Design and Analysis of Algorithms

Lecture-21

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Binomial Heap

Binomial Tree

Definition

The binomial tree B_k is an ordered tree defined recursively.

1. The binomial tree B_0 consists of a single node.
2. The binomial tree B_k consists of two binomial trees B_{k-1} i.e.

$$B_k = B_{k-1} + B_{k-1}$$

They are linked together in the following way:

The root of one is the leftmost child of the root of the other.

Example: Some binomial trees are the following:-

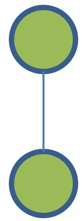
B0:

Binomial Tree

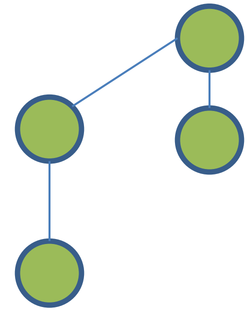
Example: Some binomial trees are the following:-

B_0:

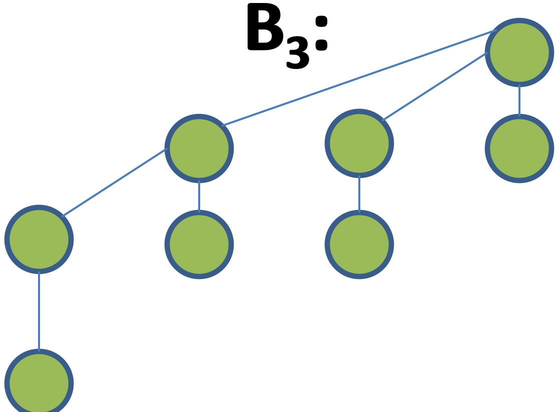
B_1:



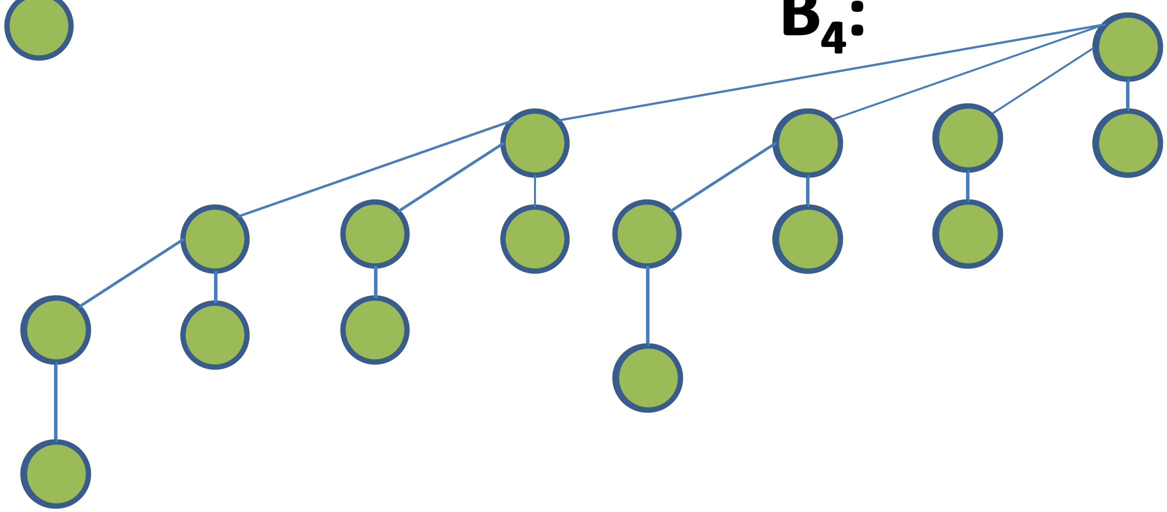
B_2:



B_3:



B_4:



Properties of binomial trees

For the binomial tree B_k,

1. There are 2^k nodes.
2. The height of the tree is k.
3. There are exactly $\binom{k}{i}$ nodes at depth i for i = 0, 1, ..., k.
4. The root has degree k, which is greater than that of any other node. Moreover if the children of the root are numbered from left to right by k−1, k−2, ..., 2, 1, 0, then child i is the root of a subtree B_i.

Properties of binomial trees(cont.)

