Chapter 5

Resummation of rapidity distributions

In this chapter, we present a derivation of the threshold resummation formula for the Drell-Yan and the prompt photon production rapidity distributions. Our arguments are valid for all values of rapidity and to all orders in perturbative QCD. For the case of the Drell-Yan process, resummation is realized in a universal way, i.e. both for the production of a virtual photon γ^* and the production of a vector boson $W^{\pm}$, Z^0. We will show that for the fixed-target proton-proton Drell-Yan experiment E866/NuSea used in current parton fits, the NLL resummation corrections are comparable to NLO fixed-order corrections and are crucial to obtain agreement with the data. This means that the NLL resummation of rapidity distributions is necessary and turns out to give better results than high-fixed-order calculations. We consider first the resummation of the DY case and its phenomenology and then the resummation of the prompt photon case.

5.1 Threshold resummation of DY rapidity distributions

5.1.1 General kinematics of Drell-Yan rapidity distributions

We consider the general Drell-Yan process in which the collisions of two hadrons (H_1 and H_2) produce a virtual photon γ^* (or an on-shell vector boson V) and any collection of hadrons (X):

$$H_1(P_1) + H_2(P_2) \to \gamma^*(V)(Q) + X(K). \tag{1}$$

In particular, we are interested in the differential cross section $\frac{d\sigma}{dQ^2 dY}(x, Q^2, Y)$, where Q^2 is the invariant mass of the photon or of the vector boson, x is defined as usual as the fraction of invariant mass that the hadrons transfer to the photon (or to the vector boson) and Y is the rapidity of γ^* (V) in the hadronic centre-of-mass-frame:

$$x \equiv \frac{Q^2}{S}, \quad S = (P_1 + P_2)^2, \quad Y \equiv \frac{1}{2} \ln \left(\frac{E + Q_z}{E - Q_z} \right), \tag{2}$$

where E and p_z are the energy and the longitudinal momentum of $\gamma^*(V)$ respectively. In this frame, the four-vector Q of $\gamma^*(V)$ can be written in terms of its rapidity and its tranverse momentum

$$Q = (Q^0, \vec{Q}) = (\sqrt{Q^2 + Q_\perp^2}\cosh Y, \vec{Q}_\perp, \sqrt{Q^2 + Q_\perp^2}\sinh Y), \tag{3}$$

or in terms of the scattering angle θ

$$Q = (Q^0, \vec{Q}) = (\sqrt{Q^2 + |\vec{Q}|^2}, \vec{Q}_\perp, |\vec{Q}|\cos(\theta)). \tag{4}$$

For completeness, we recall that when $Q^2 = 0$ (which is not our case), the rapidity Y defined in Eq.(2) reduces to the pseudorapidity η according to Eq.(4):

$$\eta = -\ln(\tan(\theta/2)). \tag{5}$$

At the partonic level, a parton 1(2) in the hadron H_1 (H_2) carries a fraction of momentum x_1 (x_2):

$$p_1 = x_1 P_1 = x_1 \frac{\sqrt{S}}{2}(1, \vec{0}_\perp, 1)), \quad p_2 = x_2 P_2 = x_2 \frac{\sqrt{S}}{2}(1, \vec{0}_\perp, -1)). \tag{6}$$

It is clear that the hadronic center-of-mass frame does not coincide with the partonic one, because x_1 is, in general, different from x_2. Furthermore, from Eq.(2), we see that the rapidity is not an invariant. Hence, in order to define the rapidity in the partonic center-of-mass frame (y), we have to perform a boost of Y which connects the two frames. This provides us a relation between the rapidity in these two frames:

$$y = Y - \frac{1}{2}\ln(\frac{x_1}{x_2}). \tag{7}$$

In order to understand the kinematic configurations in terms of rapidity, it is convenient to define a new variable u,

$$u \equiv \frac{Q \cdot p_1}{Q \cdot p_2} = e^{-2y} = \frac{x_1}{x_2} e^{-2Y}. \tag{8}$$

With no partons radiated as in the case of the LO, the rapidity is obviously zero. Beyond the LO, one or more partons can be radiated. Now, if these partons are radiated collinear with the incoming parton 2, then the partonic rapidity reaches its maximum value and u its minimum one. Similarly the minimum value of y and the maximum value of u is achieved when the radiated partons are collinear with the incoming parton 1. To be more precise, suppose that in the first case the radiated partons (collinear with the parton 2) carry away a fraction of momentum equal to $(1-z)p_2$, so that by momentum conservation $Q = p_1 + zp_2$. In this case, we obtain immediately the lower bound for u, which is z. In the second case the collinear radiated partons have momentum $(1-z)p_1$, hence $Q = zp_1 + p_2$ and the upper bound of u is $1/z$. So, z can be interpreted as the fraction of invariant mass that incoming partons transfer to $\gamma^*(V)$. In fact:

$$z = \frac{Q^2}{2p_1 \cdot p_2} = \frac{Q^2}{(p_1 + p_2)^2} = \frac{x}{x_1 x_2}, \tag{9}$$

where we have neglected the quark masses. Therefore, we have that the kinematic constraints of u are:

$$z \leq u \leq \frac{1}{z}. \tag{10}$$

Then, since $x_1 < 1$ and $x_2 < 2$, the lower and upper bounds of z are:

$$x \leq z \leq 1. \tag{11}$$

Thanks to Eqs.(7,8), the first relation can be translated directly into a relation for the upper and lower limit of the partonic center-of-mass rapidity:

$$\frac{1}{2} \ln z \leq y \leq \frac{1}{2} \ln \frac{1}{z}. \tag{12}$$

Now, we need to obtain the boundaries of the hadronic center-of-mass rapidity. Substituting Eqs.(8, 9) into the two conditions $u \geq z$ and $u \leq 1/z$, we obtain the lower kinematical bound for x_1 and x_2:

$$x_1 \geq \sqrt{x} e^{Y} \equiv x_1^0, \quad x_2 \geq \sqrt{x} e^{-Y} \equiv x_2^0 \tag{13}$$

and the obvious requirment that $x^0_{1(2)} \leq 1$ implies that the hadronic rapidity has a lower and an upper bound:

$$\frac{1}{2} \ln x \leq Y \leq \frac{1}{2} \ln \frac{1}{x}. \tag{14}$$

5.1.2 The universality of resummation in Drell-Yan processes

According to standard factorization of collinear singularities of perturbative QCD, the expression for the hadronic differential cross section in rapidity has the form,

$$\begin{aligned} \frac{d\sigma}{dQ^2 dY} &= \sum_{i,j} \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 F_i^{H_1}(x_1, \mu^2) F_j^{H_2}(x_2, \mu^2) \\ &\times \frac{d\hat{\sigma}_{ij}}{dQ^2 dy} \left(x_1, x_2, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y \right), \end{aligned} \tag{15}$$

where y depends on Y, x_1 and x_2 according to Eq.(7). The sum runs over all possible partonic subprocesses, $F_i^{(1)}$,$F_j^{(2)}$ are respectively the parton densities of the hadron H_1 and H_2, μ is the factorization scale (chosen equal to renormalization scale for simplicity) and $d\hat{\sigma}_{ij}/(dQ^2 dy)$ is the partonic cross section. Even if the cross section Eq.(15) is μ^2-independent, this is not the case for each parton subprocess. However, the μ^2-dependence of each contribution is proportional to the off-diagonal anomalous dimensions (or splitting functions), which in the threshold limit, ($z \to 1$) are suppressed by factors of $1 - z$. Therefore, each partonic subprocess can be treated independently and is separately renormalization-group invariant. Furthermore, the suppression, in the threshold limit, of the off-diagonal splitting functions implies also that only the gluon-quark channels are suppressed. So, in order to study resummation, we will consider only the quark- anti-quark channel, which can be related to the same dimensionless coefficient function $C(z, Q^2/\mu^2, \alpha_s(\mu^2), y)$ for both, the production of a virtual photon and the production of a on-shell vector boson. In fact, if for the production of a virtual photon, we define $C(z, Q^2/\mu^2, \alpha_s(\mu^2), y)$ through the equation,

$$x_1 x_2 \frac{d\hat{\sigma}^{\gamma^*}_{q\bar{q}'}}{dQ^2 dy}\left(x_1, x_2, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) = \frac{4\pi\alpha^2 c_{q\bar{q}'}}{9Q^2 S} C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right), \tag{16}$$

where the prefactor x_1x_2 has been introduced for future convenience, we find that for the case of the production of a real vector boson,

$$\begin{aligned} x_1 x_2 \frac{d\hat{\sigma}^{V}_{q\bar{q}'}}{dQ^2 dy}\left(x_1, x_2, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) &= \frac{\pi G_F Q^2 \sqrt{2} c_{q\bar{q}'}}{3S}\delta(Q^2 - M_V^2) \\ &\times \; C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right), \end{aligned} \tag{17}$$

where G_F is the Fermi constant, M_V is the mass of the produced vector boson. The coefficients $c_{q\bar{q}'}$, for the different Drell-Yan processes, are given by:

$$c_{q\bar{q}'} = Q_q^2 \delta_{q\bar{q}} \quad \text{for } \gamma^*, \tag{18}$$

$$c_{q\bar{q}'} = |V_{qq'}|^2 \quad \text{for } W^{\pm}, \tag{19}$$

$$c_{q\bar{q}'} = 4[(g_v^q)^2 + (g_a^q)^2]\delta_{q\bar{q}} \quad \text{for } Z^0. \tag{20}$$

Here, Q_q^2 is the square charge of the quark q, $V_{qq'}$ are the CKM mixing factors for the quark flavors q, q' and

$$g_v^q = \frac{1}{2} - \frac{4}{3}\sin^2\theta_W, \quad g_a^q = \frac{1}{2} \quad \text{for an up-type quark,} \tag{21}$$

$$g_v^q = -\frac{1}{2} + \frac{2}{3}\sin^2\theta_W, \quad g_a^q = -\frac{1}{2} \quad \text{for a down-type quark,} \tag{22}$$

with θ_W the Weinberg weak mixing angle. As a consequence of these facts, resummation has to be performed only for the quark-anti-quark channels omitting the overall dimensional factors of $C(z, Q^2/\mu^2, \alpha_s(\mu^2), y)$ in the different Drell-Yan processes. Thus, we are left with the following dimensionless cross section, which has the form:

$$\begin{aligned} \sigma(x, Q^2, Y) &\equiv \int_{x_1^0}^1 \frac{dx_1}{x_1}\int_{x_2^0}^1 \frac{dx_2}{x_2} F_1^{H_1}(x_1, \mu^2) F_2^{H_2}(x_2, \mu^2) \\ &\times \; C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right), \end{aligned} \tag{23}$$

where F_1 and F_2 are quark or anti-quark parton densities in the hadron H_1 and H_2 respectively. This shows the universality of resummation in Drell-Yan processes in the sense that only the renormalization-group invariant quantity defined in Eq.(23) has to be resummed.

