Digital Signal Processing

EE3900: Linear Systems and Signal Processing

Indian Institute of Technology Hyderabad

Velma Dhatri Reddy
AI21BTECH11030

3 Aug 2022

1. Software Installation

Install the necessary packages by running the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2. Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There is a lot of background noise and the key strokes are audible. This noise is represented by the large blue and red regions spread from 440 Hz to beyond 18.9 kHz. The key tones are represented by the yellow lines that are present in the lower regions between 440 Hz and 5.1 kHz.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: Download the python code for the reduction of noise by executing the following command

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/2.3.py
```

Run the code by executing

```
python3 2.3.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_Reduced_Noise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe? **Solution:** The noise has been reduced considerably and the key strokes are not audible anymore. The blue region is restricted between 440 Hz and 5.1 kHz and there are no signals beyond this range.

3. Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$
Solution: Download the following Python code that plots Fig. 3.2.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/3.2.py
```

Run the code by executing

```
python3 3.2.py
```

3.3 Repeat the above exercise using a C code.
Solution: Download the following C code that generates the values of $y(n)$

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/3.3.c
```

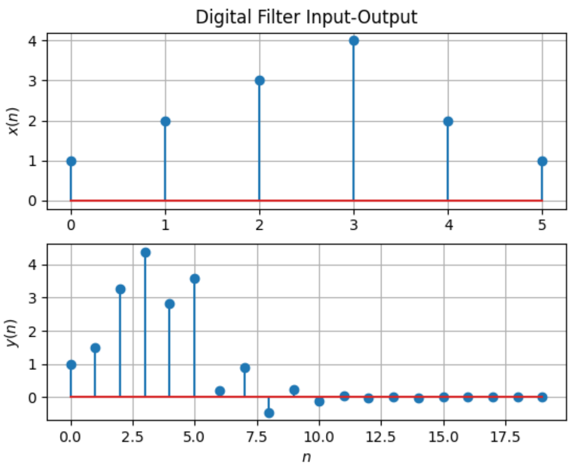

Fig. 3.2. The sketches of $x(n)$ and $y(n)$

Compile and run the C program by executing the following

```
gcc 3.3.c
./a.out
```

Download the following Python code that plots Fig. 3.3 using the data generated by the above C code

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/3.3.py
```

Run the code by executing

```
python3 3.3.py
```

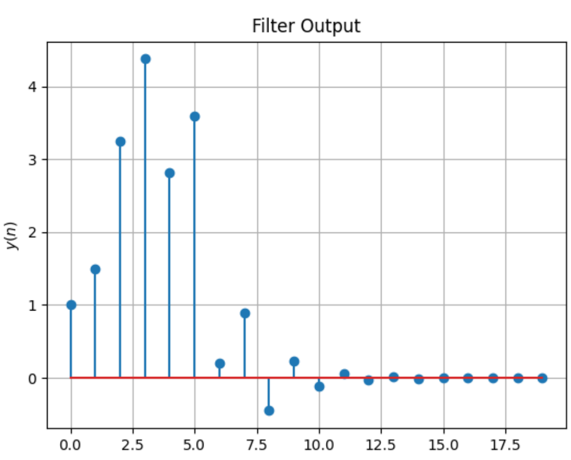

Fig. 3.3. Plot of $y(n)$

4. Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution:

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

Substitute $n-1=p$

$$\mathcal{Z}\{x(n-1)\} = \sum_{p=-\infty}^{\infty} x(p)z^{-(p+1)} \tag{4.5}$$

$$= z^{-1}\sum_{m=-\infty}^{\infty} x(p)z^{-p} \tag{4.6}$$

$$= z^{-1}\mathcal{Z}\{x(m)\} \tag{4.7}$$

$$= z^{-1}X(z) \tag{4.8}$$

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \tag{4.9}$$

$$= \sum_{m=-\infty}^{\infty} x(p)z^{-(p+k)} \tag{4.10}$$

$$= z^{-k}\sum_{m=-\infty}^{\infty} x(p)z^{-p} \tag{4.11}$$

$$= z^{-k}X(z) \tag{4.12}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.2
Solution: For the $x(n)$ given in (3.2)

$$X(z) = \mathcal{Z}\{x(n)\} \tag{4.13}$$

$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.14}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.15}$$

Also

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.16}$$

$$\mathcal{Z}\{x(n-k)\} = z^{-k} + 2z^{-(k+1)} + 3z^{-(k+2)} + 4z^{-(k+3)} + 2z^{-(k+4)} + z^{-(k+5)} \tag{4.17}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.18}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution:

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2) \tag{4.19}$$

On applying the Z-transform on both sides of the equation, we get

$$\mathcal{Z}\{y(n) + \frac{1}{2}y(n-1)\} = \mathcal{Z}\{x(n) + x(n-2)\} \tag{4.20}$$

Since we are assuming that the Z-transform is a linear operation,

$$\mathcal{Z}\{y(n)\} + \frac{1}{2}\mathcal{Z}\{y(n-1)\} = \mathcal{Z}\{x(n)\} + \mathcal{Z}\{x(n-2)\} \tag{4.21}$$

$$\implies Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.22}$$

$$\implies Y(z)\left(1 + \frac{1}{2}z^{-1}\right) = X(z)(1 + z^{-2}) \tag{4.23}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.24}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.25}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.26}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{4.27}$$

Solution:

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.28}$$

$$= \delta(0)z^{-0} \tag{4.29}$$

$$= 1 \tag{4.30}$$

$$\mathcal{Z}\{u(n)\} = \sum_{n=-\infty}^{\infty} u(n)z^{-n} \tag{4.31}$$

$$= \sum_{n=0}^{\infty} z^{-n} \tag{4.32}$$

The above sum converges when

$$\left|z^{-1}\right| < 1 \iff |z| > 1 \tag{4.33}$$

Hence,

$$U(z) = \mathcal{Z}\{u(n)\} = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{4.34}$$

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}}, |z| > |a| \tag{4.35}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.36}$$

$$= \sum_{n=0}^{\infty} a^n z{-}n \tag{4.37}$$

The above sum converges when

$$\left|az^{-1}\right| < 1 \iff |z| > |a| \tag{4.38}$$

Hence,

$$\mathcal{Z}\{a^n u(n)\} = \frac{1}{1 - az^{-1}}, |z| > |a| \tag{4.39}$$

4.6 Let

$$H(e^{\jmath\omega}) = H(z = e^{\jmath\omega}). \tag{4.40}$$

Plot $|H(e^{\jmath\omega})|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discrete-Time Fourier Transform* (*DTFT*) of $x(n)$

Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1+e^{-2\jmath\omega}}{1+\frac{1}{2}e^{-\jmath\omega}} \tag{4.41}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1+\cos 2\omega - \jmath\sin 2\omega\right|}{\left|1+\frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.42}$$

$$= \sqrt{\frac{(1+\cos 2\omega)^2+(\sin 2\omega)^2}{(1+\frac{1}{2}\cos\omega)^2+(\frac{1}{2}\sin\omega)^2}} \tag{4.43}$$

$$= \sqrt{\frac{2+2\cos 2\omega}{\frac{5}{4}+\cos\omega}} \tag{4.44}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5+4\cos\omega}} \tag{4.45}$$

$$= \frac{4\left|\cos\omega\right|}{\sqrt{5+4\cos\omega}} \tag{4.46}$$

Download the following Python code that plots Fig. 4.6.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/4.5.py
```

