Chapter 4

Renormalization group resummation of prompt photon production

In this chapter, we prove the all-order exponentiation of soft logarithmic corrections to prompt photon production in hadronic collisions, by generalizing the renormalization group approach of chapter 3. Here, we will show that all large logs in the soft limit can be expressed in terms of two dimensionful variables. Then, we use the renormalization group to resum them. The resummation formulae that we obtain are more general though less predictive than those that can be obtained with other approaches discussed in chapter 2.

4.1 Kinematics and notation

We consider the process

$$H_1(P_1) + H_2(P_2) \to \gamma(p_\gamma) + X, \tag{1}$$

of two colliding hadrons H_1 and H_2 with momentum P_1 and P_2 respectively into a real photon with momentum p_γ and any collection of hadrons X. More specifically, we are interested in the differential cross section $p_\perp^3 \frac{d\sigma}{dp_\perp}(x_\perp, p_\perp^2)$, where $p_\perp$ is the transverse momentum of the photon with respect to the direction of the colliding hadrons H_1 and H_2, and

$$x_\perp = \frac{4p_\perp^2}{S}; \qquad S = (P_1 + P_2)^2. \tag{2}$$

The scaling variable x can be viewed as the squared fraction of transverse energy that the hadrons transfer to the outgoing particles (hence $0 \le x_\perp \le 1$) and S is the hadronic center-of-mass energy. We parametrize the momentum of the photon in terms of its partonic center-of-mass pseudorapidity and its transverse momentum $\vec{p}_\perp$. The pseudorapidiry of a massless particle is defined in terms its scattering angle θ in the center-of-mass frame as follows

$$\eta = -\ln(\tan(\theta/2)). \tag{3}$$

So, in the partonic center-of-mass frame, we can write:

$$p_\gamma = (p_\perp \cosh \hat{\eta}_\gamma, \vec{p}_\perp, p_\perp \sinh \hat{\eta}_\gamma). \tag{4}$$

In the same frame, the incoming partons' momenta can be written as

$$p_1 = x_1 P_1 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, 1), \qquad p_2 = x_2 P_2 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, -1), \tag{5}$$

where $x_{1(2)}$ are the longitudinal fraction of momentum of the parton 1(2) in the hadron $H_{1(2)}$ and $s = (p_1 + p_2)^2 = x_1 x_2 S$ is the center-of-mass energy of the partonic process. The relation between the hadronic center-of-mass pseudorapidity and the partonic one is obtained performing a boost along the collision axis:

$$\hat{\eta}_\gamma = \eta_\gamma - \frac{1}{2} \ln \frac{x_1}{x_2}. \tag{6}$$

The factorized expression for this cross section in perturbative QCD is

$$\begin{aligned} p_\perp^3 \frac{d\sigma}{dp_\perp}(x_\perp, p_\perp^2) &= \sum_{a,b} \int_0^1 dx_1\, dx_2\, dz\, x_1 F_a^{H_1}(x_1, \mu^2)\, x_2 F_b^{H_2}(x_2, \mu^2) \\ &\quad \times C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \delta\left(x_\perp - z x_1 x_2\right), \end{aligned} \tag{7}$$

where $F_a^{H_1}(x_1, \mu^2)$, $F_b^{H_2}(x_2, \mu^2)$ are the distribution functions of partons a, b in the colliding hadrons. Here we have defined the perturbative scale Q^2 and the partonic scaling variable z as the squared fraction of transverse energy that the parton a, b transfer to the outgoing partons ($0 \leq z \leq 1$):

$$Q^2 = 4p_\perp^2, \tag{8}$$

$$z = \frac{Q^2}{s} = \frac{Q^2}{x_1 x_2 S}. \tag{9}$$

The coefficient function $C_{ab}(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2))$ is defined in terms of the partonic cross section for the process where partons a, b are incoming as

$$C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = p_\perp^3 \frac{d\hat{\sigma}_{ab}}{dp_\perp}. \tag{10}$$

4.2 Leading order calculation

The hard-scattering subprocesses that contribute to the amplitude of the prompt-photon prodution at the leading order are:

$$q(p_1) + \bar{q}(p_2) \rightarrow g(p') + \gamma(p_\gamma) \tag{11}$$

$$q(p_1) + g(p_2) \rightarrow q(p') + \gamma(p_\gamma) \tag{12}$$

$$\bar{q}(p_1) + g(p_2) \rightarrow \bar{q}(p') + \gamma(p_\gamma) \tag{13}$$

Figure 4.1: Feynman graphs that contribute to the amplitude of the partonic process at the leading order

The corresponding Feynman graphs are shown in figure 4.1. We now want to obtain the coefficient function for these two elementary subprocesses at the leading order. Using the QCD Feynman rules to evaluate the first amplitude in figure 4.1, we have:

$$iM_{q\bar{q}\to\gamma g} = -i\bar{v}(p_2)Q_q eg\left[\frac{\gamma^\rho \not{p}_\gamma \gamma^\mu - 2\gamma^\rho p_1^\mu}{2p_1\cdot p_\gamma} + \frac{\gamma^\mu \not{p}' \gamma^\rho - 2\gamma^\mu p_1^\rho}{2p_1\cdot p'}\right]\epsilon^*_\mu(p_\gamma)\epsilon^*_\rho(p')t^a_{ji}u(p_1), \tag{14}$$

where $e = |e|$ is the electrical charge, g is the strong charge and Q_q is the electrical charge of the quark q in units of e. Taking the square modulus of equation (14) and averaging over the two quarks polarizations and colours we obtain:

$$\frac{1}{4}\frac{1}{N_C^2}\sum_{\text{pol,col}}|\mathcal{M}_{q\bar{q}\to\gamma g}|^2 = 2\frac{C_F}{N_C}Q_q^2 e^2 g^2\left[\frac{p_1\cdot p_\gamma}{p_1\cdot p'} + \frac{p_1\cdot p'}{p_1\cdot p_\gamma}\right], \tag{15}$$

where N_C is the number of quark colours and $C_F \equiv (N_C^2-1)/2N_C$ is the Casimir operator with respect to the colour matrix t^a_{ji}. Proceeding in the same way for the Feynman graphs contributing to the second amplitude in figure 4.1 we could obtain the averaged square modulus for this other subprocess, but, we note that it can be immediately obtained using crossing simmetry. More precisely, we only have to substitute p' with p_2 and take into account the fact that we must now average not over two quarks' colours but over the colours of a quark and a gluon. Therefore we arrive at the following expression for the averaged square modulus of the second amplitude of figure 4.1:

$$\begin{aligned}\frac{1}{4}\frac{1}{2N_CC_F}\frac{1}{N_C}\sum_{\text{pol,col}}|\mathcal{M}_{q(\bar q)g\to\gamma q(\bar q)}|^2 &= \frac{1}{4}\frac{1}{2N_CC_F}\frac{1}{N_C}\sum_{\text{pol,col}}|\mathcal{M}_{q\bar q\to\gamma g}|^2\bigg|_{p'\to p_2} = \\ &= \frac{1}{N_C}Q_q^2e^2g^2\left[\frac{p_1\cdot p_\gamma}{p_1\cdot p_2}+\frac{p_1\cdot p_2}{p_1\cdot p_\gamma}\right]. \qquad (16)\end{aligned}$$

