

Assignment 5

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AI20BTECH11002

Download all python codes from

https://github.com/Dhatri-nanda/EE3900/blob/main/Assignment_5/code.py

and latex-tikz codes from

https://github.com/Dhatri-nanda/EE3900/blob/main/Assignment_5/Assignment_5.tex

1 QUADRATIC FORMS 2.37

Prove that the parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts

2 SOLUTION

Lemma 2.1. *The points of intersection of **Line** $L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m}$ with **parabola***

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

is given by:

$$\mathbf{x}_i = \mathbf{q} + \mu_i \mathbf{m} \quad (2.0.2)$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \quad (2.0.3)$$

Proof. The points of intersection must satisfy the line and parabola equation. Thus,

$$(\mathbf{q} + \mu \mathbf{m})^T \mathbf{V} (\mathbf{q} + \mu \mathbf{m}) + 2\mathbf{u}^T (\mathbf{q} + \mu \mathbf{m}) + f = 0 \quad (2.0.4)$$

On expansion, we get

$$\mu^2 \mathbf{m}^T \mathbf{V} \mathbf{m} + \mu [\mathbf{m}^T \mathbf{V} \mathbf{q} + \mathbf{q}^T \mathbf{V} \mathbf{m} + 2\mathbf{u}^T \mathbf{m}] + \mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{m} + f = 0 \quad (2.0.5)$$

Since, $\mathbf{q}^T \mathbf{V} \mathbf{m}$, $2\mathbf{u}^T \mathbf{m}$ are scalars

$$\mathbf{q}^T \mathbf{V} \mathbf{m} = \mathbf{m}^T \mathbf{V}^T \mathbf{q} \quad (2.0.6)$$

$$2\mathbf{u}^T \mathbf{m} = 2\mathbf{m}^T \mathbf{u} \quad (2.0.7)$$

Solving the above quadratic equation we get

$$\mu_i = \frac{1}{\mathbf{m}^T \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^T (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^T \mathbf{V} \mathbf{q} + 2\mathbf{u}^T \mathbf{q} + f)(\mathbf{m}^T \mathbf{V} \mathbf{m})} \right) \quad (2.0.8)$$

□

First we consider the parabola $x^2 = 4y$

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}, f = 0 \quad (2.0.9)$$

Now, we find the points of intersections with the square

The first line is

$$y = 4 \quad (2.0.10)$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (2.0.11)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.12)$$

From (2.0.3),

$$\mu_1 = 4, \mu_2 = -4 \quad (2.0.13)$$

Substituting μ_1 and μ_2 in (2.0.12), the points of intersection

$$\mathbf{M}_1 = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \mathbf{P}_1 = \begin{pmatrix} -4 \\ 4 \end{pmatrix} \quad (2.0.14)$$

The next line is

$$y = 0 \quad (2.0.15)$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (2.0.16)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.17)$$

From (2.0.3),

$$\mu = 0 \quad (2.0.18)$$

Substituting μ in (2.0.17), the point of intersection

$$\mathbf{K}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.19)$$

Next we consider the parabola $y^2 = 4x$

The matrix parameters of the parabola are

$$\mathbf{V} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{u} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, f = 0 \quad (2.0.20)$$

The first line is

$$y = 4 \quad (2.0.21)$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (2.0.22)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.23)$$

From (2.0.3),

$$\mu = 4 \quad (2.0.24)$$

Substituting μ in (2.0.23), the points of intersection

$$\mathbf{M}_2 = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (2.0.25)$$

The next line is

$$y = 0 \quad (2.0.26)$$

The parametric form is:

$$L : \mathbf{x} = \mathbf{q} + \mu \mathbf{m} \quad (2.0.27)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (2.0.28)$$

From (2.0.3),

$$\mu = 0 \quad (2.0.29)$$

Substituting μ in (2.0.28), the point of intersection

$$\mathbf{K}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.30)$$

So, both the parabolas intersect each other and the given square at the points

$$\mathbf{K} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{M} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} \quad (2.0.31)$$

Area of the square i.e., A_1

$$A_1 = Ar(KLMNK) \quad (2.0.32)$$

$$= \int_0^4 4dx \quad (2.0.33)$$

$$= 16 \quad (2.0.34)$$

Area under the parabola $y^2 = 4x$ i.e., A_2

$$A_2 = \int_0^4 \sqrt{4x} dx \quad (2.0.35)$$

$$= \frac{1}{6} ((16)^{3/2} - 0) \quad (2.0.36)$$

$$= \frac{32}{3} \quad (2.0.37)$$

Area under the parabola $x^2 = 4y$ i.e., A_3

$$A_3 = \int_0^4 \frac{x^2}{4} \quad (2.0.38)$$

$$= \frac{1}{12} ((4)^3 - 0) \quad (2.0.39)$$

$$= \frac{16}{3} \quad (2.0.40)$$

As shown in the figure, as the parabolas divide the square into 3 parts

Area of first region is

$$A = A_1 - A_2 \quad (2.0.41)$$

$$= 16 - \frac{32}{3} \quad (2.0.42)$$

$$= \frac{16}{3} \quad (2.0.43)$$

Area of the second region is

$$B = A_2 - A_3 \quad (2.0.44)$$

$$= \frac{32}{3} - \frac{16}{3} \quad (2.0.45)$$

$$= \frac{16}{3} \quad (2.0.46)$$

Area of the third region is

$$C = A_3 \quad (2.0.47)$$

$$= \frac{16}{3} \quad (2.0.48)$$

So, the 2 parabolas divide the area bounded by the square into 3 equal parts.

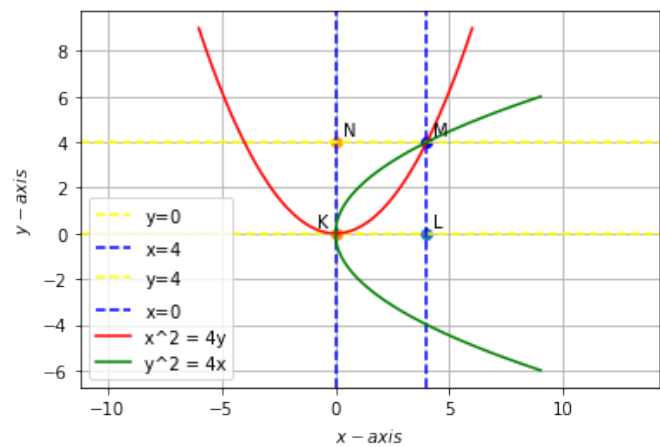


Fig. 0: plot

Our results:

Parabola	\mathbf{V}	\mathbf{u}	f	\mathbf{q}	\mathbf{m}	μ_1	μ_2	\mathbf{POI}_1	\mathbf{POI}_2
$x^2 = 4y$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$	0	$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	4	-4	$\begin{pmatrix} 4 \\ 4 \end{pmatrix}$	$\begin{pmatrix} -4 \\ 4 \end{pmatrix}$
$x^2 = 4y$	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$	0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	0	-	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-
$y^2 = 4x$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	0	$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	4	-	$\begin{pmatrix} 4 \\ 4 \end{pmatrix}$	-
$y^2 = 4x$	$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -2 \\ 0 \end{pmatrix}$	0	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	0	-	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	-