Chapter 7

Predictive power of the resummation formulae

We have shown that the renormalization group resummed expressions are less predictive than those obtained with other approaches discussed in Sec.2.3. In this Chapter we shall compare the various approaches quantitatively. We will show that all the resummation coefficients can be determined by a fixed order computation. This can be useful, because the determination of the resummation coefficients from a fixed order computation represents a possible way to check the correctness of the resummation formulae with strong factorization properties. In particular, in this chapter, we will show how the resummation coefficients g_{mnp} (for the prompt photon case), g_{mn} (for the DIS and DY cases) and $G_{mn}, \tilde{G}_m$ (for the DY transverse momentum distribution) can be determined. For the rapidity distributions of DY and DIS they are the same of the all-inclusive cases (see Chapter 5) The resummation coefficients are determined by comparing the expansion of the resummed anomalous dimension γ in powers of $\alpha_s(Q^2)$ with a fixed-order calculation, which in general has the form:

$$\gamma_{\rm FO}(N,\alpha_s) = \sum_{i=1}^{k_{\rm min}} \alpha_s^i \sum_{j=1}^{i} \gamma_j^i \ln^j \frac{1}{N} + O(\alpha_s^{k_{\rm min}+1}) + O(N^0), \tag{1}$$

where $\gamma_{\rm FO}(N,\alpha_s)$ is the physical anomalous dimension for each individual partonic subprocess for the promt photon case, for the qq channel in the DY case and for the q channel in the DIS case. For the case of the small transverse momentum DY distribution it ha the form:

$$\gamma_{\rm FO}(\bar{\mu}^2 b^2,\alpha_s) = \sum_{i=1}^{k_{\rm min}} \alpha_s^i \sum_{j=0}^{i} \tilde{\gamma}_j^i \ln^j \frac{1}{\bar{\mu} b^2} + O(\alpha_s^{k_{\rm min}+1}) + O(\frac{1}{\bar{\mu}^2 b^2}). \tag{2}$$

The number $k_{\rm min}$ is the minimum order at which the anomalous dimension must be calculated in order to determine its $N^{k-1}LL$ resummation. For prompt photon production, the number of coefficients N_k that must be determined at each logarithmic order, and the minimum fixed order which is necessary in order to determine them are summarized in Table 7.1, according to whether the coefficient function is fully factorized [Eq.(94) sec.4.4], or has factorized N-dependent and N-independent terms

	Prompt photon		
	Eq.(94) sec.4.4	Eq.(96) sec.4.4	Eq.(87) sec.4.4
N_k	$2k$	$\frac{k(k+3)}{2}$	$\frac{k(k+1)(k+5)}{6}$
$k_{\min}$	$k+1$	$2k$	$3k-1$

Table 7.1: Number of coefficients N_k and minimum order of the required perturbative calculation $k_{\min}$ for inclusive prompt photon $N^{k-1}LL$ resummation.

	DIS/DY	
	Eq.(81) sec.3.3	Eq.(80) sec.3.3
N_k	k	$\frac{k(k+1)}{2}$
$k_{\min}$	k	$2k-1$

Table 7.2: Number of coefficients N_k and minimum order of the required perturbative calculation $k_{\min}$ for inclusive DIS and DY $N^{k-1}LL$ resummation.

	DY transverse distribution	
	Eq.(84) sec.6.4	Eq.(81) sec.6.4
N_k	$2k-1$	$\frac{k^2+3k-2}{2}$
$k_{\min}$	k	$2k-1$

Table 7.3: Number of coefficients N_k and minimum order of the required perturbative calculation $k_{\min}$ for small transverse momentum DY $N^{k-1}LL$ resummation.

[Eq.(96) sec.4.4], or not factorized at all [Eq. (87) sec4.4]. In the approach of Refs.[12, 14] the coefficient function is fully factorized, and furthermore some resummation coefficients are related to universal coefficients of Altarelli-Parisi splitting functions, so that $k_{\min} = k$. For prompt-photon production, available results do not allow to test factorization, and test relation of resummation coefficients to Altarelli-Parisi coefficients only to lowest $O(\alpha_s)$. The results for DIS and Drell-Yan, according to whether the coefficient function has factorized N-dependent and N-independent terms as in Refs.[51, 2, 1] [Eq.(81) sec.3.3] or no factorization properties as in [(Eq.80) sec.3.3], are reported in table 7.2. Current fixed-order results support factorization for Drell-Yan and DIS only to the lowest nontrivial order $O(\alpha_s^2)$. In table 7.3, we report also the results for the small transverse DY resummation. We list N_k and $k_{\min}$ for the approach of Ref.[6] [Eq.(84) sec.6.4] and for the renormalization group approach [Eq.(81) sec.6.4]. If the two cases are related by factorization properties of the cross section is not yet understood even if probable. In the following, we present all the proofs of these results.

