Digital Signal Processing

EE3900: Linear Systems and Signal Processing

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1. Software Installation

Install the necessary packages by running the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2. Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There is a lot of background noise and the key strokes are audible. This noise is represented by the large blue and red regions spread from 440 Hz to beyond 18.9 kHz. The key tones are represented by the yellow lines that are present in the lower regions between 440 Hz and 5.1 kHz.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Download the python code for the reduction of noise by executing the following command

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/2.3.py
```

Run the code by executing

```
python3 2.3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_Reduced_Noise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe? **Solution:** The noise has been reduced considerably and the key strokes are not audible anymore. The blue region is restricted between 440 Hz and 5.1 kHz and there are no signals beyond this range.

3. Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$
Solution: Download the following Python code that plots Fig. 3.2.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/3.2.py
```

Run the code by executing

```
python3 3.2.py
```

3.3 Repeat the above exercise using a C code.
Solution: Download the following C code that generates the values of $y(n)$

```
wget
```

https://github.com/Dhatrireddyy/EE3900/blob/main/codes
Compile and run the C program by executing the following

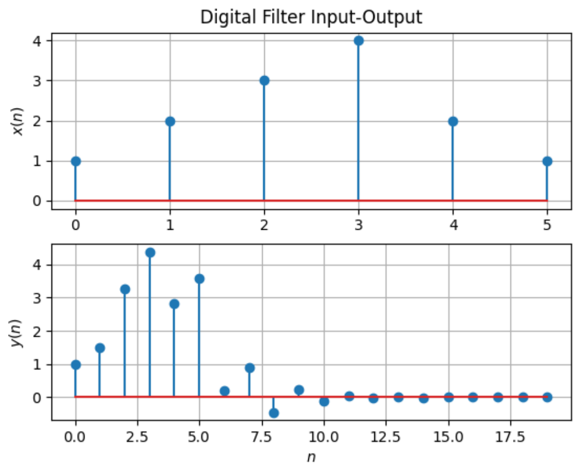

Fig. 3.2. The sketches of $x(n)$ and $y(n)$

```
gcc 3.3.c
./a.out
```

Download the following Python code that plots Fig. 3.3 using the data generated by the above C code

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/3.3.py
```

Run the code by executing

```
python3 3.3.py
```

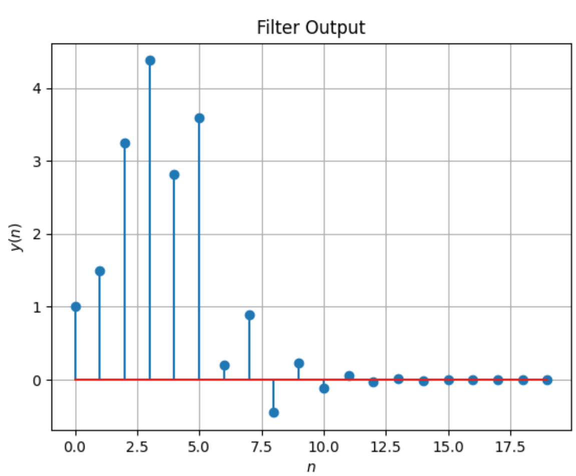

Fig. 3.3. Plot of $y(n)$

4. Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution:

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

Substitute $n-1 = p$

$$\mathcal{Z}\{x(n-1)\} = \sum_{p=-\infty}^{\infty} x(p)z^{-(p+1)} \tag{4.5}$$

$$= z^{-1}\sum_{m=-\infty}^{\infty} x(p)z^{-p} \tag{4.6}$$

$$= z^{-1}\mathcal{Z}\{x(m)\} \tag{4.7}$$

$$= z^{-1}X(z) \tag{4.8}$$

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.9}$$

$$= \sum_{m=-\infty}^{\infty} x(p)z^{-(p+k)} \tag{4.10}$$

$$= z^{-k}\sum_{m=-\infty}^{\infty} x(p)z^{-p} \tag{4.11}$$

$$= z^{-k}X(z) \tag{4.12}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.2
Solution: For the $x(n)$ given in (3.2)

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.13}$$

$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.14}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.15}$$

Also

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.16}$$

$$\mathcal{Z}\{x(n-k)\} = z^{-k} + 2z^{-(k+1)} + 3z^{-(k+2)} + 4z^{-(k+3)} + 2z^{-(k+4)} + z^{-(k+5)} \tag{4.17}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.18}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution:

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{4.19}$$

On applying the Z-transform on both sides of the equation, we get

$$\mathcal{Z}\{y(n) + \frac{1}{2}y(n-1)\} = \mathcal{Z}\{x(n) + x(n-2)\} \tag{4.20}$$

