Design and Analysis of Algorithm

Lecture-25:
Branch and Bound

Contents

Problem Definition

Let G = (V, E) be a directed graph defining an instance of the traveling sales person problem. Let C_{ij} equal the cost of edge (i, j), $C_{ij} = \infty$ if $(i, j) \neq$ E, and let $|V| = n$.

Without loss of generality, we can assume that every tour starts and ends at vertex 1

So, the solution space S is given by $S = \{1, \pi, 1 | \pi$ is a permutation of $(2,3, \ldots, n)\}$

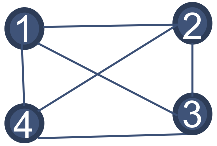
Problem Definition

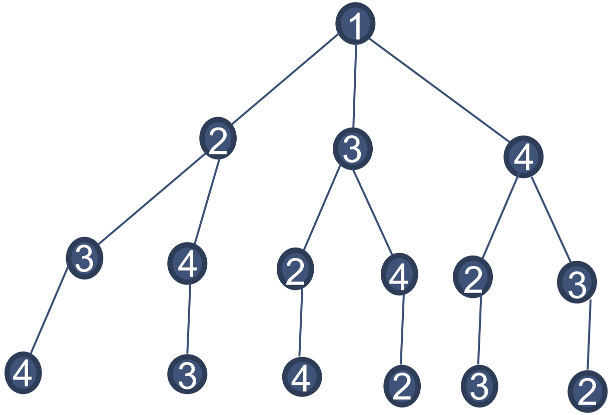
As per the definition S can have maximum of $(n-1)!$ Combination.

We can limit this to much smaller range with following general restrictions that $S \in \{i_i, i_2, i_3, \ldots . i_n\}$ $iff < i_j, i_{j+1} >$ is an edge where, $0 \leq j \leq n-1$ and $i_0 = i_n = 1$.

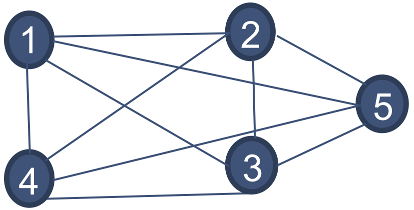
To use LCBB to search the traveling salesperson state space tree, we need to define a cost function $c(.)$ and two other functions $\hat{c}(.)$ and $u(.)$ such that $\hat{c}(r) \leq c(r) \leq u(r)$ for all nodes r. The cost $c()$ is such that the solution node with least $c(.)$ corresponds to a shortest tour in G.

General Solution





Solving TSP using Branch and Bound



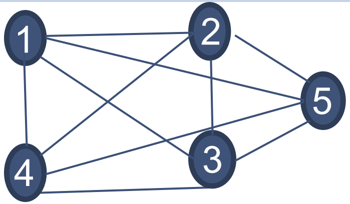
$$
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & 20 & 30 & 10 & 11 & 10 \\
2 & 15 & \infty & 16 & 4 & 2 & 2 \\
3 & 3 & 5 & \infty & 2 & 4 & 2 \\
4 & 19 & 6 & 18 & \infty & 3 & 3 \\
5 & 16 & 4 & 7 & 16 & \infty & 4
\end{array}
\longrightarrow
\begin{array}{c|ccccc|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & \infty & 10 & 20 & 0 & 1 \\
2 & 13 & \infty & 14 & 2 & 0 \\
3 & 1 & 3 & \infty & 0 & 2 \\
4 & 16 & 3 & 15 & \infty & 0 \\
5 & 12 & 0 & 3 & 12 & \infty \\
\hline
 & 1 & 0 & 3 & 0 & 0
\end{array}
$$

Step 1: Find the least bound of the TSP

$$
\begin{array}{c|ccccc|c}
 & 1 & 2 & 3 & 4 & 5 & \\
\hline
1 & \infty & 10 & 17 & 0 & 1 & 10 \\
2 & 12 & \infty & 11 & 2 & 0 & 2 \\
3 & 0 & 3 & \infty & 0 & 2 & 2 \\
4 & 15 & 3 & 12 & \infty & 0 & 3 \\
5 & 11 & 0 & 0 & 12 & \infty & 4 \\
\hline
 & 1 & 0 & 3 & 0 & 0 & \boxed{25}
\end{array}
$$