We will now rewrite equations (15) and (16) in terms of the kinematic parameters defined in section 4.1. For this purpose we first note that momentum conservation and the parametrizations defined in Eqs.(4,5) imply (in the centre of mass of the incident partons):

$$\begin{aligned} p' &= (p_\perp\cosh\hat\eta_\gamma, -\vec{p}_\perp, -p_\perp\sinh\hat\eta_\gamma) && (17)\\ s &= (p_1+p_2)^2 = (p_\gamma+p')^2 = 4p_\perp^2\cosh^2\hat\eta_\gamma && (18)\\ z &= \frac{4p_\perp^2}{s} = \frac{1}{\cosh^2\hat\eta_\gamma}. && (19)\end{aligned}$$

From the last equation we see that in the limit $z\to 1$, $\hat\eta_\gamma\to 0$. Physically this is because in this limit all the centre of mass energy is transverse and so the photon cannot have a non zero pseudorapidity. Now, using again Eqs.(4,5) and the equation for p' (17) we have:

$$\begin{aligned} p_1\cdot p_2 &= \frac{s}{2}\\ p_1\cdot p' &= \frac{\sqrt{s}}{2}p_\perp e^{\hat\eta_\gamma} \qquad (20)\\ p_1\cdot p_\gamma &= \frac{\sqrt{s}}{2}p_\perp e^{-\hat\eta_\gamma}\end{aligned}$$

A combination of equations (19) and (20) yields our final result for the averaged square modulus of the amplitudes:

$$\begin{aligned}\frac{1}{4}\frac{1}{N_C^2}\sum_{\text{pol,col}}|\mathcal{M}_{q\bar q\to\gamma g}|^2 &= 4\frac{C_F}{N_C}Q_q^2e^2g^2\frac{(2-z)}{z} && (21)\\ \frac{1}{4}\frac{1}{2N_CC_F}\frac{1}{N_C}\sum_{\text{pol,col}}|\mathcal{M}_{q(\bar q)g\to\gamma q(\bar q)}|^2 &= \frac{1}{2N_C}Q_q^2e^2g^2\Bigg[1\pm\sqrt{1-z} \\ &\qquad +\frac{4}{1\pm\sqrt{1-z}}\Bigg], && (22)\end{aligned}$$

where the plus sign has to be chosen for positive values of the pseudorapidity $\hat\eta_\gamma$ and the minus sign for negative values. The two-body phase space is:

$$\begin{aligned} d\phi(p_1+p_2;p_\gamma,p') &= \frac{d^3p_\gamma}{(2\pi)^3 2E_{p_\gamma}}\frac{d^3p'}{(2\pi)^3 2E_{p'}}(2\pi)^4\delta_{(4)}(p'+p_\gamma-p_1-p_2)\\ &= \frac{d^3p_\gamma}{4E_{p_\gamma}E_{p'}}\frac{1}{(2\pi)^2}\delta_{(1)}(E_{p'}+E_{p_\gamma}-E_{p_1}-E_{p_2}). \qquad (23)\end{aligned}$$

Imposing the conservation of spatial momentum ($E_{p_\gamma} = E_{p'} = |\vec{p_\gamma}|$) the two-body phase space becomes:

$$\begin{aligned} d\phi(p_1+p_2;p_\gamma,p') &= \frac{d^3p_\gamma}{4|\vec{p_\gamma}|^2(2\pi)^2}\delta_{(1)}(2|\vec{p_\gamma}|-\sqrt{\hat{s}}) \\ &= \frac{1}{16\pi}d\cos\theta d|\vec{p_\gamma}|\delta_{(1)}(|\vec{p_\gamma}|-\sqrt{\hat{s}}/2) \\ &= \frac{1}{16\pi}\delta_{(1)}(p_\perp\cosh\hat{\eta}_\gamma-\sqrt{\hat{s}}/2)d\cos\theta d|\vec{p_\gamma}|, \end{aligned} \tag{24}$$

where θ is the scattering angle of the photon with respect to the collision axis. Because of the fact that we want to integrate over the pseudorapidity of the photon η_γ at fixed $p_\perp$, we must perform a change of variables. In particular we must rewrite the two-body phace space, expressed in terms of $\cos\theta$ and $|\vec{p_\gamma}|$, in terms of the new variables $\cosh\hat{\eta}_\gamma$ and $p_\perp$. This change of variables is given by the equations:

$$\begin{cases} \cos\theta = & \tanh\hat{\eta}_\gamma \\ |\vec{p_\gamma}| = & p_\perp\cosh\hat{\eta}_\gamma. \end{cases}$$

The determinant of the Jacobian matrix is easily obtained and is given by:

$$|J| = \left\| \frac{\partial(\cos\theta,|\vec{p_\gamma}|)}{\partial(\cosh\hat{\eta}_\gamma,p_\perp)} \right\| = \frac{1}{\cosh\hat{\eta}_\gamma\sqrt{\cosh^2\hat{\eta}_\gamma-1}}. \tag{25}$$

Thanks to equation (19), we obtain this determinant in term of z:

$$|J| = \frac{z}{\sqrt{1-z}} \tag{26}$$

Using this last result, equation (24) becomes:

$$\begin{aligned} d\phi(p_1+p_2;p_\gamma,p') &= \frac{1}{16\pi}\frac{z}{\sqrt{1-z}}\delta_{(1)}(\cosh\hat{\eta}_\gamma-\sqrt{s}/2p_\perp)\frac{dp_\perp}{p_\perp}d\cosh\hat{\eta}_\gamma \\ &= \frac{1}{16\pi}\frac{z}{\sqrt{1-z}}[\delta_{(1)}(\hat{\eta}_\gamma-\hat{\eta}_+)+\delta_{(1)}(\hat{\eta}_\gamma-\hat{\eta}_-)] \\ &\quad \times\frac{dp_\perp}{p_\perp}d\hat{\eta}_\gamma, \end{aligned} \tag{27}$$

where $\hat{\eta}_+$ and $\hat{\eta}_-$ are the two solutions (one positive and one negative) of the equation imposed by the delta function $p_\perp\cosh\hat{\eta}_\gamma = \sqrt{s}/2$ which are:

$$\hat{\eta}_\pm = \ln\left(\frac{\sqrt{s}}{2p_\perp}\pm\sqrt{\frac{s}{4p_\perp^2}-1}\right). \tag{28}$$

The flux factor Φ is immediately obtained from equations (5) and (19):

$$\Phi \equiv 4(p_1\cdot p_2) = 2s = \frac{8p_\perp^2}{z}. \tag{29}$$

Remembering the definition of the QED and QCD coupling constants,

$$\alpha \equiv \frac{e^2}{4\pi} \qquad ; \qquad \alpha_s \equiv \frac{g^2}{4\pi},$$

and putting together expressions (21), (22), (27), (29) and performing the integration over $\hat{\eta}_\gamma$, we obtain our final result for the coefficient function at the leading order for the two subprocesses in figure 4.1:

$$C^{(LO)}_{q\bar{q}\to\gamma g}(z,\alpha_s) = \alpha\alpha_s Q_q^2 \pi \frac{C_F}{N_C}\frac{z}{\sqrt{1-z}}(2-z) \tag{30}$$

$$C^{(LO)}_{q(\bar{q})g\to\gamma q(\bar{q})}(z,\alpha_s) = \alpha\alpha_s Q_q^2 \pi \frac{1}{2N_C}\frac{z}{\sqrt{1-z}}\left(1+\frac{z}{4}\right). \tag{31}$$

