Bases and Dimension.

Let's start with a definition.

Definition 10: Let V be a vector space over the field F. A basis for V is a linearly independent set of vectors in V which spans the space V.

Examples:

1). Let F be a field. Recall that F^n is a vector space over the field F under the vector addition and scalar multiplication defined in page 14. For each $1 \leq i \leq n$, let e_i be the vector in F^n whose i^{th} coordinate is 1 and whose other cooridinates are all 0, let $S = \{e_1, e_2, \ldots, e_n\}$. Then $S \subseteq F^n$.

Let $(\alpha_1, \ldots, \alpha_n) \in F^n$. Note that $(\alpha_1, \ldots, \alpha_n) = \alpha_1 e_1 + \cdots + \alpha_n e_n \in \text{span}(S)$. Thus, $F^n \subseteq \text{span}(S)$. Clearly $\text{span}(S) \subseteq F^n$. Hence $\text{span}(S) = F^n$. That is S spans the space F^n.

Now suppose $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$ for some $\alpha_1, \ldots, \alpha_n \in F$. Then, $(\alpha_1, \ldots, \alpha_n) = (0, \ldots, 0)$. It follows that $\alpha_i = 0$ for each $i = 1, \ldots, n$. Thus, S is linearly independent.

Therefore, $S = \{e_1, \ldots, e_n\}$ is a basis for F^n. We shall call this particular basis the <u>standard basis</u> of F^n.

In particular, when $F = \mathbb{R}$ and $n = 3$, $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $\mathbb{R}^3$ and when $F = \mathbb{R}$, and $n = 4$,

$S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is a basis for $\mathbb{R}^4$.

Did you notice that e_1 in $\mathbb{R}^3$ is not the same as e_1 in $\mathbb{R}^4$? Yet, we use the same symbol. In general e_i in F^n is not the same as e_i in F^m, where $m, n \in \mathbb{Z}^+ \setminus \{1\}$ and $m \neq n$, because e_i in F^n has n coordinates (or components) and e_i in F^m has m coordinates (or components).

2). Prove that $\mathcal{B} = \{(1,2,-1), (2,1,0), (-1,-2,4)\}$ is a basis for $\mathbb{R}^3$.

Solution: Let $(x, y, z) \in \mathbb{R}^3$. Let us check whether there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1 \cdot (1,2,-1) + \alpha_2 \cdot (2,1,0) + \alpha_3 \cdot (-1,-2,4) = (x, y, z).$$

So, consider the system of linear equations:

$$\alpha_1 + 2\alpha_2 - \alpha_3 = x \quad \text{--- (1)}$$

$$2\alpha_1 + \alpha_2 - 2\alpha_3 = y \quad \text{--- (2)}$$

$$-\alpha_1 \qquad + 4\alpha_3 = z \quad \text{--- (3)}$$

① × (−2) + ②, and ① + ③ gives the following new system of linear equations.

$$\alpha_1 + 2\alpha_2 - \alpha_3 = x \quad \text{— ①}$$

$$-3\alpha_2 = y - 2x \quad \text{— ②}'$$

$$2\alpha_2 + 3\alpha_3 = x + z \quad \text{— ③}'$$

From ②′, $\alpha_2 = \frac{y-2x}{-3} = \frac{2x-y}{3}$. Thus, by ③′, $3\alpha_3 = x + z - \frac{2}{3}(2x-y)$ $= \frac{-x+2y+3z}{3}$. Hence, $\alpha_3 = \frac{-x+2y+3z}{9}$. Therefore, from ①,

$$\alpha_1 = x - \frac{2}{3}(2x-y) + \frac{(-x+2y+3z)}{9} = \frac{-4x+8y+3z}{9}.$$

So, $\alpha_1 = \frac{-4x+8y+3z}{9}$, $\alpha_2 = \frac{2x-y}{3}$ and $\alpha_3 = \frac{-x+2y+3z}{9}$.