Run the code by executing

```
python3 4.5.py
```

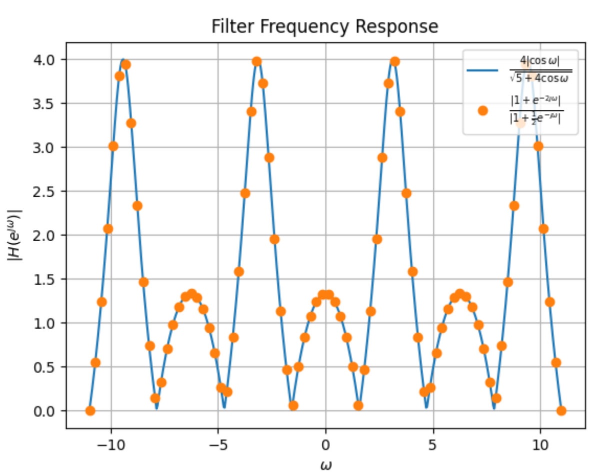

Fig. 4.6. The plot of magnitude of DTFT of $x(n)$

From the plot, it is clear that the magnitude of the DTFT of $x(n)$ is an even function and is periodic with a period of 2π.

It attains a maximum value of 4 at

$$x = (2n+1)\pi, n \in \mathbb{Z} \tag{4.47}$$

and a minimum of 0 at

$$x = (2m+1)\frac{\pi}{2}, m \in \mathbb{Z} \tag{4.48}$$

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$

Solution:

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.49}$$

$$= \int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-\jmath\omega k}e^{\jmath\omega n}d\omega \tag{4.50}$$

$$= \sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}d\omega \tag{4.51}$$

Now,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)}d\omega = \begin{cases}\int_{-\pi}^{\pi} d\omega & n-k=0\\ \frac{\exp(\jmath\omega(n-k))}{\jmath(n-k)}\Big|_{-\pi}^{\pi} & n-k\neq 0\end{cases} \tag{4.52}$$

$$= \begin{cases}2\pi & n-k=0\\ 0 & n-k\neq 0\end{cases} \tag{4.53}$$

$$= 2\pi\delta(n-k) \tag{4.54}$$

Thus,

$$\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega = 2\pi\sum_{k=-\infty}^{\infty} h(k)\delta(n-k) \tag{4.55}$$

$$= 2\pi h(n) * \delta(n) \tag{4.56}$$

$$= 2\pi h(n) \tag{4.57}$$

Therefore, $h(n)$ is given by the inverse DTFT (IDTFT) of $H\left(e^{\jmath\omega}\right)$

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{\jmath\omega})e^{\jmath\omega n}d\omega \tag{4.58}$$

5. Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for $H(z)$ in (**??**)

Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.2}$$

Substitute $z^{-1} = x$

$$
\begin{array}{r}
2x-4 \\
\frac{1}{2}x+1 \overline{\smash{\big)}\, x^2 \qquad +1} \\
\underline{-x^2-2x} \\
-2x+1 \\
\underline{2x+4} \\
5
\end{array}
$$

$$\implies 1+z^{-2} = \left(1+\frac{1}{2}z^{-1}\right)\left(-4+2z^{-1}\right)+5 \tag{5.3}$$

$$\implies H(z) = -4+2z^{-1}+\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

$$\frac{5}{1+\frac{1}{2}z^{-1}} = 5\left(1+\frac{1}{2}z^{-1}\right)^{-1} \tag{5.5}$$

$$= 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^{n} \tag{5.6}$$

$$H(z) = -4+2z^{-1}+5-\frac{5}{2}z^{-1}+\frac{5}{4}z^{-2} - \frac{5}{8}z^{-3}+\frac{5}{16}z^{-4}-\frac{5}{32}z^{-5}+\cdots \tag{5.7}$$

$$H(z) = 1-\frac{1}{2}z^{-1}+\frac{5}{4}z^{-2} - \frac{5}{8}z^{-3}+\frac{5}{16}z^{-4}-\frac{5}{32}z^{-5}+\cdots \tag{5.8}$$

But

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \tag{5.9}$$

Therefore, by comparing coefficients

$$h(n) = \begin{cases} 0 & n<0 \\ 1 & n=0 \\ -\frac{1}{2} & n=1 \\ \frac{5}{4} & n=2 \\ -\frac{5}{8} & n=3 \\ \frac{5}{16} & n=4 \end{cases} \tag{5.10}$$