5.1.3 Factorization properties and the Mellin-Fourier transform

For the case of the rapidity-integrated cross section, resummation is usually done in Mellin space transforming the variable x into its conjugate variable N, because the Mellin transformation turns convolution products into ordinary products. Furthermore, the Mellin space is the natural space where to define resummation of leading, next-to-leading and so on logarithmic contribution, because in this space momentum conservation is respected as shown in [58]. In the case of the rapidity distribution, the Mellin transformation is not sufficient. In fact, rewriting Eq.(23) in this form

$$\begin{aligned} \sigma(x, Q^2, Y) &= \int_0^1 dx_1 dx_2 dz F_1^{H_1}(x_1, \mu^2) F_2^{H_2}(x_2, \mu^2) \\ &\times C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) \delta(x - x_1 x_2 z), \end{aligned} \tag{24}$$

we see that the Mellin transform with respect to x,

$$\sigma(N, Q^2, Y) \equiv \int_0^1 dx x^{N-1} \sigma(x, Q^2, Y), \tag{25}$$

does not diagonalize the triple integral in Eq.(24). This is due to the fact that the partonic center-of-mass rapidity y depends on x_1 and x_2 through Eq.(7). The ordinary product in Mellin space can be recovered performing the Mellin transform with respect to x of the Fourier transform of Eq.(24) with respect to Y. Calling the Fourier moments M, using Eq.(7) the relations (12,14) and the identity

$$\begin{aligned} C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), Y - \frac{1}{2}\ln\frac{x_1}{x_2}\right) &= \int_{\ln\sqrt{z}}^{\ln 1/\sqrt{z}} dy C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right) \\ &\times \delta\left(y - Y + \frac{1}{2}\ln\frac{x_1}{x_2}\right), \end{aligned} \tag{26}$$

we find that

$$\sigma(N, Q^2, M) \equiv \int_0^1 dx x^{N-1} \int_{\ln\sqrt{x}}^{\ln 1/\sqrt{x}} dY e^{iMY} \sigma(x, Q^2, Y) \tag{27}$$

$$\begin{aligned} &= F_1^{H_1}(N + iM/2, \mu^2) F_2^{H_2}(N - iM/2, \mu^2) \\ &\times C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right), \end{aligned} \tag{28}$$

where

$$F_i^{H_i}(N \pm iM/2, \mu^2) = \int_0^1 dx x^{N-1 \pm iM/2} F_i^{H_i}(x, \mu^2), \tag{29}$$

$$\begin{aligned} C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right) &= \int_0^1 dz z^{N-1} \int_{\ln\sqrt{z}}^{\ln 1/\sqrt{z}} dy e^{iMy} \\ &\times C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right). \end{aligned} \tag{30}$$

Eq.(28) shows that performing the Mellin-Fourier moments of the hadronic dimensionless cross section Eq.(23), we recover an ordinary product of the Mellin-Fourier trnsform of the coefficient function and the Mellin moments of the parton densities translated outside the real axis by $\pm iM/2$. Because the coefficient function is symmetric in y, we can rewrite Eq.(30) in this way:

$$\begin{aligned} C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right) &= 2\int_0^1 dz z^{N-1} \int_0^{\ln 1/\sqrt{z}} dy \cos(My) \\ &\times C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), y\right). \end{aligned} \tag{31}$$

From this last equation and Eq.(29), we see that the dependence on M, the Fourier conjugate of the rapidity y, originates from the parton densities, that depend on $N \mp iM/2$, and from the factor of $\cos(My)$ in the integrand of Eq.(31).

5.1.4 The all-order resummation formula and its NLL implementation

In this section, we show that the resummed expression of Eq.(28) is obtained by simply replacing the coefficient function $C\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right)$ with its integral over y, resummed to the desired logarithmic accuracy. This is equivalent to saying that the factor of $\cos(My)$ in Eq.(31) is irrelevant in the large-N limit. Indeed, one can expand $\cos(My)$ in powers of y,

$$\cos(My) = 1 - \frac{M^2y^2}{2} + O(M^4y^4). \tag{32}$$

and observe that the first term of this expansion leads to a convergent integral (the rapidity-integrated cross section), while the subsequent terms are suppressed by powers of $(1-z)$, since the upper integration bound in Eq.(31) is

$$\ln\frac{1}{\sqrt{z}} = \frac{1}{2}(1-z) + O((1-z)^2). \tag{33}$$

Hence, up to terms suppressed by factors $1/N$, Eq.(30) is equal to the Mellin transform of the rapidity-integrated Drell-Yan coefficient function that we call $C_I(N, Q^2/\mu^2, \alpha_s(\mu^2))$. This completes our proof. We get

$$\begin{aligned} \sigma^{res}(N, Q^2, M) &= F_1^{H_1}(N + iM/2, \mu^2) F_2^{H_2}(N - iM/2, \mu^2) \\ &\times C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right). \end{aligned} \tag{34}$$

