Discrete Structures and Theory of Logic

Lecture-29

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Example: Show that every chain is a distributive lattice.

Solution: Consider any three elements a,b,c of a chain. There will be six different relations exist between these elements.

Case-1: ($a \preceq b \preceq c$):

In this case, $a \wedge (b \vee c) = a \wedge c =$ a

and $(a \wedge b) \vee (a \wedge c) = a \vee a =$ a

Therefore, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Similarly, $a \vee (b \wedge c) = a \vee b =$ b

and $(a \vee b) \wedge (a \vee c) = b \wedge c =$ b

Therefore, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Clearly, both properties of distributive lattice are satisfied for this case.

Case-2: ($a \preceq c \preceq b$):
In this case, $a \wedge (b \vee c) = a \wedge b = \mathrm{a}$
and $(a \wedge b) \vee (a \wedge c) = a \vee a = \mathrm{a}$
Therefore, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
Similarly, $a \vee (b \wedge c) = a \vee c = \mathrm{c}$
and $(a \vee b) \wedge (a \vee c) = b \wedge c = \mathrm{c}$
Therefore, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
Clearly, both properties of distributive lattice are satisfied for this case.
Similarly, we can show for other four cases that properties of distributive lattice are satisfied. Other four case are (3) ($b \preceq a \preceq c$) (4) ($b \preceq c \preceq a$) (5) ($c \preceq a \preceq b$) (6) ($c \preceq b \preceq a$).
Since, in all the cases, properties of distributive lattice are satisfied, therefore a chain is distributive lattice.

Lattice

Example: Let $< L, \wedge, \vee >$ be a distributive lattice. For any a,b,c $\in$ L,

$$a \wedge b = a \wedge c \text{ and } a \vee b = a \vee c \Rightarrow b = c.$$

Solution:

LHS = b = $b \wedge (b \vee a)$ (using absorption law)

$= b \wedge (a \vee b)$ (using commutative law)

$= b \wedge (a \vee c)$ (using given equality $a \vee b = a \vee c$)

$= (b \wedge a) \vee (b \wedge c)$ (using distributive law)

$= (a \wedge b) \vee (b \wedge c)$ (using commutative law)

$= (a \wedge c) \vee (b \wedge c)$ (using given equality $a \wedge b = a \wedge c$)

$= (a \wedge b) \vee c)$ (using distributive law)

$= (a \wedge c) \vee c)$ (using given equality $a \wedge b = a \wedge c$)

$=$ c (using absorption law)

$=$ RHS

Therefore, b = c

Lattice

Example: In a distributive lattice, every element has a unique complement.

Solution: Consider an element a belong into given lattice L. Suppose b and c are two complements of a in L. Therefore,

a$\wedge$b = 0, a$\vee$b = 1 , and a$\wedge$c = 0, a$\vee$c = 1. ....(1)

Now, we have to show that b and c will be equal.

b = b$\wedge$1 (using identity law)

= b$\wedge$(a$\vee$c) (using equation (1))

= $(b \wedge a) \vee (b \wedge c)$ using distributive law)

= 0$\vee(b \wedge c)$ (using equation (1))

= (a$\wedge$c)$\vee(b \wedge c)$ (using equation (1))

= $(a \wedge b) \vee c$ (using distributive law)

= 0$\vee c$ (using equation (1)) = c (using identity law)

Therefore, b = c. That is, we can say, every element has a unique complement.

Exercise

1. Find the complements of every elements of the lattice $< D(n), / >$ for n = 75.
2. Show that in a lattice with two or more elements, no element is its own complement.
3. Show that a chain of three or more elements is not complemented.
4. Which of the two lattices $< D(n), / >$ for n = 30 and n = 45 are complemented? Are these lattices are distributive?

Exercise(cont.)

5. Show that De-Morgan's law given by
$(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$
hold in complemented and distributive lattice.
6. Show that in a complemented, distributive lattice,
$a \preceq b \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a' \vee b = 1 \Leftrightarrow b' \preceq a'$
7. Show that every distributive lattice is modular, but not conversely.