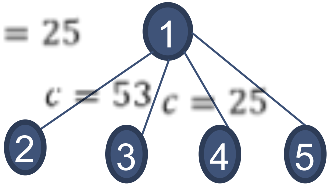
Solving TSP using Branch and Bound

$$cost = c_r + c_r' + c(1{,}2)$$



$c = 25$

$Upper = \infty$



$c = 31$

$c = 25 + 0 + 10$
$= 35$

$$\begin{array}{c|c} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}\end{array}$$

$$\begin{array}{c|c} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & 2 & 0 \\ \infty & 3 & \infty & 0 & 2 \\ 4 & 3 & \infty & \infty & 0 \\ 0 & 0 & \infty & 12 & \infty \end{bmatrix}\end{array}$$

$$\begin{array}{c|c} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{bmatrix}\end{array}$$

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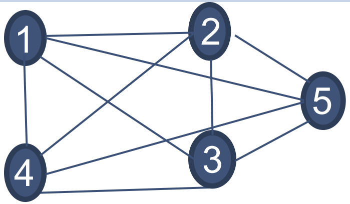
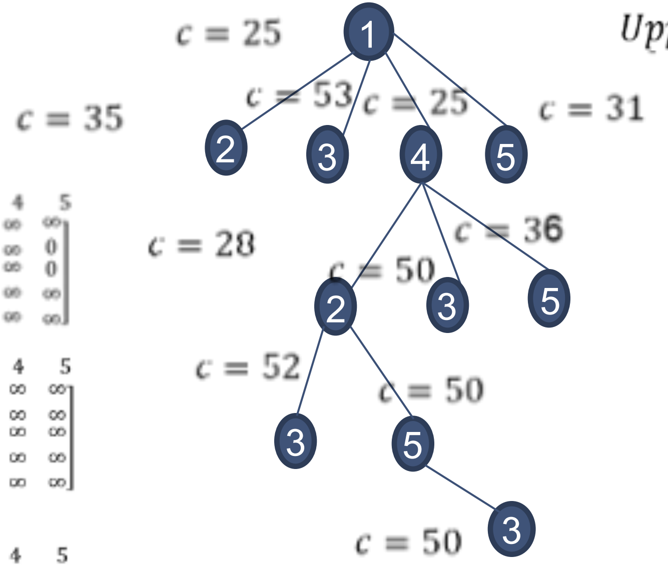
$$\begin{array}{c|c} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ 10 & \infty & 9 & 0 & \infty \\ 0 & 3 & \infty & 0 & \infty \\ 12 & 0 & 9 & \infty & \infty \\ \infty & 0 & 0 & 12 & \infty \end{bmatrix}\end{array}$$

Solving TSP using Branch and Bound

$$cost = c_r + c_r' + c(1,2)$$

$$Upper = 50$$





$$\begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\12&\infty&11&\infty&0\\0&3&\infty&\infty&2\\\infty&3&12&\infty&0\\11&0&0&\infty&\infty\end{bmatrix}\end{array}$$

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$$\begin{array}{cc} & \begin{array}{ccccc}1&2&3&4&5\end{array}\\ \begin{array}{c}1\\2\\3\\4\\5\end{array} & \begin{bmatrix}\infty&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\\0&\infty&\infty&\infty&\infty\\\infty&\infty&\infty&\infty&\infty\\\infty&\infty&0&\infty&\infty\end{bmatrix}\end{array}$$

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Introduction

Concerning the performance characteristics of branch-and-bound algorithms that find least-cost answer nodes. We might ask questions such as:

Does the use of a better starting value for upper always decrease the number of nodes generated

Is it possible to decrease the number of nodes generated by expanding some nodes with $\hat{c}()$ >upper?

Does the use of a better $\hat{c}$ always result in a decrease in the number of nodes generated?

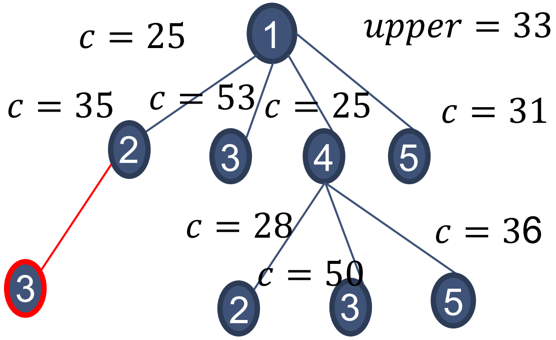
Does the use of dominance relations ever result in the generation of more nodes than would otherwise be generated.