Clearly $\frac{-4x+8y+3z}{9}$, $\frac{2x-y}{3}$, $\frac{-x+2y+3z}{9} \in \mathbb{R}$, as $x, y, z \in \mathbb{R}$. Hence $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. Hence $\mathcal{B}$ spans $\mathbb{R}^3$.

Now suppose $\alpha_1 \cdot (1,2,-1) + \alpha_2 \cdot (2,1,0) + \alpha_3 \cdot (-1,-2,4) = (0,0,0)$.

Substituting $x = y = z = 0$ in above expressions we get

$\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, $\mathcal{B}$ is linearly independent.

Therefore, $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

The above example shows that a basis for a vector space need not to be unique. In fact, there could be infinitely many number of basis for a vector space.

3). Is $B = \{(1, 2), (-2, 1), (5, 5)\}$ a basis for $\mathbb{R}^2$?

Solution : Let $(x, y) \in \mathbb{R}^2$. Let $t \in \mathbb{R}$. Put $\alpha_1 = \frac{x+2y}{5} - 3t$

and $\alpha_2 = \frac{-2x+y}{5} + t$. Clearly $\alpha_1, \alpha_2 \in \mathbb{R}$.

Observe that $\alpha_1 \cdot (1, 2) + \alpha_2 \cdot (-2, 1) + t\,(5, 5) = \left(\frac{x+2y}{5} - 3t\right) \cdot (1, 2)$

$+ \left(\frac{-2x+y}{5} + t\right) \cdot (-2, 1) + t \cdot (5, 5) = (x, y)$. Thus B spans

$\mathbb{R}^2$. However, because $(-3) \cdot (1, 2) + 1 \cdot (-2, 1) + 1 \cdot (5, 5) = (0, 0)$ and

-3 (or 1) $\neq 0$, B is not linearly independent.

Therefore, B is not a basis for $\mathbb{R}^2$.

4). Recall that $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in \mathbb{R} \right\}$ – the set of all 2×2

matrices over $\mathbb{R}$ – is a vector space over $\mathbb{R}$ under usual

addition and scalar multiplication of matrices.

Is $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ a basis for V?

Solution : Let us first show that B is linearly independent.

To this end, suppose

$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_2 + \alpha_3 \\ \alpha_3 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Equating corresponding entries of the two matrices gives;

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$
$$\alpha_2 + \alpha_3 = 0$$
$$\alpha_3 = 0.$$

Thus $\alpha_1 = \alpha_2 = \alpha_3 = 0$,

Therefore, $\mathcal{B}$ is linearly independent.

However, note that, $\mathcal{B}$ does not span V, because for each $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$,

$$\alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V.$$

Thus, $\mathcal{B}$ is not a basis for V.

5). Note that $S = \{(1, 2, 3), (-2, 1, 0), (0, 5, 6), (6, 7, 12)\}$ is not linearly independent and $\text{span}(S) \neq \mathbb{R}^3$. Thus, S is not a basis for the Euclidean space $\mathbb{R}^3$.

6). The set $\{1, x, x^2, \ldots, x^n\}$, $n \in \mathbb{N} \cup \{0\}$, is a basis for the vector space of polynomials having degree n or less over $\mathbb{R}$. (see Exercise on page 35).

7).* (Important) The infinite set $\{1, x, x^2, \ldots\}$ is a basis for the vector space of all polynomials with real coefficients over $\mathbb{R}$.

Remark: For the trivial space $Z = \{0\}$, there is no nonempty linearly independent spanning set. Consequently, the empty set, ϕ, is considered as a basis for Z. (See the remark on pg 30 and Definition 9 on pg. 32).

Theorem 10: Let V be a vector space over the field F and let $\mathcal{B}$ be a basis for V. Then any $v \in V$ has a unique representation in the form $v = \alpha_1 u_1 + \dots + \alpha_n u_n$, where $\alpha_i \in F$ and $u_i \in \mathcal{B}$ for $i = 1, 2, \dots, n$.

Proof: Follows from Lemma 1 on page 37.

Definition 11: A vector space which has a finite basis is called a finite-dimensional vector space. Also, a vector space which has an infinite basis is called an infinite-dimensional vector space.

