Chapter 2

High order QCD and resummation

2.1 When is NLO not enough?

In section 1.3, we have discussed briefly the analytic NLO calculation of the full inclusive DIS and DY cross sections. However, in many cases, the NLO pQCD computation turns out not to be enough. This is, for example, often the case at LHC where the Higgs boson production has to be distinguished from the background. A computation beyond the NLO is needed also when the NLO corrections are large and higher-order calculation permit us to test the convergence of the perturbative expansion. In figure 2.1 the total cross section of the production of the Higgs boson at LHC [36] is plotted and we note convergence in going from LO to NLO and to NNLO. This can also happen when a new parton level subprocess first appear at NLO. This is the case for example for the rapidity DY distributions at Tevatron (shown in figure 5.4) and at the fixed-target experiment E866/NuSea (shown in figure 2.3). The agreement

Figure 2.1: Total cross section for the Higgs boson production at LHC at (from bottom to top) at LO, NLO, NNLO in the gluon fusion channel [36].

Figure 2.2: DY rapidity distribution for proton anti-proton collisions at Tevatron at (from bottom to top) LO, NLO, NNLO, together with the CDF data [37].

Figure 2.3: DY rapidity distribution for proton proton collisions at fixed-target experiment E866/NuSea at (from bottom to top) LO, NLO, NNLO, together with the data [19, 38].

with the data of figure 5.4 has represented an important test of the NNLO splitting functions [29, 30]. We note also that going from the LO to the NNLO the factorization scale dependence is significantly reduced. Calculations beyond the NLO can be important also in processes which involve large logaritms when different significant scales appear. In these cases, these large logarithms should be resummed and this is the topic of this thesis. A first example of these large logarithms has appeared in section 1.3. In fact, from Eqs.(57,77)of section 1.3, we see that there are contributions that become large when $z \to 1$ from the quark-antiquark channel in the DY case and from the quark channel for the structure function F_2 in the DIS case. These are the terms proportional to

$$\alpha_s \left[\frac{\log(1-z)}{1-z}\right]_+, \quad \alpha_s \left[\frac{1}{1-z}\right]_+ . \tag{1}$$

The terms of the type of Eq.(1) arise from the infrared cancellation between virtual and real emissions. It can be shown that enhanced contributions of the same type arise at all orders. In fact, at order $O(\alpha_s^n)$ there are contributions proportional to [1, 2, 3]:

$$\alpha_s^n \left[\frac{\log^m(1-z)}{1-z}\right]_+, \quad m \leq 2n-1. \tag{2}$$

These terms become important in the limit $z \to 1$ spoiling the validity of the perturbative fixed-order QCD expansion and, thus, should be resummed to all-orders of pQCD. The limit $z \to 1$ corresponds in general to the kinematic boundary where emitted partons are all soft (as it happens in the DY case) or collinear (as in the DIS and the prompt photon case as we shall see in section 4.3). In fact, in the DY case, if we consider a contribution to the coefficient function with n radiated extra partons with momenta $k_1, \ldots, k_n$, the squaring of four-momentum conservation $(p_1 + p_2 = Q + k_1 + \cdots + k_n)$ implies

$$\begin{aligned} x_1 x_2 S(1-z) &= \sum_{i,j=1}^{n} k_i \cdot k_j + 2\sum_{i=1}^{n} Q \cdot k_i & (3)\\ &= \sum_{i,j=1}^{n} k_i^0 k_j^0 (1-\cos\theta_{ij}) + 2\sum_{i=1}^{n} k_i^0 (\sqrt{Q^2 + |\vec{Q}|^2} - |\vec{Q}|\cos\theta_i), & (4)\end{aligned}$$

where θ_{ij} is the angle between $\vec{k}_i$ and $\vec{k}_j$ and θ_i is the angle between $\vec{k}_i$ and $\vec{Q}$. Since all the terms in the first sum of Eq.(4) are positive semi-definite and the terms $(\sqrt{Q^2 + |\vec{Q}|^2} - |\vec{Q}|\cos\theta_i)$ in the second sum are positive for all possible values of θ_i, we have that the limit $z = 1$ is achieve only for $k_i^0 = 0$ for all i. This means that when z approaches 1 all the emitted partons in the Drell-Yan processes are soft and that we have reached the threshold for the production of a virtual photon or a real vector boson. In the DIS case, at the partonic level, we have

$$p + q = k_1 + \cdots + k_n + k_{n+1}, \tag{5}$$

where k_{n+1} is the LO outgoing parton. If we square this last equation, we get

$$\frac{Q^2(1-z)}{z} = \sum_{i,j=1}^{n+1} k_i^0 k_j^0 (1-\cos\theta_{ij}), \tag{6}$$

where θ_{ij} is the angle between $\vec{k}_i$ and $\vec{k}_j$. Eq.(6) tells us that in the $z \to 1$ limit, there can be not only soft radiated partons in the final state, but there can be also a set of partons collinear to each other. However, in Section 3.2 we will show with a more detailed analysis of the DIS kinematics and phase space that the collinear partons are also soft in the $z \to$ limit for the deep-inelastic process. An example of the impact of resummation can be seen in figure 2.4. There, the total cross section for the Higgs production at LHC is plotted at NNLO with its NNLL resummation improvement improvement [39]. The scale this case are of the class of the DY-like soft emissions. Resummation of another class of large logarithms plays a crucial role in transverse

Figure 2.4: Total cross section for the Higgs boson production at LHC at (from bottom to top) at NNLO and NNLO improved with NNLL resummation in the gluon fusion channel [39].

