Discrete Structures and Theory of Logic

Lecture-20

Dharmendra Kumar

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Exercise

1. In the symmetric group S_3, find all those elements a and b such that
   (a) $(a * b)^2 \neq a^2 * b^2$
   (b) $a^2 =$ e
   (c) $a^3 =$ e
2. Show that in a group (G,o), if for any a,b $\in$ G, $(aob)^2 = a^2ob^2$, then (G,o) must be abelian.
3. Show that every cyclic group of order n is isomorphic to the group $(Z_n, +_n)$.
4. Find all the subgroups of following groups:-
   (a) $(Z_{12}, +_{12})$ (b) $(Z_5, +_5)$ (c) $(Z_7^*, \times_7)$ (d) $(Z_{11}^*, \times_{11})$

Exercise's solution

(1) Let $p_1, p_2, p_3, p_4, p_5, p_6$ are the elements of S_3.

$$p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$
$$p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ p_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

The composition table for S_3 with respect to multiplication operation is the following:-

*	p_1	p_2	p_3	p_4	p_5	p_6
p_1	p_1	p_2	p_3	p_4	p_5	p_6
p_2	p_2	p_1	p_5	p_6	p_3	p_4
p_3	p_3	p_6	p_1	p_5	p_4	p_2
p_4	$p_1 4$	p_5	p_6	p_1	p_2	p_3
p_5	p_5	p_4	p_2	p_3	p_6	p_1
p_6	p_6	p_3	p_4	p_2	p_1	p_5

Exercise's solution

(1-a) In this part, we have to find elements a and b of S_3 which satisfy equation (1).
$(a*b)^2 \neq a^2 * b^2$ .... (1)
Consider, a $= p_2$ and b $= p_3$.
Now, LHS $= (a*b)^2 = (p_2 * p_3)^2 = p_5^2 = p_6$
RHS $= a^2 * b^2 = p_2^2 * p_3^2 = p_1 * p_1 = p_1$
Clearly, $(a*b)^2 \neq a^2 * b^2$ for a $= p_2$ and b $= p_3$.
Similarly, Consider, a $= p_2$ and b $= p_4$.
Now, LHS $= (a*b)^2 = (p_2 * p_4)^2 = p_6^2 = p_5$
RHS $= a^2 * b^2 = p_2^2 * p_4^2 = p_1 * p_1 = p_1$
Clearly, $(a*b)^2 \neq a^2 * b^2$ for a $= p_2$ and b $= p_4$.

Exercise's solution

Similarly, following pairs of a and b are also satisfied.

a $= p_2$ and b $= p_5$

a $= p_2$ and b $= p_6$

a $= p_3$ and b $= p_4$

a $= p_3$ and b $= p_5$

a $= p_3$ and b $= p_6$

a $= p_4$ and b $= p_5$

a $= p_4$ and b $= p_6$

(1-b) In this part, we have to find element a of S_3 which satisfy equation (2).

$a^2 = e$ ....(2)

Here, the identity element is $e = p_1$.

Consider, $a = p_1$. So, $a^2 = p_1^2 = p_1 = e$

Therefore, $a = p_1$ satisfy the equation (2).

Similarly, $a = p_2, p_3, p_4$ also satisfy the equation (2).

(1-c) In this part, we have to find element a of S_3 which satisfy equation (3).

$a^3 = e$ ....(3)

Here, the identity element is $e = p_1$.

Consider, $a = p_1$. So, $a^3 = p_1^3 = p_1 = e$

Therefore, $a = p_1$ satisfy the equation (3).

Similarly, $a = p_5, p_6$ also satisfy the equation (3).

Exercise's solution

(2) Given $(aob)^2 = a^2ob^2$, for a,b $\in$ G.

It imply that (aob)o(aob) = (aoa)o(bob)

$\Rightarrow$ ao(bo(aob)) = ao(ao(bob)) (using associative law)

$\Rightarrow$(bo(aob)) = (ao(bob)) (using left cancellation law)

$\Rightarrow$(boa)ob = (aob)ob (using associative law)

$\Rightarrow$ (boa) = (aob)(using right cancellation law)

Therefore, the group (G,o) is an abelian group.