We have obtained that

$$H(z) = 1-\frac{1}{2}z^{-1}+5\sum_{n=2}^{\infty}\left(-\frac{z^{-1}}{2}\right)^{n} \tag{5.11}$$

$$= 1-\frac{1}{2}z^{-1}+\sum_{n=2}^{\infty}5\left(-\frac{1}{2}\right)^{n}z^{-n} \tag{5.12}$$

By comparing coefficients,

$$h(n) = \begin{cases} 0 & n<0 \\ 1 & n=0 \\ -\frac{1}{2} & n=1 \\ 5\left(-\frac{1}{2}\right)^{n} & n\geq 2 \end{cases} \tag{5.13}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.14}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (**??**)
Solution:

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.15}$$

$$= \frac{1}{1+\frac{1}{2}z^{-1}}+\frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.16}$$

From (4.35),

$$\frac{1}{1-az^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} a^n u(n) \quad |z|>|a| \tag{5.17}$$

$$\implies \frac{1}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n} u(n) \quad |z|>\frac{1}{2} \tag{5.18}$$

$$\implies \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z|>\frac{1}{2} \tag{5.19}$$

Since the Z-transform is a linear operator, for $|z|>\frac{1}{2}$

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n} u(n)+\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.20}$$

Therefore,

$$h(n) = \left(-\frac{1}{2}\right)^{n} u(n)+\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.21}$$

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: Download the following Python code that plots Fig. 5.3.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/5.3.py
```

Run the code by executing

```
python 5.3.py
```

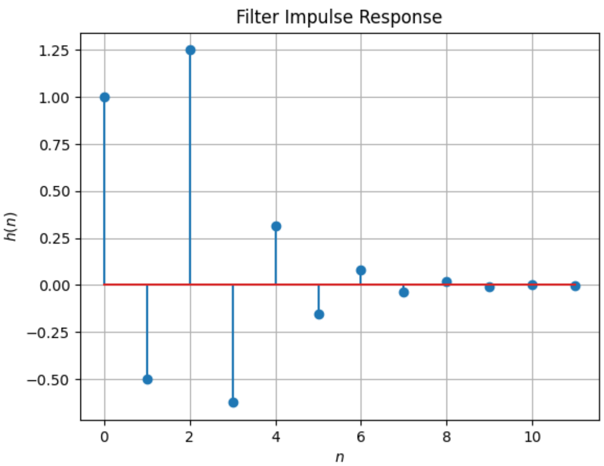

Fig. 5.3. Plot of $h(n)$

From the plot, it is clear that $h(n)$ is bounded. Theoretically,

$$|u(n)| \leq 1 \tag{5.22}$$

$$\left|\left(-\frac{1}{2}\right)^n\right| \leq 1 \tag{5.23}$$

$$\implies \left|\left(-\frac{1}{2}\right)^n u(n)\right| \leq 1 \tag{5.24}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \tag{5.25}$$

$$\implies h(n) \leq 2 \tag{5.26}$$

Therefore $h(n)$ is bounded.

5.4 Is it convergent? Justify using the ratio test.
Solution: Using the ratio test for convergence

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(\frac{1}{4}+1\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(\frac{1}{4}+1\right)}\right| \tag{5.27}$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \tag{5.28}$$

$$= \frac{1}{2} < 1 \tag{5.29}$$

Therefore, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.30}$$

Is the system defined by (**??**) stable for the impulse response in (5.14)?

Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.31}$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.32}$$

These are both sums of infinite geometric progressions with first terms 1 and common ratios $-\frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \tag{5.33}$$

$$= \frac{4}{3} < \infty \tag{5.34}$$

Therefore, the system is stable.

5.6 Verify the above result using a Python code.
Solution: The stability has been verified in the following code

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/5.6.py
```

Run the code by executing

```
python3 5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2) \tag{5.35}$$

This is the definition of $h(n)$
Solution:

$$h(0) = 1 \tag{5.36}$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \tag{5.37}$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \tag{5.38}$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \tag{5.39}$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{5}{4} \tag{5.40}$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \tag{5.41}$$

$$h(3) = \frac{5}{4}\left(-\frac{1}{2}\right) \tag{5.42}$$

$$h(4) = \frac{5}{4}\left(-\frac{1}{2}\right)^2 \tag{5.43}$$

$$\vdots \tag{5.44}$$

$$h(n) = \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} \tag{5.45}$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.46}$$

Thus, it is bounded and convergent to 0

$$\lim_{n\to\infty} h(n) = 0 \tag{5.47}$$

Download the following Python code that plots Fig. **??**.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/5.7.py
```

Run the code by executing

```
python3 5.7.py
```

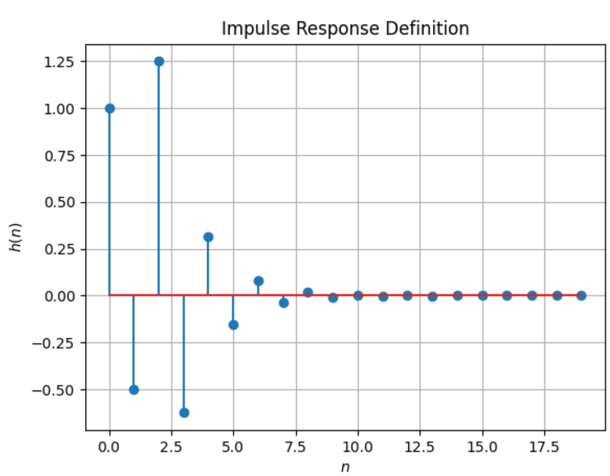

Fig. 5.7. The plot of $h(n)$ from its definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.48}$$

