Chapter 3

Renormalization group resummation of DIS and DY

In this chapter, we analyze in detail how the renormalization group approach to resummation works in the case of the all-inclusive Drell-Yan (DY) and deeep inelastic scattering (DIS). We recall that this is done only for the quark-anti-quark channel in the DY case and only for the quark channel in the DIS case as discussed in Section 2.2. First, we determine the N dependence of the regularized coefficient function in the large-N limit. Then we show that, given this form of the N-dependence of the regularized cross section, renormalization group invariance fixes the all-order dependence of the physical anomalous dimension in such a way that the infinite class of leading, next-to-leading etc. resummations can be found in terms of fixed order computations. This approach will lead us to resummation formulae valid to all logrithmic orders.

3.1 Kinematics of inclusive DIS in the soft limit

In the case of deep-inelastic scattering, the relevant parton subprocesses are:

$$\begin{aligned} \gamma^*(q) + \mathcal{Q}(p) &\to \mathcal{Q}(p') + \mathcal{X}(K) && (1) \\ \gamma^*(q) + \mathcal{G}(p) &\to \mathcal{Q}(p') + \mathcal{X}(K), && (2) \end{aligned}$$

where $\mathcal{Q}$ is a quark or an anti-quark, $\mathcal{G}$ is a gluon and $\mathcal{X}$ is any collection of quarks and gluons. We are interested in the most singular parts in the limit $z \to 1$ of the phase space and of the amplitude for the generic processes Eqs.(1,2). We treat first the tree level processes and then we will introduce the loops. Using Eq.(12) in Appendix B with $m = 1$ recursively, we can express the phase space for a generic process in terms of two-body phase space integrals. For the DIS processes Eqs.(1,2) with n extra emissions ($K = k_1 + \cdots + k_n$) we have

$$\begin{aligned} & d\phi_{n+1}(p+q; k_1, \ldots, k_n, p') \\ = & \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p+q; k_n, P_n) d\phi_n(P_n; k_1, \ldots, k_{n-1}, p') \end{aligned}$$

$$
\begin{aligned}
&= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p+q;k_n,P_n) \\
&\quad \times \int_0^{M_n^2} \frac{dM_{n-1}^2}{2\pi} d\phi_2(P_n;k_{n-1},P_{n-1}) d\phi_{n-1}(P_{n-1};k_1,\ldots,k_{n-2},p') \\
&= \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(P_{n+1};k_n,P_n) \int_0^{M_n^2} \frac{dM_{n-1}}{2\pi} d\phi_2(P_n;k_{n-1},P_{n-1}) \\
&\quad \times \cdots \times \int_0^{M_3^2} \frac{dM_2^2}{2\pi} d\phi_2(P_3;k_2,P_2) d\phi_2(P_2;k_1,P_1), \qquad (3)
\end{aligned}
$$

where we have defined $M_n^2 \equiv P_n^2$, $P_1 \equiv p'$, $P_{n+1} \equiv p+q$ and

$$
s \equiv M_{n+1}^2 = (p+q)^2 = Q^2 \frac{(1-z)}{z}; \qquad z = \frac{Q^2}{2p \cdot q}. \qquad (4)
$$

Now, according to Eq.(16) in Appendix B we have for each two-body phase space

$$
d\phi_2(P_{i+1};k_i,P_i) = N(\epsilon)(M_{i+1})^{-2\epsilon} \left(1 - \frac{M_i^2}{M_{i+1}^2}\right)^{1-2\epsilon} d\Omega_i; \quad i = 1,\ldots n, \qquad (5)
$$

where

$$
N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}} \qquad (6)
$$

and Ω_i is the solid angle in the center-of-mass frame of P_{i+1}. We perform the change of variables

$$
z_i = \frac{M_i^2}{M_{i+1}}; \quad M_i^2 = s z_n \ldots z_i; \quad i = 2,\ldots,n. \qquad (7)
$$

From the fact that $M_{i+1}^2 \geq M_i^2$ (we have one more real particle in P_{i+1} than in P_i), it follows that

$$
0 \leq z_i \leq 1. \qquad (8)
$$

From Eq.(7) we get

$$
dM_n^2 dM_{n-1}^2 \cdots dM_2^2 = \det\left(\frac{\partial M_i^2}{\partial z_j}\right) dz_n dz_{n-1} \cdots dz_2, \qquad (9)
$$

where

$$
\det\left(\frac{\partial M_i^2}{\partial z_j}\right) = \frac{\partial M_n^2}{\partial z_n} \cdots \frac{\partial M_2^2}{\partial z_2} = s^{n-1} z_n^{n-2} z_{n-1}^{n-3} \cdots z_3. \qquad (10)
$$

Furthermore, in these new variables the two-body phase space becomes

$$
d\phi_2(P_{i+1};k_i,P_i) = N(\epsilon) s^{-\epsilon} (z_n z_{n-1} \cdots z_{i+1})^{-\epsilon} (1-z_i)^{1-2\epsilon} d\Omega_i. \qquad (11)
$$

Substituting Eqs.(9,10,11) into the generic phase space Eq.(3), we finally get

$$
\begin{aligned}
&d\phi_{n+1}(p+q;k_1,\ldots,k_n,p') = 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^n s^{n-1-n\epsilon} d\Omega_n \cdots d\Omega_1 \\
&\times \int_0^1 dz_n z_n^{(n-2)-(n-1)\epsilon} (1-z_n)^{1-2\epsilon} \cdots \int_0^1 dz_2 z_2^{-\epsilon} (1-z_2)^{1-2\epsilon}. \qquad (12)
\end{aligned}
$$

