Bachelor of Science in Computer Science
University of Colombo School of Computing

SCS 1211 – Mathematical Methods I (Linear Algebra)

Topic -1: Introduction

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Introduction

The fundamental problem of linear algebra is to solve m linear equations in n unknowns:

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\vdots \\
&a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m;
\end{aligned}
$$

In this first lecture, we view this problem in three different ways.

Solving System of Linear equations: Example 2.1 (Gaussian Elimination)

Solve the following system of linear equations:

$$x_1 + 2x_2 + x_3 = 4 -----(1)$$

$$2x_1 + 2x_2 + 3x_3 = 7 -----(2)$$

$$x_1 + x_2 + 4x_3 = 6 -----(3)$$

Solution:

Step 1: Choose to pivot x_1 in (1), and eliminate x_1 in (2) & (3).

$$x_1 + 2x_2 + x_3 = 4 -----(1')$$

$$(1) \times -2 + (2) \Longrightarrow \quad -2x_2 + x_3 = -1 ---- (2')$$

$$(1) \times -1 + (3) \Longrightarrow \quad -x_2 + 3x_3 = 2 -----(3')$$

Example 2.1 Cont.

$$x_1 + 2x_2 + x_3 = 4 ------(1')$$
$$-2x_2 + x_3 = -1 ----- (2')$$
$$-x_2 + 3x_3 = 2 ------(3')$$

Step 2: Choose to pivot x_2 in (2'), and eliminate x_2 in (3').

$$x_1 + 2x_2 + x_3 = 4 ------(1'')$$
$$-2x_2 + x_3 = -1 ----- - (2'')$$
$$(2') \times -\frac{1}{2} + (3') \Longrightarrow \quad \frac{5}{2}\, x_3 = \frac{5}{2} ------(3'')$$

Triangular System

Example 2.1 (Back-substitution)

$$x_1 + 2x_2 + x_3 = 4 ----- -(1'')$$

$$-2x_2 + x_3 = -1 ---- - - (2'')$$

$$\frac{5}{2}x_3 = \frac{5}{2} ----- -(3'')$$

Step 3: Back-substitution

$$(3'') \implies x_3 = 1$$

$$(2'') \implies x_2 = 1$$

$$(1'') \implies x_1 = 1\,.$$

Example 1.2 (Singular system)

$$x_1 + x_2 + x_3 = 4 ------(1)$$
$$2x_1 + 2x_2 + 5x_3 = 7 ------(2)$$
$$4x_1 + 4x_2 + 8x_3 = 6 ------(3)$$

$$x_1 + x_2 + x_3 = 4 ------(1')$$
$$(1) \times -2 + (2) \Longrightarrow \quad +3x_3 = -1 ----- (2')$$
$$(1) \times -4 + (3) \Longrightarrow \quad +4x_3 = 10 ------(3')$$

This system is unsolvable since $3x_3 = -1$ & $4x_3 = 10$ is not possible.

Representation in Matrix Form:

Solve the following system of linear equations:

$$x_1 + 2x_2 + x_3 = 4 -----(1)$$

$$2x_1 + 2x_2 + 3x_3 = 7 -----(2)$$

$$x_1 + x_2 + 4x_3 = 6 -----(3)$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}.$$