This theorethical result is very important: it shows that, near threshold, the Mellin-Fourier transform of the coefficient function does not depend on the Fourier moments and that this is valid to all orders of QCD perturbation theory. Furthermore this result remains valid for all values of hadronic center-of-mass rapidity, because we have introduced a suitable integral transform over rapidity. The resummed rapidity-integrated Drell-Yan coefficient function to NLL order has been studied in Section 3.4. It is given by

$$C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) = \left[\exp\{\ln N g_1(\lambda, 2) + g_2(\lambda, 2)\}\right]_{\mu_r^2=\mu^2} \tag{35}$$

where $\lambda = b_0\alpha_s(\mu_r^2)\ln N$ and where the resummation functions $g_1(\lambda, 2)$ and $g_2(\lambda, 2)$ are given in Eqs.(101,102) of Section 3.4 with the resummation coefficients in the $\overline{MS}$ scheme given in Eq.(46) of Section 2.3.1. Now, we want to arrive to a NLO and NLL expression of the rapidity-dependent dimensional cross section. This is achieved firstly taking the Mellin and Fourier inverse transforms of $\sigma^{res}(N, Q^2, M)$ Eq.(34) in order to turn back to the variables x and Y:

$$\sigma^{res}(x, Q^2, Y) = \int_{-\infty}^{\infty} \frac{dM}{2\pi} e^{-iMY} \int_{C-i\infty}^{C+i\infty} \frac{dN}{2\pi i}\, x^{-N} \sigma^{res}(N, Q^2, M). \tag{36}$$

In principle the contour in the complex N-space of the inverse Mellin transform in Eq.(36) has to be chosen in such a way that the intersection of C with the real axis lies to the right of all the singularities of the integrand. In practice, this is not possible, because the resummed coefficient function Eqs.(101,102) of section 3.4 has a branch cut on the real positive axis for

$$N \geq N_L \equiv e^{\frac{1}{ab_0\alpha_s(Q^2)}}, \tag{37}$$

which corresponds to the Landau singularity of $\alpha_s(Q^2/N^a)$ (see Eq.(41) in Appendix 1.2). This is due to the fact that if the N-space expression is expanded in powers of α_s, and the Mellin inversion is performed order by order, a divergent series is obtained. The “Minimal Prescription” proposed in [58] gives a well defined formula to obtain the resummed result in x-space to which the divergent series is asymptotic and is simply obtained choosing $C = C_{MP}$ in such a way that all the poles of the integrand are to the left, except the Landau pole Eq.(37). Recently, another method has been prposed in Ref.[59]. Here, we will adopt the “Minimal Prescription ” formula, deforming the contour in order to improve numerical convergence and to avoid the singularities of the parton densities of Eq.(34) which are transated out of the real axis by $\pm iM/2$. Hence, we perform the N-integral in Eq.(36) over a curve Γ given by:

$$\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 \tag{38}$$

$$\Gamma_1(t) = C_{MP} - i\frac{M}{2} + t(1+i), \quad t \in (-\infty, 0) \tag{39}$$

$$\Gamma_2(s) = C_{MP} + is\frac{M}{2}, \quad s \in (-1, 1) \tag{40}$$

$$\Gamma_3(t) = C_{MP} + i\frac{M}{2} - t(1-i), \quad t \in (0, +\infty) \tag{41}$$

The double inverse tranform of Eq.(36) over the curve Γ then becomes:

$$\sigma^{res}(x, Q^2, Y) = \frac{1}{\pi}\int_0^1 \frac{dm}{m} \cos(-Y\ln m)\sigma^{res}(x, Q^2, -\ln m), \tag{42}$$

where we have done the change of variable$M = -\ln m$. The factor $\sigma^{res}(x, Q^2, M)$ of the integrand in Eq.(42) is given by

$$\sigma^{res}(x, Q^2, M) = \tag{43}$$
$$\frac{1}{\pi}\int_0^1 \frac{ds}{s}\Re\Bigg[x^{-C_{MP}-\ln s+i(M/2+1)}\sigma^{res}(C_{MP}+\ln s - i(M/2+1), Q^2, M)$$
$$\times(1-i) + \frac{sM}{2}x^{-C_{MP}-isM/2}\sigma^{res}(C_{MP}+isM/2, Q^2, M)\Bigg],$$

where we have done another change of variables ($t = -\ln s$). Eqs.(42,43) are the expressions that we use to evaluate numerically the resummed adimensional cross section in the variables x and Y Eq.(36). Furthermore, we need to know the analytic continuations to all the complex plane of the parton densities at the scale μ^2 in Eq.(34). Here, we need to evolve up a partonic fit taken at a certain scale solving the DGLAP evolution equations in Mellin space. The solution of the evolution equations is given in Section 1.5. Finally, we want to obtain a NLO determination of the cross section improved with NLL resummation. In order to do this, we must keep the resummed dimensionaless part of the cross section Eq.(42), multiply it by the correct dimensional prefactors Eqs.(57,19,20) and parton densities, add to the resummed part the full NLO cross section and subtract the double-counted logarithmic enhanced contributions. Thus, we have

$$\frac{d\sigma}{dQ^2dY} = \frac{d\sigma^{NLO}}{dQ^2dY} + \frac{d\sigma^{res}}{dQ^2dY} - \left[\frac{d\sigma^{res}}{dQ^2dY}\right]_{\alpha_s=0} - \alpha_s\left[\frac{\partial}{\partial\alpha_s}\left(\frac{d\sigma^{res}}{dQ^2dY}\right)\right]_{\alpha_s=0}. \tag{44}$$

