Chapter 6

Renormalization group resummation of tranverse distributions

We prove the all-order exponentiation of soft logarithmic corrections at small transverse momentum to the distribution of Drell-Yan process. We apply the renormalization group approach developed in the context of integrated cross sections. We show that all large logs in the soft limit can be expressed in terms of a single dimensional variable, and we use the renormalization group to resum them. The resummed result that we obtain is, beyond the next-to-leading log accuracy, more general and less predictive than those previously released. The origin of this could be due to factorization properties of the cross section. The understanding of this point is a work in progress.

6.1 Drell-Yan distribution at small transverse momentum

We consider the Drell-Yan process

$$H_1(P_1) + H_2(P_2) \to \gamma^*(Q) + X(K), \tag{1}$$

and, in particular, the differential cross section $\frac{d\sigma}{dq_\perp^2 dY}(Q^2, q_\perp^2, x_1, x_2)$, where $q_\perp$ is the transverse momentum with respect to colliding axis of the hadrons H_1 and H_2, Q^2 is the virtuality of photon and x_1, x_2 are useful dimensionless variables, that, in terms of the hadronic center-of-mass squared energy $S = (P_1 + P_2)^2$ and the photon center-of-mass rapidity Y, are given by

$$x_1 = \sqrt{\frac{Q^2 + q_\perp^2}{S}} e^{Y}; \qquad x_2 = \sqrt{\frac{Q^2 + q_\perp^2}{S}} e^{-Y}. \tag{2}$$

The relation between these two variables and the fraction of energy carried by the virtual photon is

$$\frac{x_1 + x_2}{2} = \frac{E_{\gamma^*}}{\sqrt{S}}. \tag{3}$$

According to standard factorization of perturbative QCD, the expression for the differential cross section is

$$\begin{aligned} \frac{d\sigma}{dq_\perp^2 dY}(Q^2, q_\perp^2, x_1, x_2) &= \int_{z_1^{min}}^1 dz_1 \int_{z_2^{min}}^1 dz_2 f_1(z_1,\mu^2) f_2(z_2,\mu^2) \\ &\times \frac{d\hat{\sigma}}{dq_\perp^2 dy}(Q^2, q_\perp^2, s, y, \mu^2, \alpha_s(\mu^2)), \end{aligned} \tag{4}$$

where $f_1(z_1,\mu^2), f_2(z_2,\mu^2)$ are the parton distribution functions of the colliding quark and anti-quark in the hadrons H_1 and H_2 respectively. The arbitrary scale μ^2 is the factorization scale, which, for simplicity, is chosen to be equal to the renormalization scale. The condition that the invariant mass of the emitted particles K^2 cannot be negative, imposes that $(z_1 - x_1)(z_2 - x_2) \geq \frac{q_\perp^2}{S}$ and, taking the small $q_\perp^2$ limit, we obtain that

$$z_1^{min} = x_1, \qquad z_2^{min} = x_2. \tag{5}$$

The partonic center-of-mass squared energy s and rapidity y are related to the hadronic ones by a scaling and a boost along the collision axis with respect to the longitudinal momentum fraction z_1, z_2 of the incoming partons:

$$s = z_1 z_2 S; \qquad y = Y - \frac{1}{2}\ln\frac{z_1}{z_2}. \tag{6}$$

We define analogous variables to that of Eqs.(2) at the partonic level

$$\xi_1 \equiv \frac{x_1}{z_1} = \sqrt{\frac{Q^2 + q_\perp^2}{s}} e^{y}; \qquad \xi_2 \equiv \frac{x_2}{z_2} = \sqrt{\frac{Q^2 + q_\perp^2}{s}} e^{-y}, \tag{7}$$

with inverse relations

$$s = \frac{Q^2 + q_\perp^2}{(x_1/z_1)(x_2/z_2)}; \qquad y = \frac{1}{2}\ln\frac{x_1/z_1}{x_2/z_2}. \tag{8}$$

Now, thanks to these equations, we can define a dimensionless differential cross section and coefficient function

$$W(q_\perp^2/Q^2, x_1, x_2) = \frac{Q^4}{x_1 x_2}\frac{d\sigma}{dq_\perp^2 dY}(Q^2, q_\perp^2, x_1, x_2), \tag{9}$$

$$\hat{W}\left(\frac{Q^2}{\mu^2}, \frac{q_\perp^2}{Q^2}, \frac{x_1}{z_1}, \frac{x_2}{z_2}, \alpha_s(\mu^2)\right) = \frac{Q^4}{(x_1/z_1)(x_2/z_2)}\frac{d\hat{\sigma}}{dq_\perp^2 dy}(Q^2, q_\perp^2, s, y, \mu^2, \alpha_s(\mu^2)), \tag{10}$$

in such a way that Eq.(4), together with the conditions Eqs.(5), takes the useful form of a convolution product

$$\begin{aligned} W(q_\perp^2/Q^2, x_1, x_2) &= \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} f_1(z_1,\mu^2) f_2(z_2,\mu^2) \\ &\times \hat{W}(Q^2/\mu^2, q_\perp^2/Q^2, x_1/z_1, x_2/z_2, \alpha_s(\mu^2)), \end{aligned} \tag{11}$$

which is valid only for small $q_\perp^2$.

6.2 The role of standard factorization

It is known that this expression (or equivalently Eq.(4)) is originated by the factorization of collinear divergences in the impact parameter ($\vec{b}$) which is conjugate upon Fourier transformation to the transverse momentum ($\vec{q}_\perp$):

$$W(Q^2b^2, x_1, x_2) = \int d^2q_\perp e^{i\vec{q}_\perp\vec{b}} W(q_\perp^2/Q^2, x_1, x_2). \tag{12}$$

In $d = 4 - 2\epsilon$ dimensions this factorization has the form

$$\hat{W}(Q^2/\mu^2, Q^2b^2, x_1, x_2, \alpha_s(\mu^2)) = \\ = \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} Z(z_1, \alpha_s(\mu^2), \epsilon) Z(z_2, \alpha_s(\mu^2), \epsilon) \hat{W}^{(0)}(Q^2, b^2, \frac{x_1}{z_1}, \frac{x_2}{z_2}, \alpha_0, \epsilon). \tag{13}$$

Note that the universal function Z that extracts the collinear divergences from the bare coefficient function doesn't depend on the Fourier conjugate (b) of the transverse momentum Ref.[11]. In Fourier space Eq.(11), becomes

$$\begin{aligned} W(Q^2b^2, x_1, x_2) \quad &= \quad \int_{x_2}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} f_1(z_1, \mu^2) f_2(z_2, \mu^2) \\ &\times \hat{W}(Q^2/\mu^2, Q^2b^2, x_1/z_1, x_2/z_2, \alpha_s(\mu^2)) \end{aligned} \tag{14}$$

Furthermore, Eq.(11) tells us that the differential cross section is a convolutional product which is diagonalized by a double Mellin transform. Thus, performing the double Mellin and Fourier transform, the coefficient function takes the simple factorized form:

$$W(Q^2b^2, N_1, N_2) = f_1(N_1, \mu^2) f_2(N_2, \mu^2) \hat{W}(Q^2/\mu^2, Q^2b^2, N_1, N_2, \alpha_s(\mu^2)). \tag{15}$$

