) $(P \wedge Q) \Rightarrow (P \rightarrow Q)$

(b) $P \Rightarrow (Q \rightarrow P)$

(c) $(P \rightarrow (Q \rightarrow R)) \Rightarrow (P \rightarrow Q) \rightarrow (P \rightarrow R)$

2. Show the following equivalences.

(a) $P \rightarrow (Q \rightarrow P) \Leftrightarrow \neg P \rightarrow (P \rightarrow \neg Q)$

(b) $P \rightarrow (Q \vee R) \Leftrightarrow (P \rightarrow Q) \vee (P \rightarrow R)$

(c) $(P \rightarrow Q) \wedge (R \rightarrow Q) \Leftrightarrow (P \vee R) \rightarrow Q$

(d) $\neg(P \leftrightarrow Q) \Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$

Exercise

3. Show the following implications without constructing truth tables.

(a) $(P \to Q) \Rightarrow P \to (P \wedge Q)$

(b) $(P \to Q) \to Q)) \Rightarrow (P \vee Q)$

(c) $((P \vee \neg P) \to Q) \to ((P \vee \neg P) \to R) \Rightarrow (Q \to R)$

(d) $(Q \to (P \wedge \neg P)) \to (R \to (P \wedge \neg P)) \Leftrightarrow (R \to Q)$

Formulas with distinct truth tables

A statement formula containing n variables must have as its truth table one of the 2^{2^n} possible truth table, each of them having 2^n rows.

Functionally complete set of connectives

A set of connectives by which every formula can be expressed in terms of an equivalent formula containing the connectives from this set, is called a functionally complete set of connectives.

Minimal functionally complete set:
A functional complete set is said to be minimal functionally complete set if its proper subset is not functionally complete.

Example:

1. Are the sets $\{\wedge, \vee, \neg\}, \{\wedge, \neg\}, and \{\vee, \neg\}$ functionally complete?
2. Is the set $\{\wedge, \vee, \neg\}$ minimal functionally complete?
3. Are the sets $\{\wedge, \neg\}$, and $\{\vee, \neg\}$ minimal functionally complete?
4. Are the sets $\{\neg\}, \{\wedge\}, \{\vee\} and \{\wedge, \vee\}$ functionally complete?

Example: Write the formulas which are equivalent to the formulas given below and which contain the connectives $\wedge$ and $\neg$.

1. $\neg(p \leftrightarrow (Q \rightarrow (R \vee P))$
2. $((P \vee Q) \wedge R) \rightarrow (P \vee R)$