Proof: The proof is by induction on k. For each property, the basis is the binomial tree B_0. Verifying that each property holds for B_0 is trivial. For the inductive step, we assume that all the properties holds for B_{k-1}.

1. Since binomial tree B_k consists of two copies of binomial tree B_{k-1}, therefore

   Number of nodes in $B_k = 2^{k-1} + 2^{k-1} = 2.2^{k-1} = 2^k$

2. Since in B_k, one B_{k-1} is child of the other B^{k-1}, therefore

   the height of B_k = the height of B_{k-1} +1

   $= (k-1) + 1$

   $= k$

Properties of binomial trees(cont.)

3.Let D(k, i) be the number of nodes at depth i of binomial tree B_k. Since B_k is composed of two copies of B_{k-1} linked together, a node at depth i in B_{k-1} appears in B_k once at depth i and i in B_k is the number of nodes at depth i in B_{k-1} plus the number of nodes at depth i−1 in B_{k-1}. Thus,

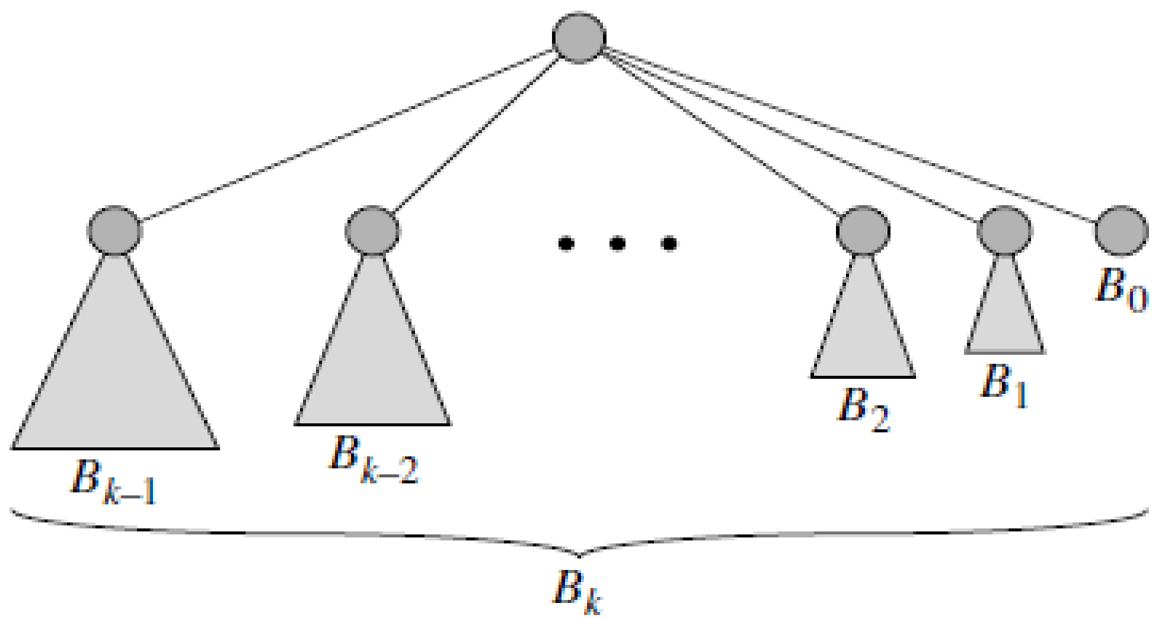
D(k,i) = D(k-1,i) + D(k-1, i-1)

$$= \binom{k-1}{i} + \binom{k-1}{i-1}$$

$$= \binom{k}{i}$$

Properties of binomial trees(cont.)

4. The only node with greater degree in B_k than in B_{k-1} is the root, which has one more child than in B_{k-1}. Since the root of B_{k-1} has degree k−1, therefore the root of B_k has degree k.



Now, by the inductive hypothesis, and as figure shows, from left to right, the children of the root of B_{k-1} are roots of B_{k-2}, B_{k-3}, ..., B_0. When B_{k-1} is linked to B_{k-1}, therefore, the children of the resulting root are roots of B_{k-1}, B_{k-2}, ..., B_0.

Binomial Tree

Lemma: The maximum degree of any node in an *n*-node binomial tree is lg *n*.

Proof: Let the maximum degree of any node is k.

According to property (4), the root node has a maximum degree . Therefore degree of root node is k. This imply that the binomial tree will be B_k.