Our goal is to resum the large logarithms $\ln Q^2b^2$ to all logarithmic orders. These logs are present to all orders in the contributions to this differential cross section. They come from the kinematical region of soft and collinear emissions. However, we know from Eq.(13) that collinear divergences that arises in the limit $q_\perp \to 0$ are absorbed in parton distribution function evolution. Consequently, we will resum only the large logarithms $\ln Q^2b^2$ that come from soft contributions.
We define the usual physical anomalous dimension:

$$Q^2 \frac{\partial}{\partial Q^2} W(Q^2b^2, N_1, N_2) = \gamma_{(W)}(Q^2b^2, N_1, N_2, \alpha_s(Q^2)) W(Q^2b^2, N_1, N_2). \tag{16}$$

It is clear that $\gamma_{(W)}(Q^2b^2, N_1, N_2, \alpha_s(Q^2))$ is a renormalization group invariant and we will show that it is also independent of N_1 and N_2 when choosing the arbitrary scale μ^2 equal to $1/b^2$ and taking into account only soft contributions. Thus, in the soft limit and with the convenient choice $\mu^2 = 1/b^2$, we can write

$$\gamma_{(W)}^{SOFT}(Q^2b^2, N_1, N_2, \alpha_s(Q^2)) = \gamma(1, Q^2b^2, \alpha_s(Q^2)) \tag{17}$$

So, the resummed expression for the cross section Eq.(11) in Fourier space, in which the collinear contributions to the large $\ln Q^2b^2$ are separated from the soft ones, has the general form:

$$\begin{aligned} W^{res}(Q^2b^2, x_1, x_2) &= \int_{x_2}^{1}\frac{dz_1}{z_1}\int_{x_2}^{1}\frac{dz_2}{z_2} f_1(z_1, 1/b^2) f_2(z_2, 1/b^2) K^{res}(b^2, Q_0^2, Q^2) \\ &\times \hat{W}^{res}(Q_0^2b^2, Q_0^2b^2, x_1/z_1, x_2/z_2, \alpha_s(1/b^2)), \end{aligned} \tag{18}$$

where

$$K^{res}(Q^2b^2, Q_0^2, Q^2) = \exp\left\{\int_{Q_0^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2b^2, \alpha_s(\bar{\mu}^2))\right\}, \tag{19}$$

The scale Q_0^2, must be larger than the lower limit of the perturbative analysis ($Q_0^2 > \Lambda^2_{QCD}$). Hence, in order to absorb the possible large correction of the type $\ln Q_0^2b^2$ we will always choose $Q_0^2 = 1/b^2$. Accordingly, the condition $Q_0^2 > \Lambda^2_{QCD}$ becomes $b^2 < 1/\Lambda^2_{QCD}$ and the resummed exponent of Eq.(19) is related to the resummed physical anomalous dimension γ^{res} through the logarithmic derivative:

$$\gamma^{res}(1, Q^2b^2, \alpha_s(Q^2)) = Q^2\frac{\partial}{\partial Q^2}\int_{1/b^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2b^2, \alpha_s(\bar{\mu}^2)). \tag{20}$$

It is now clear that resummation of collinear emissions is realized by the parton distribution evolution thanks to the fact that the factorization scale μ^2 is arbitrary. Resummation of soft gluon emissions can be achieved by the resummation of the exponent that appears in this expression. This is the subject of the next section.

6.3 The $q_\perp^2$ singularities of soft gluon contributions

We now proceed through the calculation of the resummed exponent using kinematics analysis and renormalization group improvement. According to the Appendix B, the phase space measure in $d = 4 - 2\epsilon$ dimensions for n extra emissions of the partonic Drell-Yan subprocess can be written for $n = 0$ and $n \geq 1$ respectively as:

$$\frac{d\phi_1(p_1+p_2; q)}{dq_\perp^2 dy} = \frac{1}{Q^4}\delta(1-\xi_1)\delta(1-\xi_2)\delta(\hat{q}_\perp^2) \tag{21}$$

$$\begin{aligned} \frac{d\phi_{n+1}(p_1+p_2; q, k_1, \ldots, k_n)}{dq_\perp^2 dy} &= N(\epsilon)(q_\perp^2)^{-\epsilon}\int_0^{(\sqrt{s}-\sqrt{Q^2})^2}\frac{dM^2}{2\pi}d\phi_n(k; k_1, \ldots, k_n) \\ &\times \delta(M^2 - M_0^2); \end{aligned}$$

$$k^2 = M^2; \qquad M_0^2 = \frac{Q^2}{\xi_1\xi_2}[(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)], \tag{22}$$

where $N(\epsilon) = 1/(2(4\pi)^{2-2\epsilon})$, $\xi_i = x_i/z_i$, $\hat{q}_\perp^2 = q_\perp^2/Q^2$ and $d\Omega^{n-1}(\epsilon)$ stands for the integration of $n-1$ dimensionless variables $(z_i, i = 1, \ldots, n-1)$. ξ_1 and ξ_2 are related to the partonic center-of-mass rapidity (y) and energy (s) by the relations:

$$s = \frac{Q^2 + q_\perp^2}{\xi_1 \xi_2}; \qquad y = \frac{1}{2} \ln \frac{\xi_1}{\xi_2}. \tag{23}$$

The phase space measure $d\phi_n(k; k_1, \ldots, k_n)$ is the same as the phase space measure of the DIS process with an incoming momentum with a nonzero invariant mass ($k^2 = M^2$) and n outgoing massless particles. This phase space has been analyzed in Section 3.1 and is given by,

$$d\phi_1(k; k_1) = 2\pi\delta(M^2), \qquad n = 0 \tag{24}$$

$$d\phi_n(k; k_1, \ldots, k_n) = 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^{n-1} (M^2)^{n-2-(n-1)\epsilon} d\Omega^{n-1}(\epsilon), \qquad n \geq 1. \tag{25}$$

According to this, we can rewrite Eqs.(21),(22) in this form:

$$\frac{d\phi_1(p_1 + p_2; q)}{dq_\perp^2 dy} = \frac{1}{Q^4}\delta(1-\xi_1)\delta(1-\xi_2)\delta(\hat{q}_\perp^2) \tag{26}$$

$$\frac{d\phi_2(p_1 + p_2; q, k_1)}{dq_\perp^2 dy} = (q_\perp^2)^{-\epsilon} N(\epsilon)\delta(M_0^2) \tag{27}$$

$$M_0^2 = \frac{Q^2}{\xi_1\xi_2}[(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)]. \tag{28}$$

for $n = 0, 1$ respectively, and

$$\frac{d\phi_{n+1}(p_1 + p_2; q, k_1, \ldots, k_n)}{dq_\perp^2 dy} = (q_\perp^2)^{-\epsilon} 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^n (M_0^2)^{n-2-(n-1)\epsilon} d\Omega^{n-1}(\epsilon) \tag{29}$$

for $n \geq 2$. The dependence of the phase space on $\hat{q}_\perp^2$ comes entirely from the factors:

$$(q_\perp^2)^{-\epsilon}\delta(M_0^2), \qquad n = 1 \tag{30}$$

$$(M_0^2)^{n-1}(q_\perp^2)^{-\epsilon}(M_0^2)^{-(n-1)\epsilon-1}, \qquad n \geq 1. \tag{31}$$