7.1 Prompt photon production in the strongest factorization case

This is the case of Eq.(94) in section 4.4. In this case there are $N_k = 2k$ non-vanishing coefficients g_{00i} and g_{0i0} $i = 1, 2, \dots, k$. The resummed expression of the anomalous dimension at $N^{k-1}LL$ is given by:

$$\gamma(N, \alpha_s(Q^2)) = \int_1^N \frac{dn}{n} \left(\sum_{i=1}^k g_{0i0} \alpha_s^i (Q^2/n^2) \right) + \int_1^N \frac{dn}{n} \left(\sum_{i=1}^k g_{00i} \alpha_s^i (Q^2/n) \right). \quad (3)$$

We consider first the second integral in Eq.(3). Noting that:

$$\frac{dn}{n} = -\frac{d\alpha_s(Q^2/n)}{\beta(\alpha_s)}, \quad (4)$$

where

$$\beta(\alpha_s) = -b_0 \alpha_s^2 - b_1 \alpha_s^3 + O(\alpha_s^4) \quad (5)$$

$$b_0 \equiv \frac{\beta_0}{4\pi} \qquad b_1 \equiv \frac{\beta_1}{(4\pi)^2}, \quad (6)$$

with β_0 and β_1 given in Eq.(38) in section 1.2, we can rewrite it in the form:

$$\int_{\alpha_s(Q^2)}^{\alpha_s(Q^2/N)} d\alpha_s \frac{\sum_{i=1}^k g_{00i} \alpha_s^i}{\beta_0 \alpha_s^2 \left(1 + \frac{\beta_1}{\beta_0}\alpha_s + \frac{\beta_2}{\beta_0}\alpha_s^2 + \cdots \right)}. \quad (7)$$

Now, we expand up to order α_s^{k-2} each term that compares in the integrand of this last expression and collect all the coefficients that correspond to the same power of α_s. Doing this, we have that the integral (7) can be rewritten in the following form:

$$\frac{1}{b_0} \left\{ \int_{\alpha_s(Q^2)}^{\alpha_s(Q^2/N)} d\alpha_s \left[\frac{g_{001}}{\alpha_s} + (b_1^1 g_{001} + g_{002}) + (b_2^1 g_{001} + b_2^2 g_{002} + g_{003})\alpha_s + \cdots + \right.\right.$$
$$\left.\left. + (b_{k-1}^1 g_{001} + \cdots + b_{k-1}^{k-1} g_{00k-1} + g_{00k}) \alpha_s^{k-2} \right] \right\}, \quad (8)$$

where $k > 1$ and the numbers b_i^j are build up with the coefficients of the β function. Now, we perform the integral over α_s. We get:

$$\int_1^N \frac{dn}{n} \left(\sum_{i=1}^k g_{00i} \alpha_s^i (Q^2/n) \right) =$$
$$= \frac{1}{b_0} \left\{ g_{001} \ln \left(\frac{\alpha_s(Q^2/N)}{\alpha_s(Q^2)} \right) + (b_1^1 g_{001} + g_{002})[\alpha_s(Q^2/N) - \alpha_s(Q^2)] + \right.$$
$$+ \frac{1}{2}(b_2^1 g_{001} + b_2^2 g_{002} + g_{003})[\alpha_s^2(Q^2/N) - \alpha_s^2(Q^2)] + \cdots +$$
$$\left. + \frac{1}{k-1}(b_{k-1}^1 g_{001} + \cdots + b_{k-1}^{k-1} g_{00k-1} + g_{00k})[\alpha_s^{k-1}(Q^2/N) - \alpha_s^{k-1}(Q^2)] \right\} \quad (9)$$

To perform the first integral of Eq.(3), we it is sufficient to note that in this case

$$\frac{dn}{n} = -\frac{d\alpha_s(Q^2/n^2)}{2\beta(\alpha_s)}. \tag{10}$$

and proceed as before. The result that we obtain is:

$$\begin{aligned}
&\int_1^N \frac{dn}{n}\left(\sum_{i=1}^{k} g_{0i0}\alpha_s^i(Q^2/n^2)\right) = \\
= &\frac{1}{2b_0}\Bigg\{ g_{010}\ln\left(\frac{\alpha_s(Q^2/N^2)}{\alpha_s(Q^2)}\right) + (b_1^1 g_{010} + g_{020})[\alpha_s(Q^2/N^2) - \alpha_s(Q^2)] + \\
&+\frac{1}{2}(b_2^1 g_{010} + b_2^2 g_{020} + g_{030})[\alpha_s^2(Q^2/N^2) - \alpha_s^2(Q^2)] + \cdots + \\
&+\frac{1}{k-1}(b_{k-1}^1 g_{010} + \cdots + b_{k-1}^{k-1} g_{0k-10} + g_{0k0})[\alpha_s^{k-1}(Q^2/N^2) - \alpha_s^{k-1}(Q^2)]\Bigg\}
\end{aligned} \tag{11}$$

At this point, we take the first term of Eq.(9) together with the first term of Eq.(11) in order to isolate the first contributions of Eq.(3). We have:

$$\frac{1}{2b_0}\left[g_{010}\ln\left(\frac{\alpha_s(Q^2/N^2)}{\alpha_s(Q^2)}\right) + 2g_{001}\ln\left(\frac{\alpha_s(Q^2/N)}{\alpha_s(Q^2)}\right)\right] \tag{12}$$

From this contribution, we want to extract the firs two LL terms. Hence, using the one loop running of $\alpha_s(Q^2/N^a), a = 1,2$ (Eq.34 of section 1.2), we obtain for this contribution:

$$\begin{aligned}
\alpha_s(Q^2)\ln\frac{1}{N}[-(g_{001} + g_{010})] \quad + \quad & \alpha_s^2(Q^2)\ln^2\frac{1}{N}[b_0/2(g_{001} + 2g_{010})] \\
& + O(\alpha_s^2 \ln(N)) + O(\alpha_s^{3+i}\ln^j(N)),
\end{aligned} \tag{13}$$

where $i \geq 0, 1 \leq j \leq i+3$. Now, we take the second term of Eq.(9) together with the second term of Eq.(11) in order to keep the second the second contributions of Eq.(3) and we have:

$$\frac{1}{2b_0}\left[(b_1^1 g_{010} + g_{020})(\alpha_s(Q^2/N^2) - \alpha_s(Q^2)) + 2(b_1^1 g_{001} + g_{002})(\alpha_s(Q^2/N) - \alpha_s(Q^2))\right]. \tag{14}$$