4.3 The soft limit

We will study the cross section Eq. (7) in the threshold transverse limit, when the transverse transverse energy of outgoing particles is close to its maximal value ($x_\perp \to 1$ at the hadronic level or, equivalently, $z \to 1$ at the partonic level). The convolution in Eq. (7) is turned into an ordinary product by Mellin transformation:

$$\sigma(N,Q^2) = \sum_{a,b}\sigma_{ab}(N,Q^2) \tag{32}$$

$$= \sum_{a,b} F_a^{H_1}(N+1,\mu^2)\, F_b^{H_2}(N+1,\mu^2) \times C_{ab}\left(N,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right). \tag{33}$$

As discussed in Section 2.2, in the large N limit each parton subprocess can be treated independently, specifically, each C_{ab} is separately renormalization-group invariant. At this point, it is interesting to discuss the differences in the large N behavior of the partonic subprocesses. The cross sections for the partonic channels with two quarks of different flavors ($ab = q\bar{q}'$, $\bar{q}q'$, qq, qq', $\bar{q}\bar{q}$, $\bar{q}\bar{q}'$) vanish at LO and are hence suppressed by a factor of α_s with respect to the subprocesses with $ab = q\bar{q}$, qg, $\bar{q}g$. Moreover, in the large N limit this relative suppression is further enhanced by a factor of $O(1/N)$ because the processes with two different quark flavors involve the off-diagonal Alatarelli-Parisi splitting functions. Therefore, these partonic channels do not contribute in the large N limit. The partonic channel $ab = gg$ has a different large N behavior. It begins to contribute at NLO via the partonic process $g + g \to \gamma + q + \bar{q}$, which again leads to a suppression effect of $O(1/N)$ with respect to the LO suprocesses. However, owing to the photon-gluon coupling through a fermion box, the partonic subprocess $g + g \to \gamma + g$ is also permitted. This subprocess is logarithmically-enhanced by multiple soft-gluon radiation in the final state, but it starts to contribute only at NNLO in perturbation theory. It follows that the partonic channel $ab = gg$ is suppressed by a factor α_s^2 with respect to the LO partonic channels $ab = q\bar{q}$, qg, $\bar{q}g$ and it will enter the resummed cross secion only at NNLL order. In conclusion, the partonic channels that should be resummed are $ab = q\bar{q}$, qg, $\bar{q}g$, gg, where the last channel is that coupled to the gluon via a fermion box and eneters resummation only at NNLL. Furthermore, on top of Eqs. (7, 33) the physical process Eq. (1) receives another factorized contribution, in which the final-state photon is produced by fragmentation of a primary parton produced in the partonic sub-process. However, the cross section for this process is also suppressed by a factor of $\frac{1}{N}$ in the large N limit. This is due to the fact that the fragmentation function carries this suppression, for the same reason why the anomalous dimensions γ_{qg} and γ_{gq} are suppressed. Therefore, we will disregard the fragmentation contribution. According to Eqs.(19-22) of Section 2.2, the cross section can be written in terms of the physical anomalous dimensions:

$$\begin{aligned}\sigma(N,Q^2) &= \sum_{a,b} K_{ab}(N;Q_0^2,Q^2)\,\sigma_{ab}(N,Q_0^2) &(34)\\ &= \sum_{a,b} \exp\left[E_{ab}(N;Q_0^2,Q^2)\right]\sigma_{ab}(N,Q_0^2), &(35)\end{aligned}$$

where

$$\begin{aligned}E_{ab}(N;Q_0^2,Q^2) &= \int_{Q_0^2}^{Q^2}\frac{dk^2}{k^2}\gamma_{ab}(N,\alpha_s(k^2)) &(36)\\ &= \int_{Q_0^2}^{Q^2}\frac{dk^2}{k^2}[\gamma_{aa}^{\mathrm{AP}}(N,\alpha_s(k^2))+\gamma_{bb}^{\mathrm{AP}}(N,\alpha_s(k^2))]\\ &\quad+\ln C_{ab}(N,1,\alpha_s(Q^2))-\ln C_{ab}(N,1,\alpha_s(Q_0^2)). &(37)\end{aligned}$$

In the large-$x_\perp$ limit, the order-n coefficient of the perturbative expansion of the hadronic cross section is dominated by terms proportional to $\left[\frac{\ln^k(1-x_\perp)}{1-x_\perp}\right]_+$, with $k \leq 2n-1$, that must be resummed to all orders. Upon Mellin transformation, these lead to contributions proportional to powers of $\ln\frac{1}{N}$. In the sequel, we will consider the resummation of these contributions to all logarithmic orders, and disregard all contributions to the cross section which are suppressed by powers of $(1-x_\perp)$, i.e., upon Mellin transformation, by powers of $\frac{1}{N}$. The resummation is performed in two steps as in chapter 3. First, we show that the origin of the large logs is essentially kinematical: we identify the configurations which contribute in the soft limit, we show by explicit computation that large Sudakov logs are produced by the phase-space for real emission with the required kinematics as logs of two dimensionful variables, and we show that this conclusion is unaffected by virtual corrections. Second, we resum the logs of these variables using the renormalization group. The l-th order correction to the leading $O(\alpha_s)$ partonic process receives contribution from the emission of up to $l+1$ massless partons with momenta $k_1,\ldots,k_{l+1}$. Four-momentum conservation implies:

$$p_1+p_2 = p_\gamma + k_1 + \ldots k_{l+1}. \tag{38}$$

In the partonic center-of-mass frame, according to Eqs.(4,5), we have

$$(p_1+p_2-p_\gamma)^2 = \frac{Q^2}{z}(1-\sqrt{z}\cosh\hat{\eta}_\gamma) = \sum_{i,j=1}^{l+1} k_i^0 k_j^0(1-\cos\theta_{ij}) \geq 0, \tag{39}$$

where θ_{ij} is the angle between $\vec{k}_i$ and $\vec{k}_j$. Hence,

$$1 \le \cosh \hat{\eta}_\gamma \le \frac{1}{\sqrt{z}}. \tag{40}$$

Therefore,

$$\sum_{i,j=1}^{l+1} k_i^0 k_j^0 (1 - \cos\theta_{ij}) = \frac{Q^2}{2}(1-z) + O\left[(1-z)^2\right]. \tag{41}$$

Equation (41) implies that in the soft limit the sum of scalar products of momenta k_i of emitted partons Eq. (39) must vanish. However, contrary to the case of deep-inelastic scattering or Drell-Yan, not all momenta k_i of the emitted partons can be soft as $z \to 1$, because the three-momentum of the photon must be balanced. Assume thus that momenta $k_i, i = 1, \dots, n; n < l+1$ are soft in the $z \to 1$ limit, while momenta $k_i, i > n$ are non-soft. For the sake of simplicity, we relabel non-soft momenta as

$$k'_j = k_{n+j}; \qquad 1 \le j \le m+1; \qquad m = l - n. \tag{42}$$

The generic kinematic configuration in the $z = 1$ limit is then

$$\begin{aligned} &k_i = 0 \qquad 1 \le i \le n \\ &\theta_{ij} = 0; \qquad \sum_{j=1}^{m+1} k_j^{\prime 0} = p_\perp \qquad\qquad 1 \le i, j \le m+1. \end{aligned} \tag{43}$$

for all n between 1 and l, namely, the configuration where at least one momentum is not soft, and the remaining momenta are either collinear to it, or soft. With this labelling of the momenta, the phase space can be written, using twice the phase space decomposition of Eq.(12) in Appendix B, as

$$\begin{aligned} &d\phi_{n+m+2}(p_1+p_2; p_\gamma, k_1, \dots, k_n, k'_1, \dots, k'_{m+1}) \\ = &\int_0^s \frac{dq^2}{2\pi} d\phi_{n+1}(p_1+p_2; q, k_1, \dots, k_n) \\ &\times \int_0^{q^2} \frac{dk'^2}{2\pi}\, d\phi_{m+1}(k'; k'_1, \dots, k'_{m+1})\, d\phi_2(q; p_\gamma, k'). \end{aligned} \tag{44}$$