Figure 2.5: Total cross section for the transverse momentum Higgs boson production at LHC at LO and LO improved with NLL resummation in the gluon fusion channel [40].

momentum distributions. Indeed, in figure 2.5, we observe that resummation changes substantially the behavior of the cross section for the production of the Higgs boson at small transverse momentum. In these case the large logarithms of $q_\perp^2/M_H^2$ with M_H the Higgs mass are resummed. The state of art of QCD predictions for Higgs boson production at LHC is reported in figure 2.6 as it was summarized by Laura Reina at the CTEQ summer shool 2006 on QCD analysis and phenomenology, where also the Monte Carlo event generators are indicated. Furthermore, at LHC, multi-particles/jet production will be the inescapable background to Higgs searches and searches for new physics. This means that we should have a precise knowledge of the QCD background. As seen previously, we know many QCD processes up to the NNLO. However, we have at the moment limited NLO knowledge of some important

Figure 2.6: State of art of QCD predictions for Higgs boson production at hadron colliders.

final states that will constitute background. They are

$$\begin{aligned}
&\rightarrow \quad W/Z + \text{jets} \quad (2j) \\
&\rightarrow \quad WW/ZZ/WZ + \text{jets} \quad (0j) \\
&\rightarrow \quad WWW/ZZZ/WZZ + \text{jets} \quad (0j) \\
&\rightarrow \quad Q\bar{Q} + \text{jets} \quad (0j) \\
&\rightarrow \quad \gamma + \text{jets} \quad (1j) \\
&\rightarrow \quad \gamma\gamma + \text{jets} \\
&\rightarrow \quad Z\gamma\gamma + \text{jets},
\end{aligned}$$

where in parenthesis is indicated the NLO knowledge. Finally, we also note that in higher order contributions to the splitting functions (P^1_{gg},P^1_{gq} for example), it can be shown that there can appear also terms proportional to

$$\alpha_s^n \ln^m \frac{1}{z}; \qquad m \leq n. \tag{7}$$

These contributions spoil the convergence when $z \to 0$ and, in order to have reliable predictions, must be resummed. The inclusion of the terms with $m = n$ defines a LL_z resummation, the inclusion of also the terms with $m = n - 1$ defines a NLL_z resummation. This resummation is realized by the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation. Anyway, in this thesis, we will not concentrate on this resummation. We will give a briefly description of the various techniques to resum the large soft logs giving attention to the renormalization group approach and studying in detail its applications.

2.2 The renormalization group approach to resummation

The aim of resummation is to include all the logarithmic enhanced terms of the form of Eq.(2) of Section 2.1 with a certain hierarchy of logarithms that we shall define in the current section. From Eqs.(56,76) of Section 1.3 we see that the QCD cross section (up to dimensional overall factors) can in general be written as a convolution of the parto densities $F_a^{H_i}(x_i,\mu^2)$ and of the dimensionless partonic cross section, i.e. the coefficient function $C(z,Q^2/\mu^2,\alpha_s(\mu^2))$:

$$\sigma_{\rm DY}(x,Q^2)=\sum_{a,b}\left[F_a^{H_1}(\mu^2)\otimes F_b^{H_2}(\mu^2)\otimes C_{ab}(Q^2/\mu^2,\alpha_s(\mu^2))\right](x), \tag{8}$$

for the DY case; and

$$\sigma_{\rm DIS}(x,Q^2)=\sum_{a}\left[F_a^{H}(\mu^2)\otimes C_{a}(Q^2/\mu^2,\alpha_s(\mu^2))\right](x), \tag{9}$$

for the DIS case. The convolution product $\otimes$ has been defined in Eq.(85) of Section 1.4. Performing the Mellin transformation

$$\sigma(N,Q^2)=\int_0^1 dx\, x^{N-1}\sigma(x,Q^2) \tag{10}$$

we turn the convolution products of Eqs.(8,9) into ordinary products:

$$\begin{aligned}\sigma_{\rm DY}(N,Q^2) &\equiv \sum_{a,b}\sigma_{ab}(N,Q^2)\\ &= \sum_{a,b}F_a^{H_1}(N,\mu^2)F_b^{H_2}(N,\mu^2)C_{ab}(N,Q^2/\mu^2,\alpha_s(\mu^2)), \qquad (11)\\ \sigma_{\rm DIS}(N,Q^2) &\equiv \sum_a\sigma_a(N,Q^2)=\sum_a F_a^H(N,\mu^2)C_a(N,Q^2/\mu^2,\alpha_s(\mu^2)), \qquad (12)\end{aligned}$$