$$A\,x = b.$$

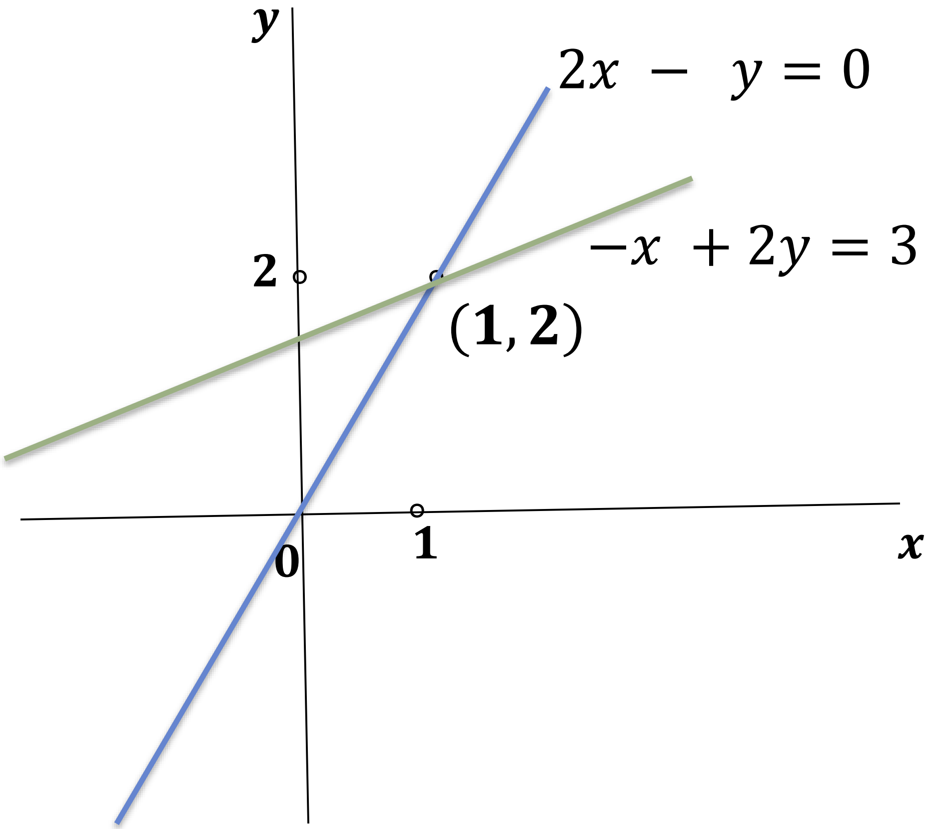
The matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$ is called the coefficient matrix. The vector $\mathrm{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the vector of unknowns.

The Geometry of Linear Equations Raw Picture – Example 1.2

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point (1, 2).

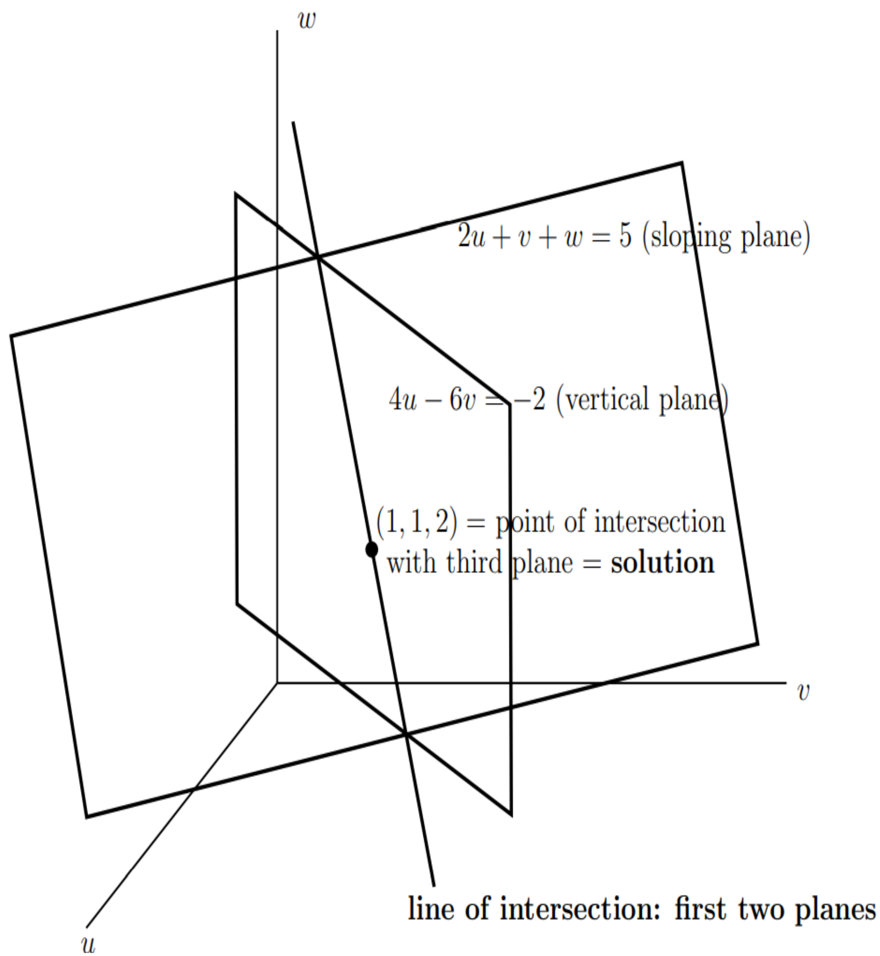
Hence $x = 1$, and $y = 2$ is the only solution to the above system of linear equations.



