Chapter 1

General aspects of perturbative QCD

1.1 Quarks, Gluons and QCD

The quarks (the constituents of hadrons) are Dirac fermions. In the Standard Model (SM), as far as the electroweak interactions are concerned, the properties of quarks and leptons are similar. Indeed, as for the leptons, the six quark flavors are grouped into three $SU_L(2)$ left-handed doublets

$$\begin{pmatrix} u \\ d \end{pmatrix}_L, \quad \begin{pmatrix} c \\ s \end{pmatrix}_L, \quad \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{1}$$

and six $SU(2)_L$ singlets, which are the right-handed parts of each flavor. Both, quarks and leptons, interact in a similar way with the electroweak gauge bosons (γ, $W^{\pm}$ and Z^0) of the group $SU_L(2) \times U_Y(1)$. The main difference is that each quark flavor eigenstate is a unitary mixing of the quark mass eigenstate, while, according to the SM, this is not the case for charged leptons and massless neutrinos. However, in this decade it has been proven that neutrinos have mass and that there is also mixing in the neutrino sector thanks to observations of their oscillations. The peculiarity of quarks is that they have a specific property, the color charge, which is absent for leptons. Indeed, a quark of a given flavor has three different color states with equal masses and electroweak charges. The interaction of the quarks is mediated by the eight gauge bosons (gluons g) of the color group $SU_C(3)$. So, the quarks belong to the fundamental representation of $SU_C(3)$ and the gluons to the adjoint one. The gauge theory of this non-abelian group is called Quantum Chromodynamics (QCD) and is the current theory of strong interactions. Specifically, a gauged $SU_C(3)$ transformation of a quark ($q_a(x)$ with $a = 1, 2, 3$) is given by

$$q_a(x) \rightarrow q'_a(x) = U_{ab}(x) q_b(x) \tag{2}$$

$$\bar{q}_a(x) \rightarrow \bar{q}'_a(x) = \bar{q}_b U^{\dagger}_{ba}(x), \tag{3}$$

where the 3×3 matrix $U_{ik}(x)$ is the fundamental representation of the $SU_C(3)$ group that acts on an internal space defined at each space-time coordinate x. It satisfies

$$UU^{\dagger} = U^{\dagger}U = 1, \qquad \det(U) = 1. \tag{4}$$

In this section the sum over all the repeated indices is implicit and the sum over spinor indices is omitted for brevity. The usual exponential representation of the gauge transformation matrix in terms of the basis of matrices (the generators of $SU_C(3)$) of the corresponding algebra $su_C(3)$ is:

$$U(x) = e^{-\frac{i}{2}\vec{\alpha}(x)\cdot\vec{\lambda}} = e^{-\vec{\alpha}(x)\cdot\vec{t}}, \tag{5}$$

where $\vec{\alpha}(x) = (\alpha_1(x), \ldots, \alpha_8(x))$ are the eight arbitrary parameters of the gauge transformation, $\vec{\lambda}$ are the eight elements of the basis of the algebra $su_C(2)$ (or equivalently the eight generators of the group) and $\vec{t}$ are the eight color operators (in analogy to the spin operators of the group $SU(2)$). It is clear that, in order to respect Eq.(4), the generators of the group must be hermitian and traceless. The normalization of the color operators depends on the representation r

$$\mathrm{tr}(t_r^A t_r^B) = T_r \delta^{AB}. \tag{6}$$

The form chosen by Gell-Mann for the $su_C(3)$ basis in the fundamental representation:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{7}$$

With these definitions, the color matrices satisfy the following relations

$$[t^A, t^B] = i f^{ABC} t^C \tag{8}$$
$$\mathrm{tr}(t^A t^B) = T_F \delta^{AB}, \quad T_F = \frac{1}{2} \tag{9}$$

where T_F is the normalization of the color matrices in the fundamental representation and f^{ABC} are called the structure constants of the algebra $su_C(3)$ which are totally antisymmetric in $\{A, B, C\}$. The independent non-vanishing structure constants are given by:

$$f^{123} = 1, \quad f^{147} = \frac{1}{2}, \quad f^{156} = -\frac{1}{2}, \quad f^{246} = \frac{1}{2}, \quad f^{257} = \frac{1}{2} \tag{10}$$
$$f^{345} = \frac{1}{2}, \quad f^{367} = -\frac{1}{2}, \quad f^{458} = \frac{\sqrt{3}}{2}, \quad f^{678} = \frac{\sqrt{3}}{2}. \tag{11}$$

Furthermore, the structure constants provide the adjoint representation of the $su_C(3)$ algebra (the one which has the same dimension of the algebra). Indeed, if we define

the adjoint representation as $T^A_{BC} = -if^{ABC}$, we can verify explicitly that this is the adjoint representatio because

$$[T^A, T^B] = if^{ABC}T^C \tag{12}$$
$$\mathrm{tr}(T^A T^B) = T_A \delta^{AB}, \quad T_A = 3 \tag{13}$$

The Casimir operator C_r (the one which commutes with all elements of the algebra), for a certain representation r, is constructed as

$$t^A_r t^A_r = C_r 1_{d_r \times d_r}. \tag{14}$$

Now, since the contraction of this last equation is equal to the contraction of Eq.(6), we get

$$d_r C_r = 8T_r. \tag{15}$$

In particular, we find the Casimir operators are given by

$$C_F = \frac{4}{3}, \qquad C_A = 3 \tag{16}$$

for the fundamental and the adjoint representation respectively. We shall now show that the symmetry with respect to the gauge transformations Eqs.(2,5) (together with the Lorentz invariance), can be used as guiding principle to construct the QCD Lagrangian. We start from the usual Dirac free Lagrangian for each quark mass and color eigenstate

$$\mathcal{L}_D(x) = \bar{\psi}_{fa}(x)\left(i\partial\!\!\!/ - m_f\right)\psi_{fa}(x), \tag{17}$$

where f is the flavor index and a is the color quark index. This term is not gauge invariant. In fact under the gauge transformations Eqs.(2,5), the Dirac free lagrangian Eq.(17) transform as

$$\mathcal{L}_D(x) \to \mathcal{L}_D(x) + \bar{\psi}_{fb}\left[iU^\dagger_{ba}(x)\partial_\mu U_{ac}(x)\right]\gamma^\mu \psi_{fc}(x). \tag{18}$$

To restore gauge invariance, we introduce a gauge field matrix $A_{\mu\, ab}(x)$ made up of eight gauge fields $A^A_\mu(x)$ in this way:

$$A_{\mu\, ab}(x) = t^A_{ab} A^A_\mu. \tag{19}$$

We assign to this matrix field the following interaction Lagrangian

$$\mathcal{L}_I(x) = g_s \bar{\psi}_{fa} A_{\mu\, ab} \gamma^\mu \psi_{fb}. \tag{20}$$