From this contribution, we want to extract the first two NLL terms. In order to do this, we use the one loop running of $\alpha_s(Q^2/N^a), a = 1,2$ and observe that the coefficients of g_{001} and g_{010} are modified by the last two terms of Eq.(13). We get:

$$\begin{aligned}
&\alpha_s^2(Q^2)\ln(1/N)[-(c_1^1 g_{010} + d_1^1 g_{001} + g_{020} + g_{002})] + \\
&\alpha_s^3(Q^2)\ln^2(1/N)[b_0(2\tilde{c}_1^1 g_{010} + \tilde{d}_1^1 g_{001} + 2g_{020} + g_{002})] \\
&+O(\alpha_s^3\ln(N)) + O(\alpha_s^{4+i}\ln^j(N)),
\end{aligned} \tag{15}$$

where $c_1^1, d_1^1, \tilde{c}_1^1, \tilde{d}_1^1$ are coefficients (that are of no concern to us) and $i \geq 0, 1 \leq j \leq i+3$. This procedure can be repeated for all the other contributions. So, we take the k-th term of Eq.(9) together with the k-th term of Eq.(11) in order to keep the k-th contributions of Eq.(3). For the general k-th contribution, we get:

$$\frac{1}{2b_0}\left[\frac{1}{k-1}(b^1_{k-1}g_{010}+\cdots+b^{k-1}_{k-1}g_{0k-10}+g_{0k0})(\alpha_s^{k-1}(Q^2/N^2)-\alpha_s^{k-1}(Q^2))+\right.$$
$$\left.+\frac{2}{k-1}(b^1_{k-1}g_{001}+\cdots+b^{k-1}_{k-1}g_{00k-1}+g_{00k})(\alpha_s^{k-1}(Q^2/N)-\alpha_s^{k-1}(Q^2))\right]. \qquad (16)$$

From this contribution, we want to extract the first two $N^{k-1}LL$ terms. In order to do this, again, we use the one loop running of $\alpha_s(Q^2/N^a), a=1,2$ and note that the coefficients g_{00i} and g_{0i0} with $i=1,2,\ldots,k-1$ are modified by the previous terms. For this generic term, we get:

$$\alpha_s^k \ln\frac{1}{N}[-(c^1_{k-1}g_{010}+\cdots+c^{k-1}_{k-1}g_{0k-10}+d^1_{k-1}g_{001}+\cdots+d^{k-1}_{k-1}g_{00k-1}+g_{0k0}+g_{00k})]+$$
$$+\alpha_s^{k+1}\ln^2\frac{1}{N}[b_0k/2(2\tilde{c}^1_{k-1}g_{010}+\cdots+2\tilde{c}^{k-1}_{k-1}g_{0k-10}+\tilde{d}^1_{k-1}g_{001}+\cdots+\tilde{d}^{k-1}_{k-1}g_{00k-1}$$
$$+2g_{0k0}+g_{00k})]+O(\alpha_s^{k+1}\ln(N))+O(\alpha_s^{k+2+i}\ln^j(N)), \qquad (17)$$

where $i\geq 0, 1\leq j\leq i+3$. To summarize, Eqs. (13,15,17) tell us that from the expression of the physical anomalous dimension Eq.(3), we can extract the following linear combinations of the coefficients $g_{010},\ldots,g_{0k0},g_{001},\ldots,g_{00k}$:

$$\begin{aligned}
l_1 &= -(g_{001}+g_{010})\\
l_2 &= \frac{b_0}{2}(g_{001}+2g_{010})\\
l_3 &= -(c^1_1g_{010}+d^1_1g_{001}+g_{020}+g_{002})\\
l_4 &= b_0(2\tilde{c}^1_1g_{010}+\tilde{d}^1_1g_{001}+2g_{020}+g_{002})\\
&\cdots\\
l_{2k-1} &= -(c^1_{k-1}g_{010}+\cdots+c^{k-1}_{k-1}g_{0k-10}+d^1_{k-1}g_{001}+\cdots+d^{k-1}_{k-1}g_{00k-1}+g_{0k0}+g_{00k})\\
l_{2k} &= \frac{kb_0}{2}(2\tilde{c}^1_{k-1}g_{010}+\cdots+2\tilde{c}^{k-1}_{k-1}g_{0k-10}+\tilde{d}^1_{k-1}g_{001}+\cdots+\tilde{d}^{k-1}_{k-1}g_{00k-1}+2g_{0k0}+g_{00k}),
\end{aligned}$$

with $k\geq 2$ and l_i, $i=1,\ldots,2k$ the known terms. These are $2k$ independent linear combinations that determine the $2k$ coefficients $g_{010},\ldots,g_{0k0},g_{001},\ldots,g_{00k}$ of a $N^{k-1}LL$ resummation comparing them with the correspondent terms of Eq.(2) up to order α_s^{k+1}. This is a direct consequence of the fact that the two vectors $(1,1)$ and $(1,2)$ are independent. This shows that, in order to obtain a $N^{k-1}LL$ resummation in the case of the strongest factorization (Eq.(94) in section 4.4) , we need to know a $N^{k+1}LO$ fixed order calculation of the physical anomalous dimension. Hence, in this case $k_{\min}=k+1$.