Since we are assuming that the Z-transform is a linear operation,

$$\mathcal{Z}\{y(n)\} + \frac{1}{2}\mathcal{Z}\{y(n-1)\} = \mathcal{Z}\{x(n)\} + \mathcal{Z}\{x(n-2)\} \tag{4.21}$$

$$\implies Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.22}$$

$$\implies Y(z)\left(1 + \frac{1}{2}z^{-1}\right) = X(z)(1 + z^{-2}) \tag{4.23}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.24}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.25}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.26}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{4.27}$$

Solution:

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.28}$$

$$= \delta(0)z^{-0} \tag{4.29}$$

$$= 1 \tag{4.30}$$

$$\mathcal{Z}\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)z^{-n} \tag{4.31}$$

$$= \sum_{n=0}^{\infty} z^{-n} \tag{4.32}$$

The above sum converges when

$$\left|z^{-1}\right| < 1 \iff |z| > 1 \tag{4.33}$$

Hence,

$$U(z) = \mathcal{Z}\{u(n)\} = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{4.34}$$

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}}, |z| > |a| \tag{4.35}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.36}$$

$$= \sum_{n=0}^{\infty} a^n z{-}n \tag{4.37}$$

The above sum converges when

$$\left|az^{-1}\right| < 1 \iff |z| > |a| \tag{4.38}$$

Hence,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}}, |z| > |a| \tag{4.39}$$

4.6 Let

$$H(e^{\jmath\omega}) = H(z = e^{\jmath\omega}). \tag{4.40}$$

Plot $|H(e^{\jmath\omega})|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discrete-Time Fourier Transform* ($DTFT$) of $x(n)$

Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.41}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.42}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.43}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.44}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.45}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.46}$$

Download the following Python code that plots Fig. 4.6.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/4.5.py
```

Run the code by executing

```
python3 4.5.py
```

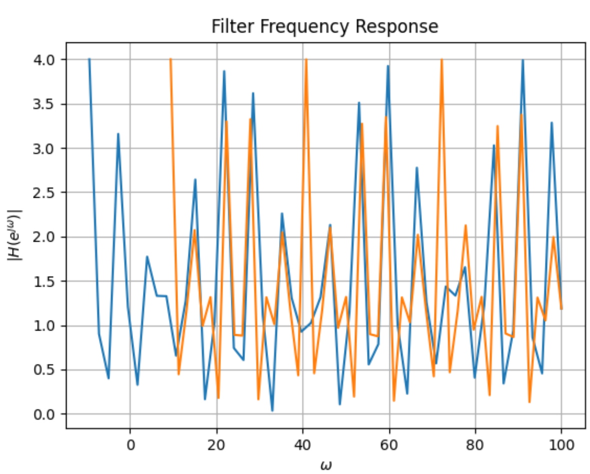

Fig. 4.6. The plot of magnitude of DTFT of $x(n)$

From the plot, it is clear that the magnitude of the DTFT of $x(n)$ is an even function and is periodic with a period of 2π.

It attains a maximum value of 4 at

$$x = (2n + 1)\pi, n \in \mathbb{Z} \tag{4.47}$$

and a minimum of 0 at

$$x = (2m + 1)\frac{\pi}{2}, m \in \mathbb{Z} \tag{4.48}$$

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$
Solution: $h(n)$ is given by the inverse DTFT (IDTFT) of $H\left(e^{\jmath\omega}\right)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{\jmath\omega n} d\omega \tag{4.49}$$

5. Impulse Response

5.1 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.1}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (**??**)
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

$$= \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.3}$$

From (4.35),

$$\frac{1}{1 - az^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} a^n u(n), |z| > |a| \tag{5.4}$$

$$\implies \frac{1}{1 + \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) \quad |z| > \frac{1}{2} \tag{5.5}$$

$$\implies \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \tag{5.6}$$

Since the Z-transform is a linear operator, for $|z| > \frac{1}{2}$

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.7}$$

Hence,

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.8}$$

5.2 Sketch $h(n)$. Is it bounded? Convergent?
Solution: Download the following Python code that plots Fig. 5.2.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/5.2.py
```

Run the code by executing

```
python3 5.2.py
```

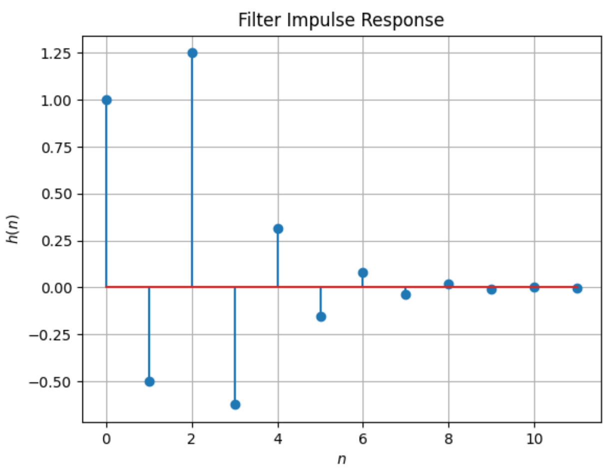

Fig. 5.2. The plot of $h(n)$

From the plot, it is clear that the sequence is convergent to 0, which implies that it is bounded as well.