Here g_s is the gauge dimensionless coupling analogous to the electric charge in QED. In order to cancel the symmetry breaking term of Eq.(18), the gauge transformation of the field matrix $A_{\mu\, ab}(x)$ has to be

$$A_{\mu\, ab}(x) \to U_{ac}(x) A_{\mu\, cd}(x) U^\dagger_{db}(x) - \frac{i}{g_s}\partial_\mu U_{ac}(x) U^\dagger_{cb}(x). \tag{21}$$

Hence, with the introduction of the field matrix $A_{\mu\,ab}$, the sum $\mathcal{L}_D + \mathcal{L}_I$ is now gauge invariant. To complete the Lagrangian one has to add the pure gauge invariant term, which is

$$\mathcal{L}_G(x) = -\frac{1}{2}\,\mathrm{tr}\,(G_{\mu\nu}(x)G^{\mu\nu}(x))\,, \tag{22}$$

where the gluon field-strength tensor is given by

$$G_{\mu\nu\,ab}(x) = \partial_\mu A_{\nu\,ab} - \partial_\nu A_{\mu\,ab} - ig_s\,[A_\mu, A_\nu]_{ab}\,. \tag{23}$$

Note that the third term of the gluon field-strength gives rise to the self interactions of gluons and that its origin stands in the fact that the gauge group $SU_C(3)$ is non-abelian. The final form of the QCD classical Lagrangian is obtained by adding the three pieces introduced above and using Eq.(9):

$$\mathcal{L}^{\mathrm{cl}}_{QCD}(x) = -\frac{1}{4}G^A_{\mu\nu}(x)G^{A\,\mu\nu}(x) + \bar{\psi}_{fa}(x)(iD_{\mu\,ab}\gamma^\mu - m_f\delta_{ab})\psi_{fb}(x), \tag{24}$$

where

$$G^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g_s f^{ABC} A^B_\mu A^C_\nu \tag{25}$$

and

$$D_{\mu ab} = \delta_{ab}\partial_\mu - ig_s A_{\mu\,ab} \tag{26}$$

is the covariant derivative in the sense that $D_{\mu\,ab}\psi_{fb}(x)$ transforms as $\psi_{fa}(x)$ under the gauge group. The quantization of the classical gauge invariant theory of QCD Eq.(24), can be done with the Fadeev-Popov procedure. This procedure takes care of the fact that the equation of motion of the gluon field A^A_μ can not be inverted and this prohibits to find the propagator. However, this is a consequence of gauge invariance which implies that the physical massless gluon has only two polarizations/spin states whereas the field A^A_μ has four components. To make things work, an additional constraint on the gluon field is introduced, the so called gauge fixing condition which uses gauge invariance to define properly the gluon propagator. In QCD, however, this constraint is not linear and one should add specially designed fictitious particles (the so called Fadeev-Popov ghosts) which are Lorentz scalar anti-commuting fields and appear only in the loops. After this procedure (usually performed in the functional formalism), we have that the QCD Lagrangian from which we can calculate directly the Feynman rules is given by (see e.g. [22] page 514):

$$\mathcal{L}^{FP}_{QCD}(x) = \mathcal{L}^{cl}_{QCD}(x) - \frac{1}{2\lambda}(\partial^\mu\partial^\nu A^A_\mu(x)A^A_\nu(x)) - \bar{c}^A\partial^\mu D^{AC}_\mu c^C, \tag{27}$$

where λ is the gauge fixing parameter, c^A is the complex colored scalar ghost field and D^{AB}_μ is the covariant derivative in the adjoint representation:

$$D^{AB}_\mu = \delta^{AB}\partial_\mu - g_s f^{ABC} A^C_\mu. \tag{28}$$

The quark, gluon and ghost propagators and vertices for QCD are collected in Figure 1.1.

Figure 1.1: Feynman rules for QCD in a covariant gauge for gluons (curly lines), quarks (solid lines) and ghosts (dotted lines). Here A, B, C, D are the color indexes in the adjoint representation, a, b, c in the fundamental one, $\alpha, \beta, \gamma, \delta$ are the gluon polarization indexes and i, j are the spinorial indexes.

1.2 Asymptotic freedom and perturbative QCD

Feynman diagrams in QCD are obtained employing the vertices and propagators as building blocks. However, the use of diagrams makes sense only if the perturbative expansion in g_s is meaningful. To respect this condition, the coupling

$$\alpha_s = \frac{g_s}{4\pi}, \tag{29}$$

the QCD analog of the electromagnetic coupling $\alpha = e^2/4\pi$, has to be sufficiently small. We shall now show that the method of perturbation theory in QCD are useful at high energy. Indeed, the coupling constant is large at low energy and becomes smaller at high energy (asymptotic freedom). The simplest way to introduce the running coupling, is to consider a dimensionless physical observable R which depends on a single energy scale $\sqrt{Q^2}$. This is the case, for example of the ratio of the annihilation cross section of electron-positron into hadron with the annihilation into muons where $Q^2 = S$ the center-of-mass energy. We assume that this scale is much bigger than the quark masses that can be therefore neglected. Now, dimensional analysis should implies that a dimensionless observable is independent of Q^2. However, higher order corrections produce divergences and so the perturbation series requires renormalization to remove ultraviolet divergences that in $d = 4 - 2\epsilon$ dimensions are regularized as $1/\epsilon$ poles. This poles can be removed defining a renormalized coupling constant at a certain renormalization scale μ_r^2. Consequently, in the finite $\epsilon \to 0$ limit, R depends in general on the ratio Q^2/μ_r^2 and the renormalized coupling α_s depends on μ_r^2; we call the renormalized coupling at the scale μ_r^2, $\alpha_{\mu_r^2}$. Since the renormalization scale is an arbitrary parameter introduced only to define the theory at the quantum level, we conclude that R has to be μ_r^2-independent. Formally this independence is expressed as follows:

$$\mu_r^2 \frac{d}{d\mu_r^2} R\left(\frac{Q^2}{\mu_r^2}, \alpha_{\mu_r^2}\right) = \left[\mu_r^2 \frac{\partial}{\partial \mu_r^2} + \beta(\alpha_{\mu_r^2}) \frac{\partial}{\partial \alpha_{\mu_r^2}}\right] R = 0, \tag{30}$$

where

$$\beta(\alpha_{\mu_r^2}) = \mu_r^2 \frac{\partial \alpha_{\mu_r^2}}{\partial \mu_r^2} \tag{31}$$

Eq.(30) is a first order differential equation with the initial condition (at $Q^2 = \mu_r^2$) $R(1, \alpha_{\mu_r^2})$. This means that if we find a solution of Eq.(30), it is the only possible solution. This solution is easily found defining a function $\alpha_s(Q^2)$ such that

$$\alpha_s(\mu_r^2) = \alpha_{\mu_r^2} \tag{32}$$

and that

$$\ln\left(\frac{Q^2}{\mu_r^2}\right) = \int_{\alpha_{\mu_r^2}}^{\alpha_s(Q^2)} \frac{dx}{\beta(x)}. \tag{33}$$