7.2 Prompt photon production in the weaker factorization case

This is the case of Eq.(96) in section 4.4. In this case in order to perform a LL resummation, we need two coefficients (g_{010},g_{001}); to perform a NLL resummation three more coefficients are added $(g_{020}, g_{002}, g_{011})$; in general to perform a $N^{k-1}LL$ resummation $k+1$ coefficients are added $(g_{0ij}\,, i+j=k)$ to those of the $N^{k-2}LL$ resummation. Hence, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k}(p+1) = \frac{k(k+3)}{2}, \tag{18}$$

coefficients. We want to determine $k_{\min}$ in Eq.(2) so that all the $k(k+3)/2$ are fixed by the same number of independent conditions obtained from the fixed order expansion of the resummed physical anomalous dimension. We note, first of all, that this happens if we can extract 2 independent conditions from the LL contributions, 3 from the NLL one and $k+1$ from the $N^{k-1}LL$. The $N^{k-1}LL$ expression of the physical anomalous dimension in this case is given by:

$$\begin{aligned}\gamma(N, \alpha_s(Q^2)) &= \int_1^N \frac{dn}{n}\left(\sum_{i=1}^{k} g_{0i0}\alpha_s^i(Q^2/n^2)\right) + \int_1^N \frac{dn}{n}\left(\sum_{i=1}^{k} g_{00i}\alpha_s^i(Q^2/n)\right) + \\ &+ \int_1^N \frac{dn}{n}\sum_{s=2}^{k}\sum_{i=1}^{s-1} g_{0is-i}\alpha_s^i(Q^2/n^2)\alpha_s^{s-i}(Q^2/n). \end{aligned}\tag{19}$$

The first two LL contributions have been already computed and are given in Eq.(13). Now, we extract the first 3 NLL contributions of Eq.(19). Recalling the derivation of Eq.(14), we obtain that these contributions are contained in the following expression:

$$\begin{aligned}&\frac{1}{2b_0}\left[(b_1^1 g_{010} + g_{020})(\alpha_s(Q^2/N^2) - \alpha_s(Q^2)) + 2(b_1^1 g_{001} + g_{002})(\alpha_s(Q^2/N) - \alpha_s(Q^2))\right] \\ &+ \int_1^N \frac{dn}{n} g_{011}\alpha_s(Q^2/n^2)\alpha_s(Q^2/n). \end{aligned}\tag{20}$$

Since

$$\begin{aligned}\alpha_s(Q^2/p^a) &= \frac{\alpha_s(Q^2)}{1 + a\beta_0\alpha_s(Q^2)\ln\frac{1}{p}} + O(\alpha_s^{2+i}\ln^j \frac{1}{p}), \quad i \geq 0, \quad 1 \leq j \leq i \\ \frac{1}{1 + ab_0\alpha_s(Q^2)\ln\frac{1}{p}} &= \sum_{j=0}^{\infty}(-)^j a^j b_0^j \alpha_s^j(Q^2)\ln^j \frac{1}{p}, \end{aligned}\tag{21}$$

and keeping in mind that corrections to the NLL come from Eq.(13), we have that Eq.(20) become:

$$\begin{aligned}&\frac{1}{2b_0}\sum_{j=1}^{\infty}(-)^j[2^j(c_{1j}^1 g_{010} + g_{020}) + 2(d_{1j}^1 g_{001} + g_{002})]b_0^j\alpha_s^{j+1}(Q^2)\ln^j\frac{1}{N} + \\ &+ \int_1^N \frac{dn}{n} g_{011}\alpha_s^2(Q^2)\left(\sum_{j=0}^{\infty}(-)^j b_0^j\alpha_s^j(Q^2)\ln^j\frac{1}{n}\right)\left(\sum_{i=0}^{\infty}(-)^i 2^i b_0^i\alpha_s^i(Q^2)\ln^i\frac{1}{n}\right). \end{aligned}\tag{22}$$

The Cauchy product of the two series in Eq.(22) is given by

$$(\sum_{j=0}^{\infty}(-)^j b_0^j \alpha_s^j(Q^2)\ln^j\frac{1}{n})(\sum_{i=0}^{\infty}(-)^i 2^i b_0^i \alpha_s^i(Q^2)\ln^i\frac{1}{n}) =$$
$$= \frac{1}{2b_0}\sum_{j=1}^{\infty}(-)^{j-1}\left(\sum_{i=1}^{j}2^i\right)b_0^j\alpha_s^{j-1}(Q^2)\ln^{j-1}\frac{1}{n} \tag{23}$$

and

$$\sum_{i=1}^{j}2^i = 2(2^j-1). \tag{24}$$

Now, because

$$\frac{dn}{n} = -d\ln\frac{1}{n}, \tag{25}$$

we can perform the integration. We get

$$\frac{1}{2b_0}\sum_{j=1}^{3}(-)^j\left[2^j(c_{1j}^1 g_{010}+g_{020})+2(d_{1j}^1 g_{001}+g_{002})+\frac{2(2^j-1)}{j}g_{011}\right]b_0^j\alpha_s^{j+1}(Q^2)\ln^j\frac{1}{n}, \tag{26}$$

where c_1^{1j}, d_1^{1j} are certain coefficients we don not need to worry about. The first three NLL contributions are given by $j = 1, 2, 3$. The last step is to extract the first $k+1$ $N^{k-1}LL$ contributions that come from Eq.(19). Recalling how Eq.((16)) was computed, we have that the desired contributions are contained in the following expression:

$$\frac{1}{2b_0}\left[\frac{1}{k-1}(b_{k-1}^1 g_{010}+\cdots+b_{k-1}^{k-1}g_{0k-10}+g_{0k0})(\alpha_s^{k-1}(Q^2/N^2)-\alpha_s^{k-1}(Q^2))+\right.$$
$$\left.+\frac{2}{k-1}(b_{k-1}^1 g_{001}+\cdots+b_{k-1}^{k-1}g_{00k-1}+g_{00k})(\alpha_s^{k-1}(Q^2/N)-\alpha_s^{k-1}(Q^2))\right]+$$
$$+\int_1^N\frac{dn}{n}\sum_{i=1}^{k-1}g_{0ik-i}\alpha_s^i(Q^2/n^2)\alpha_s^{k-i}(Q^2/n) \tag{27}$$