We shall now compute the phase space in the $z \to 1$ limit in $d = 4 - 2\epsilon$ dimensions. Consider first the two-body phase space $d\phi_2$ in Eq. (44). In the rest frame of q we have

$$\begin{aligned} d\phi_2(q; p_\gamma, k') &= \frac{d^{d-1}k'}{(2\pi)^{d-1} 2k'^0} \frac{d^{d-1}p_\gamma}{(2\pi)^{d-1} 2p_\gamma^0} (2\pi)^d \delta^{(d)}(q - k' - p_\gamma) \\ &= \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{P^{1-2\epsilon}}{\sqrt{q^2}} \sin^{-2\epsilon}\theta_\gamma\, d\,|\vec{p}_\gamma|\; d\cos\theta_\gamma\, \delta(|\vec{p}_\gamma| - P), \end{aligned} \tag{45}$$

where

$$P = \frac{\sqrt{q^2}}{2}\left(1 - \frac{k'^2}{q^2}\right). \tag{46}$$

Because momenta k_i, $i \le n$ are soft, up to terms suppressed by powers of $1-z$, the rest frame of q is the same as the center-of-mass frame of the incoming partons, in which

$$|\vec{p}_\gamma| = p_\perp \cosh \hat{\eta}_\gamma \tag{47}$$
$$\cos\theta_\gamma = \tanh \hat{\eta}_\gamma . \tag{48}$$

Hence,

$$d\phi_2(q; p_\gamma, k') = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{(Q^2/4)^{-\epsilon}}{\sqrt{q^2}} \, dp_\perp \, d\hat{\eta}_\gamma \, \delta\left(\cosh\hat{\eta}_\gamma - \frac{2P}{\sqrt{Q^2}}\right). \tag{49}$$

The conditions

$$\cosh\hat{\eta}_\gamma = \frac{2P}{\sqrt{Q^2}} \ge 1; \qquad k'^2 \ge 0, \tag{50}$$

together with Eq.(46), restrict the integration range to

$$Q^2 \le \quad q^2 \quad \le s \tag{51}$$
$$0 \le \quad k'^2 \quad \le q^2 - \sqrt{Q^2 q^2}. \tag{52}$$

It is now convenient to define new variables u, v

$$q^2 = Q^2 + u(s - Q^2) \quad = \quad Q^2\left[1 + u(1-z)\right] + O((1-z)^2) \tag{53}$$
$$k'^2 = v(q^2 - \sqrt{Q^2 q^2}) \quad = \quad Q^2 \frac{1}{2} uv(1-z) + O((1-z)^2) \tag{54}$$
$$0 \le u \le 1 \quad ; \quad 0 \le v \le 1, \tag{55}$$

in terms of which

$$P = \frac{\sqrt{Q^2}}{2}\left[1 + \frac{1}{2}u(1-v)(1-z)\right] + O\left[(1-z)^2\right]. \tag{56}$$

Thus, the two-body phase space Eq. (49) up to subleading terms is given by

$$d\phi_2(q; p_\gamma, k') = \frac{(4\pi)^\epsilon}{8\pi\Gamma(1-\epsilon)} \frac{(Q^2/4)^{-\epsilon}}{\sqrt{Q^2}} \, dp_\perp \, d\hat{\eta}_\gamma \, \frac{\delta(\hat{\eta}_\gamma - \hat{\eta}_+) + \delta(\hat{\eta}_\gamma - \hat{\eta}_-)}{\sqrt{u(1-v)(1-z)}}, \tag{57}$$

where

$$\hat{\eta}_\pm = \ln\left(\frac{2P}{\sqrt{Q^2}} \pm \sqrt{\frac{4P^2}{Q^2} - 1}\right) = \pm\sqrt{u(1-v)(1-z)}. \tag{58}$$

We now note that the phase-space element $d\phi_{n+1}(p_1 + p_2; q, k_1, \ldots, k_n)$ contains in the final state a system with large invariant mass $q^2 \ge Q^2$, plus a collection of n soft partons: this same configuration is encountered in the case of Drell-Yan pair production in the limit $z_{DY} = q^2/s \to 1$, discussed in Section 3.2 . Likewise, the phase space for the set of collinear partons $d\phi_{m+1}(k'; k'_1, \ldots, k'_{m+1})$ is the same as the phase space for deep-inelastic scattering (discussed in Section 3.1), where the invariant mass of the initial state k'^2 vanishes as $1-z$ (see Eq. (54)). We may therefore use the results obtained in chapter 3, where, in the case of deep-inelastic scattering, one of the outgoing parton momenta (k'_{m+1}, say) was identified with the momentum of the leading-order outgoing quark p'. Hence Eq. (60) is obtained from the corresponding result in chapter 3 for deep-inelastic scattering by the replacement $p' \to k'_{m+1}$:

$$\begin{aligned} d\phi_{n+1}(p_1+p_2;q,k_1,\ldots,k_n) &= 2\pi\left[\frac{N(\epsilon)}{2\pi}\right]^n (q^2)^{-n(1-\epsilon)}(s-q^2)^{2n-1-2n\epsilon} \\ &\quad \times d\Omega^{(n)}(\epsilon) \qquad (59) \\ d\phi_{m+1}(k';k'_1,\ldots,k'_{m+1}) &= 2\pi\left[\frac{N(\epsilon)}{2\pi}\right]^m (k'^2)^{m-1-m\epsilon} d\Omega'^{(m)}(\epsilon), \qquad (60) \end{aligned}$$

where $N(\epsilon) = 1/(2(4\pi)^{2-2\epsilon})$ and

$$\begin{aligned} d\Omega^{(n)}(\epsilon) &= d\Omega_1\ldots d\Omega_n \int_0^1 dz_n z_n^{(n-2)+(n-1)(1-2\epsilon)}(1-z_n)^{1-2\epsilon}\ldots \\ &\quad \times \int_0^1 dz_2 z_2^{1-2\epsilon}(1-z_2)^{1-2\epsilon} \qquad (61) \\ d\Omega'^{(m)}(\epsilon) &= d\Omega'_1\ldots d\Omega'_m \int_0^1 dz'_m z_m'^{(m-2)-(m-1)\epsilon}(1-z'_m)^{1-2\epsilon}\ldots \\ &\quad \times \int_0^1 dz'_2 z_2'^{-\epsilon}(1-z'_2)^{1-2\epsilon}. \qquad (62) \end{aligned}$$