In fact differentiating this equation, we find that

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta(\alpha_s(Q^2)) \tag{34}$$

$$\frac{\partial \alpha_s(Q^2)}{\partial \alpha_{\mu_r^2}} = \frac{\beta(\alpha_s(Q^2))}{\beta(\alpha_{\mu_r^2})} \tag{35}$$

and that $R(1, \alpha_s(Q^2))$ is the desired solution of Eq.(30), because

$$\frac{\partial}{\partial \alpha_{\mu_r^2}} = \frac{\partial \alpha_s(Q^2)}{\partial \alpha_{\mu_r^2}} \frac{\partial}{\partial \alpha_s(Q^2)}, \quad \mu_r^2 \frac{\partial}{\partial \mu_r^2} = -Q^2 \frac{\partial}{\partial Q^2} = -Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} \frac{\partial}{\partial \alpha_s(Q^2)}. \tag{36}$$

This shows that all of the scale dependence in R enters through the running of the coupling constant $\alpha_s(Q^2)$. To find explicitly this function, we need to know the β-function so that we can solve Eq.(34). The β-function can be calculate perturbatively from the counterterms of the renormalization precedure and a knowledge to order α_s^{n+1} requires a n-loop computation. The perturbative expansion of the β-function is given by:

$$\beta(\alpha_s) = -\alpha_s \sum_{n=0}^{\infty} \beta_n \left(\frac{\alpha_s}{4\pi}\right)^{n+1}. \tag{37}$$

At the moment, the QCD β-function is known to order α_s^5 [23]: where in the $\overline{MS}$ scheme,

$$\beta_0 = 11 - \frac{2}{3} N_f, \quad \beta_1 = 102 - \frac{38}{3} N_f, \tag{38}$$

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \tag{39}$$

$$\beta_3 = \left(\frac{149753}{6} + 3564\xi_3\right) - \left(\frac{1078361}{162} + \frac{6508}{27}\xi_3\right) N_f + \left(\frac{50065}{162} + \frac{6472}{81}\xi_3\right) N_f^2 + \frac{1093}{729} N_f^3, \tag{40}$$

with N_f the numbers of flavors and ξ is the Riemann zeta-function ($\xi_3 = 1.202056903\ldots$). The two loop solution of Eq.(34) is given by:

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu_r^2)}{1 + (\beta_0/4\pi)\alpha_s(\mu_r^2) \log \frac{Q^2}{\mu_r^2}} \left[1 - \frac{\beta_1}{4\pi\beta_0} \frac{\alpha_s(\mu_r^2) \log(1 + (\beta_0/4\pi)\alpha_s(\mu_r^2) \log \frac{Q^2}{\mu_r^2})}{1 + (\beta_0/4\pi)\alpha_s(\mu_r^2) \log \frac{Q^2}{\mu_r^2}}\right] + O(\alpha_s^{k+3} \log^k \frac{Q^2}{\mu_r^2}). \tag{41}$$

For simplicity, in many cases, we will use another parametrization of the coefficients β_n, which is obtained with the substitution:

$$\beta_n = b_n (4\pi)^{n+1}. \tag{42}$$

With this parametrization, the perturbative expansion of the β-function Eq.(37) becomes

$$\beta(\alpha_s) = -\sum_{n=0}^{\infty} b_n \alpha_s^{n+2}. \tag{43}$$

From Eq.(41), we see that $\alpha_s(Q^2)$ is a monotonically decreasing function of Q^2, because the coefficients β_0 and β_1 are positive (with $N_f \leq 6$). The running of $\alpha_s(Q^2)$ has been measured with great accuracy (see Figure 1.2). The fact that at high energy,

Figure 1.2: The running of the strong coupling constant. The asymptotic freedom is confirmed by the experiments

the running coupling becomes small is a peculiarity of non-abelian gauge theories and is called asymptotic freedom. Hence, perturbative QCD can be applied when the relevant scale of a certain process is high enough such that the running coupling becomes small. A typical example is given by the annihilation of two high-energy electrons into hadrons. Perturbative QCD can also be applied to processes in which hadrons are present also in the initial state thanks to the factorization theorem. According to this theorem [11], the cross section for the production of some final state with high invariant mass Q^2 (the scale at which the running coupling constant is small) with two incoming hadrons is given by:

$$\sigma(P_1, P_2, Q^2) = \sum_{a,b} \int_0^1 dx_1 dx_2 F_a^{H_1}(x_1, \mu^2) F_b^{H_2}(x_2, \mu^2) \hat{\sigma}_{ab}(x_1 P_1, x_2 P_2, \alpha_s(\mu^2), Q^2, \mu^2). \tag{44}$$

For processes with a single incoming hadron the factorization theorem is simpler. For example for the deep inelastic scattering (DIS) of a lepton that exchanges a high square momentum Q^2 with the hadron, the cross section takes the form:

$$\sigma(P, Q^2) = \sum_{a} \int_0^1 dx F_a^{H}(x, \mu^2) \hat{\sigma}_a(xP, \alpha_s(\mu^2), Q^2, \mu^2). \tag{45}$$

In Eqs.(44,45), P_i is the momentum of the incoming hadron H_i. A beam of hadrons of type H_i is equivalent to a beam of constituents (or partons) which are quarks

or gluons. These constituents carry a longitudinal momentum $x_i P_i$ characterized by the parton densities $F_a^{H_i}(x_i, \mu^2)$. That is, given a hadron H_i with momentum P_i, the probability density to find in H_i the parton a with momentum $x_i P_i$ is given by $F_a^{H_i}(x_i, \mu^2)$. Furthermore, these functions are universal in the sense that they are process independent. The parton densities depends also on the so called factorization scale μ^2. This scale is introduced to separate off the non-perturbative part of the cross section (the parton densities) from the perturbative one $\hat{\sigma}_{a(b)}$. This is exactly the cross section where the incoming particles are the partons a (and b) and can be calculated as a perturbative expansion in $\alpha_s(\mu_r^2)$. The parton densities have a mild dependence on the scale μ^2 determined by the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations (see section 1.4). Here, we have chosen the renormalization scale μ_r^2 equal to the factorization scale μ^2 for simplicity. Anyway, in order to reintroduce the scale μ_r^2, we have only to rewrite $\alpha_s(\mu^2)$ in terms of μ_r^2 (see Eq.(41)) and expand it consistently with the order of the calculation. The μ^2 dependence in the parton densities is compensated by the μ^2 dependence in the partonic cross section $\hat{\sigma}$. However, with a fixed-order computaction of the partonic coefficient function at order α_s^k the hadronic cross section will still depend on μ^2 with a dependence which should be of order α_s^{k+1}. Hence, this dependence can be used to estimate the theoretical error of a fixed-order computation. A simple discussion about the dependence of the hadronic cross section on the factorization and renormalization scale is given in Ref.[24].