The first term is the full NLO cross section given in [17, 60, 61, 62]. We report the complete expression in Appendix C. The third and the fourth terms in Eq.(44) are obtained in the same way as the second one, but with the substitutions

$$\begin{aligned} C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) &\rightarrow 1, \\ C_I^{res}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2)\right) &\rightarrow \alpha_s(\mu^2)2A_1\left\{\ln^2 N + \ln N\left[2\gamma_E - \ln\left(\frac{Q^2}{\mu^2}\right)\right]\right\}, \end{aligned} \tag{45}$$

respectively. The terms that appear in the second in the second line of Eqs.(45) are exactly the $O(\alpha_s)$ logarithmic enhanced contributions in the $\overline{MS}$ scheme. We note that the resummed cross section Eq.(44) is relevant even when the variable x is not large. In fact, the cross section can get the dominant contributions from the integral in Eq.(23) for values of z Eq.(9) that are near the threshold even when x is not close to one, because of the strong suppression of parton densities $F_i(x_i, \mu^2)$ when x_i are large.

5.1.5 NLL impact of resummation at E866 experiment

To show the importance of this resummation, we have calculated the Drell-Yan rapidity distribution for proton-proton collisions at the Fermilab fixed-target experiment

Figure 5.1: Y-dependence of $d^2\sigma/(dQ^2dY)$ in units of pb/GeV^2. The curves are, from top to bottom, the NLO result (red band), the LO+LL resummation (blue band) and the LO (black band). The bands are obtained varying the factorization scale between $\mu^2 = 2Q^2$ and $\mu^2 = 1/2Q^2$.

E866/NuSea [38]. The center-of-mass energy has been fixed at $\sqrt{S} = 38.76\,\mathrm{GeV}$ and the invariant mass of the virtual photon γ^* has been chosen to be $Q^2 = 64\,\mathrm{GeV}^2$ in analogy with [19]. Clearly the contribution of the virtual Z^0 can be neglected, because its mass is much bigger than Q^2. In this case $x = 0.04260$ and the upper and lower bound of the hadronic rapidity Y Eq.(14) are given by ± 1.57795. We have evolved up the MRST 2001 parton distributions (taken at $\mu^2 = 1\,\mathrm{GeV}^2$) in order to compare to Ref.[19], where the NNLO calculation is performed. However, results obtained using more modern parton sets should not be very different. The LO parton set is given in [63] with $\alpha_s^{LO}(m_Z) = 0.130$ and the NLO set is given in [64] with $\alpha_s^{NLO}(m_Z) = 0.119$. The evolution of parton densities at the scale μ^2 has been performed in the variable flavor number scheme. The quarks has been considered massless and, at the scale of the transition of the flavor number ($N_f \to N_f + 1$), the new flavor is generated dynamically. The resummation formula Eq.(34) together with Eqs.(101-35) has been used with the number of flavors $N_f = 4$. In figure 5.1, we plot the rapidity-dependence of the cross section at LO, NLO and LO improved with LL resummation. The effect of LL resummation is small compared to the effect of the full NLO correction. We see that, at leading order, the impact of the resummation is negligible in comparison to the NLO fixed-order correction. This means that, at leading order, resummation is not necessary. The LO, the NLO and its NLL improvement cross sections are shown in figure 5.2. The effect of the NLL resummation in the central rapidity region is almost as large as the NLO correction, but it reduces the cross section instead of enhancing it for not large values of rapidity. The origin of this suppression will be discussed in the next Section. Going from the LO result to the NLO with NLL resummation, we note a reduction of the dependence on the factorization scale i.e. a

Figure 5.2: Y-dependence of $d^2\sigma/(dQ^2dY)$ in units of pb/GeV^2. The curves are, from top to bottom, the NLO result (red band), the NLO+NLL resummation (green band) and the LO (black band). The bands are obtained as in figure 5.1.

reduction of the theoretical error. Now, we want to establish if the leading logarithmic terms that are included in the resummed exptonent represent a good approximation to the exact fixed order computation. Only if this is the case, we can believe that our resummation is reliable in perturbative QCD. In order to do this, we compare the full NLO DY rapidity cross section with the one obtained including only the large-N leading terms of the coefficient function. For simplicity, we choose the factorization scale μ^2 to the scale of the process Q^2. The leading large-N coefficient function is given by:

$$C^{\mathrm{lead}}(N,\alpha_s(Q^2)) = 1 + \alpha_s(Q^2) 2A_1 \left(\ln^2 N + 2\gamma_E \ln N + \gamma_E^2 - 2 + \frac{\pi^2}{3}\right), \qquad (46)$$

where we have added the constant terms at $O(\alpha_s)$ which are not resummed. For an explicit derivation of these constant terms see for example Ref. [17] Section 3. We plot the result in Figure 5.3. We see that the leading terms Eq.(46) represent full NLO computation for all relevant rapidities. In Figure 5.4, we plot only the $O(\alpha_s)$ correction. Here we see that the $O(\alpha_s)$ contribution of Eq.(46) represent a good approximation to the exact $O(\alpha_s)$ NLO contribution, full $O(\alpha_s)$ NLO correction for all relevant rapidities. In figure 5.5, we plot the experimental data of Ref.[38] converted to the Y variable together with our NLO and NLL resummed predictions. The data in Ref.[38] are tabulated in invariant Drell-Yan pair mass $\sqrt{Q^2}$ and Feynman x_F bins. To convert the data to the hadronic rapidity Y, we have used the definition of the Feynman x_F which is

$$x_F \equiv \frac{2Q_z}{\sqrt{S}} = \frac{2\sqrt{Q^2+Q_\perp^2}\,\sinh Y}{\sqrt{S}}, \qquad (47)$$