The phase space measure must be combined with the square modulus of the amplitude, in order to determine the logarithmic singularities in $q_\perp = 0$ which are regularized in $d = 4 - 2\epsilon$ dimensions. Studying the behavior of the invariants that can be constructed with the external momenta, we can establish in which kinematical region the square modulus of the amplitude can be singular in $\hat{q}_\perp^2 \to 0$. From the study of the DIS-like emissions (see Section 3.2) we know that

$$k_i^0 = \frac{\sqrt{M_0^2}}{2}(z_{n-1} \cdots z_{i+1})^{1/2}(1-z_i), \qquad 1 \leq i \leq n-2 \tag{32}$$

$$k_{n-1}^0 = \frac{\sqrt{M_0^2}}{2}(1-z_{n-1}) \tag{33}$$

$$k_n^0 = k_1^0. \tag{34}$$

This means that all the invariants that can appear in the function $D_G(\beta, P_E)$ in Eq.(22) of Section 3.1 can be expressed in terms of the following ones:

$$q^2 = Q^2, \qquad p_1^2 = p_2^2 = k_i^2 = 0, \qquad p_1 \cdot p_2 = \frac{s}{2} \tag{35}$$

$$k_i \cdot k_j \sim M_0^2 = \frac{Q^2}{\xi_1 \xi_2}[(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)] \tag{36}$$

$$p_1 \cdot k_i \sim p_2 \cdot k_i \sim \sqrt{s M_0^2} \tag{37}$$

$$= Q^2 \left[\frac{(1+\hat{q}_\perp^2)}{\xi_1 \xi_2}[(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)]\right]^{1/2}. \tag{38}$$

In the case $n = 1$, the single emission squared amplitude at tree level has a $1/q_\perp^2$ singularity for $q_\perp \to 0$. This can be easily seen by an explicit $O(\alpha_s)$ computation (see for example Refs.[66, 67, 68]). Hence, in the general case, we expect that in the $q_\perp \to 0$ limit the squared amplitude has the following behavior

$$|A_{n+1}|^2 \sim \frac{1}{\hat{q}_\perp^2}(M_0^2)^{n_1}(M_0^2)^{n_2\epsilon} g_{n_1 n_2}(\xi_1, \xi_2), \tag{39}$$

where N and k are integer or half-integer numbers (see Eqs.(36,37)). However, as discussed in Section 3.2, here we will assume that only the integer powers of M_0^2 contribute. Then, we know that phase space contributes the factors of Eqs.(30,31) and, hence, Eq.(39) implies that a generic contribution to the coefficient function $\hat{W}^{(0)}$ has the folowing structure:

$$(\hat{q}_\perp^2)^{-1-\epsilon}\delta(M_0^2)(M_0^2)^{n_1}(M_0^2)^{n_2\epsilon} g_{nn_2}(\xi_1,\xi_2); \qquad n = 1, \tag{40}$$

$$(\hat{q}_\perp^2)^{-1-\epsilon}(M_0^2)^{-1-(n-n_2'-1)\epsilon}(M_0^2)^{n-1+n_1} g_{nn_2'}(\xi_1,\xi_2); \qquad n > 1, \tag{41}$$

where for the moment we do not care about the overall dimensional factor. Now, we are interested in taking into account only the $1/\hat{q}_\perp^2$ singularity, because more singular terms are forbidden and less singular ones are suppressed. As proven in Appendix D, in the limit $q_\perp^2 \to 0$,

$$\delta(M_0^2) = \frac{\xi_1 \xi_2}{Q^2}\left[\frac{\delta(1-\xi_1)}{(1-\xi_2)_+} + \frac{\delta(1-\xi_2)}{(1-\xi_1)_+} - \ln \hat{q}_\perp^2 \delta(1-\xi_1)\delta(1-\xi_2)\right] + O(\hat{q}_\perp^2), \tag{42}$$

and, for $\eta = -(n - n_2' - 1)\epsilon$ (with $\epsilon < 0$ $n - n_2' - 1 > 0$),

$$\begin{aligned}[(1-\xi_1)(1-\xi_2) + \hat{q}_\perp^2(1-\xi_1-\xi_2)]^{\eta-1} = \\ = (1-\xi_1)^{\eta-1}(1-\xi_2)^{\eta-1} - \frac{(\hat{q}_\perp^2)^\eta}{\eta^2}\delta(1-\xi_1)\delta(1-\xi_2) + O(\hat{q}_\perp^2).\end{aligned} \tag{43}$$

Therefore, $n_1 = -n + 1$, $n_2' < n - 1$. Furthermore, we note that the terms not proportional to $\delta(1-\xi_1)\delta(1-\xi_2)$ are divergent (in the $q_\perp \to 0$ limit) due to collinear emissions. These divergences are absorbed and resummed by the Altarelli-Parisi evolution of the parton distributions $f_1(z_1, \mu^2)$ and $f_2(z_2, \mu^2)$ when they are evaluated at the scale $\mu^2 = 1/b^2$ in Fourier space (see Eq.(18)). As a first conclusion, we obtain that the contributions that must be resummed in the limit $\hat{q}_\perp^2 \to 0$ are those which are proportional to $\delta(1-\xi_1)\delta(1-\xi_2)$ and, thus, belong to the kinematical region of only the soft extra emissions. Hence, we obtain that the soft part of the coefficient function that must be resummed has, after the inclusion of loops, the following general form:

$$\hat{W}(Q^2, q_\perp^2, \xi_1, \xi_2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n \hat{W}_n(Q^2, q_\perp^2, \alpha_0, \epsilon)\delta(1-\xi_1)\delta(1-\xi_2), \tag{44}$$

with

$$\begin{aligned} \hat{W}_n(Q^2, q_\perp^2, \alpha_0, \epsilon) &= (Q^2)^{-n\epsilon}\big[C_{n0}^{(0)}(\epsilon)\delta(\hat{q}_\perp^2) + \sum_{k=2}^{n} C_{nk}^{(0)}(\epsilon)(\hat{q}_\perp^2)^{-1-k\epsilon} \\ &\quad + \sum_{k=1}^{n} C_{nk}'^{(0)}(\epsilon)(\hat{q}_\perp^2)^{-1-k\epsilon} \ln \hat{q}_\perp^2\big], \end{aligned} \tag{45}$$

where the factor of $(Q^2)^{-n\epsilon}$ has been introduced for dimensional reasons. We, now, perform the double Mellin transform and the Fourier transform using the fact that:

$$\int d^2\hat{q}_\perp e^{i\hat{b}\cdot\hat{q}_\perp}\delta(\hat{q}_\perp^2) = \pi, \tag{46}$$

$$\int d^2\hat{q}_\perp e^{i\hat{b}\cdot\hat{q}_\perp}(\hat{q}_\perp^2)^{-1-k\epsilon} = \pi F_k(\epsilon)(\hat{b}^2)^{k\epsilon}, \tag{47}$$

$$\int d^2\hat{q}_\perp e^{i\hat{b}\cdot\hat{q}_\perp}(\hat{q}_\perp^2)^{-1-k\epsilon}\ln \hat{q}_\perp^2 = -\pi F_k(\epsilon)(\hat{b}^2)^{k\epsilon}\ln \hat{b}^2 - \pi \frac{F_k'(\epsilon)}{k}(\hat{b}^2)^{k\epsilon}, \tag{48}$$

$$\hat{b}^2 \equiv Q^2 b^2; \qquad F_k(\epsilon) = -\frac{4^{-k\epsilon}}{k\epsilon}\frac{\Gamma(1-k\epsilon)}{\Gamma(1+k\epsilon)}. \tag{49}$$