The Geometry of Linear Equations
Raw Picture – Example 1.3

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{aligned}$$

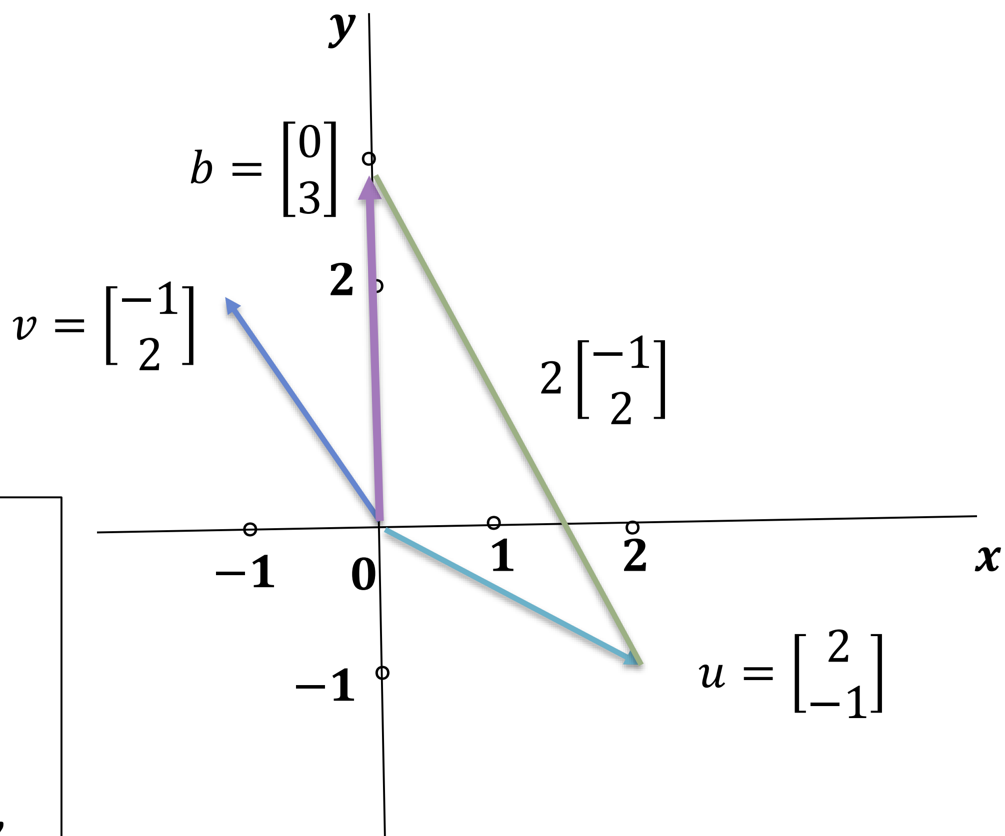
- Each equation describes a plane in three dimensions.
- The second plane is $4u - 6v = -2$. It is drawn vertically, because w can take any value.
- The figure shows the intersection of the second plane with the first.
- Finally the third plane intersects this line in a point (1,1,2).



Column Vectors and Linear Combinations
Column Picture – Example 1.2

$$\begin{aligned} 2x \; - \; y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$



Geometrically, we want to find numbers x and y so that $x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \mathrm{y}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$equals $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.
As we see from $x = 1$ and $y = 2$, agreeing with the column picture in Figure.

Multiplication of a Matrix and a Vector (Ax)

Usual way

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1\times 1 + 2\times 2 + 1\times 3 \\ 2\times 1 + 2\times 2 + 3\times 3 \\ 1\times 1 + 1\times 2 + 4\times 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 15 \end{bmatrix}$$

Other way: Considering the entries of x as the coefficients of a linear combination of the column vectors of the matrix A:

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ 15 \end{bmatrix}.$$

This shows the entries of x as the coefficients of a linear combination of the column vectors of the matrix A:

Linear Independence

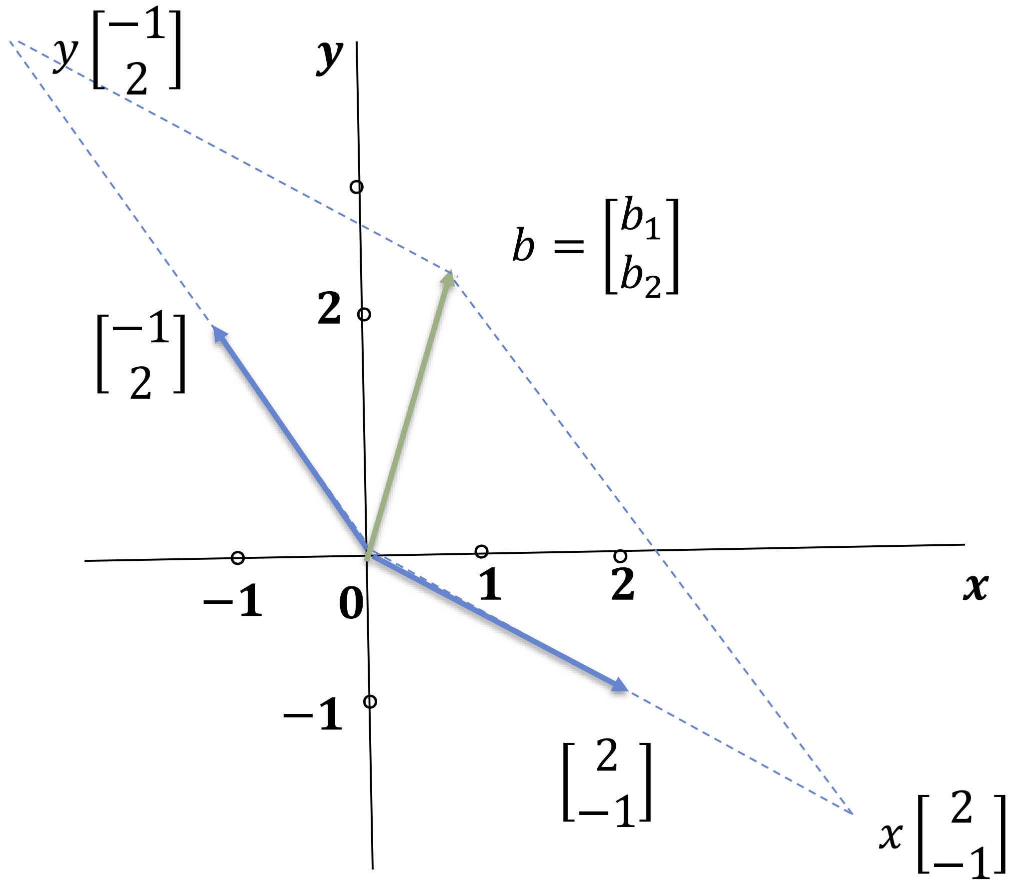
Given a matrix A, can we solve $Ax = b$ for every possible vector b?

Does $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

solvable for every b $= \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$?

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 2 \end{bmatrix} y = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

The linear combinations of the column vectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ fill the 2 dimensional xy-plane.



Multiplication of a Row Vector and a Matrix

$$\begin{bmatrix}1 & 2 & -1\end{bmatrix}_{1\times 3}\begin{bmatrix}1 & 2 & 1\\ 2 & 2 & 3\\ 1 & 1 & 4\end{bmatrix}_{3\times 3}$$

$$= 1\begin{bmatrix}1 & 2 & 1\end{bmatrix} + 2\begin{bmatrix}2 & 2 & 3\end{bmatrix} + (-2)\begin{bmatrix}1 & 1 & 4\end{bmatrix}$$

$$= \begin{bmatrix}3 & 4 & -1\end{bmatrix}.$$

Matrix × Column = Column (Combination of columns of matrix)

Row × Matrix = Row (Combination of Rows of matrix).

Solving System of Linear equations – Elimination & Back – Substitution:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ 2x_1 + 2x_2 + 3x_3 &= 7 \\ x_1 + x_2 + 4x_3 &= 6 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 6 \end{bmatrix}.$$

$$A\,x = b.$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} \xRightarrow[(E_{21})]{-2\times Row\ 1+Row\ 2} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix} \xRightarrow[(E_{31})]{-1\times Row\ 1+Row\ 3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

Elimination & Back – Substitution Cont.

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix} \xrightarrow[(E_{32})]{-\left(\frac{1}{2}\right)\times Row\ 2 + Row\ 3} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & {}^{5}/_{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -({}^{1}/_{2}) & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & {}^{5}/_{2} \end{bmatrix} = U$$

Hence,

$$E_{32}(E_{31}(E_{21}A)) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & {}^{5}/_{2} \end{bmatrix}.$$

$\therefore \quad EA = U,$ where $\; E = E_{32}\, E_{31}\, E_{21}$

Elimination & Back – Substitution Cont.