Figure 5.3: $d^2\sigma/(dQ^2dY)$ in units of pb/GeV^2 for the full NLO computation (upper red line) and for the leading terms of Eq.(46) (lower black line). It has been calculated for one value of the factorization scale $\mu^2 = Q^2$.

Figure 5.4: $d^2\sigma/(dQ^2dY)$ in units of pb/GeV^2 for $O(\alpha_s)$ correction of the full NLO computation (upper red line) and for the leading $O(\alpha_s)$ term of Eq.(46) (lower black line).

Figure 5.5: Y dependence of $d^2\sigma/(dQ^2dY)$ in units of pb/GeV^2. The curves are, from top to bottom, the NLO result (red band) and the NLO+NLL resummation (green band) together with the E866/NuSea data. The bands are obtained as in figure 5.1.

where we have used Eq.(3). Solving Eq.(47) in Y we have

$$Y = \ln\left[H + \sqrt{H^2+1}\right], \qquad H = \frac{x_F\sqrt{S}}{2\sqrt{Q^2+Q_\perp^2}}. \tag{48}$$

With this equation and with the aid of the $Q_\perp$ distribution, which is also given in Ref.[38], we have converted the data from x_F to Y. Furthermore, for each x_F bin, we have done the weighted average of three $\sqrt{Q^2}$ bins ($7.2 \leq \sqrt{Q^2} \leq 7.7$; $7.7 \leq \sqrt{Q^2} \leq 8.2$ and $8.2 \leq \sqrt{Q^2} \leq 8.7$ with the energies in GeV). The agreement with data is good and a great improvement for not large rapidity is obtained with respect to the NLO calculation. We note also that the NLL resummation gives better result than the NNLO calculation performed in [19]. The NNLO prediction has a worse agreement with data than the NLO one for not large values of rapidity. This result suggests that, for the case of rapidity distributions, NLL resummation is more important than high-fixed-order calculation and that it can be so even at higher center-of-mass energies.

5.2 The origin of suppression

In this section, we shall show that the suppression of the cross section of the NLL correction with the parameter choices of the experiment E866 is due to the shift in the complex plane of the dominant contribution of the resummed exponent. We shall do it using a simplified toy-model. Consider the collision of only two quarks with parton density

$$F(x) = (1-x)^2. \tag{49}$$

Its Mellin transform is given by:

$$F(N) = \int_0^1 dx\, x^{N-1}(1-x)^2 = \frac{\Gamma(N)\Gamma(3)}{\Gamma(N+3)} = \frac{2}{N(N+1)(N+2)}. \tag{50}$$

Furthermore, we take the double-log approximation (DLA) which is obtained performing the limit $\lambda \to 0$ in the resummed exponent Eq.(35). Thus, in this simple model, the Mellin-Fourier transform of the NLL resummed cross section Eq.(44) can be written in the following form:

$$\sigma(N,M) = \sigma^{FO}(N,M) + |F(N+iM/2)|^2 \Delta\sigma^{\rm DLA}(N), \tag{51}$$

where $\sigma^{FO}(N,M)$ are the exact NLO Mellin-Fourier moments and where

$$\Delta\sigma^{\rm DLA}(N) = \left[e^{\alpha_s 2A_1 \ln^2 N} - 1 - \alpha_s 2A_1 \ln^2 N\right]. \tag{52}$$

If there is a suppression, this means that the quantity

$$\sigma(N,M) - \sigma^{FO}(N,M) = \frac{4\Delta\sigma^{\rm DLA}(N)}{\left[N^2+\frac{M^2}{4}\right]\left[(N+1)^2+\frac{M^2}{4}\right]\left[(N+2)^2+\frac{M^2}{4}\right]}, \tag{53}$$

should produce a negative contribution in performing the inverse Mellin and Fourier transform. It is given by the integral

$$\int_{-\infty}^{\infty} \frac{dM}{2\pi} e^{-iMY} \int_{C-i\infty}^{C+i\infty} \frac{dN}{2\pi i} x^{-N} \frac{4\Delta\sigma^{\rm DLA}(N)}{\left[N^2+\frac{M^2}{4}\right]\left[(N+1)^2+\frac{M^2}{4}\right]\left[(N+2)^2+\frac{M^2}{4}\right]}. \tag{54}$$