where

$$\begin{aligned}C_{a(b)}\left(N,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right) &= \int_0^1 dz z^{N-1}C_{a(b)}\left(z,\frac{Q^2}{\mu^2},\alpha_s(\mu^2)\right), \qquad (13)\\ F_a^{H_j}(N,\mu^2) &= \int_0^1 dx x^{N-1}F_a^{H_j}(x,\mu^2), \qquad (14)\end{aligned}$$

and where the second index in brackets (b) is involved only when there are two hadrons in the initial state as the the DY case. The large logs of $1-z$ of Eq.(2) in Section 2.1 are mapped to the large logs of N by the Mellin transform. This fact and the relations between the large logs of $1-z$ and the large logs of N are shown in detail in Appendix A. Whereas the cross section $\sigma(N,Q^2)$ is clearly μ^2-independent, this is not the case for each contribution $\sigma_{a(b)}(N,Q^2)$. However, the μ^2 dependence of each contribution to the sum over $a,(b)$ in Eqs.(11,12) is proportional to the off-diagonal anomalous dimensions γ_{qg} and γ_{gq}. In the large N limit, these are suppressed by a power of $\frac{1}{N}$ in comparison to γ_{gg} and γ_{qq}, or, equivalently, the corresponding splitting functions are suppressed by a factor of $1-z$ in the large z limit (see for example Eqs.(83,84) in Section 1.4). Hence, in the large N limit each parton subprocess can be treated independently, specifically, each $C_{a(b)}$ is separately renormalization-group invariant. Because we are interested in the behaviour of $C_{a(b)}(N, Q^2/\mu^2, \alpha_s(\mu^2))$ in the limit $N \to \infty$ we can treat each subprocess independently. Because resummation takes the form of an exponentiation, we define a so-called physical anomalous dimension defined implicitly through the equation

$$Q^2 \frac{\partial \sigma_{a(b)}(N, Q^2)}{\partial Q^2} = \gamma_{a(b)}(N, \alpha_s(Q^2))\, \sigma_{a(b)}(N, Q^2). \tag{15}$$

The physical anomalous dimensions $\gamma_{a(b)}$ Eq.(15) is independent of factorization scale, and it is related to the diagonal standard anomalous dimension γ_{cc}^{AP}, defined by

$$\mu^2 \frac{\partial F_c(N, \mu^2)}{\partial \mu^2} = \gamma_{cc}^{\mathrm{AP}}(N, \alpha_s(\mu^2)) F_c(N, \mu^2), \tag{16}$$

according to

$$\gamma_{a(b)}(N, \alpha_s(Q^2)) \;=\; \frac{\partial \ln C_{a(b)}(N, Q^2/\mu^2, \alpha_s(\mu^2))}{\partial \ln Q^2} = \gamma_{aa}^{\mathrm{AP}}(N, \alpha_s(Q^2)) \tag{17}$$

$$+\gamma_{(bb)}^{\mathrm{AP}}(N, \alpha_s(Q^2)) + \frac{\partial \ln C_{a(b)}(N, 1, \alpha_s(Q^2))}{\partial \ln Q^2}. \tag{18}$$

We recall that both the standard anomalous dimensions (Altarelli-Parisi splitting functions) and the coefficient function are computable in perturbation theory. Hence, the physical anomalous dimensions differs from the standard anomalous dimensions only beyond the LO in α_s as can be seen directly from Eq.(18). In terms of the physical anomalous dimensions, the cross section can be written as

$$\sigma(N, Q^2) \;=\; \sum_{a,(b)} K_{a(b)}(N; Q_0^2, Q^2)\, \sigma_{a(b)}(N, Q_0^2) \tag{19}$$

$$= \sum_{a,(b)} \exp\left[E_{a(b)}(N; Q_0^2, Q^2)\right] \sigma_{a(b)}(N, Q_0^2), \tag{20}$$

where

$$E_{a(b)}(N; Q_0^2, Q^2) \;=\; \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma_{a(b)}(N, \alpha_s(k^2)) \tag{21}$$

$$= \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} [\gamma_{aa}^{\mathrm{AP}}(N, \alpha_s(k^2)) + \gamma_{(bb)}^{\mathrm{AP}}(N, \alpha_s(k^2))]$$

$$+ \ln C_{a(b)}(N, 1, \alpha_s(Q^2)) - \ln C_{a(b)}(N, 1, \alpha_s(Q_0^2)). \tag{22}$$

We now concentrate on the single subprocess with incoming partons $a, (b)$. Resummation of the large logs of N in the cross section is obtained performing their resummation in the physical anomalous dimension:

$$\sigma^{res}(N, Q^2) = \exp\left\{ \int_{Q_0^2}^{Q^2} \frac{dk^2}{k^2} \gamma^{res}(N, \alpha_s(k^2)) \right\} \sigma^{res}(N, Q_0^2). \tag{23}$$