Similarly, a subspace of a vector space is called a finite-(infinite) dimensional subspace if it has a finite (infinite) basis.

Example: Let A be a n-square matrix $[a_{ij}]$ over $\mathbb{R}$, which is nonsingular. Then the columns of A form a basis for the column matrices $\mathbb{R}^{n\times 1}$ $(\mathbb{R}^n)$.

Proof: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ and let $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$.

Assume $\alpha_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + \alpha_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \cdots + \alpha_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Then $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Since A is nonsingular, A^{-1} exists. Hence $A^{-1} \cdot A \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Thus, $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. That is $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$.

Hence the set of column vectors in A is linearly independent.

Now let $Y \in \mathbb{R}^{n\times 1}$ (i.e. $Y \in \mathbb{R}^n$). Let $X = A^{-1}Y$. Then $AX = Y$.

Then $x_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \cdot \begin{bmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{bmatrix} + \cdots + x_n \cdot \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{nn} \end{bmatrix} = y$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

That is, Y is a linear combination of columns of A.
In other words, columns of A span $\mathbb{R}^{n\times 1}$. Thus, columns of A form a basis for the column matrices $\mathbb{R}^{n\times 1}$ (i.e. $\mathbb{R}^n$).

So, $\mathbb{R}^{n\times 1}$ is a finite-dimensional vector space.

Exercise: Prove that the space V of all $m\times n$ matrices over the field F is finite-dimensional by exhibiting a basis for this space.

Proof: Let $\mathcal{B} = \{ A_{ij} \in F^{m\times n} \mid a_{hk} = 1$ if $h = i$ and $k = j$, $a_{hk} = 0$ otherwise with $1 \le i \le m$ and $1 \le j \le n \}$

That is $A_{11} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \cdots,$

$A_{nn} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$. It is easy to see that $\mathcal{B}$ is a basis for the space of all $m\times n$ matrices over the field F. Note that $|\mathcal{B}| = mn$.
That is, $\mathcal{B}$ is a finite basis. Hence V is finite-dimensional.

Example: In $\mathbb{R}^3$, $W = \{(x, y, 0) : x, y \in \mathbb{R}\}$ is a finite dimensional subspace. Also, the solution space of the equation $3x + y - z = 0$ is a finite-dimensional subspace of $\mathbb{R}^3$.

Example: Let V be the vector space of all polynomials over $\mathbb{R}$. Then the infinite set $\{1, x, x^2, \dots\}$ is a basis for V.

Proof: We already know that V is a vector space over $\mathbb{R}$. For each $n \in \mathbb{Z}^+$, let f_n be the polynomial in V given by $f_n(x) = x^n$, $x \in \mathbb{R}$. and f_0 be the polynomial in V given by $f_0(x) = 1$, $x \in \mathbb{R}$. Then $\{f_n : n \in \mathbb{Z}^+ \cup \{0\}\} \subseteq V$.

Let $P(x) = a_0 + a_1 x + \dots + a_m x^m \in V$. Then, $P = a_0 \cdot f_0 + a_1 \cdot f_1 + \dots + a_m f_m$. Thus, $P \in \text{span}(\{f_n : n \in \mathbb{Z}^+ \cup \{0\}\})$. In other words, the set $\{f_n : n \in \mathbb{Z}^+ \cup \{0\}\}$ spans V.

Now let us show that $\{f_n : n \in \mathbb{Z}^+ \cup \{0\}\}$ is linearly independent. Because $\{f_n : n \in \mathbb{Z}^+ \cup \{0\}\}$ is infinite, we must prove that every nonempty finite subset of $\{f_n : n \in \mathbb{Z}^+ \cup \{0\}\}$ is linearly independent. Since any subset of a linearly independent set is linearly independent, it is enough to prove that for each $n \in \mathbb{Z}^+$, $\{f_0, f_1, \dots, f_n\}$ is linearly independent.