The dependence of the phase space on $1-z$ comes entirely from the prefactor of $s^{n-1-n\epsilon}$ according to Eq.(4). Indeed the dependence on z and Q^2 has been entirely removed from the integration range thanks to the change of variables of Eq.(7). Now, the amplitude whose square modulus is integrated with the phase space Eq.(12) is in general a function:

$$A_{n+1} = A_{n+1}(Q^2, s, z_2, \ldots, z_n, \Omega_1, \ldots, \Omega_n). \tag{13}$$

The number of independent variables for a process with 2 incoming particle (one on-shell and the other virtual) and $n+1$ outgoing real particles is given by the number of parameters minus the on-shell conditions and the ten parameters of the Poincare' group:

$$4(n+3) - (n+2) - 10 = 3n. \tag{14}$$

These $3n$ variable correspond in this case to

$$Q^2, s, z_2, \ldots, z_n, \Omega_1, \ldots, \Omega_n, \tag{15}$$

where an azimutal angle is arbitrary. In the $z \to 1$ limit, $s \to 0$ and the dominant contribution is given by terms which are most singular as s vanishes. Because of cancellation of infrared singularities [48, 49], $|A_{n+1}|^2 \sim s^{-n+O(\epsilon)}$ when $s \to 0$. Indeed, a stronger singularity would lead to powerlike infrared divergences and a weaker singularity would lead to suppressed terms in the $z \to 1$ limit. Hence only terms in the square amplitude which behave as $s^{-n+O(\epsilon)}$ contribute in the $z \to 1$ limit. In d dimensions, these pick up an $s^{-1-n\epsilon+O(\epsilon)}$ prefactor from the phase space Eq.(12). Let's consider the simplest case, that is the tree level case where we have only purely real soft emission. In this case $O(\epsilon) = 0$ and thus we get that the contribution to the coefficient function from the tree level diagrams with n extra radiated partons behaves as:

$$\begin{aligned} |A_{n+1}|^2 d\phi_{n+1} \sim s^{-1-n\epsilon} \int_0^1 & dz_n \cdots dz_2 z_n^{(n-2)-(n-1)\epsilon}(1-z_n)^{1-2\epsilon} \cdots z_2^{-\epsilon}(1-z_2)^{1-2\epsilon} \\ \times & \int d\Omega_1 \cdots d\Omega_n. \end{aligned} \tag{16}$$

We note that each z integration can produce at most a $1/\epsilon$ pole from the soft region and that each angular integration can produce at most an additional $1/\epsilon$ pole from the collinear region (see Eq.(23) in Appendix B). Therefore, from the contribution of tree level diagrams with n extra radiated partons there come at most

$$\frac{1}{\epsilon^{n-1}} \frac{1}{\epsilon^n} = \frac{1}{\epsilon^{2n-1}} \tag{17}$$

poles in $\epsilon = 0$. All this means that we can write the $O(\alpha_s^n)$ to the bare coefficient function in d dimensions in the following form:

$$C_n^{\prime(0)}(z, Q^2, \epsilon) = (Q^2)^{-n\epsilon} \frac{C_{nn}^{(0)}(\epsilon)}{\Gamma(-n\epsilon)} (1-z)^{-1-n\epsilon}, \tag{18}$$

where the factor $(Q^2)^{-n\epsilon}$ is due to elementary dimensional analysis, $C_{nn}^{(0)}(\epsilon)$ are coefficients with poles in $\epsilon = 0$ of order at most of $2n$ and the Γ function factor $\Gamma^{-1}(-n\epsilon)$ has been introduced for future convenience. For the LO tree level case (see Eq.(73) of Chapter 1) we have that

$$C_0^{'(0)}(z, Q^2, \epsilon) = \delta(1-z). \tag{19}$$

Hence, the tree level coefficient function $C_{\rm tree}^{(0)}$ in the $z \to 1$ limit has the form:

$$\begin{aligned} C_{\rm tree}^{(0)}(z, Q^2, \alpha_0, \epsilon) &= \sum_{n=0}^{\infty} \alpha_0^n C_n^{'(0)}(z, Q^2, \epsilon) & (20) \\ &= \delta(1-z) + \sum_{n=1}^{\infty} \alpha_0^n (Q^2)^{-n\epsilon} \frac{C_{nn}^{'(0)}(\epsilon)}{\Gamma(-n\epsilon)} (1-z)^{-1-n\epsilon} \\ &\quad + O((1-z)^0), & (21) \end{aligned}$$

where α_0 is the bare i.e. the unrenormalized strong coupling constant. We will now study how the result of Eq.(21) is modified by the inclusion of loops. To this purpose, we notice that a generic amplitude with loops can be viewed as a tree-level amplitude formed with proper vertices. Contributions to the dimensionless coefficient function with powers of s^ϵ can only arise from loop integrations in the proper vertices. We thus consider only purely scalar loop integrals, since numerators of fermion or vector propagators and vertex factors cannot induce any dependence on s^ϵ. Let us therefore consider an arbitrary proper diagram G in a massless scalar theory with E external lines, I internal lines and V vertices. It can be shown (see e.g. section 6.2.3 of [50] and references therein) that, denoting with P the set of E external momenta and P_E the set of independent invariants, the corresponding amplitude $\tilde{A}_G(P_E)$ has the form

$$\begin{aligned} \tilde{A}_G(P_E) &= K(2\pi)^d \delta_d(P) A_G(P_E), \\ A_G(P_E) &= \frac{i^{I-L(d-1)}}{(4\pi)^{dL/2}} \Gamma(I - dL/2) \\ &\quad \times \prod_{l=1}^{I} \left[\int_0^1 d\beta_l \right] \frac{\delta\left(1 - \sum_{l=1}^{I} \beta_l\right)}{[P_G(\beta)]^{d(L+1)/2-I} [D_G(\beta, P_E)]^{I-dL/2}}. \end{aligned} \tag{22}$$

