Design and Analysis of Algorithms

Lecture-7

Dharmendra Kumar (Associate Professor)
Department of Computer Science and Engineering
United College of Engineering and Research,
Prayagraj

Master Theorem

Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n);$$

Where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ has the following asymptotic bounds:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then
   $$T(n) = \theta(n^{\log_b a}).$$
2. If $f(n) = \theta(n^{\log_b a})$, then $T(n) = \theta(n^{\log_b a} \lg n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \theta(f(n))$.

Master Theorem Method

Example: Solve the following recurrence relations using master theorem method

(a) $T(n) = 9T(n/3) + n$

(b) $T(n) = T(2n/3) + 1$

(c) $T(n) = 3T(n/4) + n \log n$

Solution:

(a) Consider $T(n) = 9T(n/3) + n$

In this recurrence relation, $a = 9$, $b = 3$ and $f(n) = n$.

Therefore, $n^{\log_b a} = n^{\log_3 9} = n^2$

Clearly, $n^{\log_b a} > f(n)$, therefore case1 can be applied.

Now determine ϵ such that $f(n) = O(n^{2-\epsilon})$. Here $\epsilon = 1$.

Therefore case 1 will be applied.

Hence solution will be $T(n) = \theta(n^2)$.

Master Theorem Method

Solution:

(b) Consider $T(n) = T(2n/3) + 1$

In this recurrence relation, $a = 1$, $b = 3/2$ and $f(n) = 1$.

Therefore, $n^{\log_b a} = n^{\log_{3/2} 1} = 0$

Clearly, $f(n) = \theta(n^{\log_b a})$, therefore case 2 will be applied.

Hence solution will be $T(n) = \theta(\log n)$.

Master Theorem Method

Solution:

(c) Consider $T(n) = 3T(n/4) + n \log n$

In this recurrence relation, $a = 3$, $b = 4$ and $f(n) = n \log n$.

Therefore, $n^{\log_b a} = n^{\log_4 3} = n^{0.793}$

Clearly, $n^{\log_b a} < f(n)$, therefore case 3 can be applied.

Now determine ϵ such that $f(n) = \Omega(n^{0.793+\epsilon})$. Here $\epsilon = 0.207$.

Now, $af(n/b) \leq cf(n)$ imply that $3f(n/4) \leq cf(n)$

$$\Rightarrow 3(n/4)\log(n/4) \leq c\, n \log n$$

$$\Rightarrow (3/4)\log(n/4) \leq c \log n$$

Clearly above inequality is satisfied for $c = 3/4$. Therefore case 3 will be applied.

Hence solution will be $T(n) = \theta(n \log n)$.

Master theorem method

Example: Solve the following recurrence relation

$$T(n) = 2T(n/2) + n \log n$$

Solution: Here, a = 2 , b=2 and f(n) = n log n.

$$n^{\log_b a} = n^{\log_2 2} = n$$

If we compare $n^{\log_b a}$ and f(n), we get f(n) is greater than $n^{\log_b a}$. Therefore, case 3 may be applied.

Now we have to determine $\epsilon > 0$ which satisfy $f(n) = \Omega(n^{\log_b a+\epsilon})$, i.e. $n \log n = \Omega(n^{1+\epsilon})$. Clearly there does not exist any ϵ which satisfy this condition. Therefore case 3 can not be applied. Other two cases are also not satisfied. Therefore Master theorem can not be applied in this recurrence relation.

Generalized Master theorem

Theorem: If $f(n) = \theta(n^{\log_b a} \lg^k n)$, where $k \geq 0$, then the solution of recurrence will be $T(n) = \theta(n^{\log_b a} \lg^{k+1} n)$.

Now, consider the previous example:-

$$T(n) = 2T(n/2) + n \log n$$

Solve it using aboe theorem,

Here a =2, b= 2, and k = 1. Therefore, the solution of this recurrence will be

$$T(n) = \theta(n^{\log_2 2} \lg^{1+1} n)$$

$$= \theta(n \lg^2 n)$$

Hence, $T(n) = \theta(n \lg^2 n)$

Recurrence relation

Exercise

1. Use the master method to give tight asymptotic bounds for the following recurrences:-

(a) $T(n) = 8T(n/2) + \theta(n^2)$

(b) $T(n) = 7T(n/2) + \theta(n^2)$

(c) $T(n) = 2T(n/4) + 1$

(d) $T(n) = 2T(n/4) + \sqrt{n}$

2. Can the master method be applied to the recurrence $4T(n/2) + n^2 \log n$? Why or why not? Give an asymptotic upper bound for this recurrence.