Comment. The operation in (5.48) is known as *convolution*
Solution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.49}$$

$$= \sum_{k=0}^{5} x(k)h(n-k) \tag{5.50}$$

since $x(k) = 0$ for $k < 0$ and $k > 5$
Download the following Python code that plots Fig. **??**.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/5.8.py
```

Run the code by executing

```
python3 5.8.py
```

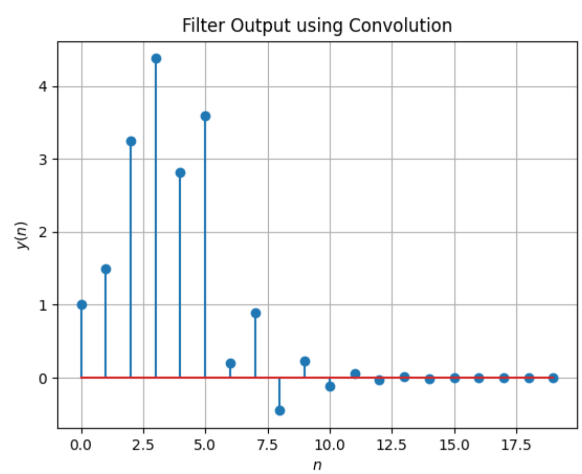

Fig. 5.8. Plot of the convolution of $x(n)$ and $h(n)$

The plot is exactly the same as that obtained in Fig. 3.2. Therefore, we can conclude that

$$y(n) = x(n) * h(n) \tag{5.51}$$

5.9 Express the above convolution using a Toeplitz matrix.

Solution: Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \tag{5.52}$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \tag{5.53}$$

and $\mathbf{x}$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.54}$$

Download the following Python code for computing the convolution by using a Toeplitz matrix and plotting Fig. 5.9

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/5.9.py
```

Run the Python code by executing

```
python3 5.9.py
```

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.55}$$

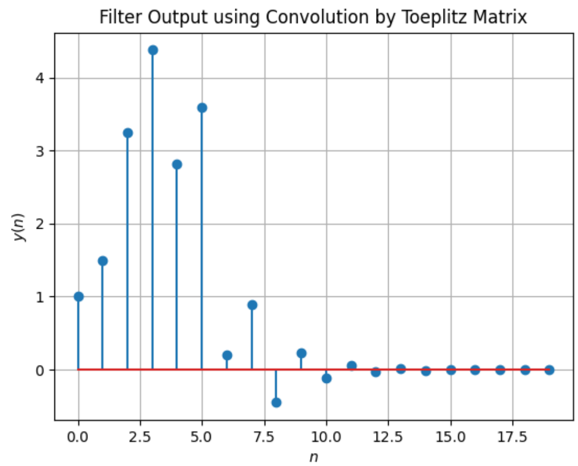

Fig. 5.9. Plot of the convolution of $x(n)$ and $h(n)$

Solution: We know that

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.56}$$

Substitute $k = n - i$

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \tag{5.57}$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \tag{5.58}$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \tag{5.59}$$

since the order of limits does not matter for a summation. Thus,

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.60}$$

$$\implies x(n) * h(n) = h(n) * x(n) \tag{5.61}$$

Therefore, convolution is commutative.

6. DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N} \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$

Solution: Download the following Python code that plots Fig. 6.1.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/6.1.py
```

Run the code by executing

```
python3 6.1.py
```

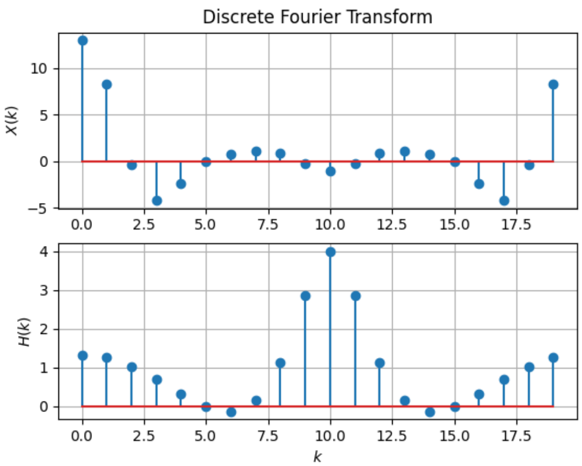

Fig. 6.1. Plots of the real parts of the discrete Fourier transforms of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Download the following Python code that plots Fig. 6.2.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/6.2.py
```

Run the code by executing

```
python3 6.2.py
```

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{\jmath 2\pi kn/N} \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: Download the following Python code that plots Fig. 6.3.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/6.3.py
```

Run the code by executing

```
python3 6.3.py
```

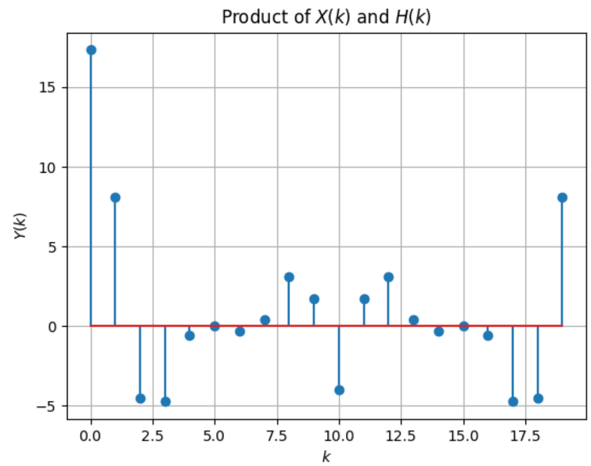

Fig. 6.2. Plot of $Y(k)$

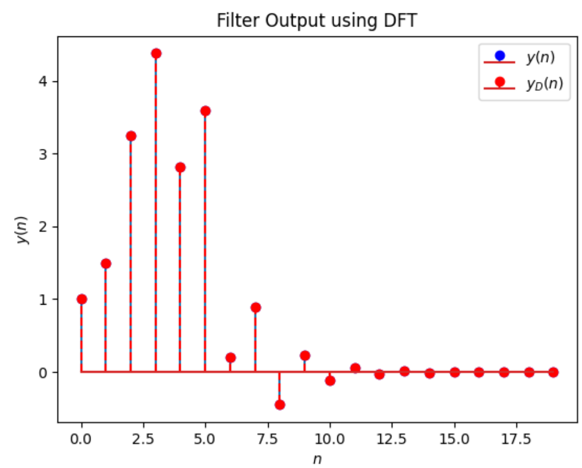

Fig. 6.3. Plot of the inverse discrete Fourier transform of $Y(k)$

The plot is exactly the same as that obtained in Fig. 3.2. Therefore, we conclude that

$$y(n) = x(n) * h(n) \tag{6.4}$$

$$\iff Y(k) = X(k)H(k) \tag{6.5}$$

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the following Python code that plots Fig. 6.4.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/6.4.py
```