To this end, let $n \in \mathbb{Z}^+$ and let $\alpha_0, \alpha_1, \dots, \alpha_n \in \mathbb{R}$. Suppose that $\alpha_0 f_0 + \alpha_1 f_1 + \dots + \alpha_n f_n = 0$. It means, for each $x \in \mathbb{R}$, $\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0$. Note that every nonzero polynomial of degree n as at most n real roots. Here, $\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n$ has infinitely many roots as $\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n = 0$ for each $x \in \mathbb{R}$. Thus, $\alpha_0 + \alpha_1 x + \dots + \alpha_n x^n$ must be the zero polynomial. Hence, $\alpha_0 = \alpha_1 = \dots = \alpha_n = 0$. Thus $\{f_0, f_1, \dots, f_n\}$ is linearly independent. Therefore $\{f_n : n \in \mathbb{Z}^+ \cup \{0\}\}$ is a basis for V.

Does this mean that V is not finite-dimensional?

We will answer this question rigorously in the next couple of pages. But for the moment assume we have finite number of polynomials $g_1, g_2, \dots, g_m$. Let $k = \max\{\deg(g_1), \deg(g_2), \dots, \deg(g_m)\}$

Observe that $f_{k+1} \notin \operatorname{span}(\{g_1, g_2, \dots, g_m\})$.

It follows that V is infinite-dimensional.

The following result is a corollary due to Theorem 9.

Corollary 3: Let $V \neq \{0\}$ be a vector space over the field F and let S be a non empty, finite spanning set for V. Then there exists $B \subseteq S$ such that B is a basis for V.

Proof: Follows from corollary 2 on page 40.

Lemma 2: If $B = \{v_1, v_2, \dots, v_n\}$ is a basis for the vector space V over F and if $\{u_1, u_2, \dots, u_m\}$ is a linearly indipendent subset of V, then $m \leq n$.

Proof: Suppose that $B = \{v_1, \dots, v_n\}$ is a basis for V and that $\{u_1, \dots, u_m\}$ is a linearly independent subset of V.

Now, because B is a basis, it spans V and hence $\{u_1, v_1, v_2, \dots, v_n\}$ is linearly dependent. Also, $\{u_1, v_1, \dots, v_n\}$ spans V. Thus, by corollary 1 (on page 38), there exists a proper subset $\{u_1, v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ with $k \leq n-1$ of $\{u_1, v_1, \dots, v_n\}$ which forms a basis for V.

Now $\{u_1, u_2, v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ is linearly dependent and spans V. Thus, again by corollary 1, there exists a proper subset $\{u_1, u_2, v_{j_1}, \dots, v_{j_s}\}$ with $s \leq n-2$ of $\{u_1, u_2, v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ which forms a basis for V.

Keeping up this procedure we eventually get down to a basis of V of the form $\{u_1, u_2, \ldots, u_{m-1}, v_x, v_y, \ldots\}$. Since u_m is not a linear combination of $u_1, u_2, \ldots, u_{m-1}$, the above basis must actually include some v in B. Note that, to get this basis we have introduced $m-1$ u's and each introduction has eliminated at least one v in B. Yet there is a v left. Thus $m-1 \leq n-1$. Hence $m \leq n$.

This completes the proof.

Remark: If $B = \{v_1, \ldots, v_n\}$ is a basis for the vector space V and $\{u_1, u_2, \ldots, u_m\}$ is linearly independent, then m vectors of B can be replaced by $u_1, \ldots, u_m$ to form another basis for V.

Theorem 11: Let V be a vector space which is spanned by a finite set of vectors $\{v_1, \ldots, v_n\}$. Then any independent set of vectors in V is finite and contains no more than n elements.

Proof: Let $\{v_1, \ldots, v_n\}$ be a spanning set for V and let S be a nonempty linearly independent set of vectors in V. To the contrary assume S has more than n elements. Then there are $n+1$ number of linearly independent vectors in S.