1.3 NLO DY and DIS cross sections

We consider for simplicity the classical Drell-Yan (DY) hadronic process for the production of a dimuon pair through a virtual photon γ^* (see Figure 1.3):

$$H_1(P_1) + H_2(P_2) = \gamma^*(Q) + X(K), \tag{46}$$

where H_1 and H_2 are the colliding hadrons with momementum P_1 and P_2 respectively, Q is the momentum of the virtual photon and X is any number of additional hadrons with total momentum K. For the process of Eq.(46), we define

$$x \equiv \frac{Q^2}{S}, \tag{47}$$

where $S = (P_1 + P_2)^2$ is the usual Mandelstam invariant, which can be viewed as the hadronic center-of-mass energy. It is clear that Eq.(47) represents the fraction of energy that the hadrons transfer to the photon and, hence, $0 \leq x \leq 1$. According to the factorized expression of the QCD cross section Eq.(44), the LO Q^2 differential cross section is given by:

$$\frac{d\sigma}{dQ^2}(x, Q^2) = \sum_i \int_0^1 dx_1 dx_2 \left[q_i(x_1, \mu^2)\bar{q}_i(x_2, \mu^2) + \bar{q}_i(x_1) q_i(x_2, \mu^2)) \right] \frac{d\hat{\sigma}_i}{dQ^2}, \tag{48}$$

$$\frac{d\hat{\sigma}_i}{dQ^2} = \sigma_0^{DY}(Q^2, x) Q_{q_i}^2 \delta(x_1 x_2 - x), \quad \sigma_0^{DY}(Q^2, x) = \frac{4\pi\alpha^2}{9Q^4} x \tag{49}$$

Figure 1.3: Drell-Yan pair production. Here $Q = M$.

where the functions $q_i(x_j, \mu^2)(\bar{q}_i(x_j, \mu^2))$are the parton densities of a quark (or an anti-quark) of flavor i in the hadron $j = 1, 2$ at the scale μ^2, α is the fine-structure constant and Q_{q_i} is the fraction of electronic charge of the quark q_i. Now, if we define the dimensionaless cross section $\sigma(x, Q^2)$ as,

$$\sigma^{DY}(x, Q^2) \equiv \frac{1}{\sigma_0^{DY}} \frac{d\sigma}{dQ^2}(x, Q^2), \tag{50}$$

and use the identity,

$$\delta(x_1 x_2 - x) = \int_0^1 dz \delta(1 - z)\delta(x_1 x_2 z - x), \tag{51}$$

then Eqs.(48,49) become:

$$\begin{aligned}
&\sigma^{DY}(x, Q^2) = \\
&= \sum_i \int_0^1 dx_1 dx_2 dz \left[q_i(x_1)\bar{q}_i(x_2) + \bar{q}_i(x_1)q_i(x_2)\right] Q_{q_i}^2 C_{qq}(z)\delta(x_1 x_2 z - x) &(52)\\
&= \sum_i \int_x^1 \frac{dx_1}{x_1} \int_{x/x_1}^1 \frac{dx_2}{x_2} \left[q_i(x_1)\bar{q}_i(x_2) + \bar{q}_i(x_1)q_i(x_2))\right] Q_{q_i}^2 C_{qq}\left(\frac{x}{x_1 x_2}\right), &(53)\\
&C_{qq}(z) = \delta(1 - z), &(54)
\end{aligned}$$

where $C_{qq}(z)$ is the LO Drell-Yan coefficient function. From Eq.(52), we see that the new variable z that we have introduced is in general given by

$$z = \frac{x}{x_1 x_2}. \tag{55}$$

This means that at the partonic level, z can be viewed as the fraction of energy that the colliding partons transfer to the virtual photon. At LO it is clear that $z = 1$ as can be explicitely seen from Eq.(54), because there is no emission but the virtual photon. Beyond the LO the extra radiated partons in the final state can carry away some energy (so $z < 1$) and the gluon channel contributes. The NLO coefficient

functions $C_{ab}(z)$ $(a, b = q, g)$ recieves contributions that have infrared and ultraviolet. Infrared singularities cancel out (see e.g. [25]). The ultraviolet ones are reabsobed by renormalization of the bare parameters of the QCD Lagrangian, thus defining a renormalized strong coupling constant $\alpha_{\mu_r^2}$ at an arbitraty renormalization scale μ_r^2 (see section 1.2). Collinear divergences are cut off by infrared physics. They can be absorbed multiplicatively in redefinition of the parton densities [26], thus reabsorbing all dependence on soft physics in the parton distributions. The parton densities at a certain scale are determined by a reference process and their scale dependence is determined by the DGLAP equations (see section 1.4). However, there is an ambiguity on how to define the reference process, related to the fact that collinear divergences can always be factorized together with finite terms. The choice of these finite terms defines a factorization scheme. The most common factorization scheme is the $\overline{MS}$ scheme in which the collinear divergence (which is in $d = 4 - 2\epsilon$ dimensions a single pole $1/\epsilon$) is factorized together the finite terms $-\gamma_E + \log 4\pi$, where $\gamma_E = 0.5772...$ is the Euler gamma. Now, in order to avoid the perturbative expansion to receive large contributions, the factorization and the renormalization scales are expected to be chosen of the same order of the scale of the process Q^2. Here, for simplicity, we choose the factorization scale μ^2 equal to the renormalization scale μ_r^2. We report the NLO Drell-Yan cross section (see e.g. [27, 28]):

$$\begin{aligned}
&\sigma^{DY}(x, Q^2) = \sum_i Q_{q_i}^2 \int_x^1 \frac{dx_1}{x_1} \int_{x/x_1}^1 \frac{dx_2}{x_2} \\
&\times \Bigg\{ \left[q_i(x_1, \mu^2) \bar{q}_i(x_2, \mu^2) + (1 \leftrightarrow 2) \right] C_{qq}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \\
&+ \left[g(x_1, \mu^2) \left(q_i(x_2, \mu^2) + \bar{q}_i(x_2, \mu^2) \right) + (1 \leftrightarrow 2) \right] C_{qg}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) \Bigg\}, \qquad (56)
\end{aligned}$$

where, in the $\overline{MS}$ scheme,

$$\begin{aligned}
C_{qq}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) &= \delta(1-z) + \frac{\alpha_s(\mu^2)}{2\pi} \Bigg\{ \frac{4}{3} \Bigg[\left(\frac{2\pi^2}{3} - 8 \right) \delta(1-z) \\
&+ 4(1+z^2) \left[\frac{\log(1-z)}{1-z} \right]_+ - 2 \frac{1+z^2}{1-z} \log z \Bigg] \\
&+ \frac{8}{3} \left[\frac{1+z^2}{[1-z]_+} + \frac{3}{2} \delta(1-z) \right] \log\left(\frac{Q^2}{\mu^2} \right) \Bigg\}, \qquad (57)
\end{aligned}$$