This shows how in general the large logs of N can be exponentiated. For the DY case only the quark-anti-quark channel should be resummed and in the DIS case only the quark one. This is a consequence of the fact that the off-diagonal splitting functions are suppressed in the large N limit as discussed before. The accuracy of resummation here depends on the accuracy at which the physical anomalous dimension γ is computed. We say that the physical anomalous dimension is resummed at the next^{k-1}-to-leading-logarithmic accuracy ($N^{k-1}LL$) when all the contributions of the form

$$\alpha_s^{n+m}(Q^2)\ln^m N; \qquad n = 0, \dots, k-1 \tag{24}$$

are included in its determination. The goal of resummation is to determine the resummed physical anomalous dimension from at most a finite fixed-order computation of it. Clearly, once the resummed physical anomalous dimension is determined, it can then predict the leading, next-to-leading...logarithmic contributions to the cross section at all orders. Here, in this Section, we shall only outline the key ideas of the renormalization group approach to resummation. However, throughout this thesis we shall show in detail how this method works and how the resummed physical anomalous dimensions obtained with such an approach can be fully determined expanding it to a certain finite fixed-order and comparing this expansion with Eq.(18) obtained from explicit computations. The renormalization group approach to resummation is essentially divided in two steps. The first is to analyze the generic phase space measure in $d = 4 - 2\epsilon$ dimensions thus finding where the large logs are originated in the coefficient function and in the physical anomalous dimension. The second consists in resumming them imposing the renormalization group invariance of the physical anomalous dimension. So, let's consider a generic phase space measure $d\phi_n$ for the emission of n massless partons with momenta $p_1, \dots, p_n$. In Appendix B, we show that this phase space can be decomposed in terms of two-body phase space. Roughly speaking, the phase spaces measure for the emission of n partons can be viewed as the emission of two partons (one with momentum p_n and the other with momentum $P_n = p_1 + \dots +_{n-1}$ and invariant mass P_n^2 times the phase space measure where the momentum P_n is incoming and the momenta $p_1, \dots, p_n$ are outgoing. The price to pay for this is the introduction of an integration over the invariant mass P_n^2. Then using recursively this procedure, we obtain that the n-body phase space measure is decomposed in $n-1$ two-body phase space measures. This means that we have reduced the study of the soft emission of n-body phase space measure to the study of the soft emission of two-body phase spaces. The two-body phase space with an incoming momentum P and two outgoing momenta Q and p in $d = 4 - 2\epsilon$ dimensions (derived explicitly in Eq.(16) of Section B) is given by

$$d\phi_2(P; Q, p) = N(\epsilon)(P^2)^{-\epsilon}\left(1 - \frac{Q^2}{P^2}\right)^{1-2\epsilon} d\Omega_{d-1}, \quad N(\epsilon) = \frac{1}{2(4\pi)^{2-2\epsilon}}, \tag{25}$$

where $d\Omega_{d-1}$ is the solid angle in $d-1$ dimensions. For definiteness, we can think that this is the phase space measure for a single DIS-like or DY-like soft emission with with momentum p. Thus, we have

$$\text{DIS-like emission:} \qquad P^2 \propto (1 - z_{_{DIS}}); \quad Q^2 = 0 \tag{26}$$

$$\text{DY-like emission:} \qquad \left(1 - \frac{Q^2}{P^2}\right) \propto (1 - z_{_{DY}}); \quad P^2 = s_{_{DY}}, \tag{27}$$

where $z_{_{DIS}}$, $z_{_{DY}}$ are close to one for a soft emission. Hence, we have that the two-body phase space measure for a single soft emission contributes with a factor $(1-z)^{-a\epsilon}$ with $a = 1$ for a DIS-like emission and $a = 2$ for DY-like emission. In the case of the prompt-photon process, we will see in Chapter 4 that there are both types of emission. The large logs of $1 - z$ are originated by the interference with the infrared poles in $\epsilon = 0$ in the square modulus amplitude in the $\epsilon \to 0$ limit. For example

$$\frac{1}{\epsilon}(1-z)^{-a\epsilon} = \frac{1}{\epsilon} - \ln(1-z)^a + O(\epsilon), \tag{28}$$

for the case of interference with a pole of order 1. Now, since each factor of $(1-z)^{-a\epsilon}$ that comes from the phase space measure is associated to a single real emission then it will appear in the coefficient function together with a power of the bare strong coupling constant α_0. In d-dimensions, the coupling constant is dimensionful, and thus on dimensional grounds each emission is accompanied by a factor

$$\alpha_0 \left[Q^2(1-z)^a\right]^{-\epsilon}, \tag{29}$$

where Q^2 is now the typical perturbative scale of a certain process. Upon Mellin transformation, this becomes

