Discrete Structures and Theory of Logic

Lecture-36

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Theory of inference for statement calculus

Let A and B be two statement formulas. We say that "B logically follows from A" or "B is a alid conclusion of the premise A" iff $A \to B$ is a tautology. That is, $A \Rightarrow B$.
A conclusion C follows from a set of premises $\{H_1, H_2, ..., H_m\}$ iff

$$H_1 \wedge H_2 \wedge .... \wedge H_m \Rightarrow C$$

Exercise

1. Determine whether the conclusion C follows logically from the premises H_1 and H_2.

(a) $H_1 : P \rightarrow Q$	$H_2 : P$	$C : Q$
(b) $H_1 : P \rightarrow Q$	$H_2 : \neg P$	$C : Q$
(c) $H_1 : P \rightarrow Q$	$H_2 : \neg(P \wedge Q)$	$C : \neg P$
(d) $H_1 : \neg P$	$H_2 : P \leftrightarrow Q$	$C : \neg(P \wedge Q)$
(e) $H_1 : P \rightarrow Q$	$H_2 : Q$	$C : P$

Exercise

2. Show that the conclusion C follows from the premises $H_1, H_2, ...$ in the following cases:-

(a) $H_1 : P \to Q$ $\quad C : P \to (P \wedge Q)$

(b) $H_1 : \neg P \vee Q$ $\quad H_2 : \neg(Q \wedge \neg R)$ $\quad H_3 : \neg R$
$C : \neg P$

(c) $H_1 : \neg P$ $\quad H_2 : P \vee Q$ $\quad C : Q$

(d) $H_1 : \neg Q$ $\quad H_2 : P \to Q$ $\quad C : \neg P$

(e) $H_1 : P \to Q$ $\quad H_2 : Q \to R$ $\quad C : P \to R$

(f) $H_1 : R$ $\quad H_2 : P \vee \neg P$ $\quad C : R$

Exercise

3. Determine whether the conclusion C is valid in the following, when $H_1, H_2, ...$ are the premises.

(a) $H_1 : P \to Q$ $H_2 : \neg Q$ $C : P$

(b) $H_1 : P \vee Q$ $H_2 : P \to R$ $H_3 : Q \to R$ $C : R$

(c) $H_1 : P \to (Q \to R)$ $H_2 : P \wedge Q$ $C : R$

(d) $H_1 : P \to (Q \to R)$ $H_2 : R$ $C : P$

(e) $H_1 : \neg P$ $H_2 : P \vee Q$ $C : P \wedge Q$

4. Without constructing a truth table, show that $A \wedge E$ is not a valid consequence of

$$A \leftrightarrow B, B \leftrightarrow (C \wedge D), C \leftrightarrow (A \vee E) \textit{ and } A \vee E$$

Also show that $A \vee C$ is not a valid consequence of

$$A \leftrightarrow (B \to C), B \leftrightarrow (\neg A \vee \neg C), C \leftrightarrow (A \vee \neg B) \textit{ and } B$$

Implication rules

1. **Simplification**
   $P \wedge Q \Rightarrow P,\ P \wedge Q \Rightarrow Q$
2. **Addition**
   $P \Rightarrow P \vee Q,\ Q \Rightarrow P \vee Q$
3. **Disjunctive Syllogism**
   $\neg P, P \vee Q \Rightarrow Q$
4. **Modus Ponens**
   $P, P \rightarrow Q \Rightarrow Q$
5. **Modus Tollens**
   $\neg Q, P \rightarrow Q \Rightarrow \neg P$
6. **Hypothetical Syllogism**
   $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$

7. **Dilemma**
   $P \vee Q, P \to R, Q \to R \Rightarrow R$
8. **Conjunction**
   $P, Q \Rightarrow P \wedge Q$

Rules of Inference

Rule P: A premise may be introduced at any point in the derivation.
Rule T: A formula S may be introduced in a derivation if S is tautology implied by any one or more of the preceding formulas in the derivation.

Example: Demonstrate that R is a valid inference from the premises $P \to Q$, $Q \to R$ and P.

Solution:

(1) $P \to Q$	By rule P
(2) $Q \to R$	By rule P
(3) $P \to R$	By rule T, (1), (2) and hypothetical syllogism
(4) P	By rule P
(5) R	By rule T, (3), (4) and modus ponens

Example: Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \to \neg H$, $\neg H \to (A \wedge \neg B)$ and $(A \wedge \neg B) \to (R \vee S)$.

Solution:

(1) $(C \vee D) \to \neg H$ By rule P

(2) $\neg H \to (A \wedge \neg B)$ By rule P

(3) $C \vee D \to (A \wedge \neg B)$ By rule T, (1), (2) and hypothetical syllogism

(4) $(A \wedge \neg B) \to (R \vee S)$ By rule P

(5) $C \vee D \to (R \wedge S)$ By rule T, (3), (4) and hypothetical syllogism

(6) $C \vee D$ By rule P

(7) $R \vee S$ By rule T, (5), (6) and modus ponens

Example: Show that $R \vee S$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

Solution:

(1) $P \vee Q$ By rule P

(2) $\neg P \rightarrow Q$ By rule T, (1) and $(P \rightarrow Q \Leftrightarrow \neg P \vee Q)$

(3) $Q \rightarrow S$ By rule P

(4) $\neg P \rightarrow S$ By rule T, (2), (3) and hypothetical syllogism

(5) $\neg S \rightarrow P$ By rule T, (4) and $(\neg P \rightarrow S \Leftrightarrow \neg S \rightarrow P)$

(6) $P \rightarrow R$ By rule P

(7) $\neg S \rightarrow R$ By rule T, (5), (6) and hypothetical syllogism

(8) $R \vee S$ By rule T, (7)and $(P \rightarrow Q \Leftrightarrow \neg P \vee Q)$