Discrete Structures and Theory of Logic

Lecture-19

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Group homomorphism

Let (G_1, o_1) and (G_2, o_2) be the two groups and f is function from G_1 to G_2. f is said to be group homomorphism from G_1 to G_2 if $\forall$ a,b$\in G_1$,

f(ao_1b)= f(a)o_2f(b)

Group Isomorphism

A group homomorphism f is said to be group isomorphism if f is bijective.

Group automorphism

An isomorphism is said to be automorphism if both groups are same i.e. $G_1 = G_2$.

Kernel of homomorphism

The kernel of homomorphism f of a group G_1 to G_2 is the set of all elements of G_1 mapped on the identity element of G_2 by f. That is, ker(f) = { a$\in$ G_1 ! f(a) = e_2, where e_2 is the identity element of G_2}

Example: Let $(G_1, o_1) = (\mathrm{Z},+)$ and $(G_2, o_2) = (\{1,\text{-}1\},\times)$ are two groups.

f: Z $\rightarrow$ $\{1,\text{-}1\}$ such that

$$f(x) = \begin{cases} 1 & , \quad \textit{if } n \textit{ is even} \\ -1 & , \quad \textit{if } n \textit{ is odd} \end{cases}$$

Find out f is a group homomprphism and isomorphism. And also find kernel of f.

Solution: Consider two integers a and b belong into Z. There will be four case for the sum a+b.

Case 1: when both a and b are even.

$f(a+b) = 1 = 1 \times 1 = f(a) \times f(b)$

Case 2: when both a and b are odd.

$f(a+b) = 1 = (-1) \times (-1) = f(a)\times f(b)$

Case 3: when a is even and b is odd.

$f(a+b) = -1 = 1 \times (-1) = f(a)\times f(b)$

Case 4: when a is odd and b is even.

$f(a+b) = -1 = (-1) \times 1 = f(a)\times f(b)$

Clearly, in all the four cases, $f(a+b) = f(a)\times f(b)$

Therefore, f is homomorphism.

Now, we have to check function is bijective or not.

Clearly, function not one-one. Because all even numbers mapped to 1 and all odd numbers mapped to -1. Therefore, this function is not bijective.

Hence the function is not isomorphism.

Now, ker(f) = The set of all even integers. Because all even integers are mapped on to identity element 1 of G_2.

Example: Let (G_1, o_1) = (R,+) and $(G_2, o_2) = (R^+, \times)$ are two groups.
f: $G_1 \rightarrow G_2$ defined by f(x) = 2^x.
Find out f is a group homomorphism and isomorphism.
Solution: Consider any two elements a and b of R.
Now, f(a+b) = $2^{(a+b)}$
$= 2^a \times 2^b$
= f(a)×f(b)
Clearly, f(a+b) = f(a)×f(b). Therefore f is homomorphism.
Clearly, for each distinct real number a, there will be distinct positive real number 2^a. Therefore the function is one-one.
Clearly, the function is onto because each element of R^+ is the image of some element of R.
Therefore the function f is bijective. Hence the function is isomorphism.

Theorem: Let (G_1, o_1) and (G_2, o_2) are two groups and let f be a homomorphism from G_1 to G_2. Then, prove the following:-

(1) $f(e_1) = e_2$, where e_1 is the identity of G_1 and e_2 is the identity of G_2.

(2) $f(a^{-1}) = (f(a))^{-1}, \forall a \in G_1$

(3) If H is a subgroup of G_1, then $f(H) = \{f(h) \; ! \; h \in H\}$ is a subgroup of G_2.

Proof: (1) $f(e_1) = f(e_1 \; o_1 \; e_1) = f(e_1) \; o_2 \; f(e_1)$

$$\Rightarrow f(e_1) = f(e_1) \; o_2 \; f(e_1) \quad \text{..........................(1)}$$

Since $f(e_1)$ is the element of G_2, therefore using identity property

$$e_2 \; o_2 \; f(e_1) = f(e_1) \quad \text{.......................(2)}$$

From (1) and (2), $f(e_1) o_2 f(e_1) = e_2 o_2 f(e_1)$

$$\Rightarrow f(e_1) = e_2 \text{ (using right cancellation law)}$$

It is proved.

Homomorphism and Isomorphism of groups

(2) $\mathrm{f}(e_1) = \mathrm{f}(a\ o_1\ a^{-1}) = f(a)\ o_2\ f(a^{-1})$

$\Rightarrow \mathrm{f}(e_1) = f(a)\ o_2\ f(a^{-1})$ ....(3)

Now, $f(a)\ o_2\ (f(a))^{-1} = e_2$ ....(4)

From part (1), we know that $\mathrm{f}(e_1) = e_2$, therefore from (3) and (4)

$f(a)\ o_2\ f(a^{-1}) = f(a)\ o_2\ (f(a))^{-1}$

$\Rightarrow \mathrm{f}(a^{-1}) = (f(a))^{-1}$ (using left cancellation law)

It is proved.

(3) Let h is subgroup of G_1.
a∈H, b∈H $\Rightarrow ao_1 b^{-1} \in$ H
Now, we have to show that f(H) is a subgroup of G_2.
Since a,b∈ H, therefore f(a), f(b) ∈ f(H).
f(a)∈ f(H), f(b)∈ f(H) ⇒ a∈H, b∈H
$\Rightarrow a\ o_1\ b^{-1} \in$ H
$\Rightarrow$ f($a\ o_1\ b^{-1}$) ∈ f(H)
$\Rightarrow f(a)\ o_2\ f(b^{-1}) \in$ f(H)
$\Rightarrow f(a)\ o_2\ (f(b))^{-1} \in$ f(H) (using part (2),f(a^{-1}) $= (f(a))^{-1}$)
Therefore, f(H) is a subgroup of G_2.
It is proved.

Factor or Quotient group

If H is a normal subgroup of group G, then the set of all left cosets of G forms a group with respect to the multiplication of left coset defined as $(aH)(bH) = (ab)H$, called the factor group of G by H. It is denoted by G/H.

$G/H = \{ gH ! g \in G \}$