Here, the variables z_i, z'_i are defined as in chapter 3 for the Drell-Yan and deep-inelastic scattering respectively. Equations (53,54) imply that the phase space depends on $(1-z)^{-\epsilon}$ through the two variables

$$k'^2 \propto Q^2(1-z) \qquad (63)$$

$$\frac{(s-q^2)^2}{q^2} \propto Q^2(1-z)^2, \qquad (64)$$

where the coefficients of proportionality are dimensionless and z-independent. By explicitly combining the two-body phase space Eq. (57) and the phase spaces for soft radiation Eq. (59) and for collinear radiation Eq. (60) we get

$$\begin{aligned} &d\phi_{n+m+2}(p_1+p_2;p_\gamma,k_1,\ldots,k_n,k'_1,\ldots,k'_{m+1}) \\ &= (Q^2)^{n+m-(n+m+1)\epsilon}\frac{dp_\perp}{p_\perp}\,\frac{(1-z)^{2n+m-(2n+m)\epsilon}}{\sqrt{1-z}} \\ &\times 2^{-m+m\epsilon}\,\frac{(16\pi)^{-1+\epsilon}}{\Gamma(1-\epsilon)}\left[\frac{N(\epsilon)}{2\pi}\right]^{n+m} d\hat\eta_\gamma\, d\Omega^{(n)}(\epsilon)\, d\Omega'^{(m)}(\epsilon) \\ &\times \int_0^1 du\,\frac{u^{m-m\epsilon}(1-u)^{2n-1-2n\epsilon}}{\sqrt{u}} \int_0^1 dv\,\frac{v^{m-1-m\epsilon}}{\sqrt{1-v}}\,[\delta(\hat\eta_\gamma-\hat\eta_+)+\delta(\hat\eta_\gamma-\hat\eta_-)]. \end{aligned} \qquad (65)$$

In the limiting cases $n = 0$ and $m = 0$ we have

$$d\phi_1(p_1 + p_2; q) = 2\pi\delta(s - q^2) = \frac{2\pi}{Q^2(1-z)}\,\delta(1-u) \tag{66}$$

$$d\phi_1(k'; p') = 2\pi\delta(k'^2) = \frac{4\pi}{Q^2 u(1-z)}\,\delta(v); \tag{67}$$

the corresponding expressions for the phase space are therefore obtained by simply replacing

$$(1-u)^{-1}\, d\Omega^{(n)}(\epsilon) \to \delta(1-u); \qquad v^{-1}\, d\Omega'^{(m)}(\epsilon) \to \delta(v) \tag{68}$$

in Eq. (65) for $n = 0$, $m = 0$ respectively. The logarithmic dependence of the four-dimensional cross section on $1-z$ is due to interference between powers of $(1-z)^{-\epsilon}$ and $\frac{1}{\epsilon}$ poles in the d-dimensional cross section. Hence, we must classify the dependence of the cross section on powers of $(1-z)^{-\epsilon}$. We have established that in the phase space each real emission produces a factor of $\left[Q^2(1-z)^2\right]^{1-\epsilon}$ if the emission is soft and a factor of $\left[Q^2(1-z)\right]^{1-\epsilon}$ if the emission is collinear. The squared amplitude can only depend on $(1-z)^{-\epsilon}$ because of loop integrations. This dependence for a generic proper Feynman diagram G will in general appear, as discussed in chapter 3, through the coefficient (see Eq.(22) of Section 3.1)

$$\left[D_G(P_E)\right]^{dL/2-I}, \tag{69}$$

where L and I are respectively the number of loops and internal lines in G, and $D_G(P_E)$ is a linear combination of all scalar products P_E of external momenta. In the soft limit all scalar products which vanish as $z \to 1$ are either proportional to $Q^2(1-z)$ or to $Q^2(1-z)^2$ as shown in Eqs.(47,53,54) of Section 3.2 . Equation (69) then implies that each loop integration can carry at most a factor of $\left[Q^2(1-z)^2\right]^{-\epsilon}$ or $\left[Q^2(1-z)\right]^{-\epsilon}$. This then proves that the perturbative expansion of the bare coefficient function, for each sub-process which involves partons a, b, takes the form

$$C^{(0)}(z, Q^2, \alpha_0, \epsilon) = \alpha\alpha_0 (Q^2)^{-\epsilon} \sum_{l=0}^{\infty} \alpha_0^l\, C_l^{(0)}(z, Q^2, \epsilon) \tag{70}$$

$$C_l^{(0)}(z, Q^2, \alpha_0, \epsilon) = \frac{(Q^2)^{-l\epsilon}}{\Gamma(1/2)\sqrt{1-z}} \sum_{k=0}^{l} \sum_{k'=0}^{l-k} C_{lkk'}^{(0)}(\epsilon)(1-z)^{-2k\epsilon - k'\epsilon}, \tag{71}$$

where the factor $1/\Gamma(1/2)$ was introduced for later convenience and terms $C_{lkk'}^{(0)}$ with $k + k' < l$ at order α_s^l are present in general because of loops. The coefficients $C_{lkk'}^{(0)}$ have poles in $\epsilon = 0$ up to order $2l$. To understand this, we have to count the independent variables for the prompt photon process. We have 2 incoming particles and $l+2$ outgoing partons (a leading-order parton, the photon and l extra emissions). Therefore, imposing the on-shell conditions and the constraints due to Poincarvariance, we get

$$4(l+4) - (l+4) - 10 = 3l + 2, \tag{72}$$

independent variables. Now, we need to understand which are these independent variables: the general expression of the phase space in the threshold limit Eq.(65) is written in terms of $3l+3$ variables which are

$$s, 4p_\perp^2, u, v, z_2, \ldots, z_n, z_2', \ldots, z_m', \Omega_1, \ldots, \Omega_n, \Omega_1', \ldots, \Omega_m', \hat{\eta}_\gamma, \tag{73}$$

where $n+m=l$ and each solid angle depends on two parameters. Clearly, one of them must be a function of some of the others because of Eq. 72. In fact, from Eq.(58), we know that $\hat{\eta}_\gamma$ depends on $u, v, 4p_\perp^2, s$. Thus, the $3l+2$ independent variables on which depends the square modulus amplitude can be chosen as

$$s, 4p_\perp^2, u, v, z_2, \ldots, z_n, z_2', \ldots, z_m', \Omega_1, \ldots, \Omega_n, \Omega_1', \ldots, \Omega_m'. \tag{74}$$

Now, each of the l integrations over a solid angle can produce a pole $1/\epsilon$ from the collinear region. Furthermore, each of the l integrations over a dimensionless variable $u, v, z_2, \ldots, z_n, z_2', \ldots, z_m'$ can produce a pole $1/\epsilon$ from the soft region. This explains why the coefficients $C^{(0)}_{lkk'}$ can have poles in $\epsilon=0$ up to order $2l$.