Here, β_l are the usual Feynman parameters, $P_G(\beta)$ is a homogeneous polynomial of degree L in the β_l, $D_G(\beta, P_E)$ is a homogeneous polynomial of degree $L+1$ in the β_l with coefficients which are linear functions of the scalar products of the set P_E, i.e. with dimensions of (mass)^2, and K collects all overall factors, such as couplings and symmetry factors. The amplitude $\tilde{A}_G(P_E)$ Eq.(22) depends on s only through $D_G(\beta, P_E)$, which, in turn is linear in s. We can determine in general the dependence of $A_G(P_E)$ by considering two possible cases. The first possibility is that $D_G(\beta, P_E)$ is independent of all invariants except s, i.e. $D_G(\beta, P_E) = s d_G(\beta)$. In such case, $A_G(P_E)$ depends on s as

$$A_G(P_E) = \left(\frac{1}{s}\right)^{I-dL/2} a_G = \left(\frac{1}{s}\right)^{I-2L+L\epsilon} a_G, \tag{23}$$

where a_G is a numerical constant, obtained performing the Feynman parameters integrals. The second possibility is that $D_G(\beta, P_E)$ depends on some of the other invariants. In such case, $A_G(P_E)$ is manifestly an analytic function of s at $s = 0$, and thus it can be expanded in Taylor series around $s = 0$, with coefficients which depend on the other invariants. In the former case, Eq.(23) implies that the s dependence induced by loops integration in the square amplitude is given by integer powers of $s^{-\epsilon}$. In the latter case, the s dependence induced by loops integrations in the square amplitude is given by integer positive powers of s. Therefore, we conclude that each loop integration can carry at most a factor of $s^{-\epsilon}$ and that Eq.(21), after the inclusion of loops, becomes:

$$
\begin{aligned}
C^{(0)}(z, Q^2, \alpha_0, \epsilon) &= \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(z, Q^2, \epsilon), && (24)\\
C_n^{(0)}(z, Q^2, \epsilon) &= (Q^2)^{-n\epsilon} \left[C_{n0}^{(0)}(\epsilon)\delta(1-z) + \sum_{k=1}^{n} \frac{C_{nk}^{(0)}(\epsilon)}{\Gamma(-k\epsilon)}(1-z)^{-1-k\epsilon} \right] \\
&\quad + O((1-z)^0), && (25)
\end{aligned}
$$

where again for future convenience we have defined the coefficients of $(1-z)^{-1-k\epsilon}$ in terms of $\Gamma^{-1}(-k\epsilon)$ and where $O((1-z)^0)$ denotes terms which are not divergent as $z \to 1$ in the $\epsilon \to 0$ limit. Using the identity

$$
\int_0^1 dz\, z^{N-1}(1-z)^{-1-k\epsilon} = \frac{\Gamma(N)\Gamma(-k\epsilon)}{\Gamma(N-k\epsilon)} \tag{26}
$$

and the Stirling expansion Eq.(5) of Appendix A we get that the Mellin transform of Eqs.(55,25) in the large-N limit is given by

$$
C^{(0)}(N, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(N, Q^2, \epsilon), \tag{27}
$$

$$
C_n^{(0)}(N, Q^2, \epsilon) = \sum_{k=0}^{n} C_{nk}^{(0)}(\epsilon)(Q^2)^{-(n-k)\epsilon} \left(\frac{Q^2}{N}\right)^{-k\epsilon} + O\left(\frac{1}{N}\right). \tag{28}
$$

The content of this result is that, in the large-N limit, the dependence of the regularized cross section on N only goes through integer powers of the dimensionful variable $(Q^2/N)^{-\epsilon}$.

3.2 Kinematics of inclusive DY in the soft limit

In the Drell-Yan case the argument follows in an analogous way with minor modification which account for the different kinematics. In this case the relevant parton subprocesses are:

$$
\begin{aligned}
\mathcal{Q}(p) + \mathcal{Q}(p') &\to \gamma^*(Q) + \mathcal{X} && (29)\\
\mathcal{Q}(p) + \mathcal{G}(p') &\to \gamma^*(Q) + \mathcal{X} && (30)\\
\mathcal{G}(p) + \mathcal{G}(p') &\to \gamma^*(Q) + \mathcal{X}. && (31)
\end{aligned}
$$

The recursive application of Eq.(12) in Appendix B with $m = 1$, in this case gives:

$$\begin{aligned} & d\phi_{n+1}(p+p'; Q, k_1, \dots, k_n) \\ = & \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p+p'; k_n, P_n) d\phi_n(P_n; Q, k_1, \dots, k_{n-1}) \end{aligned}$$

$$\begin{aligned} = & \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(p+p'; k_n, P_n) \\ & \times \int_0^{M_n^2} \frac{dM_{n-1}^2}{2\pi} d\phi_2(P_n; k_{n-1}, P_{n-1}) d\phi_{n-1}(P_{n-1}; Q, k_1, \dots, k_{n-2}) \\ = & \int_0^s \frac{dM_n^2}{2\pi} d\phi_2(P_{n+1}; k_n, P_n) \int_0^{M_n^2} \frac{dM_{n-1}}{2\pi} d\phi_2(P_n; k_{n-1}, P_{n-1}) \\ & \times \cdots \times \int_0^{M_3^2} \frac{dM_2^2}{2\pi} d\phi_2(P_3; k_2, P_2) d\phi_2(P_2; k_1, P_1), \end{aligned} \tag{32}$$

where now we have defined $P_{n+1} \equiv p + p'$, so $M_{n+1}^2 = s$ and $P_1 \equiv Q$. The change of variables which separates off the dependence on $(1 - z)$, where now $z = Q^2/s$, is

$$z_i = \frac{M_i^2 - Q^2}{M_{i+1}^2 - Q^2}; \qquad i = 2, \dots, n \tag{33}$$

$$M_i^2 - Q^2 = (s - Q^2) z_n \cdots z_i. \tag{34}$$

Also here all z_i range between 0 and 1, because $M_i^2 \leq M_{i+1}^2 \leq Q^2$. From Eq.(33), we get:

$$dM_n^2 dM_{n-1}^2 \cdots dM_2^2 = \det\left(\frac{\partial(M_i^2 - Q^2)}{\partial z_j}\right) dz_n dz_{n-1} \cdots dz_2, \tag{35}$$

where

$$\det\left(\frac{\partial(M_i^2 - Q^2)}{\partial z_j}\right) = \frac{\partial(M_n^2 - Q^2)}{\partial z_n} \cdots \frac{\partial(M_2^2 - Q^2)}{\partial z_2} = (s - Q^2)^{n-1} z_n^{n-2} z_{n-1}^{n-3} \cdots z_3. \tag{36}$$