All the following theorems assume that the branch-and-bound algorithm is to find a minimum-cost solution node. Consequently, $c(x)$ = cost of minimum-cost solution node in subtree x.

Theorem 1

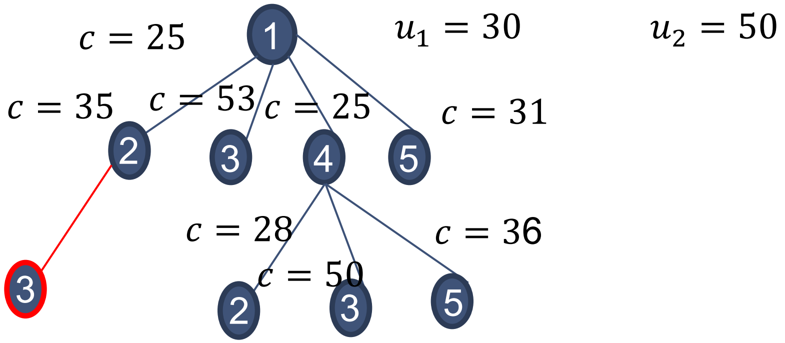
Let t be a state space tree. The number of nodes of t generated by $FIFO$, $LIFO$, and LC branch-and-bound algorithms cannot be decreased by the expansion of any node x with $c(x) >$ upper ,where upper is the current upper bound on the cost of a minimum-cost solution node in the tree t.

The theorem follows from the observation that the value of upper cannot be decreased by expanding x (as $c(x) > upper$). Hence, such an expansion cannot affect the operation of the algorithm on the remainder of the tree



Theorem 2

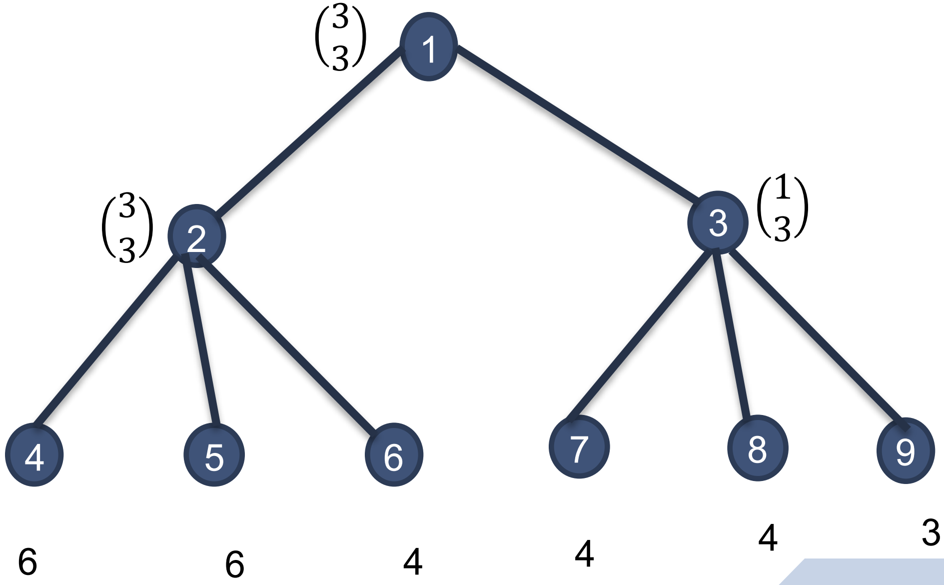
Let U_1 and U_2, $U_1 < U_2$, be two initial upper bounds on the cost of a minimum-cost solution node in the state space tree t. Then FIFO, LIFO, and LC branch-and-bound algorithms beginning with U_1 will generate no more nodes than they would if they started with U_2 as the initial upper bound



Theorem 3

The use of a better $\hat{c}$ function in conjunction with FIFO and LIFO branch-and-bound algorithms does not increase the number of nodes generated