According to this, Eq.(44) has, after Mellin and Fourier transform, this structure:

$$\begin{aligned} \hat{W}^{(0)}(Q^2, b^2, \alpha_0, \epsilon) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \tilde{C}_{nk}^{(0)}(\epsilon)[(Q^2)^{-\epsilon}\alpha_0]^{n-k}[(1/b^2)^{-\epsilon}\alpha_0]^k + \\ &\quad + \ln Q^2 b^2 \sum_{n=1}^{\infty}\sum_{k=1}^{n} \tilde{C}_{nk}'^{(0)}(\epsilon)[(Q^2)^{-\epsilon}\alpha_0]^{n-k}[(1/b^2)^{-\epsilon}\alpha_0]^k \end{aligned} \tag{50}$$

6.4 The resummed exponent in renormalization group approach

At this point, we calculate the resummed exponent that appears in Eq.(19):

$$\begin{aligned}
\int_{1/b^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\Gamma^{(0)}(Q^2,\bar\mu^2,b^2,\alpha_0,\epsilon) &= \ln\left(\frac{\hat W^{(0)}(Q^2,b^2,\alpha_0,\epsilon)}{\hat W^{(0)}(1/b^2,b^2,\alpha_0,\epsilon)}\right) = \\
&= \left[\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}E_{nk}^{(0)}(\epsilon)[(\bar\mu^2)^{-\epsilon}\alpha_0]^{n-k}[(1/b^2)^{-\epsilon}\alpha_0]^k\right]_{\bar\mu^2=1/b^2}^{\bar\mu^2=Q^2} \\
&\quad +\ln\left(1+\ln Q^2b^2\sum_{n=1}^{\infty}\sum_{k=1}^{n}\tilde E_{nk}^{(0)}(\epsilon)[(Q^2)^{-\epsilon}\alpha_0]^{n-k}\right. \\
&\quad \left.\times[(1/b^2)^{-\epsilon}\alpha_0]^k\right),
\end{aligned} \tag{51}$$

where the last term has not been expanded because we must take into account that in the bare coefficient function Eq.(44) there is only one explicit logarithm. From the explicit calculation to order $O(\alpha_0)$, we find that

$$E_{10}^{(0)}(\epsilon) = \frac{2\pi}{3}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}\left(-\frac{2}{\epsilon^2}-\frac{3}{\epsilon}+\pi^2-8\right), \tag{52}$$

$$\tilde E_{11}^{(0)}(\epsilon) = \frac{4\pi}{3}\frac{(4\pi)^\epsilon}{\Gamma(1-\epsilon)}F_1(\epsilon). \tag{53}$$

Now, we want to rewrite Eq.(51) in a renormalized form. To do this, we use, as expalined in Chapter 3, the fact that $(Q^2)^{-\epsilon}\alpha_0$ and $(1/b^2)^{-\epsilon}\alpha_0$ are renormalization group invariant. Consequently, we may write:

$$(Q^2)^{-\epsilon}\alpha_0 = \alpha_s(Q^2)Z^{(\alpha_s)}(\alpha_s(Q^2),\epsilon), \tag{54}$$

$$(1/b^2)^{-\epsilon}\alpha_0 = \alpha_s(1/b^2)Z^{(\alpha_s)}(\alpha_s(1/b^2),\epsilon), \tag{55}$$

where $Z^{(\alpha_s)}(\alpha_s(\mu^2),\epsilon)$ has multiple poles at $\epsilon=0$ and μ^2 is the renormalization scale which for simplicity has been chosen equal to the factorization scale. Furthermore we note that the universal functions $Z^{(\hat W)}(N_1,\alpha_s(\mu^2),\epsilon)Z^{(\hat W)}(N_2,\alpha_s(\mu^2),\epsilon)$ that extract the collinear poles from the coefficient function simplify in the first line of Eq.(51). Thus the renormalized expression of Eq.(51) has the form:

$$\begin{aligned}
\int_{1/b^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\Gamma(\frac{Q^2}{\bar\mu^2},\bar\mu^2b^2,\alpha_s(\bar\mu^2),\epsilon) &= \left[\sum_{m=1}^{\infty}\sum_{n=0}^{n-1}E_{mn}^{R}(\epsilon)\alpha_s^{m-n}(\bar\mu^2)\alpha_s^n(1/b^2)\right]_{\bar\mu^2=1/b^2}^{\bar\mu^2=Q^2} \\
&\quad +\ln\left(1+\ln Q^2b^2\sum_{m=1}^{\infty}\sum_{n=1}^{m}\tilde E_{mn}^{R}(\epsilon)\right. \\
&\quad \left.\times\alpha_s(Q^2)^{m-n}\alpha_s^n(1/b^2)\right).
\end{aligned} \tag{56}$$

The resummed exponent is clearly pole-free and so we can exploit the cancellation of the poles that could be present in the coefficients $E^R_{mn}(\epsilon)$ and $\tilde{E}^R_{mn}(\epsilon)$. Furthermore we want to perform a comparison with previously released resummation formulae given in [6, 67]. In order to do these two things we rewrite Eq.(56) in terms of the renormalized physical anomalous dimension $\Gamma(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon)$ and therefore, we calculate, according to Eq.(20), the logarithmic derivative of Eq.(56):

$$\begin{aligned}
&\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) = \qquad (57)\\
&= \beta^{(d)}(\alpha_s(\bar{\mu}^2)) \frac{\partial}{\partial \alpha_s(\bar{\mu}^2)} \sum_{m=1}^{\infty} \sum_{n=0}^{n-1} E^R_{mn}(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2) \\
&+ \frac{\sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}^R_{mn}(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)}{1 + \ln \bar{\mu}^2 b^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}^R_{mn}(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)} \\
&+ \frac{\ln \bar{\mu}^2 b^2 \beta^{(d)}(\alpha_s(\bar{\mu}^2)) \partial/\partial \alpha_s \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}^R_{mn}(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)}{1 + \ln \bar{\mu}^2 b^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}^R_{mn}(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2)},
\end{aligned}$$

where

$$\beta^{(d)}(\alpha_s(\bar{\mu}^2)) = -\epsilon \alpha_s(\bar{\mu}^2) + \beta(\alpha_s(\bar{\mu}^2)), \qquad (58)$$

and $\beta(\alpha_s(\bar{\mu}^2)) = -\beta_0^2 \alpha_s(\bar{\mu}^2) + O(\alpha_s^3)$ is the usual four-dimensional β-function. Now, in order to isolate the terms which contain the explicit logs from the rest we add and subtract the term

$$\sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}^R_{mn}(\epsilon) \alpha_s(\bar{\mu}^2)^{m-n} \alpha_s^n(1/b^2). \qquad (59)$$