Run the code by executing

```
python3 6.4.py
```

The plot is exactly the same as that obtained in Fig. 3.2

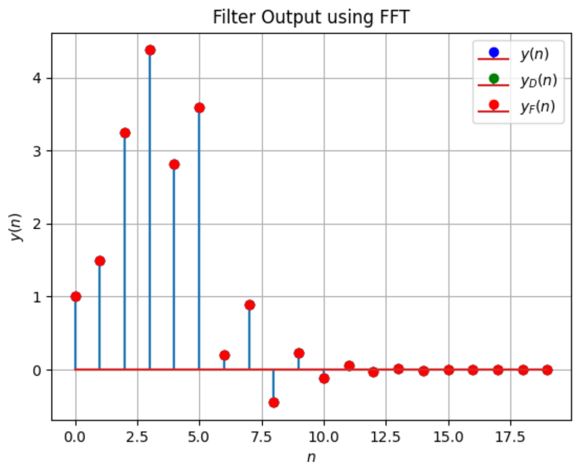

Fig. 6.4. Plot of $y(n)$ by fast Fourier transform

7. FFT

7.1 The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$

7.2 Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

7.3 Let

$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$

be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$

7.4 The 4 point *DFT diagonal matrix* is defined as

$$\mathbf{D}_4 = \text{diag}\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$

7.5 Show that

$$W_N^2 = W_{N/2} \tag{7.7}$$

Solution:

$$W_N^2 = \left(\exp\left(-j\frac{2\pi}{N}\right)\right)^2 \tag{7.8}$$
$$= \exp\left(-j\frac{2\pi}{N}\cdot 2\right) \tag{7.9}$$
$$= \exp\left(-j\frac{2\pi}{N/2}\right) \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$

7.6 Show that

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.12}$$

Solution:

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \tag{7.13}$$
$$= \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.14}$$
$$= \begin{bmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & -\jmath \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} & -\begin{pmatrix} 1 & 0 \\ 0 & -\jmath \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{bmatrix} \tag{7.15}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -\jmath & \jmath \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \jmath & -\jmath \end{bmatrix} \tag{7.16}$$

because $W_2^0 = 1$ and $W_2^1 = e^{-\jmath\pi} = -1$
Now

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.17}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -\jmath & \jmath \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \jmath & -\jmath \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.18}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\jmath & -1 & \jmath \\ 1 & -1 & 1 & -1 \\ 1 & \jmath & -1 & -\jmath \end{bmatrix} \tag{7.19}$$
$$= \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \tag{7.20}$$
$$= \mathbf{F}_4 \tag{7.21}$$

because

$$W_4^0 = 1 \tag{7.22}$$
$$W_4^1 = e^{-\jmath\frac{\pi}{2}} = -\jmath \tag{7.23}$$
$$W_4^2 = e^{-\jmath\pi} = -1 \tag{7.24}$$
$$W_4^3 = e^{-\jmath\frac{3\pi}{2}} = \jmath \tag{7.25}$$
$$W_4^n = W_4^{n-4} \qquad \forall n \geq 4 \tag{7.26}$$

7.7 Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.27}$$

Solution:

$$\begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \tag{7.28}$$
$$= \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \tag{7.29}$$

Now

$$\mathbf{D}_{N/2}\mathbf{F}_{N/2} \tag{7.30}$$
$$= \begin{bmatrix} W_N^0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_N^{N/2-1} \end{bmatrix} \begin{bmatrix} W_{N/2}^0 & \cdots & W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_{N/2}^0 & \cdots & W_{N/2}^{(N/2-1)^2} \end{bmatrix} \tag{7.31}$$
$$= \begin{bmatrix} W_N^0 W_{N/2}^0 & \cdots & W_N^0 W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_N^{N/2-1} W_{N/2}^0 & \cdots & W_N^{N/2-1} W_{N/2}^{(N/2-1)^2} \end{bmatrix} \tag{7.32}$$

Thus

$$(\mathbf{D}_{N/2}\mathbf{F}_{N/2})_{ij} = W_N^i W_{N/2}^{ij} \tag{7.33}$$
$$= W_N^i W_N^{2ij} \tag{7.34}$$
$$= W_N^{i(2j+1)} \tag{7.35}$$

where $i, j = 0, \ldots, N/2 - 1$
Therefore, $\mathbf{D}_{N/2}\mathbf{F}_{N/2}$ forms the first $N/2$ rows of the odd-indexed columns of $\mathbf{F}_N$

$$W_N^{(i+N/2)(2j+1)} = \exp\left(-\jmath\frac{2\pi}{N}(2j+1)\left(i + \frac{N}{2}\right)\right) \tag{7.36}$$
$$= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j+1)i + (2j+1)\pi\right)\right) \tag{7.37}$$
$$= -\exp\left(-\jmath\frac{2\pi}{N}(2j+1)i\right) \tag{7.38}$$
$$= -W_N^{i(2j+1)} \tag{7.39}$$