By corollary 3 (on page 51); $\{v_1, \ldots, v_n\}$ contains a basis $\mathcal{B}$ for V. Thus $\mathcal{B}$ is finite and $|\mathcal{B}| \leq n$. Now, $\mathcal{B}$ is a basis for V and we have $n+1$ linearly independent vectors in S. Thus by Lemma 2, $n+1 \leq |\mathcal{B}| \leq n$. It follows that $1 \leq 0$. This is a contradiction. Therefore S is finite and contains no more than n elements.

Corollary 4: Let V be a finite-dimensional vector space over the field F. Then any two bases of V have the same (finite) number of elements.

Proof: Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{B}' = \{u_1, \ldots, u_m\}$ be two bases for V. Then both $\mathcal{B}$ and $\mathcal{B}'$ are linearly independent and span V. By Theorem 11, because $\mathcal{B}$ is a basis and $\mathcal{B}'$ is linearly independent, $m \leq n$. Similarly, $n \leq m$. Hence $m = n$.

The above corollary allows us to assign a dimension to a finite-dimensional vector space.

Definition 12: Let V be a finite-dimensional vector space. The number of vectors in any basis for V is called the dimension of the finite-dimensional vector space V. It is denoted by $\dim V$.

Examples: 1). $\dim(\mathbb{R}) = 1$, $\dim(\mathbb{R}^2) = 2$. Indeed, for any field F and $n \in \mathbb{Z}^+$, $\dim F^n = n$.

2). $\dim(F^{m\times n}) = mn$, where $F^{m\times n}$ is the set of all $m\times n$ matrices over the field F.

3). For the subspace W in the first example on page 49, $\dim W = 2$. Note that the general solution of the equation $2x + y - z = 0$ is given by

$(x, y, z) = (t, s, 2t+s) = t(1, 0, 2) + s(0, 1, 1)$, where $s, t \in \mathbb{R}$. It is easy to see that $\{(1, 0, 2), (0, 1, 1)\}$ is a basis for the solution space.

Corollary 5: Let V be a finite-dimensional vector space and let $n = \dim V$. Then

(a). any subset of V which contains more than n vectors is linearly dependent.

(b). no subset of V which contains less than n vectors can span V.

Proof: (a) Let S be a subset of V that contains more than n elements. Let $\{w_1, w_2, \dots, w_m\} \subseteq S$ be such that $w_i \neq w_j$ for $i \neq j$, where $i, j \in \{1, 2, \dots, m\}$, with $m > n$.

Since $\dim V = n$ is finite, V is spanned by a finite set of vectors. Now, $\{w_1, w_2, \dots, w_m\}$ is finite and contains $m > n$ elements. Thus, by Theorem 11, $\{w_1, w_2, \dots, w_m\}$ is linearly dependent. Therefore, S is linearly dependent.

(b). Let $m \in \mathbb{Z}^+$ be such that $m < n$. Assume there exist $w_1, w_2, \dots, w_m$ in V such that $\text{span}(\{w_1, w_2, \dots, w_m\}) = V$.

Then, by Theorem 11, any independent set of vectors in V is finite and contains no more than m elements. Since $\dim V = n$, V contains a linearly independent set of n vectors. Then $n \leq m$. This is a contradiction as $m < n$. Thus, no subset of V which contains less than n vectors can span V.

This completes the proof.

One can ask "how do we find a spanning set, or ideally a linearly independent spanning set (i.e. a basis) for a given vector space?". One intuitive way ... start with an arbitrary non-zero vector in the vector space V. Let's call it v_1. Then,

$W_1 = \operatorname{span}(\{v_1\})$ is a one dimensional subspace of V. If $W_1 = V$, then we have succeeded in finding a linearly independent spanning set. If $W_1 \neq V$, then there exists $v_2 \in V \setminus \operatorname{span}(\{v_1\})$. Let $W_2 = \operatorname{span}(\{v_1, v_2\})$. Clearly $W_1 \subseteq W_2$. The next lemma guarantees that $\{v_1, v_2\}$ is linearly independent. By continuing in this way we can find, in finite number of steps if V is finite dimensional, a linearly independent spanning set for V.