After this, we re-expand the various terms in powers of $\alpha_s(\bar{\mu}^2)$ and $\alpha_s(1/b^2)$, but not in powers of $\ln \bar{\mu}^2 b^2$. The result that we find in this way has the following structure:

$$\begin{aligned}
\int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Gamma(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) &= \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left(\sum_{m=1}^{\infty} \sum_{n=0}^{m} \Gamma^R_{mn}(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2) \right) \\
&+ \int_{1/b^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \left(\frac{\ln \bar{\mu}^2 b^2 \sum_{m=2}^{\infty} \sum_{n=1}^{m-1} \tilde{\Gamma}^R_{mn}(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2)}{1 + \ln \bar{\mu}^2 b^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m} \tilde{E}^R_{mn}(\epsilon) \alpha_s^{m-n}(\bar{\mu}^2) \alpha_s^n(1/b^2)} \right) \qquad (60)
\end{aligned}$$

To show the cancellation of divergences we rewrite the integrand separating off the b^2-independent terms as in Chapter 3 and Chapter 4:

$$\begin{aligned}
\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) &= \hat{\Gamma}^{(c)}(\alpha_s(\bar{\mu}^2), \epsilon) + \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) \\
&+ \hat{\Gamma}'^{(l)}(\ln \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon),
\end{aligned}$$

where

$$\hat{\Gamma}^{(c)}(\alpha_s(\bar{\mu}^2), \epsilon) = \sum_{m=1}^{\infty} \Gamma^R_m(\epsilon) \alpha_s^m(\bar{\mu}^2) \qquad (61)$$

$$\hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \Gamma^R_{m+nn}(\epsilon) \alpha_s^m(\bar{\mu}^2) \alpha_s^n(1/b^2) \qquad (62)$$

$$\hat{\Gamma}'^{(l)}(\ln\bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) = \frac{\ln\bar{\mu}^2 b^2 \sum_{m=2}^{\infty}\sum_{n=1}^{m-1}\tilde{\Gamma}^R_{mn}(\epsilon)\alpha_s^{m-n}(\bar{\mu}^2)\alpha_s^n(1/b^2)}{1+\ln\bar{\mu}^2 b^2 \sum_{m=1}^{\infty}\sum_{n=1}^{m}\tilde{E}^R_{mn}(\epsilon)\alpha_s^{m-n}(\bar{\mu}^2)\alpha_s^n(1/b^2)} \tag{63}$$

We know that

$$\alpha_s(1/b^2) = f(\alpha_s(\bar{\mu}^2), \ln\bar{\mu}^2 b^2), \tag{64}$$

so, in principle, it is possible to invert this relation in order to obtain:

$$\ln\bar{\mu}^2 b^2 = g(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2)). \tag{65}$$

The function g has the following perturbative expression:

$$g(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2)) = \frac{1}{\beta_0}\left(\frac{1}{\alpha_s(\bar{\mu}^2)} - \frac{1}{\alpha_s(1/b^2)}\right)\left(1+\sum_{r=1}^{\infty} l_r \alpha_s^r(\bar{\mu}^2)\right). \tag{66}$$

Substituting this expression in Eq.(63) and re-expanding in powers of $\alpha_s(\bar{\mu}^2)$ and $\alpha_s(1/b^2)$, we obtain that:

$$\hat{\Gamma}'^{(l)}(\ln\bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) = \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) - \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon), \tag{67}$$

where

$$\bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2), \epsilon) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\bar{\Gamma}^R_{m+nn}(\epsilon)\alpha_s^m(\bar{\mu}^2)\alpha_s^n(\mu^2). \tag{68}$$

We choose as counterterm,

$$Z^{(\Gamma)}(\alpha_s(\bar{\mu}^2), \epsilon) = \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon). \tag{69}$$

With this choice we obtain:

$$\begin{aligned}\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2), \epsilon) \;=\; & \hat{\Gamma}^{(c)}(\alpha_s(\bar{\mu}^2), \epsilon) + \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon) + \\ & + \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) - \hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon) + \\ & + \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon) - \bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\bar{\mu}^2), \epsilon).\end{aligned} \tag{70}$$

The first line is a power series with coefficients which are pole-free for each b^2, because the second and the third lines vanish when $b^2 = 1/\bar{\mu}^2$. Hence, the sum of the second and the third line must be finite at $\epsilon = 0$, but it is not necessarily analytic in $\alpha_s(\mu^2)$. To find its perturbative expression in powers of $\alpha_s(\mu^2)$ we rewrite the last two lines of Eq.(70) as

$$\int_{1/b^2}^{\bar{\mu}^2}\frac{d\mu^2}{\mu^2}\left(\frac{\partial}{\partial\ln\mu^2}\hat{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2), \epsilon) + \frac{\partial}{\partial\ln\mu^2}\bar{\Gamma}^{(l)}(\alpha_s(\bar{\mu}^2), \alpha_s(1/b^2), \epsilon)\right). \tag{71}$$

There could be other residual cancellations of $\epsilon = 0$ poles between these two terms, but their sum must be finite at $\epsilon = 0$ and analytic in $\alpha_s(\mu^2)$ and $\alpha_s(1/b^2)$. Thus, we get a perturbative expression of the physical anomalous dimension with finite coefficient:

$$\begin{aligned}\gamma(1,\bar{\mu}^2 b^2,\alpha_s(\bar{\mu}^2)) &= -\int_{1/b^2}^{\bar{\mu}^2}\frac{d\mu^2}{\mu^2}A(\alpha_s(\mu^2)) - B(\alpha_s(\bar{\mu}^2)) - \\ &\quad -\int_{1/b^2}^{\bar{\mu}^2}\frac{d\mu^2}{\mu^2}C(\alpha_s(\bar{\mu}^2),\alpha_s(\mu^2)),\end{aligned} \tag{72}$$

where

$$A(\alpha_s(\mu^2)) = \sum_{n=1}^{\infty} A_n \alpha_s^n(\mu^2) \tag{73}$$

$$B(\alpha_s(\bar{\mu}^2)) = \sum_{m=1}^{\infty} B_m \alpha_s^m(\bar{\mu}^2) \tag{74}$$

$$C(\alpha_s(\bar{\mu}^2),\alpha_s(\mu^2)) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\alpha_s^m(\bar{\mu}^2)\alpha_s^n(\mu^2) \tag{75}$$

$$\tag{76}$$

After an integration by parts of the first term we obtain an expression for the all-orders resummed exponent:

$$\begin{aligned}\int_{1/b^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2},\bar{\mu}^2 b^2,\alpha_s(\bar{\mu}^2)) &= -\int_{1/b^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\Bigg[\ln\frac{Q^2}{\bar{\mu}^2}A(\alpha_s(\bar{\mu}^2)) + B(\alpha_s(\bar{\mu}^2)) + \\ &\quad + \int_{1/b^2}^{\bar{\mu}^2}\frac{d\mu^2}{\mu^2}C(\alpha_s(\bar{\mu}^2),\alpha_s(\mu^2))\Bigg]\end{aligned} \tag{77}$$