Thus, the remaining $N/2$ rows will be the negatives of the first $N/2$ rows

$$(\mathbf{F}_{N/2})_{ij} = W_{N/2}^{ij} \tag{7.40}$$
$$= W_N^{i(2j)} \tag{7.41}$$

where $i, j = 0, \ldots, N/2 - 1$
Therefore, $\mathbf{F}_{N/2}$ forms the first $N/2$ rows of the even-indexed columns of $\mathbf{F}_N$

$$W_N^{(i+N/2)(2j)} = \exp\left(-\jmath\frac{2\pi}{N}(2j)\left(i + \frac{N}{2}\right)\right) \tag{7.42}$$
$$= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j)i + (2j)\pi\right)\right) \tag{7.43}$$
$$= \exp\left(-\jmath\frac{2\pi}{N}(2j)i\right) \tag{7.44}$$
$$= W_N^{i(2j)} \tag{7.45}$$

Thus, the remaining $N/2$ rows will be the same as the first $N/2$ rows
Therefore

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{F}_{N/2} & \mathbf{D}_{N/2}\mathbf{F}_{N/2} \\ \mathbf{F}_{N/2} & -\mathbf{D}_{N/2}\mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.46}$$

where

$$\mathbf{P}_N = \begin{pmatrix} \mathbf{e}_N^1 & \mathbf{e}_N^3 & \cdots & \mathbf{e}_N^{N-1} & \mathbf{e}_N^2 & \mathbf{e}_N^4 & \cdots & \mathbf{e}_N^N \end{pmatrix} \tag{7.47}$$

Hence

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.48}$$

for even N

7.8 Find

$$\mathbf{P}_4\mathbf{x} \tag{7.49}$$

Solution: Let $\mathbf{x} = \begin{pmatrix} x(0) & x(1) & x(2) & x(3) \end{pmatrix}^\top$

$$\mathbf{P}_4\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} \tag{7.50}$$
$$= \begin{bmatrix} x(0) \\ x(2) \\ x(1) \\ x(3) \end{bmatrix} \tag{7.51}$$

7.9 Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.52}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution:

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.53}$$

$$\implies \mathbf{X} = \begin{bmatrix} \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(0)/N} \\ \vdots \\ \sum_{n=0}^{N-1} x(n)e^{-j2\pi n(N-1)/N} \end{bmatrix} \tag{7.54}$$

$$= \begin{bmatrix} x(0) + \cdots + x(N-1) \\ \vdots \\ x(0) + \cdots + x(N-1)e^{-j2\pi(N-1)^2/N} \end{bmatrix} \tag{7.55}$$

$$\mathbf{X} = x(0)\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + \cdots + x(N-1)\begin{bmatrix} 1 \\ \vdots \\ e^{-j2\pi(N-1)^2/N} \end{bmatrix} \tag{7.56}$$

$$= \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & e^{-j2\pi(N-1)^2/N} \end{bmatrix} \begin{bmatrix} x(0) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.57}$$

$$= \mathbf{F}_N \mathbf{x} \tag{7.58}$$

7.10 Derive the following step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.60}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.62}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.63}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.64}$$

$$\mathbf{P}_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.65}$$

$$\mathbf{P}_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.66}$$

$$\mathbf{P}_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.67}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.68}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.69}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.70}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.71}$$

Solution:

$$X(k) = \sum_{n=0}^{7} x(n)e^{-j2\pi kn/8}, \quad k = 0, \ldots, 7 \tag{7.72}$$

$$= \sum_{n=0}^{7} x(n)W_8^{kn} \tag{7.73}$$

$$= \sum_{n \text{ is even}} x(n)W_8^{kn} + \sum_{n \text{ is odd}} x(n)W_8^{kn} \tag{7.74}$$

$$= \sum_{m=0}^{3} x(2m)W_8^{2km} + \sum_{m=0}^{3} x(2m+1)W_8^{2km+k} \tag{7.75}$$

Now substitute $W_8^2 = W_4$

$$X(k) = \sum_{m=0}^{3} x(2m)W_4^{km} + W_8^k \sum_{m=0}^{3} x(2m+1)W_4^{km} \tag{7.76}$$

Consider

$$x_1(n) = \{x(0), x(2), x(4), x(6)\} \tag{7.77}$$
$$x_2(n) = \{x(1), x(3), x(5), x(7)\} \tag{7.78}$$

Thus

$$X(k) = X_1(k) + W_8^k X_2(k) \qquad k = 0, \dots, 7 \tag{7.79}$$

Now, $X_1(k)$ and $X_2(k)$ are 4-point DFTs which means they are periodic with period 4

$$X(k+4) = X_1(k+4) + W_8^{k+4} X_2(k+4) \tag{7.80}$$
$$= X_1(k) + e^{-\jmath 2\pi(k+4)/8} X_2(k) \tag{7.81}$$
$$= X_1(k) + e^{-\jmath(2\pi k/8+\pi)} X_2(k) \tag{7.82}$$
$$= X_1(k) - e^{-\jmath 2\pi k/8} X_2(k) \tag{7.83}$$
$$= X_1(k) - W_8^k X_2(k) \tag{7.84}$$

Therefore, for $k = 0, 1, 2, 3$

$$X(k) = X_1(k) + W_8^k X_2(k) \tag{7.85}$$
$$X(k+4) = X_1(k) - W_8^k X_2(k) \tag{7.86}$$

which is the same as

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.87}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.88}$$

Similarly, we can divide $x_1(n)$ into

$$x_3(n) = \{x(0), x(4)\} \tag{7.89}$$
$$x_4(n) = \{x(2), x(6)\} \tag{7.90}$$

i.e.,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.91}$$
$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.92}$$

to get

$$X_1(k) = X_3(k) + W_4^k X_4(k) \tag{7.93}$$
$$X_1(k+2) = X_3(k) - W_4^k X_4(k) \tag{7.94}$$

for $k = 0, 1$

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.95}$$
$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.96}$$

And on dividing $x_2(n)$ into

$$x_5(n) = \{x(1), x(5)\} \tag{7.97}$$
$$x_6(n) = \{x(3), x(7)\} \tag{7.98}$$

i.e.,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.99}$$
$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = \mathbf{F}_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.100}$$

to get

$$X_2(k) = X_5(k) + W_4^k X_6(k) \tag{7.101}$$
$$X_2(k+2) = X_5(k) - W_4^k X_6(k) \tag{7.102}$$

for $k = 0, 1$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.103}$$
$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.104}$$

7.11 For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.105}$$

compte the DFT using (7.52)
Solution: Download the following Python code that plots Fig. 7.11.