We use the following relations

$$\alpha_s^r(Q^2/p^a) = \frac{\alpha_s^r(Q^2)}{(1+ab_0\alpha_s(Q^2)\ln\frac{1}{p})^r}+O(\alpha_s^{2+i}\ln^j\frac{1}{p}),\quad i\geq 0, 1\leq j\leq i+1,$$
$$\frac{1}{(1+ab_0\alpha_s(Q^2)\ln\frac{1}{p})^r} = \sum_{m=0}^{\infty}(-)^m\binom{r+m-1}{m}a^m b_0^m\alpha_s^m(Q^2)\ln^m\frac{1}{p}, \tag{28}$$

where $\binom{r}{m}$ are the usual binomial coefficients. With this, we can compute the integral in Eq.(27) performing the Cauchy product of the two series expansion of $\alpha_s(Q^2/n^2)$

and of $\alpha_s(Q^2/n)$ and performing explicitly the integral using the change of variable Eq.(3). We get:

$$\int_1^N \frac{dn}{n} \sum_{i=1}^{k-1} g_{0ik-i}\alpha_s^i(Q^2/n^2)\alpha_s^{k-i}(Q^2/n) = \sum_{i=1}^{k-1} g_{0ik-i} \sum_{m=0}^{\infty} C_m^{(i,k-i)} b_0^m \alpha_s^{k+m}(Q^2) \ln^{m+1}\frac{1}{N}, \tag{29}$$

where

$$C_m^{(i,j)} = \frac{(-1)^{m+1}}{m+1} \sum_{l=0}^{m} 2^l \begin{pmatrix} l+i-1 \\ i-1 \end{pmatrix} \begin{pmatrix} m-l+j-1 \\ j-1 \end{pmatrix}, \tag{30}$$

and where

$$\begin{pmatrix} n \\ -1 \end{pmatrix} = \frac{\Gamma(n+1)}{\Gamma(0)\Gamma(n+2)} \tag{31}$$

is equal to 1 for $n=-1$ and 0 otherwise. Thefore, keeping in mind the calculation of Eq.(16) and taking the first $k+1$ $N^{k-1}LL$ contributions of Eq.(27), we have

$$\begin{aligned}
&\sum_{m=0}^{k} \Bigg\{ \frac{(-)^{m+1}}{k-1} \begin{pmatrix} k+m-1 \\ m+1 \end{pmatrix} [2^m (c_{k-1m}^1 g_{010} + \cdots + c_{k-1m}^{k-1} g_{0k-10} + g_{0k0}) + \\
&+(d_{k-1m}^1 g_{001} + \cdots + d_{k-1j}^{k-1} g_{00k-1} + g_{00k}) + \sum_{t=2}^{k-1}\sum_{i=1}^{t-1} g_{0it-i} f_{itm}^{(k-1)}] + \\
&+\sum_{i=1}^{k-1} g_{0ik-i} C_m^{(i,k-i)} \Bigg\} b_0^m \alpha_s^{k+m}(Q^2) \ln^{m+1}\frac{1}{N},
\end{aligned} \tag{32}$$

where $c_{k-1m}^i, d_{k-1m}^i, f_{itm}^{(k-1)}$ are certain coefficients we do not have to worry about. At this point, we can make some simplifications. In fact, since

$$C_m^{(0,k)} = \frac{(-1)^{m+1}}{m+1} \begin{pmatrix} m+k-1 \\ k-1 \end{pmatrix} = \frac{(-1)^{m+1}}{k-1} \begin{pmatrix} m+k-1 \\ m+1 \end{pmatrix} \tag{33}$$

and (see Appendix E)

$$C_m^{(k,0)} = 2^m C_m^{(0,k)}, \tag{34}$$

we can write Eq.(32) in the following form:

$$\begin{aligned}
&\sum_{m=0}^{k} \Bigg\{ [C_m^{(k,0)} (c_{k-1m}^1 g_{010} + \cdots + c_{k-1m}^{k-1} g_{0k-10}) + \\
&+C_m^{(0,k)} (d_{k-1m}^1 g_{001} + \cdots + d_{k-1j}^{k-1} g_{00k-1}) + C_m^{(0,k)} \sum_{t=2}^{k-1}\sum_{i=1}^{t-1} g_{0it-i} f_{itm}^{(k-1)}] + \\
&+\sum_{i=0}^{k} g_{0ik-i} C_m^{(i,k-i)} \Bigg\} b_0^m \alpha_s^{k+m}(Q^2) \ln^{m+1}\frac{1}{N},
\end{aligned} \tag{35}$$

