Discrete Structures and Theory of Logic

Lecture-12

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Peano Axioms

These axioms are

(1) $0 \in$ N (where $0 = \phi$)

(2) If n $\in$ N, then $n^+ \in$ N, where $n^+ =$ n$\cup\{$n$\}$

(3) If a subset S $\subseteq$ N possesses the properties

(a) $0 \in$ S, and

(b) If n $\in$ S, then $n^+ \in$ S

Then S $=$ N.

Principle of Mathematical Induction

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.

The technique involves two steps to prove a statement, as stated below:-

Step 1(Base step): It proves that a statement is true for the initial value.

Step 2(Inductive step): It proves that if the statement is true for the number n, then it is also true for the number n+1.

Some examples

Example: Show that $n < 2^n$, by principle of induction method.

Solution:

Base step: For $n = 0$.

$$0 < 2^0 \Leftrightarrow 0 < 1$$

This is true. Therefore, the given statement is true for $n = 0$.

Now, for n=1.

$$1 < 2^1 \Leftrightarrow 1 < 2$$

This is true. Therefore, the given statement is also true for $n = 1$.

Therefore, the statement is true for base step.

Inductive Step: Now suppose the statement is true for n=k. We shall prove it for n=k+1.

Since statement is true for n=k, therefore k $< 2^k$ ....(1)

For n = k+1.

$$\begin{aligned} k+1 &< 2^k + 1 && \text{Using equation (1)} \\ &< 2^k + 2^k \\ &= 2.2^k \\ &= 2^{k+1} \end{aligned}$$

Therefore, k+1 $< 2^{k+1}$.

Therefore, statement is also true for inductive step.

Hence the given statement is proved.

Example: Show that $2^n < n!, \forall$ n $\geq$ 4 by principle of induction method.

Solution:

Base step: For n = 4.

$$2^0 < 4! \Leftrightarrow 16 < 24$$

This is true. Therefore, the given statement is true for n = 4.

Now, for n=5.

$$2^5 < 5! \Leftrightarrow 32 < 120$$

This is true. Therefore, the given statement is also true for n = 5.

Therefore, the statement is true for base step.

Principle of Mathematical Induction

Inductive Step: Now suppose the statement is true for n=k. We shall prove it for n=k+1.

Since statement is true for n=k, therefore $2^k < k!$ ....(1)

For n = k+1.

$$\begin{aligned} 2^{k+1} &= 2.2^k \\ &< 2.k! && \text{Using equation (1)} \\ &< (k+1).k! \\ &= (k+1)! \end{aligned}$$

Therefore $2^{k+1} < (k+1)!$

Therefore, statement is also true for inductive step.

Hence the given statement is proved.

Example: Show that $n^3 + 2\mathrm{n}$ is divisible by 3, by principle of induction method.

Solution:

Base step: For n = 1.

$n^3 + 2\mathrm{n} = 1^3 + 2\mathrm{x}1 = 1+2 = 3$

Clearly $n^3 + 2\mathrm{n}$ is divisible by 3, therefore it is true for n = 1.

For n = 2.

$n^3 + 2\mathrm{n} = 2^3 + 2\mathrm{x}2 = 8+4 = 12$

Clearly $n^3 + 2\mathrm{n}$ is divisible by 3. Therefore it is also true for n = 2.

Therefore, the statement is true for base step.

Principle of Mathematical Induction

Inductive Step: Now suppose the statement is true for n = k. We shall prove it for n = k+1.

Since statement is true for n = k, therefore k^3 + 2k is divisible by 3. It can be written as k^3 + 2k = 3m. . . ..(1)

For n = k+1.

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 3k + 1 + 2(k+1)\\ &= (k^3 + 2k) + 3(k^2 + k + 1)\\ &= 3m + 3(k^2 + k + 1) \quad \text{Using equation (1)}\\ &= 3(m + (k^2 + k + 1)\ )\end{aligned}$$

Clearly it is divisible by 3. Therefore it is also true for n = (k+1).

Therefore, statement is also true for inductive step.

Hence the given statement is proved.

Exercise

1. Show that S(n) = 1+2+3+.......................+n = $\frac{n(n+1)}{2}$
2. Prove that $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots$......................+ $\frac{1}{n.(n+1)} = \frac{n}{(n+1)}$
3. Show that $2+2^2+2^3+$..............................+ $2^n = 2^{n+1}$ - 2
4. Show that 3^n - 1 is a multiple of 2, for n = 1, 2, 3,................

Exercise(cont.)

1. Show that 1+3+5+.........+(2n - 1) = n^2, for n = 1,2, 3, ...
2. Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} +$ ....+ $\frac{1}{\sqrt{n}} > \sqrt{n}$, for n $\geq$ 2 using principle of mathematical induction.
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3. Prove by using mathematical induction that
   7+77+777+............+777...........7 = $\frac{7}{81}$[10^{n+1} - 9n - 10] $\forall$ n $\in$ N. AKTU(2019)