According to property (1), the total number of nodes in binomial tree B_k is 2^k. Since the number of nodes in binomial tree is n, therefore

$$2^k = n \qquad \Rightarrow \qquad k = \lg n$$

It is proved.

Binomial Heap

Definition: A binomial heap H is a set of binomial trees that satisfies the following binomial heap properties.

1. Each binomial tree in H obeys the min-heap property: the key of a node is greater than or equal to the key of its parent. We say that each such tree is min-heap-ordered.

2. For any non-negative integer k, there is at most one binomial tree in H whose root has degree k.

Note: An n-node binomial heap H consists of at most $\lfloor \lg n \rfloor + 1$ binomial trees.

Example: Construct binomial heap for 27 nodes.

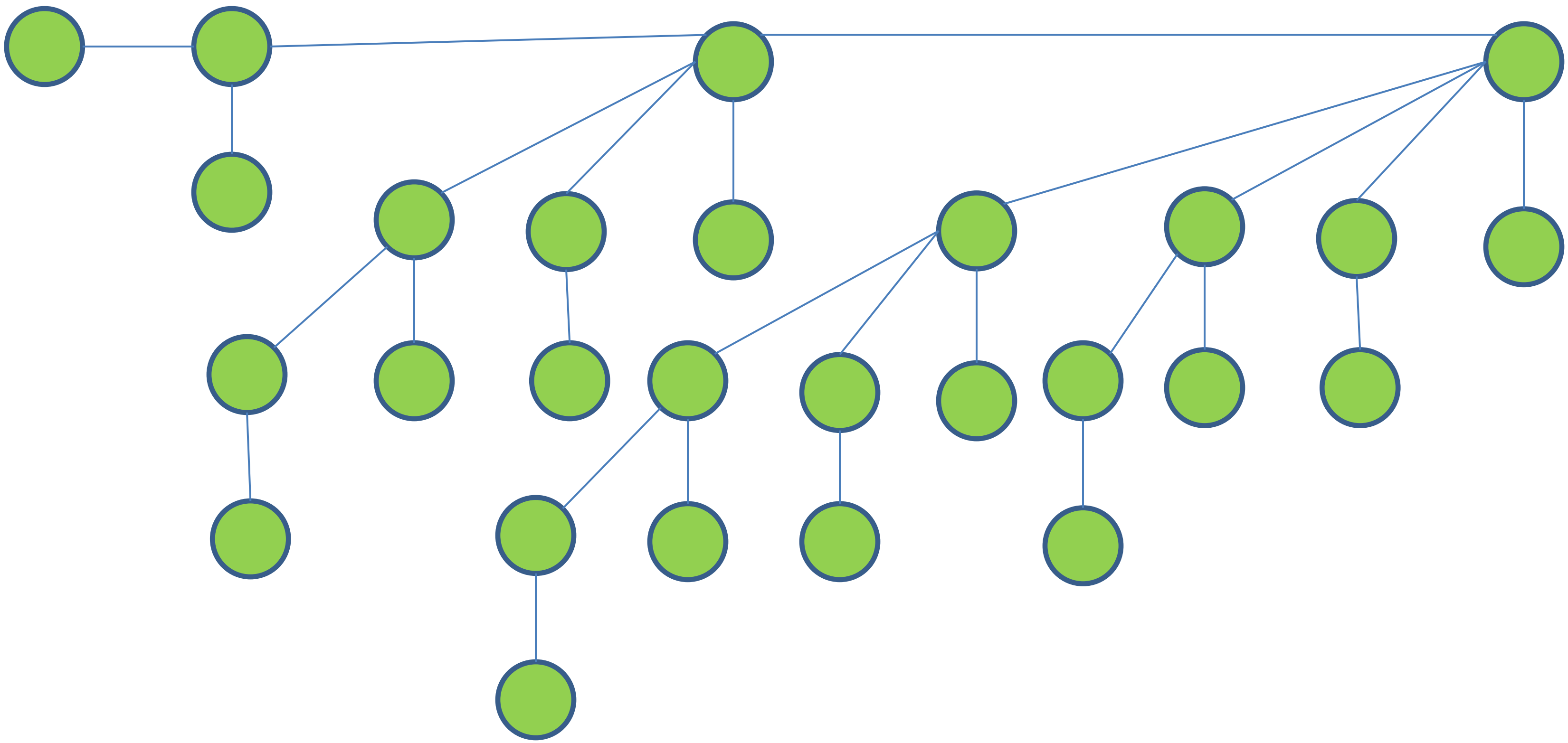
Solution: First we find binary number of 27. After it, we compare this number with $B_4B_3B_2B_1B_0$. If the corresponding binary number is 1, then we use the corresponding binomial tree in the Binomial heap.

