Discrete Structures and Theory of Logic

Lecture-28

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Complete lattice

A lattice is called complete if each of its non-empty subsets has a least upper bound and a greatest lower bound.

Bounded lattice

Bounds: The least and greatest elements of a lattice, if they exists, are called the bounds of the lattice and are denoted by 0 and 1 respectively.

Definition: A lattice which has both least and greatest elements i.e. 0 and 1, is called a bounded lattice.

Note: The bounds 0 and 1 of a lattice satisfy the following identities:-
For any a $\in$ L, a$\wedge$0 $=$ 0, a$\wedge$1 $=$ a
a$\vee$0 $=$ a, a$\vee$1 $=$ 1.

Complemented lattice

In a bounded lattice, an element b $\in$ L is said to be complement of an element a $\in$ L
if a$\wedge$b $=$ 0 and a$\vee$b $=$ 1.
A lattice $< L, \wedge, \vee, 0, 1 >$ is said to be a complemented lattice if every element of L has at least one complement.

Types of lattice

Example: Is the lattice $< P(\{a, b, c\}), \subseteq >$ a complemented?

Solution: This lattice will be complemented if eery element has complement in this lattice.

P({a,b,c}) = {ϕ,{a},{b},{c},{a,b},{b,c},{a,c},{a,b,c}}

In this lattice, least element i.e. 0 = ϕ and greatest element i.e. 1 = {a,b,c}

The complement of ϕ will be {a,b,c}, because ϕ∧{a,b,c} = ϕ and ϕ∨{a,b,c} = {a,b,c}. Similarly, the complement of {a,b,c} will be ϕ.

Similarly, {a}' = {b,c}, {b}' = {a,c}, {c}' = {a,b}.

{a,b}' = {c}, {a,c}' = {b}, and {b,c}' = {a}

Clearly each element has a complement, therefore this lattice is complemented.

Example: Is the lattice $< D(30), / >$ a complemented?

Solution: Here, D(30) = {1,2,3,5,6,10,15,30}

Two elements a and b will be complement of each other iff a$\wedge$b = 0 and a$\vee$b = 1.

In this eample, 0(least element) = 1 and 1(greatest element) = 30.

Since 2$\wedge$15 = 1 and 2$\vee$15 = 30, therefore 2 and 15 are complement of each other.

Since 3$\wedge$10 = 1 and 3$\vee$10 = 30, therefore 3 and 10 are complement of each other.

Since 5$\wedge$6 = 1 and 5$\vee$6 = 30, therefore 5 and 6 are complement of each other.

Since 1$\wedge$30 = 1 and 1$\vee$30 = 30, therefore 1 and 30 are complement of each other.

Clearly each element has a complement, therefore this lattice is complemented.

Example: Is the lattice $< D(12), / >$ a complete?

Solution: Here, D(12) = $\{$1,2,3,4,6,12$\}$

Since this lattice is finite, therefore every subset of this set has a least upper bound and greatest lower bound. Clearly, consider the set $\{$2,3,4$\}$. The least upper bound of this set is 12 because each elements of this set divides 12 and no other elements in this. The greatest lower bound will be 1 because 1 diides to each elements of this set. Similarly, we can check for any subset of the given lattice. Therefore this lattice is complete.

Distributive lattice

A lattice $< L, \wedge, \vee >$ is called a distributive lattice if for any a,b,c $\in$ L,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

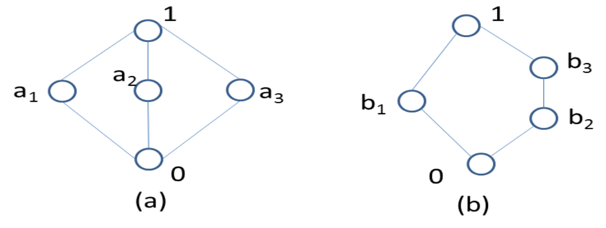
and $$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Modular lattice

A lattice $< L, \wedge, \vee >$ is called a modular lattice if for any a,b,c $\in$ L,

$$a \preceq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c.$$

Example: Check the following lattices to be modular or distributive.



Solution:

(a) For modular lattice:

Consider three elements a,b,c belongs into the lattice such that a$\preceq$c.

Let a $= a_1$, b $= a_2$, and c $= 1$.

Therefore, $a \vee (b \wedge c) = a_1 \vee (a_2 \wedge 1) = a_1 \vee a_2 = 1$

and $(a \vee b) \wedge c = (a_1 \vee a_2) \wedge 1 = 1 \wedge 1 = 1$

Therefore, a$\preceq$c $\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$ for a = a_1, b = a_2, and c = 1.

Similarly, we can show that a$\preceq$c $\Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$ for any a,b,c belongs into lattice such that a$\preceq$c. Therefore, this lattice is modular lattice.

For distributive lattice:

Consider three elements a,b,c belongs into the lattice.

Let a = a_1, b = a_2, and c = a_3.

Therefore, $a \wedge (b \vee c) = a_1 \wedge (a_2 \vee a_3) = a_1 \wedge 1 = a_1$

and $(a \wedge b) \vee (a \wedge c) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3) = 0 \vee 0 = 0$

Clearly, $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$ for a = a_1, b = a_2, and c = a_3.

Therefore this lattice is not distributive.