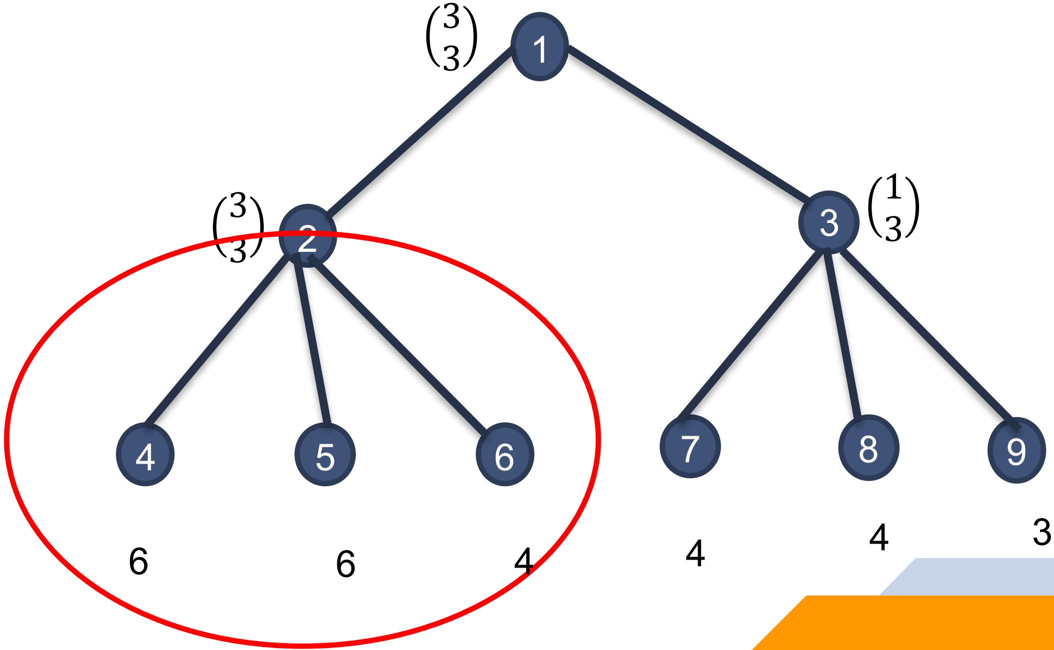
A $\hat{c}_2$ is better than $\hat{c}_1$ iff $\hat{c}_1 \le \hat{c}_2 \le c(x)$



Theorem 4

If a better $\hat{c}$ function is used in a LC branch-and-bound algorithm, the number of nodes generated may increase

A $\hat{c}_2$ is better than $\hat{c}_1$ iff $\hat{c}_1 \le \hat{c}_2 \le c(x)$



Dominance Relation

Formally, a dominance relation D is given by a set of tuples, $D = \{(i_1, i_2), (i_3, i_4), \ldots.\}$.

If $(i, j) \in D$ then node i is said to dominate node j.

By this we mean that subtree i contains a solution node with cost no more than the cost of a minimum-cost solution node in subtree j.

Dominated nodes can be killed without expansion

Since every node dominates itself, $(i, i) \in D$ for all i and D. The relation (i, i) should not result in the killing of node i.

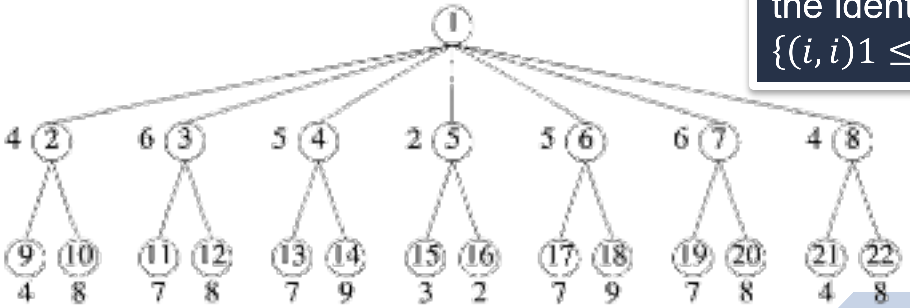
Theorem 5

The number of nodes generated during a FIFO or LIFO branch-and-bound search for a least-cost solution node may increase when a stronger dominance relation is used.

A dominance relation D_2 is said to be stronger than another dominance relation D_1 iff $D_1 \subset D_2$

Consider the state space tree. The only solution nodes are leaf nodes. Their cost is written outside the node. For the remaining nodes the number outside each node is its c value. The two dominance. Relations to use are $D_1 = \{I\}$ and $D_2 = I\ U\{(5,2),(5,8)\}$.

Clearly, D_2 is stronger than D_1 and fewer nodes are generated using D_1 rather than D_2



In the following theorems denotes the identity relation $\{(i,i)1 \leq i \leq n)$.