4.4 Resummation from renormalization group improvement

The Mellin transform of Eq. (70) can be performed using

$$\int_0^1 dz z^{N-1}(1-z)^{-1/2-2k\epsilon-k'\epsilon} = \frac{\Gamma(1/2)}{\sqrt{N}} N^{2k\epsilon} N^{k'\epsilon} + O\left(\frac{1}{N}\right), \tag{75}$$

with the result

$$\begin{aligned} &C^{(0)}(N, Q^2, \alpha_0, \epsilon) \\ &= \frac{\alpha\alpha_0 (Q^2)^{-\epsilon}}{\sqrt{N}} \sum_{l=0}^{\infty} \sum_{k=0}^{l} \sum_{k'=0}^{l-k} C^{(0)}_{lkk'}(\epsilon) \left[\left(\frac{Q^2}{N^2}\right)^{-\epsilon} \alpha_0\right]^k \left[\left(\frac{Q^2}{N}\right)^{-\epsilon} \alpha_0\right]^{k'} \\ &\times \left[(Q^2)^{-\epsilon}\alpha_0\right]^{l-k-k'} + O\left(\frac{1}{N}\right). \end{aligned} \tag{76}$$

Equation (76) shows that indeed as $N \to \infty$, up to $\frac{1}{N}$ corrections, the coefficient function depends on N through the two dimensionful variables $\frac{Q^2}{N^2}$ and $\frac{Q^2}{N}$. The argument henceforth follows in the same way as in chapter 3, in this more general situation. The argument is based on the observation that, because of collinear factorization, the physical anomalous dimension

$$\gamma(N, \alpha_s(Q^2)) = Q^2 \frac{\partial}{\partial Q^2} \ln C(N, Q^2/\mu^2, \alpha_s(\mu^2)) \tag{77}$$

is renormalization-group invariant and finite when expressed in terms of the renormalized coupling $\alpha_s(\mu^2)$, related to α_0 by

$$\alpha_0(\mu^2, \alpha_s(\mu^2)) = \mu^{2\epsilon}\alpha_s(\mu^2)\, Z^{(\alpha_s)}(\alpha_s(\mu^2), \epsilon), \tag{78}$$

where $Z^{(\alpha_s)}(\alpha_s(\mu^2),\epsilon)$ is a power series in $\alpha_s(\mu^2)$. Because α_0 is manifestly independent of μ^2, Eq. (78) implies that the dimensionless combination $(Q^2)^{-\epsilon}\alpha_0(\alpha_s(\mu^2),\mu^2)$ can depend on Q^2 only through $\alpha_s(Q^2)$:

$$(Q^2)^{-\epsilon}\alpha_0(\alpha_s(\mu^2),\mu^2) = \alpha_s(Q^2)\, Z^{(\alpha_s)}(\alpha_s(Q^2),\epsilon). \tag{79}$$

Using Eq. (79) in Eq. (76), the coefficient function and consequently the physical anomalous dimension are seen to be given by a power series in $\alpha_s(Q^2)$, $\alpha_s(Q^2/N)$ and $\alpha_s(Q^2/N^2)$:

$$\gamma(N,\alpha_s(Q^2),\epsilon) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\gamma^R_{mnp}(\epsilon)\,\alpha_s^m(Q^2)\,\alpha_s^n(Q^2/N^2)\,\alpha_s^p(Q^2/N). \tag{80}$$

Even though the anomalous dimension is finite as $\epsilon\to 0$ for all N, the individual terms in the expansion Eq. (57) are not separately finite. However, if we separate N-dependent and N-independent terms in Eq. (57):

$$\gamma(N,\alpha_s(Q^2),\epsilon) = \hat\gamma^{(c)}(\alpha_s(Q^2),\epsilon) + \hat\gamma^{(l)}(N,\alpha_s(Q^2),\epsilon), \tag{81}$$

we note that the two functions

$$\begin{aligned}\gamma^{(c)}(\alpha_s(Q^2),\epsilon) &\equiv \hat\gamma^{(c)}(\alpha_s(Q^2),\epsilon) + \hat\gamma^{(l)}(1,\alpha_s(Q^2),\epsilon) &(82)\\ \gamma^{(l)}(N,\alpha_s(Q^2),\epsilon) &\equiv \hat\gamma^{(l)}(N,\alpha_s(Q^2),\epsilon) - \hat\gamma^{(l)}(1,\alpha_s(Q^2),\epsilon) &(83)\end{aligned}$$

must be separately finite. In fact,

$$\gamma(N,\alpha_s(Q^2),\epsilon) = \gamma^{(c)}(\alpha_s(Q^2),\epsilon) + \gamma^{(l)}(N,\alpha_s(Q^2),\epsilon), \tag{84}$$

is finite for all N and $\gamma^{(l)}$ vanishes for $N=1$. This implies that $\gamma^{(c)}(\alpha_s(Q^2),\epsilon)$ is finite in $\epsilon=0$ and that $\gamma^{(l)}(N,\alpha_s(Q^2),\epsilon)$ is also finite in $\epsilon=0$ for all N because of the N-independence of $\gamma^{(c)}(\alpha_s(Q^2),\epsilon)$. We can rewrite conveniently

$$\gamma^{(l)}(N,\alpha_s(Q^2),\epsilon) = \int_1^N \frac{dn}{n}\, g(\alpha_s(Q^2),\alpha_s(Q^2/n^2),\alpha_s(Q^2/n),\epsilon), \tag{85}$$

where

$$g(\alpha_s(Q^2),\alpha_s(Q^2/n^2),\alpha_s(Q^2/n),\epsilon) = n\frac{\partial}{\partial n}\hat\gamma^{(l)}(n,\alpha_s(Q^2),\epsilon). \tag{86}$$

is a Taylor series in its arguments whose coefficients remain finite as $\epsilon\to 0$. In four dimension we have thus

$$\begin{aligned}\gamma(N,\alpha_s(Q^2)) &= \gamma^{(l)}(N,\alpha_s(Q^2),0) + \gamma^{(c)}(N,\alpha_s(Q^2),0) + O\left(\frac{1}{N}\right)\\ &= \gamma^{(l)}(N,\alpha_s(Q^2),0) + O\left(N^0\right)\\ &= \int_1^N \frac{dn}{n}\, g(\alpha_s(Q^2),\alpha_s(Q^2/n^2),\alpha_s(Q^2/n)) + O\left(N^0\right),\end{aligned} \tag{87}$$

where

$$g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) \equiv \lim_{\epsilon\to 0} g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n), \epsilon) \qquad (88)$$

is a generic Taylor series of its arguments. Renormalization group invariance thus implies that the physical anomalous dimension γ Eq. (77) depends on its three arguments Q^2, Q^2/N and Q^2/N^2 only through α_s. Clearly, any function of Q^2 and N can be expressed as a function of $\alpha_s(Q^2)$ and $\alpha_s(Q^2/N)$ or $\alpha_s(Q^2/N^2)$. The nontrivial statement, which endows Eq. (87) with predictive power, is that the log derivative of γ, $g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n))$ Eq. (86), is analytic in its three arguments. This immediately implies that when γ is computed at (fixed) order α_s^k, it is a polynomial in $\ln\frac{1}{N}$ of k-th order at most. In order to discuss the factorization properties of our result we write the function g as

$$\begin{aligned} g(\alpha_s(Q^2), \alpha_s(Q^2/n^2), \alpha_s(Q^2/n)) \;=\;& g_1(\alpha_s(Q^2), \alpha_s(Q^2/n)) \\ &+\; g_2(\alpha_s(Q^2), \alpha_s(Q^2/n^2)) \\ &+\; g_3(\alpha_s(Q^2), \alpha_s(Q^2/n), \alpha_s(Q^2/n^2)), \end{aligned} \qquad (89)$$

where

$$g_1(\alpha_s(Q^2), \alpha_s(Q^2/n)) = \sum_{m=0}^{\infty}\sum_{p=1}^{\infty} g_{m0p}\, \alpha_s^m(Q^2)\, \alpha_s^p(Q^2/n) \qquad (90)$$

$$g_2(\alpha_s(Q^2), \alpha_s(Q^2/n^2)) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} g_{mn0}\, \alpha_s^m(Q^2)\, \alpha_s^n(Q^2/n^2) \qquad (91)$$

$$\begin{aligned} g_3(\alpha_s(Q^2), \alpha_s(Q^2/n), \alpha_s(Q^2/n^2)) \;=\;& \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} g_{mnp}\, \alpha_s^m(Q^2) \\ &\times \alpha_s^n(Q^2/n^2)\, \alpha_s^p(Q^2/n). \end{aligned} \qquad (92)$$