To obtain a LL resummation we need only a coefficient (A_1) and to obtain a NLL resummation we need four coefficients A_1, A_2, B_1, C_{11}. However it has been demonstrated by explicit calculations [67, 68] that

$$C_{11} = 0. \tag{78}$$

Therefore for the resummation at the NLL level we need only three coefficients A_1, A_2, B_1. Our general formula reduces to that of Ref.[6] when

$$C_{mn} = 0. \tag{79}$$

This restriction could be a consequence of the factorization of soft emissions from the hard part of the coefficient function, but this remains unproven. The result for the anomalous dimension in Eq.(72) can be rewritten in our formalism performing the change of variable

$$n' = \frac{\bar{\mu}^2}{\mu^2}. \tag{80}$$

We get

$$\gamma(1,\bar{\mu}^2 b^2,\alpha_s(\bar{\mu}^2)) = -\int_1^{\bar{\mu}^2 b^2}\frac{dn'}{n'}G(\alpha_s(\bar{\mu}^2),\alpha_s(\bar{\mu}^2/n')) + \tilde{G}(\alpha_s(\bar{\mu}^2)), \tag{81}$$

where

$$G(\alpha_s(\bar{\mu}^2), \alpha_s(\mu^2)) = \sum_{m=0}^{\infty} G_{mn} \alpha_s^m(\bar{\mu}^2) \alpha_s^n(\mu^2) \tag{82}$$

$$\tilde{G}(\alpha_s(\bar{\mu}^2)) = \sum_{m=1}^{\infty} \tilde{G} \alpha_s^m(\bar{\mu}^2). \tag{83}$$

The case of the resummation formula of Ref.[6] is obtained when G_{mn} is non-vanishing only when $m = 0$. In this case, we have

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) = -\int_1^{\bar{\mu}^2 b^2} \frac{dn'}{n'} G(\alpha_s(\bar{\mu}^2/n')) + \tilde{G}(\alpha_s(\bar{\mu}^2)). \tag{84}$$

6.5 Logs of $q_\perp^2$ vs. logs of b^2 to all logarithmic orders

Large logarithms of $q_\perp$ appear in the perturbative coefficients in the form of plus distributions. We define

$$\left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2}\right]_+ \tag{85}$$

in such a way that

$$\int_0^1 d\hat{q}_\perp^2 \left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2}\right]_+ = 0. \tag{86}$$

Let us consider the Fourier transforms

$$I_p(Q^2b^2) = \frac{1}{Q^2}\int d^2q_\perp e^{i\vec{q}_\perp\cdot\vec{b}} \left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2}\right]_+ = 2\pi \int_0^\infty \hat{q}_\perp d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \left[\frac{\ln^p(\hat{q}_\perp^2)}{\hat{q}_\perp^2}\right]_+, \tag{87}$$

where we have used the definition of the 0-order Bessel function J_0:

$$J_0(z) = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{iz\cos\theta}. \tag{88}$$

We now exploit the definition of the plus distribution:

$$I_p(Q^2b^2) = 2\pi \int_0^1 d\hat{q}_\perp [J_0(\hat{q}_\perp \hat{b}) - 1] \frac{\ln^p \hat{q}_\perp^2}{\hat{q}_\perp} + 2\pi \int_1^\infty d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \frac{\ln^p \hat{q}_\perp^2}{\hat{q}_\perp}. \tag{89}$$

Writing $\ln^p \hat{q}_\perp^2$ as the p^{th} α-derivative of $(\hat{q}_\perp^2)^\alpha$ at $\alpha = 0$, we get

$$\begin{aligned} I_p(Q^2b^2) &= 2\pi \frac{\partial^p}{\partial\alpha^p} \left\{ \int_0^1 d\hat{q}_\perp [J_0(\hat{q}_\perp \hat{b}) - 1] \hat{q}_\perp^{2\alpha-1} + \int_1^\infty d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \hat{q}_\perp^{2\alpha-1} \right\} \\ &= 2\pi \frac{\partial^p}{\partial\alpha^p} \left[\int_0^\infty d\hat{q}_\perp J_0(\hat{q}_\perp \hat{b}) \hat{q}_\perp^{2\alpha-1} - \int_0^1 d\hat{q}_\perp \hat{q}_\perp^{2\alpha-1} \right] \\ &= \pi \frac{\partial^p}{\partial\alpha^p} \left[\left(\frac{Q^2b^2}{4}\right)^{-\alpha} \frac{\Gamma(\alpha)}{\Gamma(1-\alpha)} - \frac{1}{\alpha} \right], \end{aligned} \tag{90}$$

where the last equality follows from the identity

$$\int_0^\infty dx x^\mu J_\nu(ax) = 2^\mu a^{-\mu-1}\frac{\Gamma(1/2+\nu/2+\mu/2)}{\Gamma(1/2+\nu/2-\mu/2)} \tag{91}$$

$$a > 0; \qquad -\mathrm{Re}\nu - 1 < \mathrm{Re}\mu < 1/2. \tag{92}$$

From Eq.(90), we read off the generating function $G(\alpha)$ of I_p

$$G(\alpha) = \frac{\pi}{\alpha}\left[\left(\frac{Q^2b^2}{4}\right)^{-\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} - 1\right], \tag{93}$$

in the sense that

$$I_p(Q^2b^2) = \left[\frac{d^p}{d\alpha^p}G(\alpha)\right]_{\alpha=0}. \tag{94}$$

Now, the generating function of logarithms of Q^2b^2 is $(Q^2b^2)^{-\alpha}$ in the sense that

$$L_p \equiv \ln^p(1/(Q^2b^2)) = \left[\frac{d^p}{d\alpha^p}(Q^2b^2)^{-\alpha}\right]_{\alpha=0}. \tag{95}$$

Inverting Eq.(93), we find the relation between the generating function of L_p and the generating function of I_p, which is

$$(Q^2b^2)^{-\alpha} = \frac{1}{\pi}S(\alpha)[\alpha G(\alpha) + \pi], \tag{96}$$

where

$$S(\alpha) = \frac{1}{4^\alpha}\frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)}. \tag{97}$$

Performing the Taylor expansion of the r.h.s. of Eq.(96) around $\alpha = 0$ and using Eq.(94), we obtain:

$$(Q^2b^2)^{-\alpha} = \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{\alpha^m}{m!}\sum_{i=0}^{m}\binom{m}{i} iI_{i-1}S^{(m-i)}(0), \tag{98}$$

where $S^{(j)}(0)$ is the j-th derivative of $S(\alpha)$ evaluated at $\alpha = 0$. Now, using Eq.(95), we the relation between L_p and I_p:

$$L_p = \frac{1}{\pi}\sum_{i=1}^{p}\binom{p}{i} iI_{i-1}S^{(p-1)}(0) = \frac{1}{\pi}\sum_{k=1}^{p}\binom{p-1}{k-1} pS^{(k-1)}I_{p-k}. \tag{99}$$