Lemma 3 : Let S be a nonempty linearly independent subset of a vector space V over the field F. Let $v \in V$ be such that $v \notin \operatorname{span}(S)$.

Then, $S \cup \{v\}$ is linearly independent.

Proof: Let S be a linearly independent subset of the vector space V and let $v \in V$ be such that $v \notin \operatorname{span}(S)$.

Also, let $v_1, v_2, \ldots, v_n$ be any n, where $n \in \mathbb{Z}^+$, distinct vectors in S. Let $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in F$.

Suppose $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n + \beta v = 0$. If $\beta \neq 0$, then $v = [-(\beta^{-1}\alpha_1)]v_1 + [-(\beta^{-1}\alpha_2)]v_2 + \cdots + [-(\beta^{-1}\alpha_n)]v_n$. It follows that $v \in \text{span}(\{v_1, \ldots, v_n\}) \subseteq \text{span}(S)$. Thus $\beta = 0$. Then $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$. Because S is linearly independent, $\{v_1, \ldots, v_n\}$ is also linearly independent. Hence $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Therefore, $S \cup \{v\}$ is linearly independent.

Theorem 12: Let W be a subspace of a finite-dimensional vector space V. Then, every linearly independent subset of W is finite and is part of a (finite) basis for W.

Proof: Let $n = \dim V$ and let S be a linearly independent subset of W. Since $W \subseteq V$, S is a linearly independent subset of V as well. Also, V is spanned by n vectors. Hence, by Theorem 11 on page 52, S is finite and contains no more than n elements (i.e. $|S| \leq n$).

Observe that $\text{span}(S) = W$ or $\text{span}(S) \neq W$. If $\text{span}(S) = W$, then S is a basis for W. Suppose $\text{span}(S) \neq W$. Then there exists $v_1 \in V$ such that $v_1 \in W \setminus \text{span}(S)$. By Lemma 3 $S \cup \{v_1\}$ is linearly independent. Now let $S_1 = S \cup \{v_1\}$. Clearly $\text{span}(S_1) = W$ or $\text{span}(S_1) \neq W$. If $\text{span}(S_1) = W$, then we are done. Suppose $\text{span}(S_1) \neq W$. Let $v_2 \in W \setminus \text{span}(S_1)$.

By Lemma 3, $S_1 \cup \{v_2\} = S \cup \{v_1, v_2\}$ is linearly independent.

Let $S_2 = S \cup \{v_1, v_2\}$. If $\text{span}(S_2) = W$, then we are done.

If $\text{span}(S_2) \neq W$, then we proceed as above. Notice that in not more than n steps, we reach a set

$S_m = S \cup \{v_1, v_2, \dots, v_m\}$, $m \leq n$, which is a basis for W.

Remark: In the above proof we have used the fact $\text{span}(\phi) = \{0\}$, for the case where $S = \phi$.

Corollary 6: Let W be a nonzero proper subspace of a finite-dimensional vector space V. Then $\dim W < \dim V$.

Proof: Since $W \neq \{0\}$, there exists $w \in W$ such that $w \neq 0$. Then $\{w\}$ is linearly independent. Thus, by Theorem 12, there exists a basis $\mathcal{B}$ for W which contains $\{w\}$. Since $\mathcal{B}$ is linearly independent subset of V, by Theorem 11 (pg: 52), $\dim W = |\mathcal{B}| \leq \dim V$. Now, because W is a proper subspace of V, there exists $v \in V$ such that $v \in V \setminus W$. By Lemma 3, $\mathcal{B} \cup \{v\}$ is a linearly independent subset of V. Therefore, again by Theorem 11, $|\mathcal{B} \cup \{v\}| \leq n$. Hence, $\dim W = |\mathcal{B}| < \dim V$. (What if $\dim W = |\mathcal{B}| = \dim V = n$?). ▨

Corollary 7: In a finite-dimensional vector space V every non-empty linearly independent set of vectors is part of a basis for V.

Proof: Note that V itself is a subspace of V. ▨

Exercise: Prove Corollary 7 using Corollary 1 on page 38. (Tutorial problem).