Binary number of 27 = 11011

Therefore, binomial tree in binomial heap will be B_4, B_3, B_1, B_0.

Binomial heaps

Therefore, binomial heap for 27 nodes will be



Representation of binomial heaps

- Each binomial tree within a binomial heap is stored in the left-child, right-sibling representation.
- Each node x in binomial heap consists of following fields:-

Key[x] → value stored in the node x

p[x] → pointer representing parent of node x

child[x] → pointer representing left most child of node x

sibling[x] → pointer representing immediate right sibling of node x

degree[x] → the number of children of node *x*

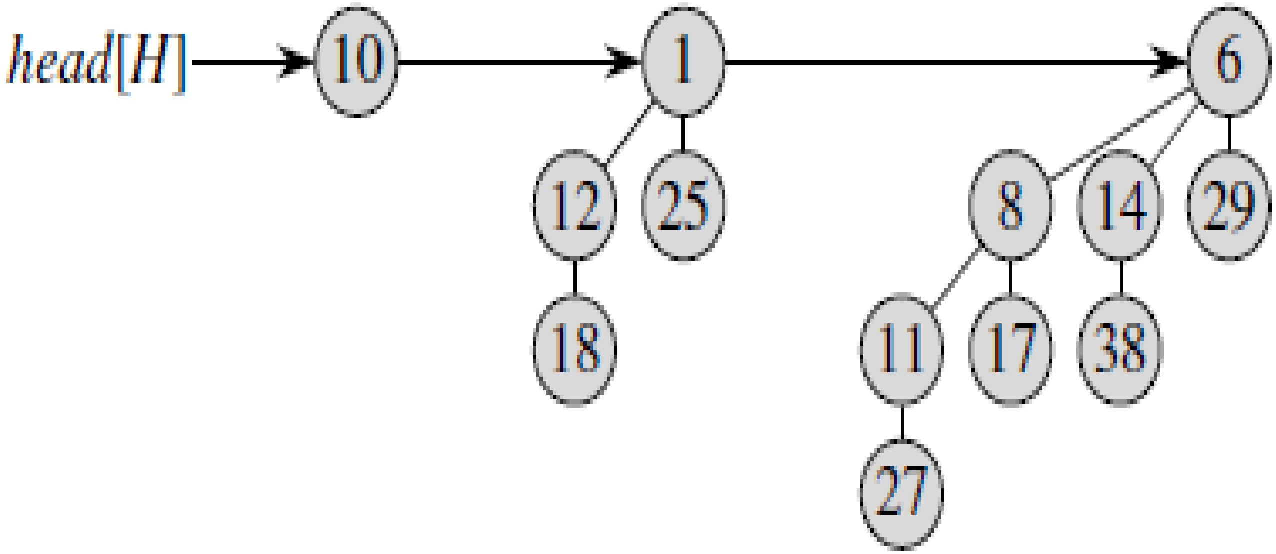
<table>
<tr><td colspan="2">p[x]</td></tr>
<tr><td colspan="2">Key[x]</td></tr>
<tr><td colspan="2">Degree[x]</td></tr>
<tr><td>Child[x]</td><td>Sibling[x]</td></tr>
</table>

Representation of binomial heaps

- The roots of the binomial trees within a binomial heap are organized in a linked list, which we refer to as the root list. The degrees of the roots strictly increase as we traverse the root list.
- The sibling field has a different meaning for roots than for non roots. If x is a root, then sibling[x] points to the next root in the root list.
- A given binomial heap H is accessed by the field head[H], which is simply a pointer to the first root in the root list of H. If binomial heap H has no elements, then head[H] = NIL.

Representation of binomial heaps

Example: Consider the following binomial heap:-



Find the representation of this binomial heap.

Representation of binomial heaps

Representation of binomial heap is