Discrete Structures and Theory of Logic

Lecture-25

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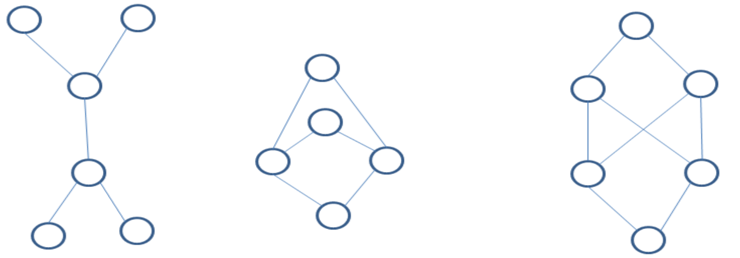
Exercise

1. Find out the following POSETs are lattices or not.



(a) (b) (c)

2. Draw the diagram of lattices $< S_n, D >$ for n = 4, 6, 10, 12, 15, 45, 60, 75 and 210. For what values of n, do you expect $< S_n, D >$ to be a chain?

Exercise

3. Let R be the set of real numbers in [0,1] and $\preceq$ be the usual operation of "less than or equal" on R. Show that $< R, \preceq >$ is a lattice. What are the operations of meet and join on this lattice?

4. Let the sets $S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7$ be given by

$S_0 = \{a,b,c,d,e,f\}$ $S_1 = \{a,b,c,d,e\}$ $S_2 = \{a,b,c,e,f\}$
$S_3 = \{a,b,c,e\}$ $S_4 = \{a,b,c\}$ $S_5 = \{a,b\}$
$S_6 = \{a,c\}$ $S_7 = \{a\}$

Draw the diagram of $< L, \subseteq >$, where L $= \{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$.

Principle of Duality

Any statement about lattices involving the operations $\wedge$ and $\vee$ remains true if $\wedge$ is replaced by $\vee$ and $\vee$ is replaced by $\wedge$.
The operations $\wedge$ and $\vee$ are said to be dual of each other. For example $a \wedge b$ is the dual of $a \vee b$.

Properties of lattices

(1) Idempotent law

$$a \wedge a = a, \qquad a \vee a = a$$

(2) Commutative law

$$a \wedge b = b \wedge a, \qquad a \vee b = b \vee a$$

(3) Associative law

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c, \qquad a \vee (b \vee c) = (a \vee b) \vee c$$

(4) Absorption law

$$a \wedge (a \vee b) = a, \qquad a \vee (a \wedge b) = a$$

Theorem: Let $< L, \preceq >$ be a lattice. For any a,b $\in$ L,

$$a \preceq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$$

Proof: In this theorem,we have to prove many parts.

First part: In this part, we will prove $a \preceq b \Leftrightarrow a \wedge b = a$.

Suppose $a \preceq b$. Since $a \preceq a$, therefore a = lower bound(l.b.) of a and b.

Since a is lower bound therefore, a$\preceq$ greatest lower bound(g.l.b.) of a and b.

hence $a \preceq a \wedge b$ ....(1)

By the definition of glb, $a \wedge b \preceq a$ ....(2)

from (1) and (2), $a \wedge b = a$.

Conversely, suppose $a \wedge b = a$.

By the definition of glb,

$a \wedge b \preceq b$

Since $a \wedge b = a$, therefore $a \preceq b$.

Lattice

Second part: In this part, we will prove $a \preceq b \Leftrightarrow a \vee b = b$.
Suppose $a \preceq b$. Since $b \preceq b$, therefore b = upper bound(u.b.) of a and b.
Since b is an upper bound therefore, least upper bound(l.u.b.) of a and $b \preceq b$.
hence $a \vee b \preceq b$ ....(1)
By the definition of lub, $b \preceq a \vee b$ ....(2)
from (1) and (2),
$a \vee b = b$.
Conversely, suppose $a \vee b = b$.
By the definition of lub,
$a \preceq a \vee b$
Since $a \vee b = b$, therefore $a \preceq b$.

Lattice

Third part: In this part, we will prove $a \wedge b = a \Leftrightarrow a \vee b = b$.

Suppose $a \wedge b = a$.

Now, $a \vee b = (a \wedge b) \vee b = b$, by absorption law.

Suppose $a \vee b = b$.

Now, $a \wedge b = a \wedge (a \vee b) = a$, by absorption law.