What this result tells us, is that passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ resummation, $k+1$ new resummation coefficients are added. In Eq.(35), we have $k+1$

conditions for this coefficients (one for each m) to be set equal to the corresponding fixed order contribution of Eq.(2) . We shall now show that this conditions are independent. This is equivalent to showing that the $k+1$ linear combinations

$$\sum_{i=0}^{k} g_{0ik-i}\tilde{C}_m^{(i,k-i)} \qquad m = 0, 1, \ldots, k \tag{36}$$

with

$$\tilde{C}_m^{(i,j)} \equiv (-)^{m+1}(m+1)C_m^{(i,j)} = \sum_{l=0}^{m} 2^l \binom{l+i-1}{i-1}\binom{m-l+j-1}{j-1} \tag{37}$$

are independent. Moreover, this is equivalent to showing that for each k the columns of the $(k+1)\times(k+1)$ matrix $A_{mi}^k \equiv \tilde{C}_m^{(i,k-1)}$ are independent vectors. To show this, we need to use two identities proved in Appendix E:

$$\tilde{C}_m^{(i,0)} = 2^m \tilde{C}_m^{(0,i)} \tag{38}$$

$$\tilde{C}_m^{(i,j)} = 2\tilde{C}_m^{(i,j-1)} - \tilde{C}_m^{(i-1,j)}; \qquad i, j \geq 1, \tag{39}$$

$$\tilde{C}_m^{(0,i)} = \binom{m+i-1}{i-1} \tag{40}$$

which allows us to to compute the columns of the matrix $A_{mj}^{(k)}$ explicitly for all k. To show their independence, we use induction on k, i.e. we demonstrate the independence of the columns for $k=1$ and then we assume that the property is valid for $k-1$ to prove that it remains valid for k. In the case $k=1$, we have a 2×2 matrix:

$$(\tilde{C}_m^{(0,1)} \quad \tilde{C}_m^{(1,0)}) = (1 \quad 2^m), \tag{41}$$

and now it is clear the two columns are independent, because 1 and 2^m are independent functions of m. Now, using the induction hypothesis

$$\sum_{i=0}^{k-1} \alpha_i \tilde{C}_m^{(i,k-1-i)} = 0 \Longleftrightarrow \alpha_i = 0, \quad i = 0, \ldots, k-1, \tag{42}$$

we want to show that it is sufficient to prove that:

$$\sum_{i=0}^{k} \beta_i \tilde{C}_m^{(i,k-i)} = 0 \Longleftrightarrow \beta_i = 0, \quad i = 0, \ldots, k. \tag{43}$$

Using the relations (38,39), we have:

$$\begin{aligned}
\sum_{j=0}^{k}\beta_j\tilde{C}_m^{(j,k-j)} &= (\beta_0+2^m\beta_k)\tilde{C}_m^{(0,k)}+\sum_{j=1}^{k-1}\beta_j\tilde{C}_m^{(j,k-j)} \\
&= (\beta_0+2^m\beta_k)\tilde{C}_m^{(0,k)}+2\sum_{j=1}^{k-1}\beta_j\tilde{C}_m^{(j,k-1-j)}-\sum_{j=1}^{k-1}\beta_j\tilde{C}_m^{(j-1,k-j)} \\
&= (\beta_0+2^m\beta_k)\tilde{C}_m^{(0,k)}+2\sum_{j=1}^{k-1}\beta_j\tilde{C}_m^{(j,k-1-j)}-\sum_{j'=0}^{k-2}\beta_{j'+1}\tilde{C}_m^{(j',k-1-j')} \\
&= (\beta_0+2^m\beta_k)\tilde{C}_m^{(0,k)}-\beta_1\tilde{C}_m^{(0,k-1)}+\sum_{j=1}^{k-2}(2\beta_j-\beta_{j+1})\tilde{C}_m^{(j,k-1-j)} \\
&\quad +2\beta_{k-1}\tilde{C}_m^{(k-1,0)}=0 \qquad (44)
\end{aligned}$$

Now, from Eq.(40), we know that $\tilde{C}_m^{(0,k)}$ is a degree-$(k-1)$ polynomial in m. Furthermore, from Eq.(39), we know that the vectors $\tilde{C}_m^{(j,k-1-j)}$ with $j=0,\ldots,k-1$ are at most polynomials of degree $k-2$ in m. Consequently, Eq.(44) can be satisfied if and only if

$$(\beta_0+2^m\beta_k)\tilde{C}_m^{(0,k)}=0, \qquad (45)$$

$$-\beta_1\tilde{C}_m^{(0,k-1)}+\sum_{j=1}^{k-2}(2\beta_j-\beta_{j+1})\tilde{C}_m^{(j,k-1-j)}+2\beta_{k-1}\tilde{C}_m^{(k-1,0)}=0. \qquad (46)$$

From the first, it follows that:

$$\beta_0=\beta_k=0, \qquad (47)$$

while from the second, thanks to the induction hypothesis Eq.(42), we have that

$$\beta_1=\beta_{k-1}=0, \qquad (48)$$

and that

$$\beta_1=\frac{1}{2}\beta_2=\frac{1}{4}\beta_3=\cdots=\frac{1}{2^{k-2}}\beta_{k-1}. \qquad (49)$$

In conclusion from Eqs.(47,48,49) it follows that

$$\beta_j=0, \qquad j=0,\ldots,k. \qquad (50)$$

This completes the proof that the columns of the squared $(k+1)\times(k+1)$ matrices $A_{mj}^{(k)}\equiv\tilde{C}_m^{(j,k-j)}$ are independent for all k. This shows that, in order to obtain a $N^{k-1}LL$ resummation in the case of the weaker factorization (Eq.(96) in section 4.4), we need to know a $N^{2k}LO$ fixed order calculation of the physical anomalous dimension. Hence, in this case $k_{\min}=2k$.

