Discrete Structures and Theory of Logic

Lecture-30

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Boolean algebra

A lattice is said to be boolean algebra if it is both complemented and distributive. It is denoted by $< B, \wedge, \vee,', 0, 1 >$.

Some examples

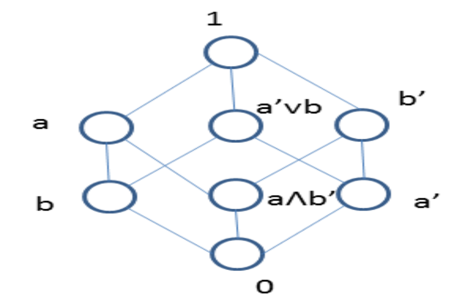
Example: $< P(S), \cap, \cup,', \phi, S >$ is a boolean algebra. Because it is complemented and distributive both. In this lattice, 0 is ϕ and 1 is S. Each elements of P(S) has a complement.
Example: $< D(30), gcd, lcm,', 1, 30 >$ is a boolean algebra.

Sub-boolean algebra

Let $< B, \wedge, \vee,' , 0, 1 >$ be a boolean algebra and S $\subseteq$ B. If S contains the elements 0 and 1 and is closed under the operations $\wedge, \vee$ and ', then $< S, \wedge, \vee,' , 0, 1 >$ is called a sub-boolean algebra.

Boolean Algebra

Example: Consider the following boolean algebra.



Let the subsets of this boolean algebra are:-

$S_1 = \{$a,a',0,1$\}$ $S_2 = \{$a'$\vee$b ,a$\wedge$b',0,1$\}$ $S_3 = \{$a$\wedge$b' ,b',a,1$\}$

$S_4 = \{$b' ,a$\wedge$b',a',0$\}$ $S_5 = \{$a,b',0,1$\}$

Find out which are sub-boolean algebra.

Solution: Consider the subset S_1. Since S_1 is subset of B, therefore it satisfies distributive property. Now, 0 and 1 are also belongs into this subset. Complement of each element is also belong into S_1. Since a and a' are complement of each other. we know 0 and 1 are complement of each other. Therefore, S_1 is sub-boolean algebra.
Consider the subset S_2. Since S_1 is subset of B, therefore it satisfies distributive property. Now, 0 and 1 are also belongs into this subset. Clearly from diagram, complement of a'$\vee$b = a$\wedge$b'. complement of 0 = 1. Therefore, S_2 is sub-boolean algebra.
Consider the subset S_3. In this subset 0 is not belong, therefore this subset is not a boolean algebra.
Consider the subset S_4. In this subset 1 is not belong, therefore this subset is not a boolean algebra.
Consider the subset S_5. In this subset, complement of a and b' do not exists, therefore this subset is also not a boolean algebra.

Boolean expression

Definition: A Boolean expression always produces a Boolean value. A Boolean expression is composed of a combination of the Boolean constants (True or False), Boolean variables and logical connectives. Each Boolean expression represents a Boolean function.

A Boolean expression in n variables $x_1, x_2, ..., x_n$ is any finite string of symbols formed in the following manner:-

(1) 0 and 1 are Boolean expressions.

(2) $x_1, x_2, ..., x_n$ are Boolean expressions.

(3) If α_1 and α_2 are Boolean expressions, then $\alpha_1 \wedge \alpha_2$ and $\alpha_1 \vee \alpha_2$ are also Boolean expressions.

(4) If α is a Boolean expression then α' is also a Boolean expression.

(5) All the expressions formed by step 1 to 4, are also Boolean expressions.

Boolean Algebra

Example: x_1, $x_1' \vee x_2$, $(x_2' \vee x_1)' \wedge (x_3 \vee x_1)$, and $(x_1' \vee x_1) \wedge x_2 \wedge x_3'$ are all Boolean expressions.

Equivalent Boolean expressions: Two Boolean expressions α and β are said to be equivalent if one can be obtained from the other by a finite number of applications of the identities of a Boolean algebra.