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/7.11.py
```

Run the code by executing

```
python3 7.11.py
```

7.12 Repeat the above exercise using the FFT after zero padding **x**.
Solution: Download the following Python code that plots Fig. 7.12.

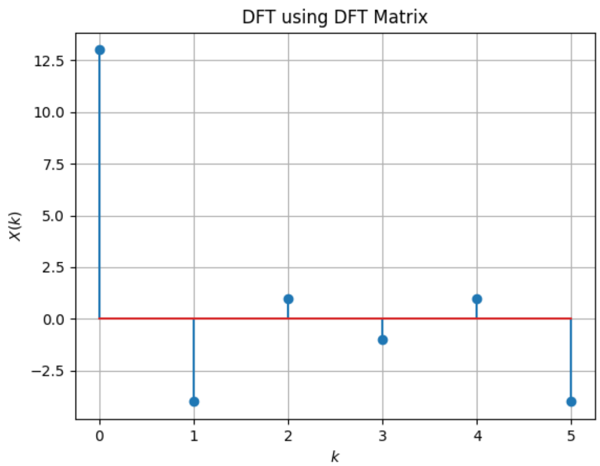

Fig. 7.11. Plot of the discrete fourier transform of **x** using the DFT matrix

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/7.12.py
```

Run the code by executing

```
python3 7.12.py
```

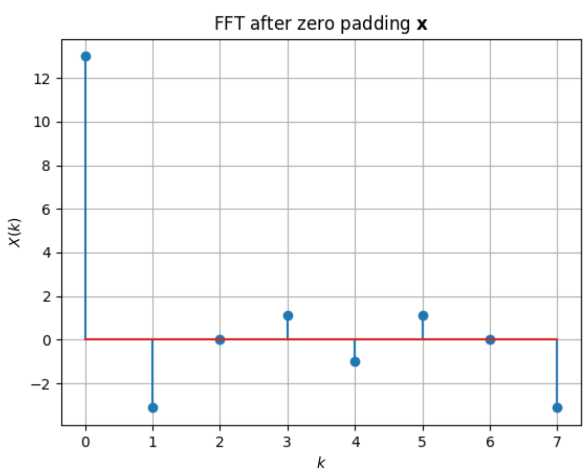

Fig. 7.12. Plot of the fast fourier transform of **x** after zero padding

7.13 Write a C program to compute the 8-point FFT.
Solution: Download the following C codes that generate the values of $X(k)$ using 8-point FFT

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/header.h
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/7.13.c
```

Compile and run the C program by executing the following

```
cc -lm 7.13.c
./a.out
```

Download the following Python code that plots Fig. 7.13 using the data generated by the above C code

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/7.13.py
```

Run the code by executing

```
python3 7.13.py
```

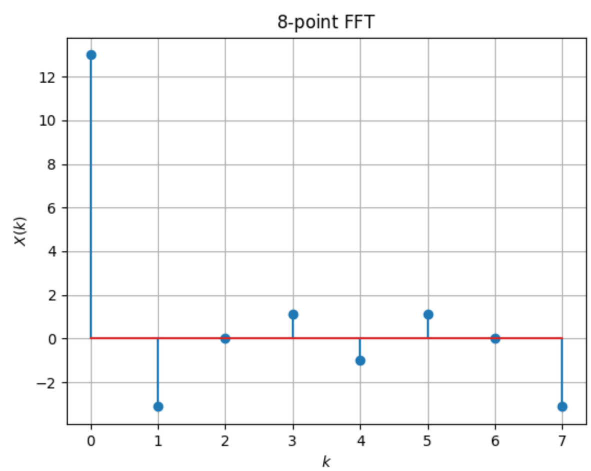

Fig. 7.13. Plot of **X** by 8-point FFT

7.14 Compare and determine the running time complexities of FFT/IFFT and convolution graphically
Solution: Download the following C codes that measure the running times of both the algorithms

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/header.h
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/7.14.c
```

Compile and run the C program by executing the following

```
cc -lm 7.14.c
./a.out
```

Download the following Python code that plots Fig. 7.14 using the running times generated by the C code and fits them to appropriate functions of the input size

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/7.14.py
```

Run the code by executing

```
python 7.14.py
```

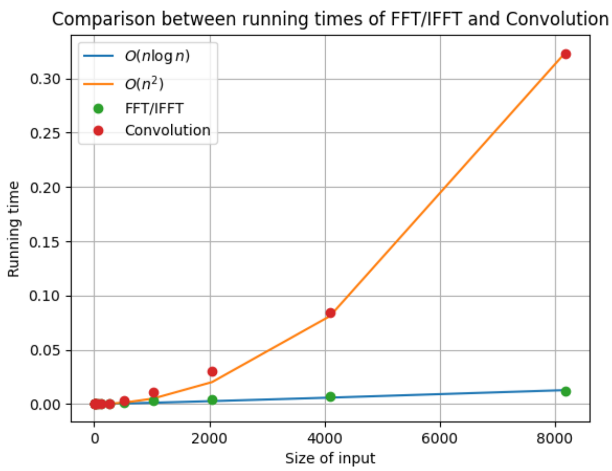

Fig. 7.14. Plot of the running times of FFT/IFFT and convolution

From the plot, it is evident that the time complexity of FFT/IFFT is $O(n \log n)$ and that of convolution is $O(n^2)$