$$\alpha_0 \left[\frac{Q^2}{N^a}\right]^{-\epsilon}. \tag{30}$$

Furthermore an analysis of the structure of diagrams shows that in the soft (large N) limit, all dependence on N appears through the variable Q^2/N^a also in the amplitude. Finally, a renormalization group argument shows that all this dependence can be reabsorbed in the running of the strong coupling. Indeed, the first order renormalization of the bare coupling constant at the renormalization scale μ

$$\alpha_0 = \mu^{2\epsilon}\alpha_s(\mu^2) + O(\alpha_s^2) \tag{31}$$

and the renormalization group invariance of the physical anomalous dimension imply that

$$\begin{aligned} \alpha_0 \left[\frac{Q^2}{N^a}\right]^{-\epsilon} &= \alpha_s(\mu^2)\left[\frac{Q^2}{\mu^2 N^a}\right]^{-\epsilon} + O(\alpha_s^2) \\ &= \alpha_s(Q^2/N^a) + O(\alpha_s^2), \end{aligned} \tag{32}$$

where α_0 is the bare coupling, $\alpha_s(\mu^2)$ the renormalized coupling and the higher order terms contain divergences which cancel those in the cross section. Following this line of argument one may show that the finite expression of the renormalized cross section in terms of the renormalized coupling is a function of $\alpha_s(Q^2)$ and $\alpha_s(Q^2/N^a)$ with numerical coefficients, up to $O(1/N)$ corrections. We shall see this in detail in Chapter 3.

2.3 Alternative approaches

The exponentiation of the large soft logs and their resummation has been demonstrated in QCD with the eikonal approximation [2] or thanks to strong non-standard factorization properties of the cross section in the soft limit [1]. Recently, also the effective field theoretic (EFT) approach has been applied to QCD resummation in Refs.[9, 10] for DIS and DY and in Ref.[41] for the B meson decay $B \to X_s \gamma$. In this Section, we shall only give a brief description of these alternative approaches to the resummation of the large perturbative logarithms.

2.3.1 Eikonal approach

We first consider the simpler case of QED. In QED the exponentiation of the large soft logs has been proved thanks to the eikonal approximation in Ref.[7]. We report the basic steps of the proof for the QED case and a brief description of the generalization to the QCD case. Consider a final fermion line with momentum p' of a generic QED Feynman diagram. We attach n soft photons to this fermion line with momenta $k_1, \ldots, k_n$. For the moment we do not care whether these are external photons, virtual photons connected to each other, or virtual photons connected to vertices on other fermion lines. The amplitude for such a diagram has the following structure in the soft limit:

$$\bar{u}(p')(-ie\gamma^{\mu_1})\frac{i\not{p}'}{2p'\cdot k_1}(-ie\gamma^{\mu_2})\frac{i\not{p}'}{2p'\cdot(k_1+k_2)}\cdots(-ie\gamma\mu_n)\frac{i\not{p}'}{2p'\cdot(k_1+\cdots+k_n)}i\mathcal{M}_h, \quad (33)$$

where $e = -|e|$ is the electron charge and $i\mathcal{M}_h$ is the amplitude of the hard part of the process without the final fermion line we are considering. We note that here we have neglected the electron mass. Then we can push the factors of $\not{p}'$ to the left and use the Dirac equation $\bar{u}(p')\not{p}' = 0$:

$$\bar{u}(p')\gamma^{\mu_1}\not{p}'\gamma^{\mu_2}\not{p}'\cdots\gamma^{\mu_n}\not{p}' = \bar{u}(p')2p'^{\mu_1}\gamma^{\mu_2}\not{p}'\cdots\gamma^{\mu_n}\not{p}' = \bar{u}(p')2p'^{\mu_1}2p'^{\mu_2}\cdots 2p'^{\mu_n}. \quad (34)$$

Thus Eq.(33) becomes

$$e^n\bar{u}(p')\left(\frac{p'^{\mu_1}}{p'\cdot k_1}\right)\left(\frac{p'^{\mu_2}}{p'\cdot(k_1+k_2)}\right)\cdots\left(\frac{p'^{\mu_n}}{p'\cdot(k_1+\cdots+k_n)}\right)i\mathcal{M}_h. \quad (35)$$

Still working with only a final fermion line, we must now sum over all possible orderings of momenta $k_1, \ldots, k_n$. There are $n!$ different diagrams to sum, corresponding to the $n!$ permutations of the n photon momenta. Let P denote one such permutation, so that $P(i)$ is the number between 1 and n that i is taken to. Now, using the identity

$$\sum_P \frac{1}{p\cdot k_{P(1)}}\frac{1}{p\cdot(k_{P(1)}+k_{P(2)})}\cdots\frac{1}{p\cdot(k_{P(1)})+\cdots+k_{P(n)})} = \frac{1}{p\cdot k_1}\cdots\frac{1}{p\cdot k_n}, \quad (36)$$

the sum over all the permutations of the photons of Eq.(35) is:

$$e^n\bar{u}(p')\left(\frac{p'^{\mu_1}}{p'\cdot k_1}\right)\left(\frac{p'^{\mu_2}}{p'\cdot k_2}\right)\cdots\left(\frac{p'^{\mu_n}}{p'\cdot k_n}\right)i\mathcal{M}_h. \quad (37)$$