Exercise's solution

(3) Let cyclic group(G,o) of order n be generated by an element a $\in$ G. So the elements of G are $a, a^2, a^3, ..., a^n = e$.
Define g : $Z_n \to G$ such that g([1]) = a. [1] is the generator of $(Z_n, +_n)$. Then g([j]) = a^j, for all j = 0,1,2,3,..............,n-1.
Clearly this function is bijective because each element j is mapped to unique element a^j.
Now, g([j]+[k]) = $a^{[j]+[k]}$

$$= a^{[j]} o a^{[k]}$$

= g[j] o g[k]

Clearly, g([j]+[k]) = g[j] o g[k]
Therefore, g is homoomorphism. Since g is bijective and homomorphism, so g is isomorphism.
Therefore, every cyclic group of order n is isomorphic to the group $(Z_n,\ +_n)$.

Exercise's solution

(4) In this question, we have to find all the subgroups of given groups. In these questions, $Z_n = \{$0,1,2,3,4,....................,n-1$\}$ and $+_n$ and $\times_n$ are addition and multiplication modulo n operations.
According to Lagrange's theorem, order of each subgroup is the divisor of the order of the group. We will use this theorem to find all the subgroups.
(4-a) Here group is $(Z_{12}, +_{12})$.
Therefore $Z_{12} = \{$0,1,2,3,4,5,6,7,8,9,10,11$\}$. Clearly, the order of this group is 12. Using Lagrange's theorem, the number of subgroups of Z_{12} = number of positive divisors of 12.
The positive divisors of 12 are 1,2,3,4,6,12. Since the number of divisors is 6, therefore number of subgroups will be 6 with orders 1,2,3,4,6,12.

Exercise's solution

These subgroups are the following:-

Now, $H_1 = \{0\}$, this is a subgroup with order 1.

$H_2 = \{0,6\}$, this is a subgroup with order 2.

$H_3 = \{0,4,8\}$, this is a subgroup with order 3.

$H_4 = \{0,3,6,9\}$, this is a subgroup with order 4.

$H_5 = \{0,2,4,6,8,10\}$, this is a subgroup with order 6.

$H_6 = \{0,1,2,3,4,5,6,7,8,9,10,11\}$, this is a subgroup with order 12.

Exercise's solution

(4-b) Here group is $(Z_5, +_5)$. Therefore $Z_5 = \{0,1,2,3,4\}$. Clearly, the order of this group is 5. The positive divisors of 5 are 1,5. Since the number of divisors is 2, therefore number of subgroups will be 2 with orders 1,5. These subgroups are the following:-

$H_1 = \{0\}$, this is a subgroup with order 1.

$H_2 = \{0,1,2,3,4\}$, this is a subgroup with order 5.

(4-c) Here group is $(Z_7^*, \times_7)$. Therefore $Z_7^* = \{1,2,3,4,5,6\}$. Clearly, the order of this group is 6. The positive divisors of 6 are 1,2,3,6. Since the number of divisors is 4, therefore number of subgroups will be 4 with orders 1,2,3,6. These subgroups are the following:-

$H_1 = \{1\}$, this is a subgroup with order 1.

$H_2 = \{1,6\}$, this is a subgroup with order 2.

$H_3 = \{1,2,4\}$, this is a subgroup with order 3.

$H_4 = \{1,2,3,4,5,6\}$, this is a subgroup with order 6.

Exercise's solution

(4-d) Here group is $(Z_{11}^{*}, \times_{11})$. Therefore $Z_{11}^{*} = \{1,2,3,4,5,6,7,8,9,10\}$. Clearly, the order of this group is 10. The positive divisors of 10 are 1,2,5,10. Since the number of divisors is 4, therefore number of subgroups will be 4 with orders 1,2,5,10. These subgroups are the following:-

$H_1 = \{1\}$, this is a subgroup with order 1.
$H_2 = \{1,10\}$, this is a subgroup with order 2.
$H_3 = \{1,3,4,5,9\}$, this is a subgroup with order 5.
$H_4 = \{1,2,3,4,5,6,7,8,9,10\}$, this is a subgroup with order 10.