Design and Analysis of Algorithm

Lecture-3:

Contents

Master Theorem

The master method provides a “cookbook” method for solving recurrences of the form

$T(n) = a\,T(n/b) + f(n)$, where, $a \geq 1$ and $b > 1$ and f(n) should always be asymptotically positive function

1. if $f(n) = O(n^{\log_b a - \epsilon})$ where $\epsilon > 0$

   then $T(n) = \theta(n^{\log_b a})$
2. if $f(n) = \theta(n^{\log_b a})$

   then $T(n) = \theta(n^{\log_b a}\lg(n))$
3. if $f(n) = \Omega(n^{\log_b a + \epsilon})$

   if $af\left(\frac{n}{b}\right) \leq cf(n)$ for some c<1

   then $T(n) = \theta(f(n))$

Numerical on Master Theorem

$T(n) = 9T(n/3) + n$

Step 1: $a = 9 \quad and \quad b = 3 \quad and \quad f(n) = n$
Step 2: compute $\log_b a \Rightarrow \log_3 9$
Step 3: compute $n^{\log_b a} \Rightarrow n^{\log_3 9} \Rightarrow n^2$
Step 4: compare $f(n)$ with $n^{\log_b a} \Rightarrow n < n^2$
Step 5: Choose the case: Case 1
Step 6: $T(n) = \theta(n^2)$

$T(n) = T(2n/3) + 1$

Step 1: $a = 1 \quad and \quad b = 3/2 \quad and \quad f(n) = 1$
Step 2: compute $\log_b a \Rightarrow \log_{3/2} 1$
Step 3: compute $n^{\log_b a} \Rightarrow n^{\log_{3/2} 1} \Rightarrow 1$
Step 4: compare $f(n)$ with $n^{\log_b a} \Rightarrow 1 = 1$
Step 5: Choose the case: Case 2
Step 6: $T(n) = \theta(\log(n))$

$T(n) = 3T(n/4) + nlog(n)$

Step 1: $a = 3 \quad and \quad b = 4 \quad and \quad f(n) = n\, log\, (n)$
Step 2: compute $\log_b a \Rightarrow \log_4 3$
Step 3: compute $n^{\log_b a} \Rightarrow n^{\log_4 3} \Rightarrow n^{0.793}$
Step 4: compare $f(n)$ with $n^{\log_b a} \Rightarrow nlog(n) > n^{.793}$
Step 5: Choose the case: Case 3
Step 6: $af\left(\frac{n}{b}\right) = 3\left(\frac{n}{4}\right)\lg\left(\frac{n}{4}\right) \leq \frac{3}{4}nlog(n) = cf(n); c = 3/4$

$T(n) = \theta(nlog(n))$

Practice Problems

$$T(n) = 4T(n/2) + n^2$$

$a = 4,\qquad b = 2,\qquad f(n) = n^2$

$\log_b a = \log_2 4 = 2$

$n^{\log_b a} = \mathrm{n}^2$

Compare $f(n)$ with $n^{\log_b a}$

$f(n) = n^{\log_b a}$

Case 2 $\qquad T(n) = \theta(n^2 \log n)$

Practice Problems

$T(n) = 16T\left(\frac{n}{4}\right) + n!$

$a = 16, \qquad b = 4, \qquad f(n) = n!$

$\log_b a = \log_4 16 = 2$

$n^{\log_b a} = \mathrm{n}^2$

Compare $f(n)$ with $n^{\log_b a}$

$f(n) > n^{\log_b a}$

Case 3 $\quad a\left(f\left(\frac{n}{b}\right)\right) = cf(n) \Rightarrow 16\left(\frac{n}{4}\right)! \leq 0.5\, n! \quad this\ imples\ c = 0.5\ which\ is\ less\ than\ 1$

$T(n) = \theta(n!)$

Practice Problems

$$T(n) = 3T(n/2) + n^2$$

$T(n) = 3T(\frac{n}{2}) + n^2$

$a = 3$ $b = 2$ $f(n) = n^2$

$\log_b a = \log_2 3$

$n^{\log_b a} = n^{\log_2 3}$

$f(n) > n^{\log_b a}$

Case - 3

$a f(\frac{n}{b}) = 3(\frac{n}{2})^2 = \frac{3}{4} n^2$

$c f(n) = c n^2$

$\frac{3}{4} n^2 = c n^2$ $c = \frac{3}{4} > 0$

$$T(n) = \theta(f(n) = \theta(n^2)$$

Practice Problems

a) $T(n) = 2^n T\left(\frac{n}{2}\right) + n^n$

Master method does not apply as a is not constant

$b)\ T(n) = T(n/2) + n$

T(n) =$\theta(n)$

$c)\ T(n) = 2T(n/2) + 2$

$T(n) \ = \theta(n)$

$d)\ T(n) = 4T(n/2) + n$

$T(n) \ = \theta(n^2)$

e) $T(n) = 0.5T\left(\frac{n}{2}\right) + 1/n$

Master method does not apply as a is less than

f) $T(n) = 8T(n/2) + n^2$

$T(n) = \theta(n^3)$

Practice Problems

$$(a)\quad T(n) = \sqrt{2}\, T\left(\frac{n}{2}\right) + \log n$$

$$(b)\quad T(n) = 3\, T\left(\frac{n}{2}\right) + n$$

$$(c)\quad T(n) = 3\, T\left(\frac{n}{3}\right) + \sqrt{n}$$

$$(d)\quad T(n) = 7\, T\left(\frac{n}{3}\right) + n^2$$

$$(e)\quad T(n) = 2T(\sqrt{n}) + c$$

$$(f)\quad T(n) = T(\sqrt{n}) + \log(n)$$

$$(g)\quad T(n) = T(\sqrt{n}) + c$$