Assignment 5

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AI20BTECH11002

Download all python codes from

```
https://github.com/Dhatri−nanda/EE3900/blob/main
/Assignment_5/code.py
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and latex-tikz codes from

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https://github.com/Dhatri−nanda/EE3900/blob/main
/Assignment_5/Assignment_5.tex
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1 Quadratic Forms 2.37

Prove that the parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts

2 Solution

Lemma 2.1. *The points of intersection of* ***Line*** L : $\mathbf{x} = \mathbf{q} + \mu\mathbf{m}$ *with* ***parabola***

$$\mathbf{x}^{\mathbf{T}}\mathbf{V}\mathbf{x} + 2\mathbf{u}^{\mathbf{T}}\mathbf{x} + f = 0 \tag{2.0.1}$$

is given by:

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{2.0.2}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\Big) \tag{2.0.3}$$

Proof. The points of intersection must satisfy the line and parabola equation. Thus,

$$(\mathbf{q} + \mu\mathbf{m})^{\mathbf{T}}\mathbf{V}(\mathbf{q} + \mu\mathbf{m}) + 2\mathbf{u}^{\mathbf{T}}(\mathbf{q} + \mu\mathbf{m}) + f = 0 \tag{2.0.4}$$

On expansion, we get

$$\mu^2\mathbf{m}^{\mathbf{T}}\mathbf{V}\mathbf{m} + \mu\left[\mathbf{m}^{\mathbf{T}}\mathbf{V}\mathbf{q} + \mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{m} + 2\mathbf{u}^{\mathbf{T}}\mathbf{m}\right] + \mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{q} + 2\mathbf{u}^{\mathbf{T}}\mathbf{m} + f = 0 \tag{2.0.5}$$

Since, $\mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{m}, 2\mathbf{u}^{\mathbf{T}}\mathbf{m}$ are scalars

$$\mathbf{q}^{\mathbf{T}}\mathbf{V}\mathbf{m} = \mathbf{m}^{\mathbf{T}}\mathbf{V}^{\mathbf{T}}\mathbf{q} \tag{2.0.6}$$

$$2\mathbf{u}^{\mathbf{T}}\mathbf{m} = 2\mathbf{m}^{\mathbf{T}}\mathbf{u} \tag{2.0.7}$$

Solving the above quadratic equation we get

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q} + \mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\Big) \tag{2.0.8}$$

□

First we consider the parabola $x^2 = 4y$

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \tag{2.0.9}$$

Now, we find the points of intersections with the square

The first line is

$$y = 4 \tag{2.0.10}$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.11}$$

$$\implies \mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.12}$$

From (2.0.3),

$$\mu_1 = 4, \mu_2 = -4 \tag{2.0.13}$$

Substituting μ_1 and μ_2 in (2.0.12),the points of intersection

$$\mathbf{M_1} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{P_1} = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \tag{2.0.14}$$

The next line is

$$y = 0 \tag{2.0.15}$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.16}$$

$$\implies \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.17}$$

From (2.0.3),

$$\mu = 0 \tag{2.0.18}$$

Substituting μ in (2.0.17),the point of intersection

$$\mathbf{K_1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.19}$$

Next we consider the parabola $y^2 = 4x$

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \tag{2.0.20}$$

The first line is

$$y = 4 \tag{2.0.21}$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.22}$$

$$\implies \mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.23}$$

From (2.0.3),

$$\mu = 4 \tag{2.0.24}$$

Substituting μ in (2.0.23),the points of intersection

$$\mathbf{M_2} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{2.0.25}$$

The next line is

$$y = 0 \tag{2.0.26}$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.27}$$

$$\implies \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.0.28}$$

From (2.0.3),

$$\mu = 0 \tag{2.0.29}$$

Substituting μ in (2.0.28),the point of intersection

$$\mathbf{K_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.30}$$

So, both the parabolas intersect each other and the given square at the points

$$\mathbf{K} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \tag{2.0.31}$$

Area of the square i.e., A_1

$$A_1 = Ar(KLMNK) \tag{2.0.32}$$

$$= \int_0^4 4dx \tag{2.0.33}$$

$$= 16 \tag{2.0.34}$$

Area under the parabola $y^2 = 4x$ i.e., A_2

$$A_2 = \int_0^4 \sqrt{4x}dx \tag{2.0.35}$$

$$= \frac{1}{6}((16)^{3/2} - 0) \tag{2.0.36}$$

$$= \frac{32}{3} \tag{2.0.37}$$

Area under the parabola $x^2 = 4y$ i.e., A_3

$$A_3 = \int_0^4 \frac{x^2}{4} \tag{2.0.38}$$

$$= \frac{1}{12}((4)^3 - 0) \tag{2.0.39}$$

$$= \frac{16}{3} \tag{2.0.40}$$

As shown in the figure, as the parabolas divide the square into 3 parts

Area of first region is

$$A = A_1 - A_2 \tag{2.0.41}$$

$$= 16 - \frac{32}{3} \tag{2.0.42}$$

$$= \frac{16}{3} \tag{2.0.43}$$

Area of the second region is

$$B = A_2 - A_3 \tag{2.0.44}$$

$$= \frac{32}{3} - \frac{16}{3} \tag{2.0.45}$$

$$= \frac{16}{3} \tag{2.0.46}$$

Area of the third region is

$$C = A_3 \tag{2.0.47}$$

$$= \frac{16}{3} \tag{2.0.48}$$

So, the 2 parabolas divide the area bounded by the square into 3 equal parts.

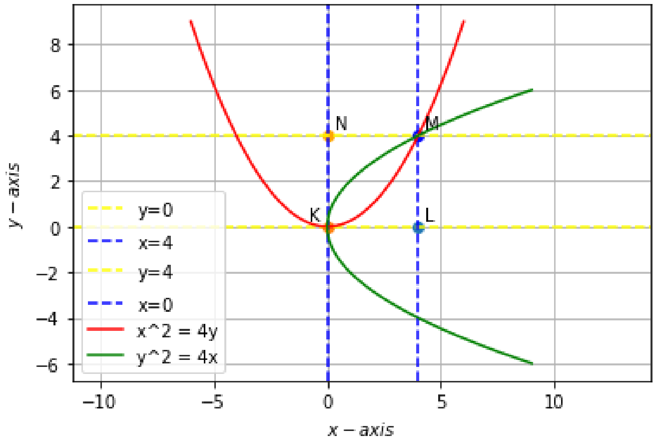

Fig. 0: plot