7.3 Prompt photon in the general case

Let us now consider the most general case, in which the coefficient function does not satisfy any factorization property. This is the case of Eq.(87) in section 4.4. In this case, in order to perform a LL resummation we need 2 coefficients (g_{001}, g_{010}); to perform a NLL resummation 5 coefficients are added $(g_{002}, g_{101}, g_{020}, g_{110}, g_{011})$ and to perform a $N^{k-1}LL$ resummation $k(k+3)/2$ coefficients are added (g_{mnp} with $m+n+p=k$ but without $n=p=0$). Thus, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k} \frac{p(p+3)}{2} = \frac{k(k+1)(k+5)}{6} \tag{51}$$

coefficients. The $N^{k-1}LL$ expression of the physical anomalous dimension is given, in this case, by

$$\gamma(N, \alpha_s(Q^2)) = \int_1^N \frac{dn}{n} \sum_{s=1}^{k} \sum_{i=0}^{s-1} \sum_{j=0}^{s-i} g_{ijs-i-j} \alpha_s^i(Q^2) \alpha_s^j(Q^2/n^2) \alpha_s^{s-i-j}(Q^2/n). \tag{52}$$

Now, we proceed in the same way as we have done in section 7.2 and we find that the $k(k+3)/2$ new coefficients that are added passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ contributions appear only in the following combinations:

$$\sum_{i=0}^{k-1} \sum_{j=0}^{k-i} g_{ijk-i-j} \sum_{m=0}^{\infty} C_m^{(j,k-i-j)}\, b_0^m \alpha_s(Q^2)^{k+m} \ln^{m+1} \frac{1}{N}. \tag{53}$$

Each term with fixed m in the expansion Eq. (66) provides a new condition on these coefficients. However, these conditions are not linearly independent for all choices of m. Indeed, let us define the matrix $C_m^{(j,k-i-j)} \equiv D^{(k)}_{m(i,j)}$, where the lines are labelled by the index m and the columns by the multi-index (i,j). This matrix gives the linear combination of the coefficients g_{mnp} in Eq.(66) to be determined and it turns out to be of rank

$$rg(D^{(k)}_{m(i,j)}) = 2k \leq \frac{k(k+3)}{2}. \tag{54}$$

We shall now prove this statement: $D^{(k)}_{m(i,j)}$ is a $M \times \frac{k(k+3)}{2}$ matrix , whose columns are the M-component vectors

$$D^{(k)}_m = C_m^{(j,k-i-j)}; \qquad 0 \leq i \leq k-1; \quad 0 \leq j \leq k-i; \qquad 0 \leq m \leq M. \tag{55}$$

We use induction on k. For $k=1$, $D^{(1)}$ is a 2×2 matrix with columns

$$D^{(1)}_m = \left(C_m^{(0,1)}, \quad C_m^{(1,0)}\right) = \frac{(-1)^{m+1}}{m+1} (1, \quad 2^m), \tag{56}$$

that are linearly independent; the rank of $D^{(1)}$ is 2. Let us check explicitly also the case $k=2$. In this case

$$D^{(2)}_m = \left(C_m^{(0,1)}, \quad C_m^{(1,0)}, \quad C_m^{(0,2)}, \quad C_m^{(1,1)}, \quad C_m^{(2,0)}\right). \tag{57}$$

The first two columns are the same as in the case $k = 1$: they span a 2-dimensional subspace. The last three columns are independent as a consequence of Eq.(43) with $k = 1$. Furthermore, $C_m^{(0,2)}$ and $C_m^{(2,0)} = 2^m C_m^{(0,2)}$ are independent of all other columns, because they are the only ones that are proportional to a degree-1 polynomial in m. Finally, $C_m^{(1,1)}$ is a linear combination of the first two columns, as a consequence of Eqs.(37,39) with $i = j = 1$. Thus, the rank of $D^{(2)}$ is $2+2 = 4$. We now assume that $D^{(k-1)}$ has rank $2(k-1)$, and we write the columns of $D^{(k)}$ as

$$D_m^{(k)} = \left(C_m^{(j,k-1-i-j)}, \quad C_m^{(l,k-l)}\right) \tag{58}$$

$$0 \le i \le k-2, \qquad 0 \le j \le k-1-i \qquad\qquad 0 \le l \le k. \tag{59}$$

By the induction hypothesis, only $2(k-1)$ of the columns $C_m^{(j,k-1-i-j)}$ are independent. The columns $C_m^{(l,k-l)}$ are all independent as a consequence of Eq.(43); among them, those with $1 \le l \le k-1$ can be expressed as linear combinations of $C_m^{(j,k-1-i-j)}$ by Eq. (39). Only $C_m^{(0,k)}$ and $C_m^{(k,0)}$ are independent of all other columns because they are proportional to a degree-$(k-1)$ polynomial in m, while all others are at most of degree $(k-2)$. Hence, only two independent vectors are added to the $2(k-1)$-dimensional subspace spanned by $C_m^{(j,k-1-i-j)}$, and the rank of $D^{(k)}$ is

$$2(k-1) + 2 = 2k. \tag{60}$$

It follows that each individual terms in the sum over m in Eq. (66) depends only on $2k$ independent linear combinations of the coefficients $g_{ijk-i-j}$, $0, \le i \le k-1$, $0 \le j \le k-i$. This means that the N^{k-1}LL order resummed result depends only on $2k$ independent linear combinations of the $k(k+3)/2$ new coefficients that are added passing from the $N^{k-2}LL$ resummation to the $N^{k-1}LL$ one and that the remaining coefficients are arbitrary. Because a term with fixed m in Eq.(66) is of order α_s^{k+m}, this implies that a computation of the anomalous dimension up to fixed order $k_{\min} = 3k-1$ is sufficient for the N^{k-1}LL resummation, because $m = 0, 1, \ldots, k-1$. Note that when going from N^{k-1}LL to N^{k}LL, at this higher order, in general some new linear combinations of the $k(k+3)/2$ coefficients, that we have added from the $N^{k-2}LL$ to the $N^{k-1}LL$, will appear through terms depending on β_1. Hence, some of the combinations of coefficients that were left undetermined in the N^{k-1}LL resummation will now become determined. However, this does not affect the value $k_{\min}$ of the fixed-order accuracy needed to push the resummed accuracy at one extra order. In conclusion, even in the absence of any factorization, despite the fact that now the number of coefficients which must be determined grows cubically according to Eq. (51), the required order in α_s of the computation which determines them grows only linearly.