In this case in the new variables Eq.(33) the two-body phase space becomes

$$\begin{aligned} & d\phi_2(P_{i+1}; k_i, P_i) \\ & = N(\epsilon)(M_{i+1}^2)^{-1+\epsilon} \left[(M_{i+1}^2 - Q^2) - (M_i^2 - Q^2)\right]^{1-2\epsilon} d\Omega_i \\ & = N(\epsilon)(Q^2)^{-1+\epsilon}(s - Q^2)^{1-2\epsilon}(z_n \cdots z_{i+1})^{1-2\epsilon}(1 - z_i)^{1-2\epsilon} d\Omega_i, \end{aligned} \tag{37}$$

where in the last step we have replaced $(M_{i+1}^2)^{-1+\epsilon}$ by $(Q^2)^{-1+\epsilon}$ in the $z \to 1$ limit as can be seen comparing with Eq.(34). Now, substituting Eqs.(35,36,37) into Eq.(32), we finally get

$$\begin{aligned} d\phi_{n+1}(p+p'; Q, k_1, \dots, k_n) = 2\pi \left[\frac{N(\epsilon)}{2\pi}\right]^n (Q^2)^{-n(1-\epsilon)}(s - Q^2)^{2n-1-2n\epsilon} d\Omega_n \cdots d\Omega_1 \\ \times \int_0^1 dz_n z_n^{(n-2)-(n-1)\epsilon}(1 - z_n)^{1-2\epsilon} \cdots \int_0^1 dz_2 z_2^{-\epsilon}(1 - z_2)^{1-2\epsilon}. \end{aligned} \tag{38}$$

The dependence on $1-z$ is now entirely contained in the phase space prefactor

$$(Q^2)^{-n(1-\epsilon)}(s-Q^2)^{2n-1-2n\epsilon} = \frac{z^{1-2n+2n\epsilon}}{Q^2(1-z)}\left[Q^2(1-z)^2\right]^{n-n\epsilon}. \quad (39)$$

As before, this proves that the coefficient function for real emission at tree level is given by

$$\begin{aligned} C^{(0)}_{\rm tree}(z,Q^2,\alpha_0,\epsilon) &= \sum_{n=0}^{\infty}\alpha_0^n C_n^{'(0)}(z,Q^2,\epsilon) && (40)\\ &= \delta(1-z)+\sum_{n=1}^{\infty}\alpha_0^n(Q^2)^{-n\epsilon}\frac{C_{nn}^{'(0)}(\epsilon)}{\Gamma(-2n\epsilon)}(1-z)^{-1-2n\epsilon} \\ &\quad +O((1-z)^0), && (41)\end{aligned}$$

In this case the introduction of loops requires more care than for the deep-inelastic-scattering case. We shall now show the main difference between the DIS case and the DY case as far as the introduction of loops is concerned. The two-body kinematics (see Eq.(14) in Appendix B together with Eqs.(3,7) states that the radiated partons in the DIS case are all soft:

$$\begin{aligned} k_i^0 &= \frac{M_{i+1}}{2}\left(1-\frac{M_i^2}{M_{i+1}^2}\right) = \frac{\sqrt{s}}{2}(z_n\cdots z_{i+1})^{1/2}(1-z_i); \quad 1\le i\le n-1 && (42)\\ k_n^0 &= \frac{\sqrt{s}}{2}\left(1-\frac{M_n^2}{s}\right) = \frac{\sqrt{s}}{2}(1-z_n); \quad s=Q^2\frac{1-z}{z}. && (43)\end{aligned}$$

This confirms the validity of the argument of Section 3.1 for the introduction of loops, because all the invariants that can appear in the function $D_G(\beta,P_E)$ in Eq.(22) can be expresses in terms of the following ones

$$\begin{aligned} q^2 &= -Q^2 && (44)\\ p^2 = p^{'2} = k_i^2 &= 0 && (45)\\ p\cdot p' \sim p\cdot k_i &\sim Q^2 && (46)\\ k_i\cdot k_j &\sim Q^2(1-z), && (47)\end{aligned}$$

which are either constant or proportional to $1-z$, i.e. to s. Here we have used Eqs.(42,43) of this Section, Eq.(6) in Chapter 2 and the fact that the definition of z in the DIS case Eq.(4) implies that

$$(p^0)^2 = \frac{Q^2}{4z(1-z)}. \quad (48)$$

In the DY case things are quite different, because in this case two-body kinematics (see Eq.(14) in Appendix B together with Eqs.(32,33) gives

$$\begin{aligned} k_i^0 &= \frac{\sqrt{s}}{2}(1-z)z_n\cdots z_{i+1}(1-z_i)+O((1-z)^2); \quad 1\le i\le n-1 && (49)\\ k_n^0 &= \frac{\sqrt{s}}{2}(1-z)(1-z_n); \quad s=\frac{Q^2}{z} && (50)\end{aligned}$$

and this implies that all the invariants that can appear in the function $D_G(\beta, P_E)$ in Eq.(22) can be expresses in terms of the following ones

$$q^2 = Q^2 \tag{51}$$
$$p \cdot p' = s/2 \tag{52}$$
$$p \cdot k_i \sim p' \cdot k_i \sim s(1-z) \tag{53}$$
$$k_i \cdot k_j \sim s(1-z)^2. \tag{54}$$