5.3 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.9}$$

Is the system defined by (**??**) stable for the impulse response in (**??**)?

Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.10}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \tag{5.11}$$

These are both sums of infinite geometric progressions with first terms 1 and common ratios $-\frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \tag{5.12}$$

$$= \frac{4}{3} < \infty \tag{5.13}$$

Therefore, the system is stable.

5.4 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \tag{5.14}$$

This is the definition of $h(n)$

Solution:

$$h(0) = 1 \tag{5.15}$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \tag{5.16}$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \tag{5.17}$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \tag{5.18}$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{3}{2} \tag{5.19}$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \tag{5.20}$$

$$h(3) = \frac{3}{2}\left(-\frac{1}{2}\right) \tag{5.21}$$

$$h(4) = \frac{3}{2}\left(-\frac{1}{2}\right)^2 \tag{5.22}$$

$$\vdots \tag{5.23}$$

$$h(n) = \frac{3}{2}\left(-\frac{1}{2}\right)^{n-2} \tag{5.24}$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{3}{2}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.25}$$

Thus, it is bounded and convergent to 0

$$\lim_{n\to\infty} h(n) = 0 \tag{5.26}$$

Download the following Python code that plots Fig. 5.4.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/5.4.py
```

Run the code by executing

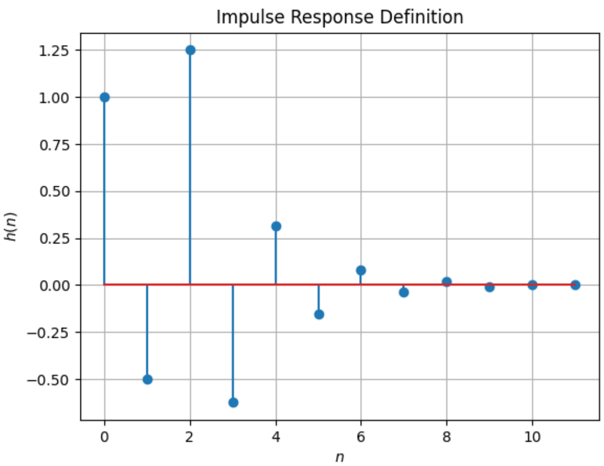

Fig. 5.4. Plot of $h(n)$

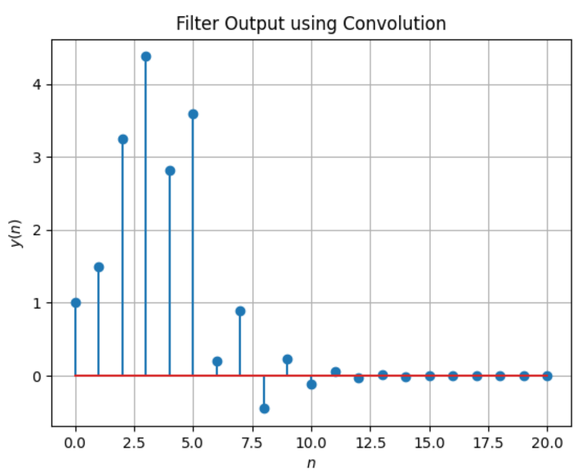

Fig. 4. Plot of the convolution of $x(n)$ and $h(n)$

```
python3 5.4.py
```

The plot is exactly the same as that obtained in Fig. 3.2. Therefore, we can conclude that

$$y(n) = x(n) * h(n) \tag{5.27}$$

Download the following code for computing the convolution by using a Toeplitz matrix

```
wget https://github.com/Ankit-Saha-2003/EE3900
    /raw/main/Assignment_1/codes/5.5-toeplitz.py
```

Run the code by executing

```
python 5.5-toeplitz.py
```

figs/5.5-toeplitz.png

Fig. 4. Plot of the convolution of $x(n)$ and $h(n)$

Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.28}$$

Solution: We know that

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.29}$$

Substitute $k = n - i$

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \tag{5.30}$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \tag{5.31}$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \tag{5.32}$$

since the order of limits does not matter for a summation. Thus,

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.33}$$

$$\implies x(n) * h(n) = h(n) * x(n) \tag{5.34}$$

Therefore, convolution is commutative.