Thanks to the first equality in Eq.(87) and to the fact that

$$\ln^{p-k}\hat{q}_\perp^2 = \frac{1}{p(p-1)\cdots(p-k+1)}\frac{d^k}{d\ln^k\hat{q}_\perp^2}\ln^p\hat{q}_\perp^2, \tag{100}$$

we arrive at a relation to all logarithmic orders between the logs of b^2 and the logs of $q_\perp^2$:

$$L_p = \frac{1}{\pi}\sum_{k=1}^{p}\frac{S^{(k-1)}(0)}{(k-1)!}\int\frac{d^2q_\perp}{Q^2}e^{i\vec{q}_\perp\cdot\vec{b}}\left[\frac{1}{\hat{q}_\perp^2}\frac{d^k}{d\ln^k\hat{q}_\perp^2}\ln^p\hat{q}_\perp^2\right]_+ . \tag{101}$$

This relation allows us to derive the relation between a generic function of $\ln(1/Q^2b^2)$ and a function of $\ln\hat{q}_\perp^2$. Indeed, given a function

$$h\left(\ln\frac{1}{Q^2b^2}\right) = \sum_{p=0}^{\infty}h_p\ln^p\frac{1}{Q^2b^2}, \tag{102}$$

Eq.(101) implies:

$$h\left(\ln\frac{1}{Q^2b^2}\right) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{S^{(k-1)}(0)}{(k-1)!}\int\frac{d^2q_\perp}{Q^2}e^{i\vec{q}_\perp\cdot\vec{b}}\left[\frac{1}{\hat{q}_\perp^2}\frac{d^k}{d\ln^k\hat{q}_\perp^2}h(\ln\hat{q}_\perp^2)\right]_+ . \tag{103}$$

The r.h.s. of Eq.(103) can be viewed as the Fourier transform of a function (more properly a distribution) $\hat{h}(\ln\hat{q}_\perp^2)$:

$$\begin{aligned} h\left(\ln\frac{1}{Q^2b^2}\right) &= \int\frac{d^2q_\perp}{Q^2}e^{i\vec{q}_\perp\cdot\vec{b}}\hat{h}(\ln\hat{q}_\perp^2), \\ \hat{h}(\ln\hat{q}_\perp^2) &= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{S^{(k-1)}(0)}{(k-1)!}\left[\frac{1}{\hat{q}_\perp^2}\frac{d^k}{d\ln^k\hat{q}_\perp^2}h(\ln\hat{q}_\perp^2)\right]_+ . \end{aligned} \tag{104}$$

6.6 Resummation in $q_\perp$-space

In this section, we investigate the consequences of our general result Eq.(104) for the resummation at the NLL level of logarithmic accuracy. According to eq(19) and the discussion below and according to Eq.(77), we have that our resummation factor formula in Fourier space is:

$$K^{res}(Q^2b^2, 1/b^2, Q^2) = \exp\left\{E^{res}(Q^2b^2, 1/b^2, Q^2)\right\}, \tag{105}$$

where

$$E^{res}(Q^2b^2, 1/b^2, Q^2) = \int_{1/b^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\Gamma^{res}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2b^2, \alpha_s(\bar{\mu}^2)), \tag{106}$$

and where at NLL level

$$\begin{aligned} \Gamma^{res}_{NLL}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2b^2, \alpha_s(\bar{\mu}^2)) &= -\ln\frac{Q^2}{\bar{\mu}^2}\left[A_1\alpha_s(\bar{\mu}^2) + A_2\alpha_s^2(\bar{\mu}^2)\right] - B_1\alpha_s(\bar{\mu}^2) \\ &\quad -\frac{C_{11}}{\beta_0}\alpha_s(\bar{\mu}^2)\ln\frac{\alpha_s(1/b^2)}{\alpha_s(\bar{\mu}^2)}, \end{aligned} \tag{107}$$

where we have used the definition of the β-function:

$$\mu^2\frac{d}{d\mu^2}\alpha_s(\mu^2) = \beta\alpha_s = -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 + O(\alpha_s^4) \tag{108}$$

and where we have used the change of variable

$$\frac{d\mu^2}{\mu^2} = \frac{d\alpha_s}{\beta(\alpha_s)} \tag{109}$$

to compute the integral that appears in the last term of Eq.(77). Now, thanks to Eq.(104), we can rewrite the resummed exponent in b-space Eq.(106) in terms of a resummed exponent defined in $q_\perp$-space. Thus, up to NNLL terms, we obtain:

$$E^{res}_{NLL}(Q^2b^2, 1/b^2, Q^2) = \int d^2q_\perp e^{i\vec{q}_\perp\cdot\vec{b}} \left[\frac{\hat{\Gamma}^{res}_{NLL}(\hat{q}^2_\perp, q^2_\perp, Q^2)}{q^2_\perp}\right]_+, \tag{110}$$

where

$$\begin{aligned}\hat{\Gamma}^{res}_{NLL}(\hat{q}^2_\perp, q^2_\perp, Q^2) = \;&= -\ln\hat{q}^2_\perp\left[\hat{A}_1\alpha_s(q^2_\perp) + \hat{A}_2\alpha_s^2(q^2_\perp)\right] - \hat{B}_1\alpha_s(q^2_\perp) + \\ &\quad -\frac{\hat{C}_{11}}{\beta_0}\alpha_s(q^2_\perp)\ln\frac{\alpha_s(Q^2)}{\alpha_s(q^2_\perp)}\end{aligned} \tag{111}$$

and where the relation of the constant coefficients of this last equation and the of Eq.(107) is

$$\hat{A}_1 = \frac{A_1}{\pi} \tag{112}$$

$$\hat{A}_2 = -\left(\frac{A_2}{\pi} + \frac{A_1}{\pi}\beta_0\ln\frac{e^{2\gamma_E}}{4}\right) \tag{113}$$

$$\hat{B}_1 = \frac{B_1}{\pi} - \frac{A_1}{\pi}\ln\frac{e^{2\gamma_E}}{4} \tag{114}$$

$$\hat{C}_{11} = \frac{C_{11}}{\pi}. \tag{115}$$

Here γ_E is the usual Euler gamma. Now, we want to define a resummation factor in $q_\perp$-space. Looking at Eq.(18), we note that large $\ln Q^2b^2$ of collinear nature are resummed by the parton distribution function. So, in order to define a resummation in $q_\perp$-space, we must take them into account. For simplicity, we consider the resummed part of non-singlet cross section, because the non-singlet parton distribution functions, which are defined as

$$f'_{a'}(N, \mu^2) = f_a(N, \mu^2) - f_b(N, \mu^2) \qquad a, b \neq g, \tag{116}$$

evolve independently. In particular, in Mellin moments N they satisfy the following evolution equations:

$$\mu^2\frac{\partial}{\partial\mu^2}f'_{a'}(N, \mu^2) = \gamma'(N, \alpha_s(\mu^2))f'_{a'}(N, \mu^2). \tag{117}$$