Hence, we see that, in general, both odd and even powers of $(1-z)^{-\epsilon}$ may arise adding the loops contribution to the tree level coefficient function Eq.(41). Here, in this thesis, we will assume that odd powers of $(1-z)^{-\epsilon}$ do not arise, because it can be shown by explicit computations that it is the case up to order $O(\alpha_s^2)$. However, there are possible indications that at higher orders this assumption could not be true. Anyway the investigations of these aspects is beyond the aim of this thesis. Thus, after the inclusions of loops and with our assumptions, Eq.(41) becomes

$$C^{(0)}(z, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \alpha_0^n C_n^{(0)}(z, Q^2, \epsilon), \tag{55}$$
$$C_n^{(0)}(z, Q^2, \epsilon) = (Q^2)^{-n\epsilon} \left[C_{n0}^{(0)}(\epsilon)\delta(1-z) + \sum_{k=1}^{n} \frac{C_{nk}^{(0)}(\epsilon)}{\Gamma(-2k\epsilon)}(1-z)^{-1-2k\epsilon} \right] + O((1-z)^0). \tag{56}$$

Its Mellin transform can be written in a compact way together with that of the DIS case Eq.(28):

$$C^{(0)}(N, Q^2, \alpha_0, \epsilon) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} C_{nk}^{(0)}(\epsilon)[\alpha_0 (Q^2)^{-\epsilon}]^{(n-k)} \left[\alpha_0 \left(\frac{Q^2}{N^a} \right)^{-\epsilon} \right]^k + O\left(\frac{1}{N}\right), \tag{57}$$

where $a = 1$ for the DIS case and $a = 2$ for the DY case and where the coefficients are those that could be obtained from the parton-level cross sections for the partonic subprocesses that contribute to the given process.

3.3 Resummation from renormalization group improvement

In this section, we want to impose the restrictions that renormalization group invariance imposes on the cross section. Our only assumption is that the coefficient function can be multiplicatively renormalized. This means that all divergences can be removed from the bare coefficient function $C^{(0)}(N, Q^2, \alpha_0, \epsilon)$ Eq.(57) by defining a renormalized running coupling $\alpha_s(\mu^2)$ according to the implicit equation

$$\alpha_0(\mu^2, \alpha_s(\mu^2), \epsilon) = \mu^{2\epsilon} \alpha_s(\mu^2) Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \tag{58}$$

and a renormalized coefficient function

$$C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2, \epsilon)\right) = Z^{(C)}(N, \alpha_s(\mu^2), \epsilon) C^{(0)}(N, Q^2, \alpha_0, \epsilon), \tag{59}$$

where μ is the renormalization scale (here chosen equal to the factorization one) and $Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon)$ and $Z^{\mathrm{C}}(N, \alpha_s(\mu^2), \epsilon)$ are computable in perturbation theory and have multiple poles in $\epsilon = 0$. The renormalized coefficient function $C(N, Q^2/\mu^2, \alpha_s(\mu^2, \epsilon))$ is finite in $\epsilon = 0$ and it can only depend on Q^2 through Q^2/μ^2, because $\alpha_s(\mu^2)$ is dimensionless. The physical anomalous dimension is given by

$$\begin{aligned}\gamma(N, \alpha_s(Q^2), \epsilon) &= Q^2 \frac{\partial}{\partial Q^2} \ln C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = Q^2 \frac{\partial}{\partial Q^2} \ln C^{(0)}\left(N, Q^2, \alpha_0, \epsilon\right) \\ &= -\epsilon(\alpha_0 Q^{-2\epsilon}) \frac{\partial}{\partial(\alpha_0 Q^{-2\epsilon})} \ln C^{(0)}\left(N, Q^2, \alpha_0, \epsilon\right),\end{aligned} \tag{60}$$

where we have exploited the fact that $C^{(0)}$ Eq.(57) depends on Q^2 through the combination $\alpha_0 Q^{-2\epsilon}$. This implies that the physical anomalous dimension γ has the following perturbative expression:

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \sum_{i=0}^{\infty} \sum_{j=0}^{n} \gamma_{ij}(\epsilon) [\alpha_0 (Q^2)^{-\epsilon}]^{(i-j)} \left[\alpha_0 \left(\frac{Q^2}{N^a}\right)^{-\epsilon}\right]^j + O\left(\frac{1}{N}\right). \tag{61}$$

The renormalized expression of the physical anomalous dimension is found expressing in this equation the bare coupling constant in terms of the renormalized one by means of Eq.(58). Now, the functions

$$(Q^2)^{-\epsilon}\alpha_0 = \left(\frac{Q^2}{\mu^2}\right)^{-\epsilon} \alpha_s(\mu^2) Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \tag{62}$$

$$(Q^2/N^a)^{-\epsilon}\alpha_0 = \left(\frac{Q^2/N^a}{\mu^2}\right)^{-\epsilon} \alpha_s(\mu^2) Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon) \tag{63}$$

are manifestly renormalization group invariant, i.e. μ^2-independent. Thus, it follows that

$$(Q^2)^{-\epsilon}\alpha_0 = \alpha_s(Q^2) Z^{(\alpha_s)}(\alpha_s(Q^2), \epsilon) \tag{64}$$

$$(Q^2/N^a)^{-\epsilon}\alpha_0 = \alpha_s(Q^2/N^a) Z^{(\alpha_s)}(\alpha_s(Q^2/N^a), \epsilon). \tag{65}$$

The renormalized physical anomalous dimension is then found by substituting Eqs.(64,65) into Eq.(61) and re-expanding $Z^{(\alpha_s)}$ in powers of the renormalized coupling. We obtain:

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \sum_{m=1}^{\infty} \sum_{n=0}^{m} \gamma^{\mathrm{R}}_{mn}(\epsilon) \alpha_s^{m-n}(Q^2) \alpha_s^n(Q^2/N^a) + O\left(\frac{1}{N}\right). \tag{66}$$

At this point, we cannot yet conclude that the four-dimensional physical anomalous dimension admits an expression of the form of Eq.(66), because the coefficients $\gamma^{\mathrm{R}}_{mn}(\epsilon)$ are not necessarily finite as $\epsilon \to 0$. In order to understand this, it is convenient to separate off the N-independent terms in the renormalized physical anomalous dimension, i.e. the terms with $n = 0$ in the internal sum in Eq.(66). Namely, we write

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + O\left(\frac{1}{N}\right), \quad (67)$$

where we have defined

$$\hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \gamma^{\rm R}_{m+nn}(\epsilon)\alpha_s^m(Q^2)\alpha_s^n(Q^2/N^a) \quad (68)$$

$$\hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) = \sum_{m=1}^{\infty} \gamma^{\rm R}_{m0}(\epsilon)\alpha_s^m(Q^2). \quad (69)$$

Whereas $\gamma(N, \alpha_s(Q^2), \epsilon)$ is finite in the limit $\epsilon \to 0$, where it coincides with the four-dimensional physical anomalous dimension, $\hat{\gamma}^{(l)}$ and $\hat{\gamma}^{(c)}$ are not necessarily finite as $\epsilon \to 0$. However, Eq.(67) implies that $\hat{\gamma}^{(l)}$ and $\hat{\gamma}^{(c)}$ can be made finite by adding and subtracting the counterterm

$$Z^{(\gamma)}(\alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon). \quad (70)$$

In this way the physical anomalous dimension Eq.(67) becomes

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + \gamma^{(c)}(\alpha_s(Q^2), \epsilon) + O\left(\frac{1}{N}\right), \quad (71)$$

where

$$\gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) = \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + -\hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon), \quad (72)$$

$$\gamma^{(c)}(\alpha_s(Q^2), \epsilon) = \hat{\gamma}^{(c)}(\alpha_s(Q^2), \epsilon) + \hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2), \epsilon). \quad (73)$$

Now, $\gamma^{(c)}$ is clearly finite in $\epsilon = 0$, because at $N = 1$ $\gamma^{(l)}$ vanishes and it is N-independent. This also implies that $\gamma^{(l)}$ is finite for all N, because γ shoul be finite for all N. Therefore, $\gamma^{(l)}$ provides an expression of the resummed physical anomalous dimension in the large N limit, up to non-logarithmic terms:

$$\gamma(N, \alpha_s(Q^2), \epsilon) = \gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) + O(N^0). \quad (74)$$

It is apparent from Eq.(73) that $\gamma^{(c)}$ is a power series in $\alpha_s(Q^2)$ with finite coefficients in the $\epsilon \to 0$ limit. In order to understand the perturbative structure of $\gamma^{(l)}$ as well, define implicitly the function $g(\alpha_s(Q^2), \alpha_s(Q^2/n), \epsilon)$ as

$$\gamma^{(l)}(\alpha_s(Q^2), \alpha_s(Q^2/N^a), \epsilon) = \int_1^{N^a} \frac{dn}{n}\, g(\alpha_s(Q^2), \alpha_s(Q^2/n), \epsilon), \quad (75)$$

where

$$g(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon) \equiv -\mu^2 \frac{\partial}{\partial \mu^2}\hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon) \quad (76)$$

$$= -\beta^{(d)}(\alpha_s(\mu^2), \epsilon)\frac{\partial}{\partial \alpha_s(\mu^2)}\hat{\gamma}^{(l)}(\alpha_s(Q^2), \alpha_s(\mu^2), \epsilon), \quad (77)$$

with $\beta^{(d)}(\alpha_s)$ is the d-dimensional beta function

$$\beta^{(d)}(\alpha_s(\mu^2),\epsilon) - \epsilon\alpha_s(\mu^2) + \beta(\alpha_s(\mu^2)) \tag{78}$$

and where we have performed the change of variable $n = Q^2/\mu^2$. It immediately follows from Eqs.(68-77) that g is a power series in $\alpha_s(Q^2)$ and $\alpha_s(\mu^2)$ with finite coefficients in the limit $\epsilon \to 0$:

$$\begin{aligned}\lim_{\epsilon\to 0} g(\alpha_s(Q^2),\alpha_s(\mu^2),\epsilon) &\equiv g(\alpha_s(Q^2),\alpha_s(\mu^2)) \\ &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{mn}\alpha_s^m(Q^2)\alpha_s^n(\mu^2).\end{aligned} \tag{79}$$

Hence, our final result for the four-dimensional all-rder resummed physical anomalous dimension is given by

$$\gamma^{\mathrm{res}}(N,\alpha_s(Q^2)) = \int_1^{N^a} \frac{dn}{n}\, g(\alpha_s(Q^2),\alpha_s(Q^2/n)) + O(N^0). \tag{80}$$