At this point, we consider an initial fermion line with momentum p. In this case the photon momenta in the denominators of the fermion propagators have an opposite sign. Therefore, if we sum over all the diagrams containing a total of n soft photons, connected in any possible order to an arbitrary number of initial and final fermion lines, Eq.(37) becomes:

$$e^n i\mathcal{M}_0 \prod_{r=1}^{n} \sum_i \frac{\eta_i p^{\mu_i}}{p_i \cdot k_r}, \tag{38}$$

where $i\mathcal{M}_0$ is the full amplitude of the hard part of the process and where the index r runs over the radiated photons and the index j runs over the initial and final fermion lines with

$$\eta_i = \begin{cases} 1 & \text{for a final fermion line} \\ -1 & \text{for an initial fermion line} \end{cases} \tag{39}$$

If only a real soft photon is radiated, we must multiply by its polarization vector, sum over polarizations, and integrate the squared matrix element over the photon phase space. In the Feynman gauge this gives a factor

$$Y = \int \frac{d^3k}{(2\pi)^3 2k^0} e^2 \left(\sum_i \frac{\eta_i p_i}{p_i \cdot k} \right)^2 \tag{40}$$

in the final cross section. If n real photons are emitted, we get n such Y factors Eq.(40), and also a symmetry factor $1/n!$ since there are n identical bosons in the final state. The cross section resummed for the emission of any number of soft photons is therefore

$$\sigma^{res}(i \to f) = \sigma_0(i \to f) \sum_{n=0}^{\infty} \frac{Y^n}{n!} = \sigma_0(i \to f) e^Y, \tag{41}$$

where $\sigma_0(i \to f)$ is the cross section for the hard process without extra soft emissions. This result shows that all the possible soft real emissions exponentiate and that only the single emission contributes to the exponent. However, this is not the end of the story, because the exponent Y Eq.(40) is infrared divergent. Indeed, to obtain a reliable finite result, we must include also loop corrections to all orders. For a detailed analysis about the inclusion of loops see for example Ref.[42]. Here, we just give the final result which reads:

$$\sigma^{res}(i \to f) = \sigma_0(i \to f) e^{\sigma^{(1)}}, \tag{42}$$

where $\sigma^{(1)}$ is the cross section relative to the single soft emission from the hard process. Clearly, the accuracy of this resummation formula for soft photon emission Eq.(42) depends on the accuracy at which the exponent for the single emission is computed. In Ref.[2] the exponentiation of the soft emissions, here outlined for QED, is generalized to the QCD case. Differently from QED, QCD is a non-abelian gauge theory and this implies that this generalization is highly non-trivial. Indeed, the gluons can interact with each other. This fact makes the exponentiation mechanism much more difficult since the three gluon vertex color factor is different from that of the quark-gluon vertex. In order to exponentiate the single emission cross section (as it happens in

QED), one should prove that these gluon correlations cancels out order by order in perturbation theory. This is shown for example in Ref.[43]. According to this result, it has been shown in Ref.[2] how the the exponentiation of soft emission works in QCD resummation. We report here the result for the NLL resummed coefficient function in Mellin space for inclusive DIS and DY processes in the $\overline{MS}$ scheme in a compact form:

$$\begin{aligned} C_{\rm NLL}(N, Q^2/\mu^2, \alpha_s(\mu^2)) &= \exp\left\{ a \int_0^1 dx \, \frac{x^{N-1}-1}{1-x} \left[\int_{\mu^2}^{Q^2(1-x)^a} \frac{dk^2}{k^2} A(\alpha_s(k^2)) \right.\right. \\ &\quad \left.\left. + B^{(a)}(\alpha_s(Q^2(1-x)^a)) \right] \right\}, \end{aligned} \tag{43}$$

where

$$A(\alpha_s) = A_1 \alpha_s + A_2 \alpha_s^2 + \dots \tag{44}$$

$$B^{(a)}(\alpha_s) = B_1^{(a)} \alpha_s + \dots \tag{45}$$

with

$$A_1 = \frac{C_F}{\pi}, \qquad A_2 = \frac{C_F}{2\pi^2}\left[C_A \left(\frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{5}{9} N_f \right], \qquad B_1^{(a)} = -\frac{(2-a)3C_F}{4\pi}. \tag{46}$$

Here $a = 1$ for the DIS structure functio F_2 and $a = 2$ for the DY case. How this result is strictly connected to the resummed results that can be obtained with the renormalization group approach will be discussed in Section 3.4.