7.4 DIS and DY in all cases

This is the case of Eq.(80) in section 3.3. Here we discuss the case without assuming any factorization property, because the factorized case will be recovered as a particular case. So, in the general case, at LL we need to determine 1 coefficient (g_{01}); at

NLL 2 coefficients are added (g_{02}, g_{11}) and at $N^{k-1}LL$, k coefficients (g_{ik-i} with $i = 0, 1, \ldots, k-1$) are added to the $N^{k-2}LL$ ones . Thus, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k} p = \frac{k(k+1)}{2} \tag{61}$$

coefficients. The $N^{k-1}LL$ expression of the physical anomalous dimension is given by

$$\gamma(N, \alpha_s(Q^2)) = a \int_1^N \frac{dn}{n} \sum_{s=1}^{k} \sum_{i=0}^{s-1} g_{is-i} \alpha_s^i(Q^2) \alpha_s^{s-i}(Q^2/n^a), \tag{62}$$

where $a = 1$ for DIS and $a = 2$ for DY. Now, we proceed in the same way as we have done in the previous sections and we find that the k new coefficients that are added passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ contributions appear only in the following combinations:

$$\sum_{i=0}^{k-1} g_{ik-i} \sum_{m=0}^{\infty} a^{m+1} C_m^{(0,k-i)}\, b_0^m \alpha_s(Q^2)^{k+m} \ln^{m+1} \frac{1}{N}. \tag{63}$$

Again, each term with fixed m in this expansion provides a new condition on these coefficients. These conditions are all linearly independent for any choice of m, because the $k \times k$ matrix $A_{mi}^{(k)} \equiv C_m^{0,k-i}$ with $m, i = 0, 1, \ldots, k-1$ has all independent columns. This is a direct consequence of the fact that each column is a polynomial with different degree in m. This implies that, in order to determine a $N^{k-1}LL$ resummation, a computation of the physical anomalous dimension up to order $k_{\min} = 2k-1$ is sufficient. In the more restrictive case Eq.(81) in Section 3.3, the only non-vanishing coefficients are $g_{0,k}$. This means that going from the $N^{k-2}LL$ to the $N^{k-1}LL$ we need to take only one combination of the expansion Eq.(63). Hence, in this more restrictive case, $N_k = k$ and $k_{\min} = k$.

7.5 DY small transverse momentum distribution

This is the case of Eq.(81) in section 6.4 obtained with the renormalization group approach. The result of the approach of Ref.[6] (reported in Eq.(81) of section 6.4) will be recovered as a particular case. In the most general case, at LL we need to determine 1 coefficient (G_{01}); at NLL 3 more coefficients are added ($G_{11}, G_{02}, \tilde{G}_1$) and at $N^{k-1}LL$, $k+1$ coefficients (G_{ik-i} with $i = 0, 1, \ldots, k-1$ and $\tilde{G}_{k-1}$) are added. Therefore, in order to perform a $N^{k-1}LL$ resummation, we need to determine

$$N_k = \sum_{p=1}^{k} (p+1) - 1 = \frac{k^2 + 3k - 2}{2} \tag{64}$$

coefficients. The $N^{k-1}LL$ expression of the physical anomalous dimension is given by

$$\gamma(1, \bar{\mu}^2 b^2, \alpha_s(\bar{\mu}^2)) = -\int_0^{\bar{\mu}^2 b^2} \frac{dn'}{n'} \sum_{s=1}^{k} \sum_{i=0}^{s-1} G_{is-i} \alpha_s^i(\bar{\mu}^2) \alpha_s^{s-i}(\bar{\mu}^2/n') + \sum_{i=1}^{k-1} \tilde{G}_i \alpha_s^i(\bar{\mu}^2). \tag{65}$$

Now, we proceed as before and we find that the k new coefficients that are added passing from the $N^{k-2}LL$ to the $N^{k-1}LL$ contributions appear only in the following combinations:

$$-\sum_{i=0}^{k-1} G_{ik-i} \sum_{m=0}^{\infty} C_m^{(0,k-i)}\, b_0^m \alpha_s(\bar{\mu}^2)^{k+m}\, \ln^{m+1}\left(\frac{1}{\bar{\mu}^2 b^2}\right) + \tilde{G}_{k-1}\alpha_s^{k-1}(\bar{\mu}^2). \qquad (66)$$

As before, each term with fixed m in this expansion provides a new independent condition on this coefficients. This implies that, in order to determine a $N^{k-1}LL$ resummation, a computation of the physical anomalous dimension up to order $k_{\min} = 2k-1$ is sufficient. In the more restrictive case of Eq.(84 sec.6.4), the only non-vanishing coefficients are $G_{0,k}$ and $\tilde{G}_{k-1}$. This means that going from the $N^{k-2}LL$ to the $N^{k-1}LL$ only 2 coefficients are added but the LL where we have only one coefficient. Hence, in this more restrictive case, $N_k = 2k-1$ and $k_{\min} = k$.