ASSIGNMENT 4

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AI20BTECH11002

Download all python codes from

https://github.com/Dhatri−nanda/EE3900/blob/main/Assignment_4/code.py

and latex-tikz codes from

https://github.com/Dhatri−nanda/EE3900/blob/main/Assignment_4/Assignment_4.tex

1 Linear Forms 2.35

Find the shortest distance between the lines

$$x = \begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \tag{1.0.1}$$

and

$$x = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \tag{1.0.2}$$

2 Solution

We have,

$$L_1 : \mathbf{x} = \mathbf{a_1} + \lambda_1 \mathbf{b_1} \tag{2.0.1}$$

$$L_2 : \mathbf{x} = \mathbf{a_2} + \lambda_2 \mathbf{b_2} \tag{2.0.2}$$

where $\mathbf{a_i}, \mathbf{b_i}$ are positional vector, slope vector of line L_i respectively.
As $\mathbf{b_1} \neq k\mathbf{b_2}$, the lines are not parallel to each other.
Let us assume that L_1 and L_2 intersect at a point. Therefore,

$$\begin{pmatrix} 6 \\ 2 \\ 2 \end{pmatrix} + \lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \tag{2.0.3}$$

$$\lambda_1 \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -2 \\ -3 \end{pmatrix} \tag{2.0.4}$$

$$\begin{pmatrix} 1 & -3 \\ -2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} -10 \\ -2 \\ -3 \end{pmatrix} \tag{2.0.5}$$

The augmented matrix for above equation in row reduced form

$$\begin{pmatrix} 1 & -3 & -10 \\ -2 & 2 & -2 \\ 2 & 2 & -3 \end{pmatrix} \xleftrightarrow[R_3 \leftarrow R_3 - 2R_1]{R_2 \leftarrow R_2 + 2R_1} \begin{pmatrix} 1 & -3 & -10 \\ 0 & -4 & -22 \\ 0 & 8 & 17 \end{pmatrix}$$

$$\xleftrightarrow{R_3 \leftarrow R_3 + 2R_2} \begin{pmatrix} 1 & -3 & -10 \\ 0 & -4 & -22 \\ 0 & 0 & -27 \end{pmatrix}$$

$\therefore$ The rank of the matrix = 3. Hence the lines do not intersect.
L_1 and L_2 are skew lines.
Let d be the shortest distance and $\mathbf{p_1}, \mathbf{p_2}$ be positional vectors of its end points. For d to be shortest, we know that,

$$\mathbf{b_1}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{2.0.6}$$

$$\mathbf{b_2}^\top (\mathbf{p_2} - \mathbf{p_1}) = 0 \tag{2.0.7}$$

$$\mathbf{b_1}^\top ((\mathbf{a}_2 - \mathbf{a}_1)) + \begin{pmatrix} \mathbf{b_2} & \mathbf{b}_1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{2.0.8}$$

$$\mathbf{b_2}^\top ((\mathbf{a}_2 - \mathbf{a}_1)) + \begin{pmatrix} \mathbf{b_2} & \mathbf{b_1} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} \tag{2.0.9}$$

Let

$$\mathbf{B} = \begin{pmatrix} \mathbf{b_2} & \mathbf{b}_1 \end{pmatrix} \qquad \mathbf{B}^\top = \begin{pmatrix} \mathbf{b_2}^\top \\ \mathbf{b_1}^\top \end{pmatrix} \tag{2.0.10}$$

By combining equations (2.0.8) and (2.0.9) and writing in terms of $\mathbf{B}$ and $\mathbf{B}^\top$ using (2.0.10), we get

$$\mathbf{B}^\top \mathbf{B} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \mathbf{B}^\top (\mathbf{a}_1 - \mathbf{a}_2) \tag{2.0.11}$$

By putting the values of a_1, a_2, b_1, b_2 in (2.0.11), we geT

$$\begin{pmatrix} 17 & 3 \\ 3 & 9 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 12 \end{pmatrix} \tag{2.0.12}$$

Solving (2.0.12), we get

$$\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.13}$$

Substituting the value of λ_1 and λ_2 in (2.0.1) and

(2.0.2), we get

$$\mathbf{p_1} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix} \qquad \mathbf{p_2} = \begin{pmatrix} -7 \\ 2 \\ 1 \end{pmatrix} \tag{2.0.14}$$

Hence, the shortest distance between these two skew lines is

$$d = \|\mathbf{p_2} - \mathbf{p_1}\| = 14.457 \tag{2.0.15}$$

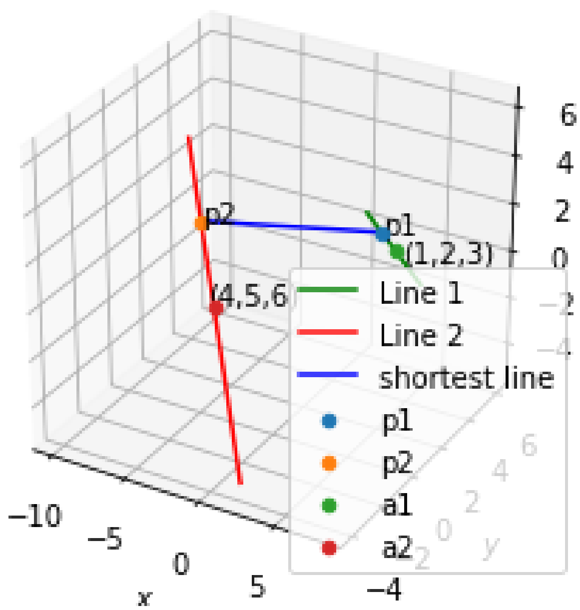

Fig. 0: plot