and

$$\begin{aligned}
C_{qg}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2) \right) &= \frac{\alpha_s(\mu^2)}{2\pi} \Bigg\{ \frac{1}{2} \Bigg[(z^2 + (1-z)^2) \log \frac{(1-z)^2}{z} + \frac{1}{2} + 3z \\
&- \frac{7}{2} z^2 \Bigg] + \frac{1}{2} [z^2 + (1-z)^2] \log\left(\frac{Q^2}{\mu^2} \right) \Bigg\}, \qquad (58)
\end{aligned}$$

where the "+" distribution is defined as follows:

$$\int_0^1 dz f(z) [g(z)]_+ \equiv \int_0^1 dz [f(z) - f(1)] g(z). \qquad (59)$$

Figure 1.4: Deep inelastic electron-proton scattering

Also for the case of the deep-inelastic scattering (DIS), we consider the simplest process in which a high energy electron scatters from a hadron exchanging with it a virtual photon γ^* (see Figure 1.4):

$$H(P) + e(k) \to e(k') + X(K), \tag{60}$$

where H is typically a proton with momentum P, e is the scattered electron and X is any collection of hadrons. The standard parametrization of DIS is done in terms of three relevant parameters:

$$\begin{aligned} Q^2 &\equiv -q^2 \equiv -(k-k')^2 && (61)\\ y &\equiv \frac{P\cdot q}{P\cdot k};\quad 0 \le y \le 1 && (62)\\ x &\equiv x_{Bj} = \frac{Q^2}{2P\cdot q} = \frac{Q^2}{(P+q)^2+Q^2};\quad 0 \le x \le 1, && (63) \end{aligned}$$

where in the last line we have neglected the proton mass. Q^2 is the virtuality of the photon exchanged between the electron and the proton and y is the fraction of energy that the incoming electron transfer to the proton. The Bjorken scaling variable x has a simple physical interpretation: it is the fraction of longitudinal momentum of the LO incoming quark of the partonic subprocess. Indeed, the most general parametrization of the Q^2 differential cross section is given by:

$$\frac{d\sigma}{dQ^2}(x,Q^2,y) = \frac{4\pi\alpha^2}{Q^4}\left[[1+(1-y)^2]F_1(x,Q^2) + \frac{(1-y)}{x}(F_2(x,Q^2) - 2xF_1(x,Q^2))\right], \tag{64}$$

The functions $F_{1(2)}$ are called structure functions and contains the information about the structure of the proton. In fact they are determined by the photon-proton subprocess in this way:

$$\begin{aligned} F_1(x,Q^2) &= \frac{Q^2}{16\pi^2\alpha x}\left[\sigma_\Sigma(\gamma^*P) + \sigma_L(\gamma^*P)\right] && (65)\\ F_2(x,Q^2) &= 2xF_1(x,Q^2) + F_L(x,Q^2) && (66)\\ F_L(x,Q^2) &= \frac{Q^2}{4\pi^2\alpha}\sigma_L(\gamma^*P), && (67) \end{aligned}$$

where $\sigma_\Sigma(\gamma^* P)$ and $\sigma_\Sigma(\gamma^* P)$ are the cross sections of the photon-proton process determined summing over all the virtual photon polarization and over only the longitudinal one respectively. At LO

$$F_1(x,Q^2) = \frac{1}{2}\sum_i Q^2_{q_i}[q_i(x,Q^2)+\bar{q}_i(x,Q^2)] \tag{68}$$

$$F_2(x,Q^2) = 2xF_1(x,\mu^2), \tag{69}$$

where q_i and $\bar{q}_i$ are the parton densities. In general the structure functions should depend on both x and Q^2, because these are the relevant kinematic variable of the photon-proton sub-process. Now we want, as we have done for the Drell-Yan case, rewrite the structure functions in terms of parton densities and of a coefficient function that can be computed in perturbative QCD. If we use the identity

$$\delta(y-x) = \int_0^1 dz\, \delta(1-z)\delta(yz-x), \tag{70}$$

we have for the LO structure functions F_2 and F_L:

$$F_2(x,Q^2) = x\sum_i \int_0^1 dydz\, \left[q_i(y,\mu^2)+\bar{q}_i(y,\mu^2)\right] Q^2_{q_i} C_q(z)\delta(yz-x) \tag{71}$$

$$= x\sum_i \int_x^1 \frac{dy}{y}\, \left[q_i(y,\mu^2)+\bar{q}_i(y,\mu^2)\right] Q^2_{q_i} C_q\left(\frac{x}{y}\right) \tag{72}$$

$$C_q(z) = \delta(1-z) \tag{73}$$

$$F_L(x,Q^2) = 0, \tag{74}$$

where $C_q(z)$ is the LO DIS coefficient function for F_2. From Eq.(71), we see that the variable z is in general given by

$$z = \frac{x}{y}. \tag{75}$$

This means that at the partonic level, z can be viewed as the longitudinal momentum of the incoming parton before it scatters with the virtual photon. At LO it is clear that $z = 1$ as can be explicitely seen from Eq.(73), because there is no extra emissions. Beyond the LO the extra radiated partons in the final state can carry some energy (so $z < 1$) and also the gluon channel contributes. At NLO we have ultraviolet, infrared and collinear singularities. They must be regularized and treated as in the Drell-Yan case. We report the NLO structure functions (see e.g. [27, 28]) with the renormalization scale equal to the factorization scale:

$$F_2(x,Q^2) = x\sum_i Q^2_{q_i} \int_x^1 \frac{dy}{y} \left\{ \left[q_i(y,\mu^2)+\bar{q}_i(y,\mu^2)\right] C_q\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \right\}$$
$$+x\sum_i Q^2_{q_i} \int_x^1 \frac{dy}{y}\, g(y,\mu^2) C_g\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) \Bigg\} \tag{76}$$

where, in the $\overline{MS}$ scheme,

$$C_q\left(z,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right) = \delta(1-z)+\frac{\alpha_s(\mu^2)}{2\pi}\left\{\frac{4}{3}\left[2\left[\frac{\ln(1-z)}{1-z}\right]_+-\frac{3}{2}\left[\frac{1}{1-z}\right]_+\right.\right.$$
$$-(1+z)\ln(1-z)-\frac{1+z^2}{1-z}\ln z-\left(\frac{\pi^2}{3}+\frac{9}{2}\right)\delta(1-z)$$
$$\left.\left.+3+2z\right]+\frac{4}{3}\left[\frac{1+z^2}{[1-z]_+}+\frac{3}{2}\delta(1-z)\right]\log\left(\frac{Q^2}{\mu^2}\right)\right\}, \quad (77)$$

$$C_g\left(z,\frac{Q^2}{\mu^2},\alpha_s(\mu^2\right) = \frac{\alpha_s(\mu^2)}{2\pi}\left\{\frac{1}{2}\left[\left((1-z)^2+z^2\right)\ln\left(\frac{1-z}{z}\right)-8z^2\right.\right.$$
$$\left.\left.+8z-1+[z^2+(1-z)^2]\log\left(\frac{Q^2}{\mu^2}\right)\right]\right\} \quad (78)$$