$$\begin{bmatrix}1 & 2 & 1 & 4\\ 2 & 2 & 3 & 7\\ 1 & 1 & 4 & 6\end{bmatrix} \xRightarrow{E_{21}} \begin{bmatrix}1 & 2 & 1 & 4\\ 0 & -2 & 1 & -1\\ 1 & 1 & 4 & 6\end{bmatrix} \xRightarrow{E_{32}} \begin{bmatrix}1 & 2 & 1 & 4\\ 0 & -2 & 1 & -1\\ 0 & -1 & 3 & 2\end{bmatrix} \xRightarrow{E_{32}} \begin{bmatrix}1 & 2 & 1 & 4\\ 0 & -2 & 1 & -1\\ 0 & 0 & {}^{5}/_{2} & {}^{5}/_{2}\end{bmatrix}$$

Augmented Matrix

$$x_1 + 2x_2 + x_3 = 4 \qquad \Longrightarrow x_1 = 1$$

$$-2\,x_2 + x_3 = -1 \qquad \Longrightarrow x_2 = 1$$

$${}^{5}/_{2}\,x_3 = {}^{5}/_{2} \qquad \Longrightarrow x_3 = 1$$

Inverse of an Elementary Matrix

What is the inverse matrix of E_{21}, where $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$A = LU$

$$E_{32}(E_{31}(E_{21}A)) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & {}^{5}/_{2} \end{bmatrix} = U.$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\left({}^{1}/_{2}\right) & 1 \end{bmatrix}$$

$$\Longrightarrow\ E_{31}(E_{21}A) = \ E_{32}^{-1}U$$

$$\Longrightarrow\ \ E_{21}A = E_{31}{}^{-1}\ E_{32}^{-1}U$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore\ \ A = E_{32}^{-1}E_{31}^{-1}\ E_{32}^{-1}U$$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & {}^{1}/_{2} & 1 \end{bmatrix} \mathrm{U}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & {}^{1}/_{2} & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & {}^{5}/_{2} \end{bmatrix} = \mathrm{LU}.$$

Relationship among $\boldsymbol{A}, \boldsymbol{E}, \boldsymbol{L}, \boldsymbol{U}$, and $\boldsymbol{b}$

$$E_{32}E_{31}E_{21}\ A = U$$

$$\implies \quad EA = U, \text{ where } E = E_{32}E_{31}E_{21}$$

$$\implies \quad A = LU, \text{ where } L = E^{-1} = E_{32}^{-1}E_{31}^{-1}\ E_{32}^{-1}.$$

$$A\boldsymbol{x} = \boldsymbol{b}$$

$$\implies \quad EA\boldsymbol{x} = E\boldsymbol{b} = \mathbf{c}, \quad \text{where } \boldsymbol{c} = E\boldsymbol{b}$$

$$\implies \quad U\boldsymbol{x} = \boldsymbol{c}.$$

Singular Case (With no solution): Example 1.4

$$\boxed{\begin{aligned} x + y + z &= 3 \\ 2x + 2y + 5z &= 8 \\ 4x + 4y + 8z &= 15 \end{aligned}} \Rightarrow \boxed{\begin{aligned} x + y + z &= 3 \\ 3z &= 2 \\ 4z &= 3 \end{aligned}}$$

This system is singular and has no solution!

Singular Case (With infinitely many solutions): Example 1.5

$$\begin{aligned} x + y + z &= 3 \\ 2x + 2y + 5z &= 9 \\ 4x + 4y + 10z &= 18 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x + y + z &= 3 \\ 3z &= 3 \\ 6z &= 6 \end{aligned}$$

This system is singular but has infinitely many solution!

Any point in the line $x + y = 2$ is a solution to the system.

Permutation Matrix

Consider the matrix $A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 4 \\ 0 & 2 & 1 \end{bmatrix}$.

Since the entry at the (2,2) position is 0, we have to interchange rows 2 and 3 of A.

What is the matrix P such that $PA = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 0 & 4 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Permutation Matrices of Order 3

How many permutation matrices of order 3 are there? List them all.

$$P_{123} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{213} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{132} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_{321} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{231} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{312} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The Cost of Gaussian Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n;$$

$$\boldsymbol{Ax = b}$$

The first stage of elimination needs $n(n-1) = n^2 - n$ operations.

The k^{th} stage of elimination needs $k(k-1) = k^2 - k$ operations.

Hence, the total operation for the left side is

$$= n^2 + (n-1)^2 + \cdots + 1^2 - (n + n - 1 + \cdots + 1)$$

$$= {}^{n}/_{6}\,(2n+1)(n+1) - {}^{n}/_{2}\,(n+1) = {}^{1}/_{3}\,(n^3 - n).$$

The total for back-substitution is $1 + 2 + \cdots + n = {}^{n}/_{2}\,(n+1)$.

If n is at all large, a good estimate for the number of operations is ${}^{1}/_{3}\ n^3$.