Assignment 11: Papoulis Chapter 7

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Outline

Question

Problem 7.24

Show that if the random variables x_i are of continuous type and independent, then, for sufficiently large n, the density of $\sin\{x_1 + \ldots + x_n\}$ is nearly equal to density of $\sin x$, where x is a random variable uniform in the interval $(-\pi, \pi)$

Solution

The density $f_z(z)$ of the sum $z = x_1 + ... + x_n$ tends to a normal curve with variance $\sigma_1^2 + ... + \sigma_n^2 \to \infty$ as $n \to \infty$.
x is uniform in the interval $(-\pi, \pi)$ and $y = \sin x$. In this case

Solving

$$\phi_y(\omega) = \int_{-\infty}^{\infty} e^{j\omega \sin x} f(x)\, dx \tag{1}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega \sin x} dx \tag{2}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\frac{-\pi}{2}} e^{j\omega \sin x} + \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{j\omega \sin x} dx + \int_{\frac{\pi}{2}}^{0} e^{j\omega \sin x} dx \tag{3}$$

$$dy = \cos x dx = \left(1 - y^2\right)^{\frac{1}{2}} dx \tag{4}$$

$$\phi_y(\omega) = \frac{1}{2\pi}\int_{0}^{-1} e^{j\omega}\left(1 - y^2\right)^{-\frac{1}{2}} dy + \int_{-1}^{1} e^{j\omega}\left(1 - y^2\right)^{-\frac{1}{2}} dy \tag{5}$$

$$+ \int_{1}^{0} e^{j\omega}\left(1 - y^2\right)^{-\frac{1}{2}} dy \tag{6}$$

Solution

This leads to the conclusion that

$$\phi_y(\omega) = \frac{1}{\pi}\int_{-1}^{1} e^{j\omega}\left(1-y^2\right)^{-\frac{1}{2}} dy \tag{7}$$

$$= \int_{-1}^{1} e^{j\omega}\frac{1}{\pi\left(1-y^2\right)^{\frac{1}{2}}} dy \tag{8}$$

$$f_y(y) = \frac{1}{\pi\left(1-y^2\right)^{\frac{1}{2}}} \tag{9}$$