The dependence on the resummation variables Q^2, Q^2/N and Q^2/N^2 is fully factorized if the bare coefficient functions has the factorized structure

$$C^{(0)}(N, Q^2, \alpha_0, \epsilon) = C^{(0,c)}(Q^2, \alpha_0, \epsilon)\, C_1^{(0,l)}(Q^2/N, \alpha_0, \epsilon)\, C_2^{(0,l)}(Q^2/N^2, \alpha_0, \epsilon). \qquad (93)$$

This is argued to be the case in the approach of Refs.[14, 12]. If this happens, the resummed anomalous dimension is given by Eq. (87) with all $g_{mnp} = 0$ except g_{0n0}, g_{00p}:

$$\gamma(N, \alpha_s(Q^2)) = \int_1^N \frac{dn}{n}\, g_1(0, \alpha_s(Q^2/n)) + \int_1^N \frac{dn}{n}\, g_2(0, \alpha_s(Q^2/n^2)). \qquad (94)$$

We recall that the coefficient function depends on the parton subprocess in which the incoming partons are a, b (compare Eq. (7)). So, the factorization Eq. (93) applies to the coefficient function corresponding to each subprocess, and the decomposition

Eq. (94) to the physical anomalous dimension computed from each of these coefficient functions. A weaker form of factorization is obtained assuming that in the soft limit the N-dependent and N-independent parts of the coefficient function factorize:

$$C^{(0)}(N,Q^2,\epsilon) = C^{(0,c)}(Q^2,\alpha_0,\epsilon)\, C^{(0,l)}(Q^2/N^2,Q^2/N,\alpha_0,\epsilon). \tag{95}$$

This condition turns out to be satisfied [8] in Drell-Yan and deep-inelastic scattering to order α_s^2. It holds in QED to all orders [53] as a consequence of the fact that each emission in the soft limit can be described by universal (eikonal) factors, independent of the underlying diagram. This eikonal structure of Sudakov radiation has been argued in Refs. [2, 14] to apply also to QCD. If the factorized form Eq. (95) holds, the coefficients g_{mnp} Eqs. (90,91,92) vanish for all $m \neq 0$, and the physical anomalous dimension takes the form

$$\begin{aligned} \gamma(N,\alpha_s(Q^2)) \;&=\; \int_1^N \frac{dn}{n}\, g_1(0,\alpha_s(Q^2/n)) + \int_1^N \frac{dn}{n}\, g_2(0,\alpha_s(Q^2/n^2)) \\ &\quad + \int_1^N \frac{dn}{n}\, g_3(0,\alpha_s(Q^2/n^2),\alpha_s(Q^2/n)). \end{aligned} \tag{96}$$

It is interesting to observe that in the approach of Refs. [14, 12] for processes where more than one colour structure contributes to the cross-section, the factorization Eq. (93) of the coefficient function is argued to take place separately for each colour structure. This means that in such case the exponentiation takes place for each colour structure independently, i.e. the resummed cross section for each parton subprocess is in turn expressed as a sum of factorized terms of the form of Eq. (93). This happens for instance in the case of heavy quark production [14, 54]. In prompt photon production different colour structures appear for the gluon-gluon subprocess which starts at next-to-next-to-leading order, hence their separated exponentiation would be relevant for next-to-next-to-leading log resummed results. When several colour structures contribute to a given parton subprocess, the coefficients of the perturbative expansion Eq. (76) for that process take the form

$$C^{(0)}_{lkk'}(\epsilon) = C^{(0)\mathbf{1}}_{lkk'}(\epsilon) + C^{(0)\mathbf{8}}_{lkk'}(\epsilon), \tag{97}$$

(assuming for definiteness that a colour singlet and octet contribution are present) so that the coefficient function can be written as a sum $C^{(0)} = C^{(0)}_{\mathbf{1}} + C^{(0)}_{\mathbf{8}}$. The argument which leads from Eq. (76) to the resummed result Eq. (87) then implies that exponentiation takes place for each colour structure independently if and only if

$$\gamma_{\mathbf{1}} \equiv \partial \ln C^{(0)}_{\mathbf{1}}/\partial \ln Q^2, \qquad \gamma_{\mathbf{8}} \equiv \partial \ln C^{(0)}_{\mathbf{8}}/\partial \ln Q^2 \tag{98}$$

are separately finite. This, however, is clearly a more restrictive assumption than that under which we have derived the result Eq. (87), namely that the full anomalous dimension γ is finite. It follows that exponentiation of each colour structure must be a special case of our result. However, this can only be true if the coefficients g_{ijk} of the expansion Eq. (90) of the physical anomalous dimension satisfy suitable relations. In particular, at the leading log level, it is easy to see that exponentiation of each colour structure is compatible with exponentiation of their sum only if the leading order coefficients are the same for the given colour structures: $g^{\mathbf{1}}_{001} = g^{\mathbf{8}}_{001}$ and $g^{\mathbf{1}}_{010} = g^{\mathbf{8}}_{010}$. This is indeed the case for heavy quark production (where $g_{001} = 0$). Note that, however, if the factorization holds for each colour structure separately it will not apply to the sum of colour structures. For instance, the weaker form of factorization Eq. (95) requires that

$$C^{(0)}_{lkk'}(\epsilon) = F_{k+k'}(\epsilon) G_{l-k-k'}(\epsilon), \tag{99}$$

but

$$F^{\mathbf{1}}_{k+k'}(\epsilon) G^{\mathbf{1}}_{l-k-k'}(\epsilon) + F^{\mathbf{8}}_{k+k'}(\epsilon) G^{\mathbf{8}}_{l-k-k'}(\epsilon) \neq F_{k+k'}(\epsilon) G_{l-k-k'}(\epsilon). \tag{100}$$

Hence, our result Eq. (87) for the sum of colour structures is more general than the separate exponentiation of individual colour structures, but it leads to results which have weaker factorization properties.