and

$$F_L(x,Q^2) = x\sum_i Q_{q_i}^2\int_x^1\frac{dy}{y}\left[q_i(y,\mu^2)+\bar{q}_i(y,\mu^2)\right]\frac{\alpha_s(\mu^2)}{2\pi}\left\{\frac{8}{3}z\right.$$
$$+\frac{4}{3}\left[\frac{1+z^2}{[1-z]_+}+\frac{3}{2}\delta(1-z)\right]\log\left(\frac{Q^2}{\mu^2}\right)\Big\}$$
$$+x\sum_i Q_{q_i}^2\int_x^1\frac{dy}{y}\,g(y,\mu^2)\frac{\alpha_s(\mu^2)}{2\pi}\left\{2z(1-z)\right.$$
$$\left.+\frac{1}{2}[z^2+(1-z)^2]\log\left(\frac{Q^2}{\mu^2}\right)\right\}, \quad (79)$$

is factorization scheme independent at the lowest non-vanishing order.

1.4 The DGLAP equations

The coefficient function and the parton densities depend on the factorization scale in such a way that the resulting hadronic cross section is μ^2-independent. The equations that fix the μ^2-dependence of parton densities (the so called Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations) can be found imposing the μ^2-independence of the DY cross section or of the DIS structure functions. For example, imposing this condition to the explicit expression for the NLO F_2 (see Eqs.(76,77,78), we find the LO DGLAP evolution equations for the quark parton densities:

$$\mu^2\frac{\partial q_i(x,\mu^2)}{\partial\mu^2} = \frac{\alpha_s(\mu^2)}{4\pi}\left\{\sum_j\int_x^1\frac{dy}{y}\left[P_{q_iq_j}^{(0)}\left(\frac{x}{y}\right)q_j(y,\mu^2)+P_{q_i\bar{q}_j}^{(0)}\left(\frac{x}{y}\right)\bar{q}_j(y,\mu^2)\right]\right.$$
$$\left.+\int_x^1\frac{dy}{y}\,P_{q_ig}^{(0)}\left(\frac{x}{y}\right)g(y,\mu^2)\right\}+O(\alpha_s^2), \quad (80)$$

where

$$P^{(0)}_{q_iq_j}(z) = \delta_{ij}P^{(0)}_{qq}(z), \tag{81}$$

$$P^{(0)}_{q_i\bar{q}_j}(z) = 0 \tag{82}$$

$$P^{(0)}_{qq}(z) = \frac{8}{3}\left[\frac{1+z^2}{[1-z]_+} + \frac{3}{2}\delta(1-z)\right], \tag{83}$$

and

$$P^{(0)}_{q_ig}(z) = \frac{1}{N_f}P^{(0)}_{qg}(z) = z^2 + (1-z)^2, \tag{84}$$

with N_f the number of active flavors. The functions $P^{(0)}_{pp'}(z)$ are called LO splitting functions. They can be viewed as the probability per unit of $\ln(\mu^2/Q^2)$ to find a parton p in a parton p'. The LO evolution equation for the gluon can be calculated from the LO splitting diagrams for a quark into another quark and a gluon and for a gluon into two gluons. Furthermore, we simplify the notation introducing the convolution product $\otimes$, defined in this way:

$$(f_1 \otimes f_2 \otimes \cdots \otimes f_n)(x) = \int_0^1 dx_1 dx_2 \cdots dx_n f_1(x_1) f_2(x_2) \cdots f_n(x_n)\delta(x_1x_2\cdots x_n - x). \tag{85}$$

We report here the full result for the DGLAP evolution equations:

$$\mu^2\frac{\partial}{\partial\mu^2}\begin{pmatrix} q_i(z,\mu^2) \\ g(z,\mu^2)\end{pmatrix} = \sum_{q_j,\bar{q}_j}\begin{pmatrix} P_{q_iq_j}(z,\mu^2) & P_{q_ig}(z,\mu^2) \\ P_{gq_j}(z,\mu^2) & P_{gg}(z,\mu^2)\end{pmatrix} \otimes \begin{pmatrix} q_j(z,\mu^2) \\ g(z,\mu^2)\end{pmatrix}, \tag{86}$$

where q_i can be also a quark or anti-quark and where the splitting functions $P_{pp'}$ have the following perturbative expansion:

$$\begin{aligned} P_{q_iq_j}(z,\mu^2) &= P_{\bar{q}_i\bar{q}_j}(z,\mu^2) = \frac{\alpha_s(\mu^2)}{4\pi}\delta_{ij}P^{V\,(0)}_{qq}(z) \\ &\quad + \sum_{k=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^{k+1}\left(\delta_{ij}P^{V\,(k)}_{qq}(z) + P^{S\,(k)}_{qq}(z)\right), \end{aligned} \tag{87}$$

$$\begin{aligned} P_{q_i\bar{q}_j}(z,\mu^2) &= P_{\bar{q}_iq_j}(z,\mu^2) = \left(\frac{\alpha_s(\mu^2)}{4\pi^2}\right)^2\left(\delta_{ij}P^{V\,(1)}_{q\bar{q}} + P^{S\,(1)}_{qq}\right) \\ &\quad + \sum_{k=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^{k+2}\left(\delta_{ij}P^{V\,(k+1)}_{q\bar{q}}(z) + P^{S\,(k+1)}_{q\bar{q}}(z)\right), \end{aligned} \tag{88}$$

$$P_{q_ig}(z,\mu^2) = P_{\bar{q}_ig}(z,\mu^2) = \frac{1}{N_f}\sum_{k=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^k P^{(k-1)}_{qg}, \tag{89}$$

$$P_{gq_i}(z,\mu^2) = P_{g\bar{q}_i} = \sum_{k=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^k P^{(k-1)}_{gq}, \tag{90}$$

$$P_{gg}(z,\mu^2) = \sum_{k=1}^{\infty}\left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^k P^{(k-1)}_{gg}, \tag{91}$$

where N_f is the number of active flavors. Eq.(86) represents a system of $2N_f + 1$ integro-differential equations equations. The solution to this system however can be calculated analytically for a certain fixed-order. In fact it can be translated into a system of ordinary differential equation performing a Mellin transform:

$$F_p(N,\mu^2) = \int_0^1 dz\, z^{N-1} F_p(z,\mu^2), \tag{92}$$

$$\mu^2 \frac{\partial F_p(N,\mu^2)}{\partial \mu^2} = \sum_{p'} \gamma_{pp'}^{AP}(N,\mu^2) F_{p'}(N,\mu^2), \tag{93}$$

where $p, p' = q_i, \bar{q}_j, g$ and

$$\gamma_{pp'}^{AP}(N,\mu^2) = \int_0^1 dz P_{pp'}(z,\mu^2). \tag{94}$$

After that, these equations can be decoupled searching linear combinations of parton densities that depends on the independent splitting functions of Eqs.(68-77) and that diagonalize the system. For example, at LO, there are 4 independent splitting functions which are $P_{qq}^{V\,(0)} = P_{qq}^{(0)}$, $P_{qg}^{(0)}$ (given in Eqs.(83,84) respectively) and

$$P_{gq}^{(0)}(z) = \frac{8}{3}\left[\frac{1+(1-z)^2}{z}\right] \tag{95}$$

$$P_{gg}^{(0)}(z) = 12\left[\frac{z}{[1-z]_+} + \frac{1-z}{z} + z(1-z)\right] + \left(\frac{44}{9} - \frac{2}{3}N_f\right)\delta(1-z). \tag{96}$$