Hence, the non-singlet parton distribution functions evaluated at $\mu^2 = 1/b^2$ are related to the ones evaluated at $\mu^2 = Q^2$ by,

$$f'_{a'}(N, 1/b^2) = \exp\left\{-\int_{1/b^2}^{Q^2}\frac{d\bar{\mu}^2}{\bar{\mu}^2}\gamma'(N, \alpha_s(\bar{\mu}^2))\right\} f'_{a'}(N, Q^2). \tag{118}$$

Thus, the resummed part of the cross section with the non-singlet parton distribution functions evaluated at $\mu^2 = Q^2$ becomes

$$\exp\left\{-\int_{1/b^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\sum_{j=1}^{2}\gamma'(N,\alpha_s(\bar\mu^2))\right\}K^{res}(b^2,1/b^2,Q^2) = \tag{119}$$

$$= \exp\left\{\int_{1/b^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\left[\Gamma^{res}(\frac{Q^2}{\bar\mu^2},\bar\mu^2 b^2,\alpha_s(\bar\mu^2)) - \sum_{j=1}^{2}\gamma'(N_j,\alpha_s(\bar\mu^2))\right]\right\}$$

The general relation between a function of $\ln Q^2 b^2$ and its Fourier anti-transform Eq.(104), immediately enables us to define a resummed exponent of the non-singlet part of the cross section in $q_\perp$-space, which is:

$$K^{res}(\hat q_\perp^2, q_\perp^2, Q^2) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{S^{(k-1)}(0)}{(k-1)!}$$

$$\times\left\{\frac{1}{\hat q_\perp^2}\frac{d^k}{d\ln^k \hat q_\perp^2}\exp\left[\int_{q_\perp^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\left(\Gamma^{res}(\frac{Q^2}{\bar\mu^2},\bar\mu^2/q_\perp^2,\alpha_s(\bar\mu^2)) - \sum_{j=1}^{2}\gamma'(N_j,\alpha_s(\bar\mu^2))\right)\right]\right\}_+$$

$$= \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{S^{(k-1)}(0)}{(k-1)!}$$

$$\times\left\{\frac{d}{d\hat q_\perp^2}\frac{d^{k-1}}{d\ln^{k-1}\hat q_\perp^2}\exp\left[\int_{q_\perp^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\left(\Gamma^{res}(\frac{Q^2}{\bar\mu^2},\bar\mu^2/q_\perp^2,\alpha_s(\bar\mu^2)) - \sum_{j=1}^{2}\gamma'(N_j,\alpha_s(\bar\mu^2))\right)\right]\right\}_+$$

$$= \lim_{\eta\to 0^+}\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{S^{(k)}(0)}{k!}\frac{d^k}{d\ln^k\hat q_\perp^2}\frac{d}{\hat q_\perp^2}\left\{\theta(\hat q_\perp^2-\eta)\exp\left[\int_{q_\perp^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\right.\right.$$

$$\left.\left.\times\left(\Gamma^{res}(\frac{Q^2}{\bar\mu^2},\bar\mu^2/q_\perp^2,\alpha_s(\bar\mu^2)) - \sum_{j=1}^{2}\gamma'(N_j,\alpha_s(\bar\mu^2))\right)\right]\right\}, \tag{120}$$

where the last equation defines implicity the $q_\perp$-space resummation exponent:

$$K^{res}(\hat q_\perp^2, q_\perp^2, Q^2) = \exp\left\{E^{res}(\hat q_\perp^2, q_\perp^2, Q^2)\right\}. \tag{121}$$

All the previously released expressions for this exponent given in [69, 70, 71] are particular cases of this general expression. They differ essentially in the criteria according to which the subleading terms are kept.
We want to calculate the NLL result in $q_\perp$-space. Thus, keeping only the terms up to NNLL in Eq.(120) we obtain

$$K^{res}_{NLL}(\hat q_\perp^2, q_\perp^2, Q^2) = \frac{1}{\pi}\frac{d}{\hat q_\perp^2}\left\{\theta(\hat q_\perp^2-\eta)\exp\left[\int_{q_\perp^2}^{Q^2}\frac{d\bar\mu^2}{\bar\mu^2}\left(\Gamma^{res}_{NLL}(\frac{Q^2}{\bar\mu^2},\bar\mu^2/q_\perp^2,\alpha_s(\bar\mu^2))\right.\right.\right.$$

$$\left.\left.\left. - \sum_{j=1}^{2}\gamma'(N_j,\alpha_s(\bar\mu^2))\right)\right]\sum_{k=0}^{\infty}\frac{S^{(k)}(0)}{k!}[-\ln\hat q_\perp^2 A_1\alpha_s(q_\perp^2)]^k\right\}. \tag{122}$$

In order to compare this result at NLL to that of [70], we define a new variable h:

$$h \equiv 2 \ln \hat{q}_\perp^2 A_1 \alpha_s(q_\perp^2). \tag{123}$$

In terms of this variable and using Eq.(97) the series that appears in Eq.(122) can be computed:

$$\sum_{k=0}^{\infty} \frac{S^{(k)}(0)}{k!} [-\ln \hat{q}_\perp^2 A_1 \alpha_s(q_\perp^2)]^k = S(-h/2) = 2^h \frac{\Gamma(1+h/2)}{\Gamma(1-h/2)}. \tag{124}$$

In conclusion, we obtain that the NLL resummation factor becomes

$$K^{res}_{NLL}(\hat{q}_\perp^2, q_\perp^2, Q^2) = \frac{1}{\pi} \frac{d}{\hat{q}_\perp^2} \Bigg\{ \theta(\hat{q}_\perp^2 - \eta) \exp\Bigg[\int_{q_\perp^2}^{Q^2} \frac{d\bar{\mu}^2}{\bar{\mu}^2} \Bigg(\Gamma^{res}_{NLL}(\frac{Q^2}{\bar{\mu}^2}, \bar{\mu}^2/q_\perp^2, \alpha_s(\bar{\mu}^2)) - \sum_{j=1}^{2} \gamma'(N_j, \alpha_s(\bar{\mu}^2)) \Bigg) \Bigg] 2^h \frac{\Gamma(1+h/2)}{\Gamma(1-h/2)} \Bigg\}, \tag{125}$$

which gives the same result given in Ref.[70] in the case that the coefficient C_{11} that appears in Eq.(107) is equal to zero and that the arbitrary constants c_1 and c_2 also defined in [70] are equal to one. It is clear that the last two terms of the exponential of our result let the non-singlet parton distribution densities, which enter the $\hat{q}_\perp^2$ derivative, evolve from the scale Q^2 to the scale $q_\perp^2$. We conclude the chapter noting that also in this case the resummed results using the renormalization group approach are less predictive than results obtained with the approach of Ref.[1], as it is shown in Ref.[6]. Furthermore the conditions that reduce our results to those of Ref.[6] in terms of factorization properties is still an interesting open question.