This result can be compared to the all-order resummation formula derived in Ref.[51]. This resummation has the form of Eq.(80), but with g a function of $\alpha_s(\mu^2)$ only, i.e. with all $g_{mn} = 0$ when $m > 0$ and our result is thus less predictive in the sense that it requires a higher fixed-order computation of the physical anomalous dimension in order to extract the resummation coefficients g_{mn}. The predictive power of the resummation formulae is analyzed in detail in Chapter 7. According to the more restrictive result of Ref.[51], Eq.(80) becomes

$$\gamma^{\mathrm{res}}(N,\alpha_s(Q^2)) = \int_1^{N^a} \frac{dn}{n}\, g(\alpha_s(Q^2/n)) + O(N^0), \tag{81}$$

where

$$g(\alpha_s(\mu^2)) = \sum_{n=1}^{\infty} g_{0n}\alpha_s^n(\mu^2). \tag{82}$$

The conditions under which the more restrictive result of Ref.[51] holds can be understood by comparing to our approach the derivation of that result. The approach of Ref.[51] is based on assuming the validity of the factorization formula Eq.(47) of Sec.2.3.2 which is more restrictive than the standard collinear factorization. This factorization was proven for a wide class of processes in Ref.[11], and implies that the perturbative coefficient function Eq.(57) in the large N limit can be factored as:

$$C^{(0)}(N,Q^2,\alpha_0,\epsilon) = C^{(0,l)}(Q^2/N^a,\alpha_0,\epsilon)C^{(0,c)}(Q^2,\alpha_0,\epsilon). \tag{83}$$

We notice that this can happen if and only if the coefficients $C^{(0)}_{nk}(\epsilon)$ in Eq.(57) can be written in the form

$$C^{(0)}_{nk}(\epsilon) = F_k(\epsilon)G_{n-k}(\epsilon). \tag{84}$$

The validity of factorization Eq.(47) of Sec.2.3.2 to all orders and for various processes is based on assumptions whose reliability will not be discussed here. Anyway, Eq.(83) implies that the physical anomalous dimension Eq.(60) becomes

$$\gamma(N,\alpha_s(Q^2),\epsilon) = \gamma^{(l)}(\alpha_s(Q^2/N^a),\epsilon) + \gamma^{(c)}(\alpha_s(Q^2),\epsilon). \tag{85}$$

Then, proceeding as before, one then ends up with the resummation formula Eq.(81).

3.4 NLL resummation

In this Section, we shall give explicit expressions of the reummation formulae at NLL for the deep-inelastic structure function F_2 and for the Drell-Yan cross section. These explicit expressions are useful for practical computations and we shall use them in Chapter 5. The expression of the resummed physical anomalous dimension in Eq.(80) can be used to compute the resummed evolution factor $K^{res}(N;Q_0^2,Q^2)$ Eq.(19) in Section 2.2. At NLL, we get

$$K_{\rm NLL}(N;Q_0^2,Q^2)=\exp\left[E_{\rm NLL}(N;Q_0^2,Q^2)\right]=\exp\int_{Q_0^2}^{Q^2}\frac{dk^2}{k^2}\gamma_{\rm NLL}^{\rm res}(N,\alpha_s(k^2)), \quad (86)$$

where

$$\gamma_{\rm NLL}^{\rm res}(N,\alpha_s(k^2))=\int_1^{N^a}\frac{dn}{n}\left[g_{01}\alpha_s(k^2/n)+g_{02}\alpha_s^2(k^2/n)\right]; \qquad g_{11}=0. \quad (87)$$

The fact that the resummation coefficient $g_{11}=0$ for both the deep-inelastic and the Drell-Yan case can be shown by explicit computations of the fixed-order anomalous dimension (see Ref.[8] in Section 4.3). This shows that at NLL level Eq.(81) holds. However this does not mean that this is the case at all logarithmic orders. Many times in literature, the resummed results are given in terms of the Mellin transform of a resummed physical anomalous dimension in z space. To rewrite Eq.(87) as the Mellin transform of a function of x, we can use the all-orders relations between the logs of N and the logs of $1-z$ that are given in Appendix A. In particular, we can use Eqs.(23,24) in Appendix A to rewrite Eq.(87) at NLL in the following form:

$$\gamma_{\rm NLL}^{\rm res}(N,\alpha_s(k^2))=a\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\left[\hat g_{01}\alpha_s(k^2(1-x)^a)+\hat g_{02}\alpha_s^2(k^2(1-x)^a)\right], \quad (88)$$

where

$$\hat g_{01}=-g_{01}, \qquad \hat g_{02}=-(g_{02}+a\gamma_E b_0 g_{01}). \quad (89)$$

and where we have used the definition of the beta function Eq.(43) in Section 1.2. As a consequence, the NLL resummed exponent in Eq.(86) can be rewritten in the following form

$$E_{\rm NLL}(N;Q_0^2,Q^2)=a\int_0^1 dx\,\frac{x^{N-1}-1}{1-x}\int_{Q_0^2(1-x)^a}^{Q^2(1-x)^a}\frac{dk^2}{k^2}\hat g(\alpha_s(k^2)), \quad (90)$$

where

$$\hat g(\alpha_s)=\hat g_{01}\alpha_s(k^2)+\hat g_{02}\alpha_s^2(k^2) \quad (91)$$

Beyond leading order the standard anomalous dimension differs from the physical one, so $\hat g_{02}$ receives a contribution both from the standard anomalous dimension and from the coefficient function. It is thus natural to rewrite the resummation formula Eq.(90) separating off the contribution the contribution which originates from the anomalous dimension $\gamma^{\rm AP}$ Eq.(18) of Section 2.2. This is done defining two functions of α_s, $A(\alpha_s)$ and $B^a(\alpha_s)$ in such a way that

$$\hat{g}(\alpha_s) = A(\alpha_s) + \frac{\partial B^{(a)}(\alpha_s(k^2))}{\partial \ln k^2}, \quad A(\alpha_s) = A_1\alpha_s + A_2\alpha_s^2, \quad B^{(a)}(\alpha_s) = B_1^{(a)}\alpha_s. \tag{92}$$