The integrand function of this expression has not only a cut on the negative real axis (as it happens in the inclusive case), but also poles that are shifted in the complex plane:

$$-n \pm \frac{iM}{2}; \qquad n = 0,1,2. \tag{55}$$

Because of the factor x^{-N} in the inverse Mellin integral in Eq.(54), its dominant contribution comes from the poles with $n=0$ in Eq.(55). The contribution of the pole at $+iM/2$ is given by

$$\int_{-\infty}^{\infty} \frac{dM}{2\pi} e^{-iM(Y+\ln\sqrt{x})} \frac{4\Delta\sigma^{\rm DLA}(N=iM/2)}{iM(iM+1)(iM+2)}, \tag{56}$$

where

$$\begin{aligned}\Delta\sigma^{\rm DLA}(N=iM/2) \;&=\; \exp\left[\alpha_s 2A_1\left(\ln^2\frac{|M|}{2} - \frac{\pi^2}{4} + i\pi\ln\frac{|M|}{2}\right)\right] + \\ &\quad -1 - \alpha_s 2A_1\left(\ln^2\frac{|M|}{2} - \frac{\pi^2}{4} + i\pi\ln\frac{|M|}{2}\right).\end{aligned} \tag{57}$$

The important thing to notice of this contribution is the fact that the imaginary pole has produced an oscillating prefactor in front of the resummed exponent which, together with the oscillating factor of the Fourier inverse integral in Eq.(56) at zero hadronic rapidity Y, is given by

$$\exp\left[i\left(\alpha_s 2\pi A_1 \ln\frac{|M|}{2} - iM\ln\sqrt{x}\right)\right]. \tag{58}$$

We note that, with the inclusion of the contribution of the other pole at $-iM/2$, the real and immaginary part of Eq.(57) contribute to the integral of Eq.(54). The real and immaginary part of Eq.(57) are given by:

$$\begin{aligned} \Re[\Delta\sigma^{\mathrm{DLA}}(iM/2)] &= e^{\alpha_s 2A_1\left(\ln^2\frac{|M|}{2}-\frac{\pi^2}{4}\right)}\cos\left(\alpha_s 2\pi A_1\ln\frac{|M|}{2}\right) \\ &\quad -1-\alpha_s 2A_1\left(\ln^2\frac{|M|}{2}-\frac{\pi^2}{4}\right) \end{aligned} \tag{59}$$

$$\begin{aligned} \Im[\Delta\sigma^{\mathrm{DLA}}(iM/2)] &= e^{\alpha_s 2A_1\left(\ln^2\frac{|M|}{2}-\frac{\pi^2}{4}\right)}\sin\left(\alpha_s 2\pi A_1\ln\frac{|M|}{2}\right) \\ &\quad -\alpha_s 2\pi A_1\ln\frac{|M|}{2} \end{aligned} \tag{60}$$

Now, to roughly estimate the effect of the oscillating factor in Eq.(56), we use the value of $M = M_0$ where the phase of Eq.(58) is stationary and is given by:

$$|M_0| = \frac{\alpha_s 2\pi A_1}{|\ln\sqrt{x}|}. \tag{61}$$

which has more or less the same value of x. We should now recall that this is a rough estimation and that there is also the contribution of the cut on the negative real axis which usually produces an enhancement. However, comparing this estimation with the result for the W boson production at RHIC (see e.g. figure 1 and 2 in reference [17]),

5.3 Resummation of prompt photon rapidity distribution

5.3.1 General kinematics of prompt photon rapidity distribution

Here, we consider the rapidity distribution of the prompt photon process discussed in chapter 4,

$$H_1(P_1) + H_2(P_2) \to \gamma(p_\gamma) + X. \tag{62}$$

Specifically, we are interested in the differential cross section $p_\perp^3\frac{d\sigma}{dp_\perp dY}(x_\perp, p_\perp^2, \eta_\gamma)$, where as in section 4.1 of chapter 4 $p_\perp$ is the transverse momentum of the photon, η_γ is its hadronic center-of-mass pseudorapidity and

$$x_\perp = \frac{4p_\perp^2}{S}. \tag{63}$$

The pseudorapidity of the direct real photon in the partonic center-of-mass frame $\hat{\eta}_\gamma$ is related to η_γ through Eq.(6) in section 4.1:

$$\hat{\eta}_\gamma = \eta_\gamma - \frac{1}{2}\ln\frac{x_1}{x_2}. \tag{64}$$

Furthermore, as in chapter 4, we use the following parametrizations of the photon and of the incoming partons' momenta

$$\begin{aligned} p_\gamma &= (p_\perp \cosh\hat{\eta}_\gamma, \vec{p}_\perp, p_\perp \sinh\hat{\eta}_\gamma), && (65)\\ p_1 &= x_1 P_1 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, 1), && (66)\\ p_2 &= x_2 P_2 = \frac{\sqrt{s}}{2}(1, \vec{0}_\perp, -1). && (67) \end{aligned}$$

The transverse energy that the partons can transfer to the outgoing partons must be less than the partonic center-of-mass energy $\sqrt{s} = \sqrt{x_1 x_2 S}$. This means that

$$z\cosh^2\hat{\eta}_\gamma \leq 1, \tag{68}$$

where we have defined the parton scaling variable

$$z = \frac{Q^2}{s} = \frac{x_\perp}{x_1 x_2}, \tag{69}$$

with, as in chapter 4, $Q^2 = 4p_\perp^2$. Eq.(68) implies that the upper and lower boundaries for the partonic center-of-mass pseudorapidity are given by

$$\hat{\eta}_- \leq \hat{\eta}_\gamma \leq \hat{\eta}_+, \tag{70}$$

where

$$\hat{\eta}_\pm = \ln\left(\frac{1}{\sqrt{z}} \pm \sqrt{\frac{1}{z} - 1}\right) = \pm\ln\left(\frac{1}{\sqrt{z}} + \sqrt{\frac{1}{z} - 1}\right). \tag{71}$$