Minterm: A Boolean expression of n variables in the following form is said to be minterm.

$$x_1^{\alpha_1} \wedge x_2^{\alpha_2} \wedge \ldots\ldots\ldots\ldots \wedge x_n^{\alpha_n}$$

where α_i is either 0 or 1, x_i^0 stands for x_i' and x_i^1 stands for x_i.

Maxterm: A Boolean expression of n variables in the following form is said to be maxterm.

$$x_1^{\alpha_1} \vee x_2^{\alpha_2} \vee \ldots\ldots\ldots\ldots \vee x_n^{\alpha_n}$$

where α_i is either 0 or 1, x_i^0 stands for x_i' and x_i^1 stands for x_i.

Canonical sum of product form: A Boolean expression is said to be in canonical sum of product form if it is the join of only minterms.

For example, for three variables, Boolean expression $(x_1' \wedge x_2' \wedge x_3') \vee (x_1' \wedge x_2 \wedge x_3')$ is in canonical sum of product form.

Canonical product of sum form: A Boolean expression is said to be in canonical product of sum form if it is the meet of only maxterms.

For example, for three variables, Boolean expression $(x_1' \vee x_2' \vee x_3') \wedge (x_1' \vee x_2 \vee x_3')$ is in canonical product of sum form.

Boolean Algebra

Example: Obtain the values of the Boolean expression (1) $x_1 \wedge (x_1' \vee x_2)$ (2) $x_1 \wedge x_2$ and (3) $x_1 \vee (x_1 \wedge x_2)$
over the ordered pairs of the two elements Boolean algebra.

Solution: Let B = $\{0,1\}$. Consider $x_1 = 0$ and $x_2 = 1$.

(1) $x_1 \wedge (x_1' \vee x_2) = (x_1 \wedge x_1') \vee (x_1 \wedge x_2)$

$= 0 \vee (x_1 \wedge x_2)$

$= x_1 \wedge x_2$

$= 0 \wedge 1$ (putting the values of $x_1 = 0$ and $x_2 = 1$)

$= 0.$

(2) $x_1 \wedge x_2 = 0 \wedge 1$ (putting the values of $x_1 = 0$ and $x_2 = 1$)

$= 0.$

(3) $x_1 \vee (x_1 \wedge x_2) = 0 \vee (0 \wedge 1)$

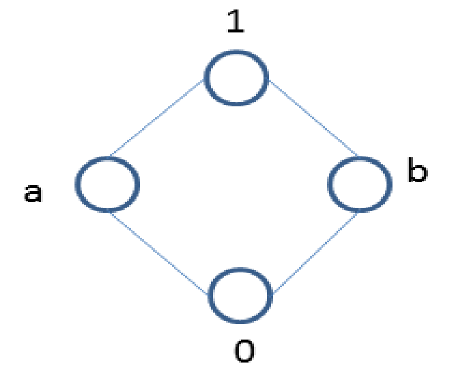
$= 0 \vee 0 = 0.$

Boolean Algebra

Example: Find the value of following Boolean expression

$$x_1 \wedge x_2 \wedge [(x_1 \wedge x_4) \vee x_2' \vee (x_3 \wedge x_1')]$$

for $x_1 = \text{a}$, $x_2 = 1$, $x_3 = \text{b}$, $x_4 = 1$, where 1,a,b $\in$ B and the Boolean algebra B is the following:-



Boolean Algebra

Solution:

$$
\begin{aligned}
f(x_1, x_2, x_3) &= x_1 \wedge x_2 \wedge [(x_1 \wedge x_4) \vee x_2' \vee (x_3 \wedge x_1')] \\
&= a \wedge 1 \wedge [(a \wedge 1) \vee 1' \vee (b \wedge a')] \\
&= a \wedge [a \vee 0 \vee (b \wedge b)] \\
&= a \wedge [a \vee b] \\
&= a \wedge 1 \\
&= \mathrm{a}
\end{aligned}
$$