At NLO there are 6 independent splitting functions which are for example $P_{qq}^{V}(z)$, $P_{qq}^{S}(z)$, $P_{q\bar{q}}^{V}(z)$, $P_{qg}(z)$, $P_{gq}(z)$ and $P_{gg}(z)$. They are given in Ref.[27] pages 111 and 112. The LO and the NLO solution to the DGLAP equations (93) in Mellin space is computed in the next Section.

1.5 NLO solution of the DGLAP evolution equations

In this Section, we want to solve the NLO DGLAP equations (Eq.(93) of section 1.4):

$$\mu^2 \frac{\partial F_p(N,\mu^2)}{\partial \mu^2} = \sum_{p'} \gamma_{pp'}(N,\mu^2) F_{p'}(N,\mu^2), \tag{97}$$

where all the splitting functions defined here (and in the following) have the same perturbative expansion:

$$\gamma_{pp'}(N,\mu^2) = \frac{\alpha_s(\mu^2)}{4\pi}\gamma_{pp'}^{(0)}(N) + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \gamma_{pp'}^{(1)}(N) + O(\alpha_s^3). \tag{98}$$

This is a system of $2N_f + 1$ coupled equations with N_f the number of active flavors. At NLO, there are 6 independent splitting functions defined through the following equations:

$$\begin{aligned}
\gamma_{gq_i} &= \gamma_{g\bar{q}_i} \equiv \gamma_{gq} && (99)\\
\gamma_{q_i g} &= \gamma_{\bar{q}_i g} \equiv 1/N_f \gamma_{qg} && (100)\\
\gamma_{q_i q_k} &= \gamma_{\bar{q}_i \bar{q}_k} \equiv \delta_{ik}\gamma^V_{qq} + \gamma^S_{qq} && (101)\\
\gamma_{q_i \bar{q}_k} &= \gamma_{\bar{q}_i q_k} \equiv \delta_{ik}\gamma^V_{q\bar{q}} + \gamma^S_{qq} && (102)\\
\gamma_{gg} &\equiv \gamma_{gg}, && (103)
\end{aligned}$$

where i, k are a flavor index. We omit the dependence on N and μ^2 for brevity of notation. Note that beyond the NLO, there is one more independent splitting function. In fact in Eq.(102) we should substitute γ^S_{qq} with $\gamma^S_{q\bar{q}}$ which are different beyond the NLO [29, 30]. We note also that at LO $\gamma^S_{qq} = \gamma^S_{q\bar{q}} = \gamma^V_{q\bar{q}} = 0$ and hence at LO there are only 4 independent splitting functions. Now, we define the $2N_f - 1$ so called non-singlet (NS) combinations

$$\begin{aligned}
q^{\pm}_{(NS)k} &= \sum_{i=1}^{k}(q_i \pm \bar{q}_i) - k(q_k \pm \bar{q}_k); \quad k = 2, \ldots, N_f && (104)\\
q^V_{(NS)} &= \sum_{i=1}^{N_f}(q_i - \bar{q}_i) && (105)
\end{aligned}$$

and the 2 so called singlet (S) combinations: g and

$$q_{(S)} = \sum_{i=1}^{N_f}(q_i + \bar{q}_i). \qquad (106)$$

With this definitions, from Eq.(97) and Eqs.(99-103), we find that for the non-singlet combinations

$$\begin{aligned}
\mu^2\frac{\partial q^{\pm}_{(NS)k}}{\partial \mu^2} &= \gamma^{\pm} q^{\pm}_{(NS)k} && (107)\\
\mu^2\frac{\partial q^V_{(NS)}}{\partial \mu^2} &= \gamma^{-} q^V_{(NS)}, && (108)
\end{aligned}$$

where

$$\gamma^{\pm} = \gamma^V_{qq} \pm \gamma^V_{q\bar{q}}. \qquad (109)$$

For the 2 remaining singlet combinations, we find in the same way that

$$\mu^2\frac{\partial}{\partial \mu^2}\begin{pmatrix} q_{(S)} \\ g \end{pmatrix} = \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}\begin{pmatrix} q_{(S)} \\ g \end{pmatrix}, \qquad (110)$$

where

$$\gamma_{qq} = \gamma^+ + \gamma_{PS}, \qquad \gamma_{PS} \equiv 2N_f\gamma^S_{q\bar{q}}. \qquad (111)$$

The NLO Mellin splitting functions can be found in Ref.[31] written in terms of harmonic sums. In many cases, however, their analytic continuation to all the complex plane is useful (see e.g. [32, 33]) For the NNLO solution of the DGLAP equations and the NNLO splitting functions we refer to [29, 30]. The techniques for the analytic continuations of the NNLO splitting functions can be found in Ref.[34]. For the NS combinations, the solution is easy to obtain. Indeed, making the change of variable

$$\frac{d\mu^2}{\mu^2} = \frac{d\alpha_s(\mu^2)}{\beta(\alpha_s(\mu^2))}, \tag{112}$$

where $\beta(\alpha_s)$ is the β function defined in section 1.2, we get:

$$\frac{q^{\pm}_{(NS)k}(\mu^2)}{q^{\pm}_{(NS)k}(\mu_0^2)} = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{-\gamma^{(0)\pm}/\beta_0} \left[1 + \left(\frac{\gamma^{(1)\pm}}{\beta_0} - \frac{\beta_1\gamma^{(0)\pm}}{\beta_0^2}\right)\left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi}\right)\right] \tag{113}$$

and

$$\frac{q^{V}_{(NS)}(\mu^2)}{q^{V}_{(NS)}(\mu_0^2)} = \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{-\gamma^{(0)-}/\beta_0} \left[1 + \left(\frac{\gamma^{(1)-}}{\beta_0} - \frac{\beta_1\gamma^{(0)-}}{\beta_0^2}\right)\left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi}\right)\right], \tag{114}$$

where we have omitted the N-dependence of the splitting functions for brevity. For the S combinations, some linear algebra is needed. We, first, define the singlet vector and the splitting matrix:

$$\vec{q}_S \equiv \begin{pmatrix} q_{(S)} \\ g \end{pmatrix}, \qquad \tilde{\gamma}_S \equiv \begin{pmatrix} \gamma_{qq} & \gamma_{qg} \\ \gamma_{gq} & \gamma_{gg} \end{pmatrix}. \tag{115}$$

