Discrete Structures and Theory of Logic

Lecture-27

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Lattices as algebraic system

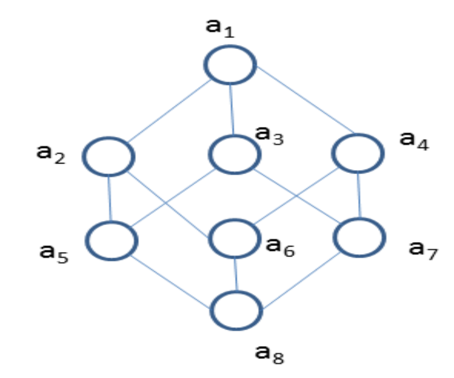
A lattice is an algebraic system $< L, \wedge, \vee >$ with two binary operations $\wedge$ and $\vee$ on L which are satisfy commutative, associative, absorption and idempotent properties.

Sublattice

Let $< L, \wedge, \vee >$ be a lattice and let S $\subseteq$ L be a subset of L. Then $< S, \wedge, \vee >$ is said to be sublattice of $< L, \wedge, \vee >$ iff $< S, \wedge, \vee >$ is also a lattice.

Lattice

Example: Consider the following lattice $L = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$.



Let $S_1 = \{a_1, a_2, a_4, a_6\}$, $S_2 = \{a_3, a_5, a_7, a_8\}$, and $S_3 = \{a_1, a_2, a_4, a_8\}$. Find out $< S_1, \prec >, < S_2, \prec >, and < S_3, \prec >$ sublattices or not.

Lattice Homomorphism

Let $< L, \otimes, \oplus >$ and $< S, \wedge, \vee >$ be two lattices. A mapping f: L $\rightarrow$ S is called lattice homomorphism from the lattice $< L, \otimes, \oplus >$ to $< S, \wedge, \vee >$ if for any (a,b) $\in$ L,

$$f(a \otimes b) = f(a) \wedge f(b) \text{ and } f(a \oplus b) = f(a) \vee f(b)$$

Lattice Isomorphism

A homomorphism f: L $\rightarrow$ S is said to be isomorphism if f is bijective. If there exists isomorphism between two lattices, then the lattices are called isomorphic.

Lattice Endomorphism

A homomorphism is said to be endomorphism if both lattices are same.

Lattice Automorphism

An isomorphism is said to be autoomorphism if both lattices are same.

Order-preserving

Let $< P, \preceq >$ and $< Q, \preceq' >$ be two POSETs. A mapping f: P→Q is said to be order-preserving relatie to the ordering $\prec$ in P and $\prec$' in Q iff for any a,b $\in$ P such that a $\prec$ b, f(a) $\prec$' f(b).

Note: If f is homomorphism, then f is order-preserving.

Order-isomorphic

Two POSETs $< P, \preceq >$ and $< Q, \preceq' >$ are called order-isomorphic if there exists a mapping f: P→Q which is bijective and if both f and f^1 are order-preserving.

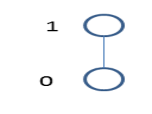
Note: It may happen that a mapping f: P→Q is bijective and order-preserving, but that f^1 is not order-preserving.

Direct product or Cartesian product

Let $< L, \otimes, \oplus >$ and $< S, \wedge, \vee >$ be two lattices. The algebraic system $< L \times S, *, + >$ in which the binary operations + and * on L×S are such that for any (a_1, b_1) and (a_2, b_2) in L×S
(a_1, b_1) * $(a_2, b_2) = (a_1 \otimes a_2, b_1 \wedge b_2)$
$(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$
is called the direct product of the lattices $< L, \otimes, \oplus >$ and $< S, \wedge, \vee >$.

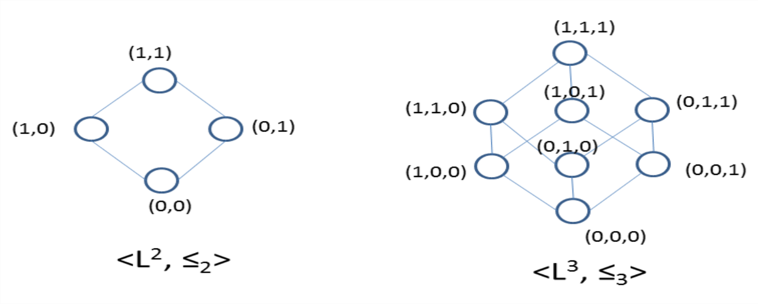
Note: $L^2 =$ L×L and $L^3 =$ L×L×L

Example: Let L $= \{0,1\}$ and the lattice $< L, \prec >$ is



Find the lattices $< L^2, \prec_2 >$ and $< L^3, \prec_3 >$.

Solution: Lattices $< L^2, \prec_2 >$ and $< L^3, \prec_3 >$ are drawn as following:-

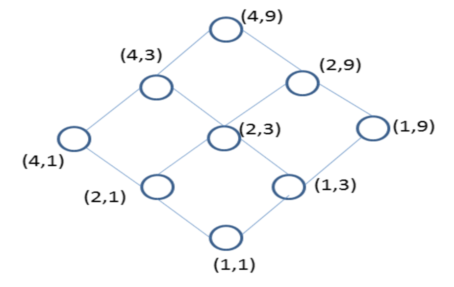


Note: The partial ordering relation $\preceq^n$ on L^n can be defined for any a,b $\in L^n$, where a $= (a_1, a_2, \ldots\ldots, a_n)$ and $(b_1, b_2, \ldots\ldots, b_n)$, as a$\prec_n$b $\Leftrightarrow a_i \preceq b_i$, $\forall$ i.

Where $\preceq$ means the relation of "less than or equal to" on {0,1}.

Example: Consider the chains of divisors of 4 and 9, that is $L_1 =$ {1,2,4} and $L_2 =$ {1,3,9}, and the partial order relation of "division" on L_1 and L_2. Draw the Hasse diagram for $L_1 \times L_2$.

Solution: Hasse diagram for this lattice can be drawn as following:-



Example: Let S be any set containing n elements and P(S) be its power set. Then the lattice $< P(S), \cap, \cup >$ or $< P(S), \subseteq >$ is isomorphic to the lattice $< L^n, \prec_n >$.

Exercise

1. Find all the sublattices of the lattice $< D(n), / >$, for n $= 12$.
2. Draw the diagram of a lattice which is the direct product of the five element lattice and a two element lattice.