Assignment 10: Papoulis Chapter 6

Dhatri Reddy

May 31, 2022

Outline

Question

Problem 6.19

The random variables x and y are independent with Rayleigh densities

$f_x(x) = \frac{x}{\alpha^2} e^{-\frac{x^2}{2\alpha^2}} U(x)$ and $f_y(y) = \frac{y}{\beta^2} e^{-\frac{y^2}{2\beta^2}} U(y)$

(a) Show that if $z = \frac{x}{y}$, then

$$f_z(z) = \frac{2\alpha^2}{\beta^2} \frac{z}{\left(z^2 + \frac{\alpha^2}{\beta^2}\right)^2} U(z) \tag{1}$$

(b) Using (1), show that for any $k > 0$, $P\{x \leq ky\} = \frac{k^2}{k^2 + \frac{\alpha^2}{\beta^2}}$

Solution

(a)

$$f_z(z) = \int_{-\infty}^{\infty} |y| \, f_x(yz) \, f_y(y) \, dy \tag{2}$$

$$= \int_{0}^{\infty} y \times \frac{yz}{\alpha^2} e^{-\frac{yz^2}{2\alpha^2}} \times \frac{y}{\beta^2} e^{-\frac{y^2}{2\beta^2}} dy \tag{3}$$

$$= \frac{z}{\alpha^2\beta^2} \int_{0}^{\infty} y^3 e^{-cy^2} dy \tag{4}$$

$$= \frac{z}{2\alpha^2\beta^2 \left(\frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2}\right)^2} \tag{5}$$

$$= \frac{2\alpha^2}{\beta^2} \frac{z}{\left(z^2 + \frac{\alpha^2}{\beta^2}\right)^2} \tag{6}$$

Solution

(b)

$$F_z(z) = \int_0^z \frac{2\alpha^2}{\beta^2} \frac{z}{\left(z^2 + \frac{\alpha^2}{\beta^2}\right)^2} dz \tag{7}$$

$$= \frac{\alpha^2}{\beta^2} \int_{\frac{\alpha^2}{\beta^2}}^{\frac{\alpha^2}{\beta^2}+z^2} \frac{dt}{t^2} \tag{8}$$

$$= \frac{z^2}{z^2 + \frac{\alpha^2}{\beta^2}} \tag{9}$$

$$= P\left\{x \le zy\right\} \tag{10}$$

Hence, proved.