Using Eq.(64), we can rewrite the transverse energy condition Eq.(68) as a condition for the lower bound of x_2:

$$x_2 \geq \frac{x_1\sqrt{x_\perp}e^{-\eta_\gamma}}{2x_1 - \sqrt{x_\perp}e^{\eta_\gamma}} \equiv x_2^0. \tag{72}$$

Now, the requirement that $x_2 \leq 1$ implies the lower bound for x_1:

$$x_1 \geq \frac{\sqrt{x_\perp}e^{\eta_\gamma}}{2 - \sqrt{x_\perp}e^{-\eta_\gamma}} \equiv x_1^0. \tag{73}$$

The upper and lower bounds of the hadronic center-of-mass pseudorapidity can be found with the obvious condition that $x_{1(2)}^0 \leq 1$. In this way, we find,

$$\eta_- \leq \eta_\gamma \leq \eta_+, \tag{74}$$

where

$$\eta_\pm = \ln\left(\frac{1}{\sqrt{x_\perp}} \pm \sqrt{\frac{1}{x_\perp} - 1}\right) = \pm\ln\left(\frac{1}{\sqrt{x_\perp}} + \sqrt{\frac{1}{x_\perp} - 1}\right). \tag{75}$$

5.3.2 Mellin-Fourier transform and all-order resummation

The expression with the factorization of collinear singularities of this cross section in perturbative QCD is

$$\begin{aligned} p_\perp^3 \frac{d\sigma}{dp_\perp d\eta_\gamma}(x_\perp, p_\perp^2, \eta_\gamma) &= \sum_{a,b} \int_{x_1^0}^1 dx_1 \int_{x_2^0}^1 dx_2 x_1 F_a^{H_1}(x_1, \mu^2) x_2 F_b^{H_2}(x_2, \mu^2) \\ &\times C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma\right) \delta(x_\perp - z x_1 x_2), \end{aligned} \tag{76}$$

where $F_a^{H_1}(x_1, \mu^2)$, $F_b^{H_2}(x_2, \mu^2)$ are the distribution functions of partons a, b in the colliding hadrons and μ^2 is the factorization scale equal to the renormalization scale. $\hat{\eta}_\gamma$ is a function of η_γ, x_1 and x_2 as defined by Eq.(64). The coefficient function $C_{ab}(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma)$ is defined in terms of the partonic cross section for the process where partons a, b are incoming as

$$C_{ab}\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma\right) = p_\perp^3 \frac{d\hat{\sigma}_{ab}}{dp_\perp d\eta_\gamma}. \tag{77}$$

To allow the Mellin transform to deconvolute Eq.(76), we first perform the Fourier transform with respect to η_γ, thus obtaining

$$\sigma(N, Q^2, M) = \int_0^1 dx_\perp x_\perp^{N-1} \int_{\eta_-}^{\eta_+} d\eta_\gamma p_\perp^3 \frac{d\sigma}{dp_\perp d\eta_\gamma}(x_\perp, p_\perp^2, \eta_\gamma) \tag{78}$$

$$\begin{aligned} &= \sum_{a,b} F_a^{H_1}(N+1+iM/2, \mu^2) F_b^{H_2}(N+1-iM/2, \mu^2) \\ &\times C_{ab}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right). \end{aligned} \tag{79}$$

where

$$F_c^{H_i}(N+1 \pm iM/2, \mu^2) = \int_0^1 dx x^{N \pm iM/2} F_i^{H_i}(x, \mu^2), \tag{80}$$

$$\begin{aligned} C_{ab}\left(N, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), M\right) &= 2 \int_0^1 dz z^{N-1} \int_0^{\hat{\eta}_+} d\hat{\eta}_\gamma \cos(M \hat{\eta}_\gamma) \\ &\times \; C\left(z, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \hat{\eta}_\gamma\right). \end{aligned} \tag{81}$$

A resummed expression of Eq.(83) in the threshold limit for the transverse energy ($z \to 1$ or equivalently $N \to \infty$) is obtained in the same way as we have done at the beginning of section 5.1.4, since the upper integration bound of $\hat{\eta}_\gamma$ in Eq.(81) is

$$\hat{\eta}_+ = \ln\left(\frac{1}{\sqrt{z}} + \sqrt{\frac{1}{z} - 1}\right) = \sqrt{1-z} - (1-z) + O((1-z)^{3/2}). \tag{82}$$

Thus, up to terms suppressed by factors $1/N$, the resummed expression of Eq.(83) is:

$$\begin{aligned}\sigma^{res}(N,Q^2,M) &= \sum_{a,b} F_a^{H_1}(N+1+iM/2,\mu^2)F_b^{H_2}(N+1-iM/2,\mu^2) \\ &\quad \times C_{I\,ab}^{res}\left(N,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right), \end{aligned} \tag{83}$$

where $C_{I\,ab}^{res}$ is the resummed pseudorapidity-integrated coefficient function for the prompt photon production for the subprocess which involves the initial partons a, b. These resummed coefficient function has been studied in chapter 4 and in Refs. [14, 65]. This result is analogous to that of the Drell-Yan rapidity distributions case, in the sense that the resummed formula is obtained through a translation of the parton densities' moments by $1 \pm iM/2$ and the pseudorapidity-integrated coefficient functions.