Using the NLO splitting matrix and the change of variable Eq.(112), we find immeditely the formal solution, which is

$$\vec{q}_S(\mu^2) = \exp\left\{-R_0 \ln\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)} + R_1\left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi}\right)\right\}\vec{q}_S(\mu_0^2), \tag{116}$$

where

$$R_0 = \frac{\tilde{\gamma}_S^{(0)}}{\beta_0}, \qquad R_1 = \frac{\tilde{\gamma}_S^{(1)}}{\beta_0} - \frac{\beta_1\tilde{\gamma}_S^{(0)}}{\beta_0^2}. \tag{117}$$

The two matrices R_0 and R_1 in Eq.(116) cannot be diagonalized simultaneously, as they do not commute. Hence, in order to extract the NLO solution from Eq.(116), we use the following Ansatz:

$$\vec{q}_S(\mu^2) = U(\alpha_s(\mu^2))\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{-R_0} U^{-1}(\alpha_s(\mu_0^2))\vec{q}_S(\mu_0^2), \tag{118}$$

where the matrix U has the perturbative expansion:

$$U(\alpha_s(\mu^2)) = 1 + \frac{\alpha_s(\mu^2)}{4\pi}U_1 + O(\alpha_s^2). \tag{119}$$

The condition that the matrix U_1 should satisfy can be easily obtained imposing that the derivative with respect to $\alpha_s(\mu^2)$ of Eq.(116) and of Eq.(118) are equal at NLO. Thus, we get

$$[U_1, R_0] = U_1 + R_1. \tag{120}$$

We write R_0 in terms of its 2 eigenvalues

$$R^{\pm} = \frac{1}{2\beta_0}\left[(\gamma_{qq}^{(0)} + \gamma_{gg}^{(0)}) \pm \sqrt{(\gamma_{qq}^{(0)} - \gamma_{gg}^{(0)})^2 + 4\gamma_{qg}^{(0)}\gamma_{gq}^{(0)}}\right] \tag{121}$$

and of the 2 corresponding eigenspaces projectors $P_{\pm}$:

$$R_0 = R^+ P_+ + R^- P_-. \tag{122}$$

The explicit expression for the projectors can be obtained using the completeness relation $P_+ + P_- = 1$. We find

$$P_{\pm} = \frac{1}{R^{\pm} - R^{\mp}}[R_0 - R^{\mp}]. \tag{123}$$

Now, writing the matrices U_1 and R_1 in terms of these projectors

$$U_1 = P_- U_1 P_- + P_- U_1 P_+ + P_+ U_1 P_- + P_+ U_1 P_+ \tag{124}$$

$$R_1 = P_- R_1 P_- + P_- R_1 P_+ + P_+ R_1 P_- + P_+ R_1 P_+, \tag{125}$$

substituting them and Eq.(122) into Eq.(120) and comparing each matrix element, we find

$$U_1 = -(P_- R_1 P_- + P_+ R_1 P_+) + \frac{P_+ R_1 P_-}{R^- - R^+ - 1} + \frac{P_- R_1 P_+}{R^+ - R^- - 1}. \tag{126}$$

Thanks to this result, we can now write the NLO solution of the singlet doublet in a form which is useful for practical calculations. Indeed, if we substitute Eq.(126) in Eq.(118), we get (at NLO)

$$\begin{aligned}\vec{q}_S(\mu^2) = \Bigg\{ & \left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{-R_-} \Bigg[P_- + \left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi}\right) P_- R_1 P_- \\ & - \left(\frac{\alpha_s(\mu_0^2)}{4\pi} - \frac{\alpha_s(\mu^2)}{4\pi}\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{R^- - R^+}\right) \frac{P_- R_1 P_+}{R^+ - R^- - 1}\Bigg] \\ & + (+ \leftrightarrow -)\Bigg\} \vec{q}_S(\mu_0^2).\end{aligned} \tag{127}$$

After the evolution of the NS and S combinations has been performed from a certain scale μ_0^2 to the scale μ^2, we need to return to the parton distributions for all the quarks but the gluon. These are obtained straightforwardly with the following relations

$$q_k + \bar{q}_k = \frac{1}{N_f} q_S - \frac{1}{k} q^+_{(NS)k} + \sum_{i=k+1}^{N_f} \frac{1}{i(i-1)} q^+_{(NS)i}, \qquad k = 1, \ldots, N_f \tag{128}$$

$$q_k - \bar{q}_k = \frac{1}{N_f} q^V_{(NS)} - \frac{1}{k} q^-_{(NS)k} + \sum_{i=k+1}^{N_f} \frac{1}{i(i-1)} q^-_{(NS)i}, \qquad k = 1, \ldots, N_f. \tag{129}$$

However, Eqs.(113,114,127) represent the NLO solution of the DGLAP equations Eq.(97) in the case when the number of active flavors N_f has been kept fixed. This is the so called fixed flavor scheme solution. If we want to take into account the thresholds of the heavy quark flavors, we can evolve up the NS a S combinations from the scale μ_0^2 (with a certain number N_f of active flavors) to the scale of production of a new flavor. Then, we can take the result of this evolution as the starting point of a second evolution (with $N_f + 1$ active flavors this time) above the production scale of the new flavor, assuming that the new flavor vanishes at threshold. This is the most simple way to generate dynamically a new flavor. Finally, we note that the procedure outlined in this appendix can be easily generalized beyond the NLO order [35]. Furthermore, in many cases, it is interesting to study the dependence on the renormalization scale, in order to estimate the theoretical error of the evolution. Here the renormalization scale μ_r^2 has been chosen equal to the factorization one μ^2 for simplicity. To restore the implicit μ_r^2-dependence in parton densities, we need only to rewrite the running coupling constant $\alpha_s(\mu^2)$ in terms of μ_r^2 (see Eq.(41) in section 1.2) in the splitting functions. Making this substitution, we have that the perturbative expansion of a generic splitting function Eq.(130) becomes

$$\gamma_{pp'}(N, \mu^2, k') = \frac{\alpha_s(k'\mu^2)}{4\pi}\gamma_{pp'}^{(0)}(N) + \left(\frac{\alpha_s(k'\mu^2)}{4\pi}\right)^2 (\gamma_{pp'}^{(1)}(N) + \beta_0\gamma_{pp'}^{(0)} \ln k') + O(\alpha_s^3), \tag{130}$$

where $k' = \mu_r^2/\mu^2$. Hence, in Eqs.(113,114,127), we should perform the following substitutions:

$$\gamma^{(1)} \to \gamma^{(1)} + \beta_0\gamma^{(0)} \ln k', \quad \alpha_s(\mu^2) \to \alpha_s(k'\mu^2), \quad \alpha_s(\mu_0^2) \to \alpha_s(k'\mu_0^2) \tag{131}$$

and use $k'\mu^2$ as reference scale for new flavors production.