It is clear that the constant A_i are obtained directly form the coefficients of the $1/[1-x]_+$ terms of the i-loop quark-quark splitting functions and that the coefficients B_i^a depends on the particular process ($a = 1$ for DIS and $a = 2$ for DY). To the NLL order, we find that

$$\hat{g}_{01} = A_1, \qquad \hat{g}_{02} = A_2 - b_0 B_1^{(a)} \tag{93}$$

and that

$$\begin{aligned} E_{\rm NLL}(N; Q_0^2, Q^2) = a \int_0^1 dx \, \frac{x^{N-1}-1}{1-x} \Bigg[\int_{Q_0^2(1-x)^a}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \\ + B^{(a)}(\alpha_s(Q^2(1-x)^a)) - B^{(a)}(\alpha_s(Q_0^2(1-x)^a)) \Bigg]. \end{aligned} \tag{94}$$

We can then rewrite the resummed cross section

$$\sigma_{\rm NLL}(N, Q^2) = \exp\left[E_{\rm NLL}(N; Q_0^2, Q^2)\right] \sigma_{\rm NLL}(N, Q_0^2) \tag{95}$$

in a factorized form according to Eqs.(11,12) in Section 2.2 by collecting all Q^2-dependent contributions to the resummation Eq.(94) into a resummed perturbative coefficient function $C_{\rm NLL}$:

$$\sigma_{\rm NLL}(N, Q^2) = C_{\rm NLL}(N, Q^2/\mu^2, \alpha_s(\mu^2)) F(N, \mu^2), \tag{96}$$

where

$$\begin{aligned} C_{\rm NLL}(N, Q^2/\mu^2, \alpha_s(\mu^2)) &= \exp\Bigg\{ a \int_0^1 dx \, \frac{x^{N-1}-1}{1-x} \Bigg[\int_{\mu^2}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \\ &\quad + B^{(a)}(\alpha_s(Q^2(1-x)^a)) \Bigg] \Bigg\}, \end{aligned} \tag{97}$$

which ha the same form of the resummed results discussed in Section 2.3.1. The precise definition of the parton distribution F and the factorization scale μ^2 will depend on the choice of factorization scheme: according to the choice of scheme, the resummed terms will be either part of the hard coefficient function $C_{\rm NLL}$, or of the evolution of the parton distribution F. In the $\overline{MS}$ scheme the NLL coefficients A_1, A_2 and $B_1^{(a)}$ are given in Eq.(46) of Section 2.3.1 and the NNLL ones are given for example in Ref.[52]. These coefficients can also be obtained comparing a fixed order computation of the physical anomalous dimension with a fixed order expansion of Eq.(80) as is shown explicitly in Section 4.3 of Ref.[8]. To compute explicitly Eq.(97), we first exploit Eq.(20) in Appendix A at NLL level, thus finding

$$C_{\rm NLL}(N, Q^2/\mu^2, \alpha_s(\mu^2)) = \exp\left\{ - \int_1^{N^a} \frac{dn}{n} \left[\int_{n\mu^2}^{Q^2} \frac{dk^2}{k^2} A(\alpha_s(k^2/n)) + \tilde{B}^{(a)}(\alpha_s(Q^2/n)) \right] \right\}, \tag{98}$$

where $\tilde{B}^{(a)}(\alpha_s) = B^{(a)}(\alpha_s) - a\gamma_E A_1 \alpha_s$. The explicit expression of Eq.(97) is then obtained performing the changes of variables,

$$\frac{dk^2}{k^2} = \frac{d\alpha_s(k^2/n)}{\beta(\alpha_s(k^2/n))}, \qquad \frac{dn}{n} = -\frac{d\alpha_s(Q^2/n)}{\beta(\alpha_s(Q^2/n))}, \tag{99}$$

to evaluate the integrals in Eq.(98) and using the two loop solution of the renormalitazion-group equation for the running of α_s given in Eq.(41) of Section 1.2, Now, after some algebra we find for the integral in Eq.(98):

$$\begin{aligned}-\int_1^{N^a} \frac{dn}{n}\left[\int_{n\mu^2}^{Q^2} \frac{dk^2}{k^2}\left(A_1\alpha_s(\frac{k^2}{n}) + A_2\alpha_s^2(\frac{k^2}{n})\right) + \tilde{B}_1^{(a)}\alpha_s(\frac{Q^2}{n})\right] \\ = \log N g_1(\lambda, a) + g_2, (\lambda, a)\end{aligned} \tag{100}$$

where $\lambda = b_0\alpha_s(\mu_r^2)\log N$ and

$$g_1(\lambda, a) = \frac{A_1}{b_0\lambda}[a\lambda + (1 - a\lambda)\log(1 - a\lambda)] \tag{101}$$

$$\begin{aligned} g_2(\lambda, a) = & -\frac{A_1 a\gamma_E - B_1^{(a)}}{b_0}\log(1 - a\lambda) + \frac{A_1 b_1}{b_0^3}[a\lambda + \log(1 - a\lambda) + \frac{1}{2}\log^2(1 - a\lambda)] \\ & -\frac{A_2}{b_0^2}[a\lambda + \log(1 - a\lambda)] + \log\left(\frac{Q^2}{\mu_r^2}\right)\frac{A_1}{b_0}\log(1 - a\lambda) \\ & +\log\left(\frac{\mu^2}{\mu_r^2}\right)\frac{A_1}{b_0}a\lambda, \end{aligned} \tag{102}$$

where $a = 1$ for the DIS case and $a = 2$ for the DY case. Evidently, in Eqs.(101,102) there is a dependence on the renormalization scale. To obtain the desired result, we simply have to keep the renormalization scale equal to the factorization scale $\mu_r^2 = \mu^2$. Thus, for the explicit analytic expression of Eq.(98), we get

$$C_{\mathrm{NLL}}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \left[\exp\{\log N g_1(\lambda, a) + g_2(\lambda, a)\}\right]_{\mu_r^2=\mu^2}, \tag{103}$$

with the resummation coefficients given in Eq.(46) in Section 2.3.1 for the $\overline{MS}$ factorization scheme choice. However, a general analysis of the factorization scheme choices and changes for the resummation formulae is given for example in Section 6 of Ref.[8].