2.3.2 Resummation from strong factorization properties

This is the approach of Ref.[1]. Also in this approach the results given in Eq.(43) of Section 2.3.1 are recovered. Here we give only a rough description of this method based on strong factorization properties of the QCD cross section. It is essentially assumed that at the boundary of the phase space, the cross section is factorized in a hard and in a soft part and eventually in an other factor associated to final collinear jets as in the DIS case where there is an outgoing emitting quark. The final result is then obtained exponentiating the soft and collinear factors. This is done solving their evolution equations. In Ref.[1] it is shown that the semi-inclusive cross section can be factorized in three factors relative to the three different regions in the momentum space of the process: the off-shell partons that participate to the partonic hard process, the collinear and soft on-shell radiated partons. The cross section is given by

$$\begin{aligned} \sigma(w) &= H\left(\frac{p_1}{\mu}, \frac{p_2}{\mu}, \zeta_i \right) \int \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{dw_3}{w_3} J_1 \left(\frac{p_1 \cdot \zeta_1}{\mu}, w_1 \left(\frac{Q}{\mu} \right)^a \right) \\ &\quad J_2 \left(\frac{p_2 \cdot \zeta_2}{\mu}, w_2 \left(\frac{Q}{\mu} \right)^a \right) S\left(w_s \frac{Q}{\mu}, \zeta_i \right) \delta(w - w_1 - w_2 - w_3), \end{aligned} \tag{47}$$

where a is the number of hadrons in the initial state, μ is the factorization scale and ζ_i are gauge-fixing parameters; the integration variables w_1, w_2 and w_3 are referred to the two collinear jets and to soft radiation respectively. Each factor of Eq.(47) is evaluated at the typical scale of the momentum space region which is associated to. The delta function imposes that

$$w = w_1 + w_2 + w_3 = \begin{cases} 1 - x_{Bj}, & \text{for the DIS case} \\ 1 - Q^2/S & \text{for the DY case} \end{cases} \tag{48}$$

The convolution of Eq.(47) is turned into an ordinary product performing the Mellin transform:

$$\begin{aligned} \sigma(N) &= \int_0^\infty dw\, e^{-Nw}\sigma(w) = H\left(\frac{p_1}{\mu}, \frac{p_2}{\mu}, \zeta_i\right) S\left(\frac{Q}{\mu N}, \zeta_i\right) \\ &\times J_1\left(\frac{p_1 \cdot \zeta_1}{\mu}, \frac{Q}{\mu N^{1/a}}\right) J_2\left(\frac{p_2 \cdot \zeta_2}{\mu}, \frac{Q}{\mu N^{1/a}}\right). \end{aligned} \tag{49}$$

Each factor H, J_i, S satisfy the following evolution equations

$$\mu^2 \frac{\partial}{\partial \mu^2} \ln H = -\gamma_H(\alpha_s(\mu^2)), \tag{50}$$

$$\mu^2 \frac{\partial}{\partial \mu^2} \ln S = -\gamma_S(\alpha_s(\mu^2)), \tag{51}$$

$$\mu^2 \frac{\partial}{\partial \mu^2} \ln J_i = -\gamma_{J_i}(\alpha_s(\mu^2)), \tag{52}$$

where the physical anomalous dimensions $\gamma_H(\alpha_s)$, $\gamma_{J_i}(\alpha_s)$ and $\gamma_S(\alpha_s)$ are calculable in perturbation theory and must satisfy, according to the renormalization group invariance of the cross section the relation

$$\gamma_H(\alpha_s) + \gamma_S(\alpha_S) + \sum_{i=1}^{2} \gamma_{J_i}(\alpha_s) = 0. \tag{53}$$

Solving Eqs.(50-52) and imposing renormalization group invariance Eq.(53), the resummed section can be written in the form

$$\begin{aligned} \sigma(N) &= \exp\left\{ D_1(\alpha_s(Q^2)) + D_2\left(\alpha_s\left(\frac{Q^2}{N^2}\right)\right) - \frac{2}{a-1}\int_{\frac{Q}{N}}^{\frac{Q}{N^{1/a}}} \frac{d\xi}{\xi} \ln\left(\frac{\xi N}{Q}\right) A(\alpha_s \xi^2) \right. \\ &\left. -2\int_{\frac{Q}{N^{1/a}}}^{Q^2} \frac{d\xi}{\xi}\left[\ln\left(\frac{Q}{\xi}\right) A(\alpha_s(\xi^2)) - B(\alpha_s(\xi^2))\right] \right\}, \end{aligned} \tag{54}$$

where the functions $A(\alpha_s)$, $B(\alpha_s)$, $D_i(\alpha_s)$ are determined in terms of the anomalous dimensions and the beta function. Finally, it can be shown that this result can be casted in the form of Eq.(43) in Section 2.3.1 for the resummed coefficient function.

