Discrete Structures and Theory of Logic

Lecture-8

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Equivalence class

Let R is an equivalence relation defined on set S. For any $a \in S$, the equivalence class of a is the set of all the elements of set S which are related from a. It is denoted by [a]. Mathematically it is defined as

$a = \{ b \in S \,!\, aRb$ i.e. $(a,b) \in R\}$.

Equivalence class

Example: Let Z be the set of integers and let R be the relation called " Congruence modulo 3 ". Determine the equivalence classes generated by the elements of Z. That is, R = { (a,b) ! a,b $\in$ Z and (a-b) is divisible by 3 }.

Solution: The equivalence classes for this relation are the followings:-

$$[0] = \{........,-12,-9,-6,-3,0,3,6,9,12,..........\}$$
$$[1] = \{........,-11,-8,-5,-2,1,4,7,10,13,..........\}$$
$$[2] = \{........,-10,-7,-4,-1,2,5,8,11,14,..........\}$$

Matrix and Graph representation of the relations

Matrix representation

Let A $= \{a_1, a_2, ..., a_m\}$, B $= \{b_1, b_2, ..., b_n\}$, and R be a relation from A to B. Then the relation matrix corresponding to relation R will be m×n order matrix. Let this matrix is M. Then

$$
\begin{aligned}
m_{ij} &= 1 \qquad \text{if } (a_i, b_j) \in R \\
&= 0 \qquad \text{if } (a_i, b_j) \notin R
\end{aligned}
$$

where m_{ij} is the element of matrix in i^th row and in j^th column.

Example: Consider a relation R $= \{(a_1, b_1), (a_2, b_1), (a_3, b_2), (a_2, b_2)\}$, and A $= \{a_1, a_2, a_3\}$, B $= \{b_1, b_2\}$. Find the relation matrix for R.

Solution:

	b_1	b_2
a_1	1	0
a_2	1	1
a_3	0	1

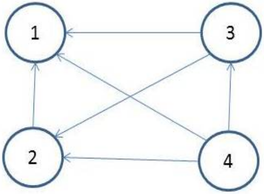
Matrix and Graph representation of the relations

Graph representation

Let R be a relation defined in a set A = $\{a_1, a_2, ..., a_m\}$. The nodes in the graph corresponds to the elements in set a. Therefore, the number of nodes in the graph will be equal to number of elements in the set A. This graph will be directed graph. If $(a_i, a_j) \in$ R, then the directed edge will be from a_i to a_j in the graph.

Example: Let A = {1,2,3,4} and R = {(a,b) ! a > b}. Draw the graph of R and also give its matrix.

Solution:



	1	2	3	4
1	0	0	0	0
2	1	0	0	0
3	1	1	0	0
4	1	1	1	0

Composition of binary relations

Let R be a relation from A to B and S be a relation from B to C. Then a relation RoS is called composition of relation R and S. It is defined as:-

RoS = $\{(a,c)$! $a \in A$ and $c \in C$ and $\exists$ $b \in B$ such that $(a,b) \in R$ and $(b,c) \in S$ $\}$

Example: Let $R = \{(1,2),(3,4),(2,2)\}$ and $S = \{(4,2),(2,5),(3,1),(1,3)\}$. Find RoS, SoR and Ro(SoR).

Solution:

RoS = $\{(1,5),(3,2),(2,5)\}$
SoR = $\{(4,2),(3,2),(1,4)\}$
Ro(SoR) = $\{(3,2)\}$

Composition of binary relations

Example: Let R and S be two relations on a set of positive integers I such that R $= \{(a, 2a) ! \ a \in I \}$ and S $= \{(a, 7a) ! \ a \in I \}$. Find RoS, RoR, RoRoR and RoSoR.

Solution:

RoS $= \{(a,14a) ! \ a \in I \}$

RoR $= \{(a,4a) ! \ a \in I \}$

RoRoR $= \{(a,8a) ! \ a \in I \}$

RoSoR $= \{(a,28a) ! \ a \in I \}$

Closure of a relation

Consider R be relation defined on a set S.

Reflexive closure

The reflexive closure of a relation R is the smallest reflexive relation that contains R as a subset. It is denoted by r(R). Mathematically, it is defined as :-

$r(R) = R \cup I_S$

Where I_S is the identity relation defined on set S.

Closure of a relation

Symmetric closure

The symmetric closure of a relation R is the smallest symmetric relation that contains R as a subset. It is denoted by s(R). Mathematically, it is defined as :-

s(R) = R$\cup R^{-1}$

Where R^{-1} is the inverse relation of R.

Transitive closure

The transitive closure of a relation R is the smallest transitive relation that contains R as a subset. It is denoted by t(R).

Closure of a relation

Example: Let S= {1,2,3,4}. Consider the following relation defined on the set S:-

R = { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}

Find reflexive, symmetric and transitive closure of R.

Solution: Reflexive closure r(R) = R $\cup$ I_S

= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}$\cup$ { (1,1),(2,2),(3,3),(4,4)}

= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2),(3,3),(4,4)}

Symmetric closure s(R) = R $\cup R^{-1}$

= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}$\cup${ (1,1),(2,2),(2,1),(1,3),(3,1

= { (1,1),(2,2),(1,2),(2,1),(1,3),(3,1),(4,2),(2,4)}

Transitive closure t(R) = R $\cup$ The set of ordered pairs to satisfy the transitive property

= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2)}$\cup$ { (3,3),(3,2)}

= { (1,1),(2,2),(1,2),(1,3),(3,1),(4,2),(3,3),(3,2)}