# Discrete Structures and Theory of Logic

## Lecture-24

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## Least and Greatest element

An element a $\in$ S is said to be the least element of the POSET $< S, \preceq >$, if a$\preceq$b, $\forall$ b $\in$ S.
An element b $\in$ S is said to be the greatest element of the POSET $< S, \preceq >$, if a$\preceq$b, $\forall$ a $\in$ S.

## Minimal and Maximal element

An element a $\in$ S is said to be the minimal element of the POSET $< S, \preceq >$, if there is no element b $\in$ S such that b$\preceq$a.
An element b $\in$ S is said to be the maximal element of the POSET $< S, \preceq >$, if there is no element a $\in$ S such that b$\preceq$a.

## Upper bound and Lower bound

Let $< S, \preceq >$ be a POSET and let A$\subseteq$S.
An element u$\in$S is said to be upper bound of set A if a$\preceq$u, $\forall$ a$\in$ A.
An element l$\in$S is said to be lower bound of set A if l$\preceq$a, $\forall$ a$\in$ A.

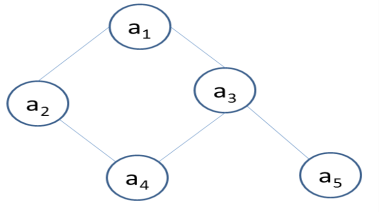
## Least upper and Greatest lower bound

An upper bound u of the set A is said to be least upper bound of set A if u$\preceq$u', $\forall$ upper bound u' of A.
An upper bound l of the set A is said to be greatest lower bound of set A if l'$\preceq$l, $\forall$ lower bound l' of A.

**Well ordered set**

A POSET is said to be well-ordered set if for every non-empty subset of it has a least element.

**Example:** Consider the following Hasse diagram:-




Find the least and greatest element of this POSET if they exist. Also find minimal and maximal elements. Find the upper and lower bounds of $\{a_2, a_3, a_4\}$, $\{a_3, a_4, a_5\}$, $\{a_1, a_2, a_3\}$. Also indicate the least upper bound and greatest lower bound of these subsets if they exists.

**Solution:**

Least element = does not exists because no element in this POSET is related to all the elements.

Greatest element = $a_1$ because all the elements of the POSET is related to $a_1$

Minimal elements = $a_4, a_5$

Maximal elements = $a_1$

**Consider the set** $\{a_2, a_3, a_4\}$**.**

lower bounds = $a_4$, because $a_4$ is related to all the elements of the set $\{a_2, a_3, a_4\}$

upper bounds = $a_1$, because all the elements of the set $\{a_2, a_3, a_4\}$ is related to $a_1$.

greatest lower bound = $a_4$

least upper bound = $a_1$

# POSET

**Consider the set $\{a_3, a_4, a_5\}$.**

lower bounds = does not exists because no element is related to all the elements of the set $\{a_3, a_4, a_4\}$

upper bounds = $a_1, a_3$, because all the elements of the set $\{a_3, a_4, a_5\}$ is related to $a_1, a_3$.

greatest lower bound = does not exists

least upper bound = $a_3$

**Consider the set $\{a_1, a_2, a_3\}$.**

lower bounds = $a_4$, because $a_4$ is related to all the elements of the set $\{a_1, a_2, a_3\}$

upper bounds = $a_1$, because all the elements of the set $\{a_1, a_2, a_3\}$ is related to $a_1$.

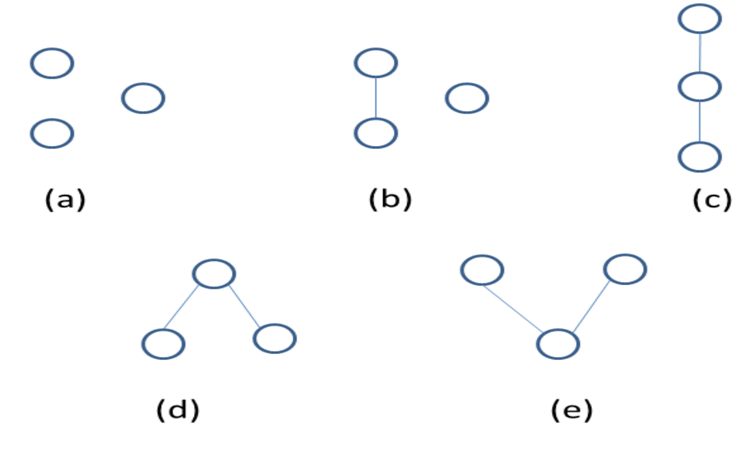
greatest lower bound = $a_4$

least upper bound = $a_1$

# POSET

**Example:** Show that there are only five distinct Hasse diagrams for partial ordered sets that contain three elements.

**Solution:** All the distinct Hasse diagrams corresponding to three elements are the followings:-



## Exercise

1. Draw the Hasse diagram of the following sets under the partial ordering relation "divides" and indicate those which are totally ordered.
   (a) $\{2,6,24\}$ (b) $\{3,5,15\}$
   (c) $\{1,2,3,6,12\}$ (d) $\{2,4,8,16\}$
   (e) $\{3,9,27,54\}$
2. Give an example of a set A such that $< P(A), \subseteq >$ is a totally ordered set.
3. Give a relation which is both partial ordering and an equivalence on a set.
4. Draw all the Hasse diagrams corresponding to four elements.

# Lattice

A POSET $< L, \subseteq >$ is said to be lattice if for every pair of elements a,b $\in$ L, its greatest lower bound and least upper bound exists.
Greatest lower bound is denoted by a$\wedge$b and least upper bound is denoted by a$\vee$b.
**Example:** Is POSET $< P(A), \subseteq >$ a lattice, where A $=$ $\{$1,2,3$\}$?
**Solution:** In this POSET, greatest lower bound of two elements of P(A) is equivalent to intersection of those elements and least upper bound of two elements of P(A) is equivalent to union of those elements.
Clearly, intersection and union of any two elements of P(A) always exists in P(A). Therefore this POSET is lattice.

# POSET

**Example:** Let $I_+$ be the set of all positive integers and D denote the relation of division in $I_+$ such that for any a,b $\in I_+$, aDb iff a divides b. Is ($I_+$,D) a lattice?

**Solution:** In this example, first we have to check this set is POSET or Not. After this, we have to check for lattice.
Clearly, this set is POSET because it satisfies all the three properties.
Clearly, for each pair of integers, its least upper bound and greatest lower bound exists because the operation is division and set is set of all positive integers. For example, consider two elements 4 and 6. Its lub = 12 and glb = 2. Both 2 and 12 belongs into $I_+$. Similarly for elements 3 and 4, lub = 12 and glb = 1. Here 1 and 12 both belong into $I_+$. Therefore this set is lattice.

**Example:** Let n be a positive integer and $S_n$ be the set of all positive divisors of n. And D is a division relation. Is ($S_6$,D), ($S_8$,D), ($S_{24}$,D), and($S_{30}$,D) lattices?

**Solution:** All these are lattices.