2.3.3 Resummation from Effective Field Theory

This is the approach of Refs.[9, 10]. This EFT methodology to resum threshold logarithms is made concrete due to the recently developed "soft collinear effective theory" (SCET) [44, 45, 46, 47]. The SCET describes interactions between soft and collinear partons. The starting point (considering the DY case as an example) is the collinearly factorized inclusive cross section in Mellin space:

$$\sigma(N,Q^2) = \sigma_0\, C(N,Q^2/\mu^2,\alpha_s(\mu^2))F_1(N,\mu^2)F_2(N,\mu^2), \tag{55}$$

where σ_0 is the born level cross section and $F_i(N,\mu^2)$ are the parton densities at the factorization scale μ. Here, the basic idea is to write the coefficient function $C(N,Q^2/\mu^2,\alpha_s(\mu^2))$ in terms of an intermediate scale μ_I:

$$C(N,Q^2/\mu^2,\alpha_s(\mu^2)) = C(N,Q^2/\mu_I^2,\alpha_s(\mu_I^2))\exp\left[-2\int_{\mu_I^2}^{\mu^2}\frac{dk^2}{k^2}\gamma_{qq}^{\mathrm{AP}}(N,\alpha_s(k^2))\right] \tag{56}$$

and then to compute $C(N,Q^2/\mu_I^2,\alpha_s(\mu_I^2))$ with the "soft collinear effective theory" with the intermediate scale μ_I^2 equal to the typical scale of the soft-collinear emission, i.e. $\mu_I = Q^2/N^2$ in the DY case and $\mu_I^2 = Q^2/N$ in the DIS case. In SCET, $C(N,Q^2/\mu_I^2,\alpha_s(\mu_I^2))$ has the following general structure:

$$C\left(N,\frac{Q^2}{\mu_I^2},\alpha_s(\mu_I^2)\right) = \left|\tilde{C}\left(\frac{Q^2}{\mu_I^2},\alpha_s(\mu_I^2)\right)\right|^2 \mathcal{M}(N,\alpha_s(\mu_I^2)), \tag{57}$$

Here $\tilde{C}(Q^2/\mu_I^2,\alpha_s(\mu_I^2))$ is the effective coupling that matches the full QCD theory currents

$$J_{QCD} = \tilde{C}(Q^2/\mu_I^2,\alpha_s(\mu_I^2))J_{eff}(\mu_I^2). \tag{58}$$

We note that the effective coupling $\tilde{C}$ contains the perturbative contribution between the scale Q^2 and μ_I^2 and $J_{eff}(\mu_I^2)$ contains the soft and collinear contributions below the scale μ_I^2. Then $\mathcal{M}(N,\alpha_s(\mu_I^2)$ is the matching coefficient that guarantees that the EFT used generates the full QCD results in the appropriate kinematical limit. In SCET the matching coefficient $\mathcal{M}$ can be computed perturbativly and is free of any logarithms. The effective coupling $\tilde{C}$ satisfy to a certain evolution equation

$$\mu^2\frac{\partial}{\partial\mu^2}\ln\tilde{C}(Q^2/\mu^2,\alpha_s(\mu^2)) = -\frac{1}{2}\gamma_1(\alpha_s(\mu^2)) \tag{59}$$

where the physical anomalous dimension γ_1 is computable perturbativly in SCET. Finally, solving the evolution equation Eq.(59) one finds that Eq.(57) becomes

$$\begin{aligned} C\left(N,\frac{Q^2}{\mu_I^2},\alpha_s(\mu_I^2)\right) &= \left|\tilde{C}(1,\alpha_s(Q^2))\right|^2 e^{E_1(Q^2/\mu_I^2,alpha_s(Q^2))}\times \\ &\quad\times\mathcal{M}(N,\alpha_s(Q^2))e^{E_2(Q^2/\mu_I^2,\alpha_s(Q^2))}, \end{aligned} \tag{60}$$

where

$$E_1\left(\frac{Q^2}{\mu_I^2},\alpha_s(Q^2)\right) = -\frac{1}{4}\int_{\mu_I^2}^{Q^2}\frac{dk^2}{k^2}\gamma_1(\alpha_s(k^2)), \tag{61}$$

$$E_2\left(\frac{Q^2}{\mu_I^2},\alpha_s(Q^2)\right) = \int_{\mu_I^2}^{Q^2}\frac{dk^2}{k^2}\beta(\alpha_s(k^2))\frac{d\ln\mathcal{M}(N,\alpha_s(k^2))}{d\ln\alpha_s(k^2)}, \tag{62}$$

with $\beta(\alpha_s)$ the beta function of Eq.(43) of Section 1.2. $\tilde{C}(1, \alpha_s(Q^2))$ contains the non-logarithmic contribution of the purely virtual diagrams and the exponent E_1 contains all the logarithms originating from the same type of diagrams. E_2 encodes all the contributions due to the running of the coupling constant between the scale μ_I^2 and Q^2. All the logarithms appear only in the exponents (see Eqs.(56,60)) and the term $|\tilde{C}(1, \alpha_s(Q^2))|^2 \mathcal{M}(N, \alpha_s(Q^2))$ is free of any large logarithms. The various approaches can be related one to the other according to factorization properties of the QCD cross section in the soft limit. In this way, it is possible to show that all the approaches are equivalent except for the renormalization group approach that produces correct but less restrictive results.