4.5 Comparison with previous results

In this section, we want to make a comparison with the resummation formula for prompt photon production previously released. In order to do this, we need to rewrite the NLL result of Ref.[14] in our formalism. The physical anomalous dimension can be obtained performing the Q^2-logarithmic derivative of the NLL resummed exponent in the $\overline{MS}$ scheme of Ref.[14]. We obtain for a particular partonic sub-process:

$$\begin{aligned} \gamma(N, \alpha_s(Q^2)) &= \int_0^1 dx \, \frac{x^{N-1} - 1}{1 - x} \big[\hat{g}_2 \alpha_s(Q^2(1-x)^2) + \hat{g}'_2 \alpha_s^2(Q^2(1-x)^2) \\ &\quad + \hat{g}_1 \alpha_s(Q^2(1-x)) + \hat{g}'_1 \alpha_s^2(Q^2(1-x-)) \big], \end{aligned} \tag{101}$$

where

$$\hat{g}_2 = \frac{A^{(1)}_d + A^{(1)}_b - A^{(1)}_d}{\pi} \tag{102}$$

$$\hat{g}'_2 = \frac{A^{(2)}_a + A^{2}_b - A^{(2)}_d}{\pi^2} - \frac{\beta_0}{4\pi} \left[\frac{A^{(1)}_a + A^{(1)}_b - A^{(1)}_d}{\pi} \right] \ln 2 \tag{103}$$

$$\hat{g}_1 = \frac{A^{(1)}_d}{\pi} \tag{104}$$

$$\hat{g}'_1 = \frac{A^{(2)}_d}{\pi^2} - \frac{\beta_0}{4\pi} \frac{B^{(1)}_d}{2\pi}. \tag{105}$$

Here, A^i_a is the coefficient of $\ln(1/N)$ in the Mellin transform of the P_{aa} Altarelli-Parisi splitting function at order α_s^i, β_0 is the α_s^2 coefficient of the β function (Eq.(38) in section 1.2) and $B^{(1)}_d$ is a constant to be determined from the comparison with the fixed-order calculation. In Eqs.(102-105) a, b are the incoming partons (on which the coefficient function implicitly depends) and d is the LO outgoing parton uniquely determined by the incoming ones. For completeness, we list explicitly these coefficients:

$$A^{(1)}_{a=q,\bar{q}} = \frac{4}{3}, \qquad A^{1}_{a=g} = 3 \tag{106}$$

$$A^{(2)}_{a} = \frac{1}{2}A^{(1)}_{a}K, \qquad K = \frac{67}{6} - \frac{\pi^2}{6} - \frac{5}{9}N_f \tag{107}$$

$$B^{(1)}_{d=q,\bar{q}} = -2, \qquad B^{(1)}_{d=g} = -\frac{11}{2} + \frac{1}{3}N_f, \tag{108}$$

where N_f is the number of active flavors. Now, performing the change of variable

$$n = \frac{1}{1-x} \tag{109}$$

in the integral Eq.(101), we obtain at NLL

$$\begin{aligned}\gamma(N, \alpha_s(Q^2)) &= \int_0^1 \frac{dn}{n}\big[g_{010}\alpha_s\left(\frac{Q^2}{n^2}\right) + g_{001}\alpha_s\left(\frac{Q^2}{n}\right) \\ &\quad + g_{020}\alpha_s^2\left(\frac{Q^2}{n^2}\right) + g_{002}\alpha_s^2\left(\frac{Q^2}{n}\right)\big],\end{aligned} \tag{110}$$

where

$$g_{010} = -\hat{g}_2, \qquad g_{020} = -\left(\hat{g}'_2 - \frac{\gamma_E\beta_0}{2\pi}\hat{g}_2\right) \tag{111}$$

$$g_{001} = -\hat{g}_1, \qquad g_{002} = -\left(\hat{g}'_1 - \frac{\gamma_E\beta_0}{4\pi}\hat{g}_1\right), \tag{112}$$

with γ_E the Euler constant. Hence, according to Ref.[14], we know exactly the value of the coefficients g_{010}, g_{020}, g_{001} and g_{002}. This enables us to compute predictions of high-order logarithmic contributions to the physical anomalous dimension performing a fixed order expansion of γ. We shall now show that the resummation formula of Ref.[14] predicts the coefficient of $\alpha_s^3 \ln^2(1/N)$ of the fixed order expansion of γ, while in our approach it is required in order to perform a NLL resummation. We need to expand Eq.(110) to order α_s^3 and this is obtained using the change of variable

$$\frac{dn}{n} = -\frac{d\alpha_s(Q^2/n^a)}{a\beta\alpha_s}, \qquad a = 1, 2 \tag{113}$$

to perform the integral and expanding the two loops running of α_s (see Eq.(43) in section 1.2). We find

$$\begin{aligned}\gamma &= [-(g_{001} + g_{010})]\alpha_s(Q^2)\ln\frac{1}{N} \\ &\quad + [-(g_{002} + g_{020}) - (b_1/b_0)(g_{001} + g_{010})]\alpha_s^2(Q^2)\ln\frac{1}{N} \\ &\quad + \left[\frac{b_0}{2}(g_{001} + 2g_{010})\right]\alpha_s^2(Q^2)\ln^2\frac{1}{N} + [-(g_{003} + g_{030})]\alpha_s^3(Q^2)\ln\frac{1}{N} \\ &\quad + [(3b_1/2)(g_{001} + 2g_{010}) + b_0(g_{002} + 2g_{020})]\alpha_s^3(Q^2)\ln^2\frac{1}{N} \\ &\quad + [-(b_0^2/3)(g_{001} + 4g_{010})]\alpha_s^3\ln^3\frac{1}{N} + O(\alpha_s^4).\end{aligned} \tag{114}$$

In our approach, in order to determine the NLL resummation coefficients g_{010}, g_{020}, g_{001} and g_{002}, we must compare this expansion to a fixed order computation of the physical anomalous dimension, which in the general has the form

$$\gamma_{\rm FO}(N,\alpha_s)=\sum_{i=1}^{k}\alpha_s^i\sum_{j=1}^{i}\gamma_j^i\,\ln^j\frac{1}{N}+O(\alpha_s^{k+1})+O(N^0), \tag{115}$$

where k is the fixed-order at which it has been computed (see Chapter 7 for a general discussion about the determination of the resummation coefficients). Hence, we determine the 4 NLL resummation coefficients through the following 4 independent conditions:

$$\begin{aligned}
g_{001}+g_{010} &= -\gamma_1^1 &(116)\\
g_{001}+2g_{010} &= \frac{2}{b_0}\gamma_2^2 &(117)\\
g_{002}+g_{020} &= -\gamma_1^2-\frac{b_1}{b_0}(g_{001}+g_{010}) &(118)\\
g_{002}+2g_{020} &= \frac{1}{b_0}\gamma_2^3-\frac{3b_1}{2b_0}(g_{001}+2g_{010}). &(119)
\end{aligned}$$

Thus, according to our formalism, all the coefficients γ_1^1, γ_2^2, γ_1^2 and γ_2^3 should be known. The first three coefficients are all known thanks to the explicit $O(\alpha_s^2)$ calculation of the prompt photon cross section [55, 56, 57]. The last one (γ_2^3), is not yet known from explicit $O(\alpha_s^3)$ calculation, but, according to the approach of Ref.[14], it is predicted using Eqs.(119,111,112):

$$\begin{aligned}
\gamma_2^3 &= -b_0\left[\frac{2(A_a^{(2)}+A_b^{(2)})-A_d^{(2)}}{\pi^2}-b_0(2\ln 2+4\gamma_E)\frac{A_a^{(1)}+A_b^{(1)}}{\pi}\right.\\
&\left.+b_0(2\ln 2+3\gamma_E)\frac{A_d^{(1)}}{\pi}-\frac{b_0B_d^{(1)}}{2\pi}\right]-b_1\frac{3}{2}\left[\frac{2(A_a^{(1)}+A_b^{(1)})-A_d^{(1)}}{\pi}\right].
\end{aligned} \tag{120}$$

The correctness of this result could be tested by an order α_s^3 calculation. If it were to fail, the more general resummation formula with g_{020} determined by Eq.(119) should be used, or one of the resummations which do not assume the factorization Eq.(93).