8. Exercises

Answer the following questions by looking at the python code in Problem **??**

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem **??** is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m)\, z^{-m} Y(z) = \sum_{k=0}^{N} b(k)\, z^{-k} X(z) \tag{8.2}$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.3}$$

For obtaining the discrete Fourier transform, put $z = \jmath^{\frac{2\pi i}{I}}$ where I is the length of the input signal and $i = 0, 1, \dots, I-1$
Download the following Python code that does the above

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/8.1.py
```

Run the code by executing

```
python3 8.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \tag{8.4}$$

The difference equation is then given by

$$\mathbf{a}^\top \mathbf{y} = \mathbf{b}^\top \mathbf{x} \tag{8.5}$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \tag{8.6}$$

We have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k)\, z^{-k}}{\sum_{m=0}^{M} a(m)\, z^{-m}} \tag{8.7}$$

By using partial fraction decomposition, we can write this as

$$H(z) = \sum_i \frac{r(i)}{1 - p(i) z^{-1}} + \sum_j k(j) z^{-j} \tag{8.8}$$

On taking the inverse Z-transform on both sides by using (4.35)

$$H(z) \overset{Z}{\rightleftharpoons} h(n) \tag{8.9}$$

$$\frac{1}{1 - p(i)z^{-1}} \overset{Z}{\rightleftharpoons} (p(i))^n u(n) \tag{8.10}$$

$$z^{-j} \overset{Z}{\rightleftharpoons} \delta(n - j) \tag{8.11}$$

Thus

$$h(n) = \sum_i r(i)\,(p(i))^n\, u(n) + \sum_j k(j)\delta(n - j) \tag{8.12}$$

Download the following Python code

```
wget https://github.com/Dhatrireddyy/EE3900/
    blob/main/codes/8.2.py
```

Run the code by executing

```
python3 8.2.py
```

The above code outputs the values of $r(i), p(i), k(i)$

$$\begin{aligned} h(n) &= \Re\left((0.24 - 0.71\jmath)(0.56 + 0.14\jmath)^n\right) u(n) \\ &+ \Re\left((0.24 + 0.71\jmath)(0.56 - 0.14\jmath)^n\right) u(n) \\ &+ 0.016\delta(n) \end{aligned} \tag{8.13}$$

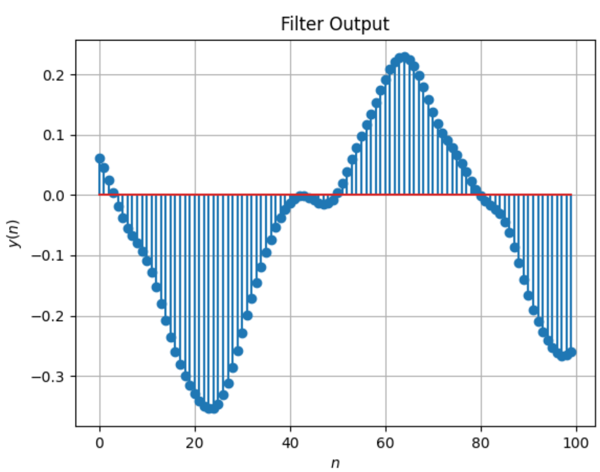

Fig. 8.2. Plot of $y(n)$

8.3 What is the sampling frequency of the input signal?
Solution: The sampling frequency of the input signal is 44 100 Hz = 44.1 kHz

8.4 What is the type, order and cutoff frequency of the above Butterworth filter?

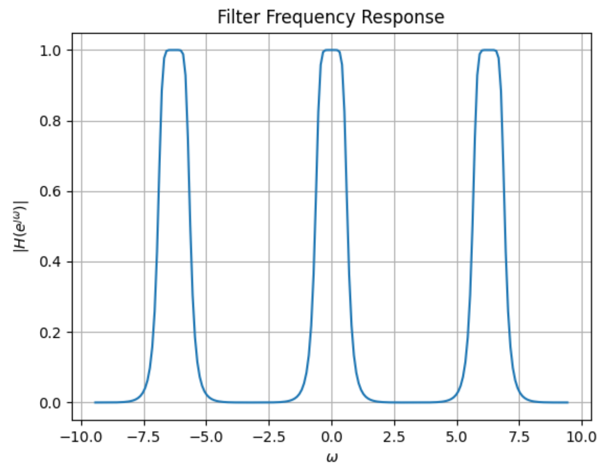

Fig. 8.2. Plot of $|H(e^{\jmath\omega})|$

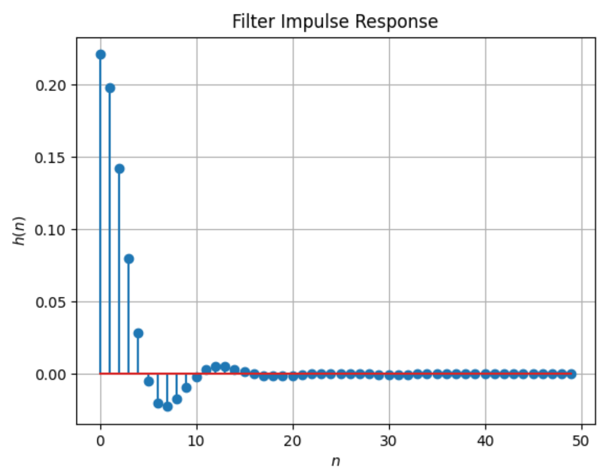

Fig. 8.2. Plot of $h(n)$

Solution:
Type: low-pass
Order: 4
Cutoff frequency: 4000 Hz = 4 kHz

8.5 Modify the code with different input parameters to get the best possible output.
Solution:
Order: 10
Cutoff frequency: 3000 Hz = 3 kHz